Shifting coordinate stars

Justus John Michael Seeck

28 December 2022

Table of Contents

1 Introduction

1.1 Foreword

In the following work only the shifting of coordinate systems in the plane (i.e. in two dimensions) is treated. However, the methodology can also be applied to the third dimension. Also the number of displacements is arbitrary, because the formula used here can be applied as often as desired. But here we will limit ourselves to one displacement. The displacement of coordinate systems is used for example in robotics. At the end of a robot arm, which is already in a coordinate system, there is usually another coordinate system, "tool system". In this system, there is no displacement of the individual points when the arm moves, but only when the tool or its load or tool moves.

1.2 Spellings and definitions

Points A point P was previously represented in the form P(x|y). This will no longer be used in the following work. Instead, the point P is represented in the form $\begin{pmatrix} x \\ y \end{pmatrix}$. This is a so-called matrix with two rows and one column. The first row contains the x-coordinate, the second row the y-coordinate. Since in the following we work with two different coordinate systems, the world system or original system U and the tool system T , points are indicated with respect to a coordinate system with index, if this is necessary. The point P in relation to the world system U is represented as follows: $[P]_U$. In relation to the tool system the representation is $[P]_T$. The representation can also be done without square brackets. Points then only have an index that indicates the coordinate system.

2 Vectors

Geometric vectors are arrows. $\vec{v}$ is a vector. Vectors are defined by their length in x-direction and y-direction (for three dimensions additionally in z-direction) and marked with an arrow above the letter. Vectors have no fixed position in the coordinate system. They can be moved arbitrarily. As long as the vector keeps all its properties (length in x-, y- and possibly z-direction), it is the same vector. Appendix: Vectors 1. Between two equal vectors a paralellogram can be drawn.

2.1 Calculating with vectors and points

Vectors can be modified by mathematical operations. They can be added, subtracted and multiplied by a scalar (a number). Appendix: Vectors 2. The following rules apply:
Totals

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

Differences

$$\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$$

(Scalar) product (explanation follows)

$$r - \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r - a \\ r - b \end{pmatrix}$$

From the graphic (Appendix: Vectors 2) it is also clear that the sum of two vectors (a juxtaposition of arrows) is the sum of the two individual vectors (coordinate vectors):

$$[\vec{w}+\vec{a}]=[\vec{w}]+[\vec{a}]$$

Points can be moved with the help of vectors. The point is added to the vector. This corresponds to the attachment of the vector to the point. The end of the vector then corresponds to the coordinate of the moved point. You can also form a parallelogram between the point and the vector. The missing point then gives the searched place of the new point. Appendix: Vectors 3.

2.2 Unit vectors in the coordinate system

A point P can be described using unit vectors in the coordinate system U. The coordinate system U has right-angled axes. Therefore, if the x-axis is rotated 90 degrees counterclockwise, the y-axis is created. There is a unit vector in x-direction and a unit vector in y-direction (with three dimensions additionally in z-direction). The unit vector in the x-direction is $\vec{x}_U = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the unit vector in the y-direction is $\vec{y}_U = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(Other names / letters are also possible for unit vectors). The length of the unit vector is 1 (one unit of length) in its respective direction. Now for example to describe the point $[P]_U = \begin{pmatrix} a \\ b \end{pmatrix}$ one needs the origin of the coordinate system 0_U, its coordinate in the x-direction (here: a), its coordinate in the y-direction (here: b) and the two unit vectors $\vec{x}_U$ and $\vec{y}_U$.

This results in the following formula:

$$[P]_U = [0]_U + a\cdot\vec{x}_U + b\cdot\vec{y}_U = \begin{pmatrix} a \\ b \end{pmatrix}$$

The point $[P]_U$ corresponds to the point $[0]_U$ to which one adds a displacement in x-direction (here: $a\cdot\vec{x}_U$) and y-direction (here: $b\cdot\vec{y}_U$). Appendix: Vectors 4.

If a vector is in a coordinate system, it can be represented by the difference of the starting and ending points of the vector. This results in the following:

$$\vec{v} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

Appendix: Vectors 5 shows the representation of a vector in a coordinate system.

3 The displacement of coordinate systems

Our goal is to transfer the known coordinates of an object, e.g. a point in the tool system T, into the coordinate system U. For this we need the displacement of the origin of the tool system $[0_T]_U$ starting from the origin of the coordinate system $[0]_U$ with respect to the origin system. This displacement can be described by a vector. This vector is called the displacement vector.

The position of $[0_T]_U$ is given.

3.1 Vector length and distance from points

To determine the displacement vector, the vector by which the point 0_U must be displaced to reach the point $[0_T]_U$, the distance between the origin of the tool system $[0_T]_U$ and the origin of the coordinate system U is required, i.e. $[0]_U$. This formula can also be used for other points.

The length of a vector is calculated by the Pythagorean formula. The length of $\vec{v}$ is represented as follows $\|\vec{v}\|$.

$$\|\vec{v}\| = \left\| \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \right\|$$

$$\|\vec{v}\| = \sqrt{\Delta x^2 + \Delta y^2}$$

The distance between points P and Q according to this formula is then: dist(P, Q) = $\|\vec{PQ}\|$.
is $[P]_U = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ and $[Q]_U = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, then it follows:

$$\begin{aligned} \text{dist}(P, Q) &= \|\vec{PQ}\| \\ &= \left\| \begin{pmatrix} v_x - u_x \\ v_y - u_y \end{pmatrix} \right\| \\ &= \sqrt{(v_x - u_x)^2 + (v_y - u_y)^2} \end{aligned}$$

3.2 The scalar product

"The scalar product is a mathematical kniipfication that assigns a number (scalar) to two vectors."
(https://www.mathebibel.de/skalarprodukt - Last call: 12/26/2022 @ 17:03:54) In this case, the scalar product is used to calculate the angle θ between two vectors. Appendix: Scalar Product 1 shows a sketch.
θ is the angle between the vectors $\vec{v}$ and $\vec{u}$. Both vectors have the same origin. The angle θ is undirected (see appendix: scalar product 1) and lies between 0 and 180 degrees.
The cosine theorem now states the following:

$$\|\vec{v} - \vec{u}\| = \|\vec{u}\|^2 + \|\vec{v}\| - 2 \cdot \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$$

From the cosine theorem follows the scalar product of $\vec{u}$ and $\vec{v}$:

$$\begin{aligned} \frac{1}{2} \cdot \left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2 \right) &= \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta \\ &= \langle \vec{u} | \vec{v} \rangle \end{aligned}$$

In coordinates, the scalar product can be represented as follows. Given are: $\vec{u}_s = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$, $\vec{v}_s = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

$$\begin{aligned} \left\langle \begin{pmatrix} u_x \\ u_y \end{pmatrix} \middle| \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right\rangle &\overset{\text{D=ef.}}{=} \frac{1}{2} \cdot \left(\left\| \begin{pmatrix} u_x \\ u_y \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} v_x - u_x \\ v_y - u_y \end{pmatrix} \right\|^2 \right) \\ &= \frac{1}{2} \cdot \left(u_x^2 + u_y^2 + v_x^2 + v_y^2 - [(v_x - u_x)^2 + (v_y - u_y)^2] \right) \end{aligned}$$

Applying the binomial formulas and then simplifying gives:

$$\begin{aligned} &= \frac{1}{2} \cdot \left(u_x^2 + u_y^2 + v_x^2 + v_y^2 - [u_x^2 + v_x^2 - 2u_x v_x + u_y^2 + v_y^2 - 2u_y v_y] \right) \\ &= \left\langle \begin{pmatrix} u_x \\ u_y \end{pmatrix} \middle| \begin{pmatrix} v_x \\ v_y \end{pmatrix} \right\rangle \\ &= u_x v_x + u_y v_y \end{aligned}$$

It should be noted that the scalar product can be formed only with vectors (arrows). It is not possible to form the scalar product with points.

Now, by rearranging the angles between two vectors given coordinates ($\vec{u}$ and $\vec{v}$):
From the above equations:

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$$
$$= \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$
$$= u_x v_x + u_y v_y$$

where $\|\vec{u}\| = \sqrt{u_x^2 + u_y^2}$ and $\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$, that θ between 0 and 180 degrees yields the following:

$$\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \cdot \|\vec{v}\|}$$
$$\theta = \cos^{-1}\left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

3.3 Transform coordinates

A coodinate system T as described in point 2.2 is placed on another coordinate system U. In the tool system T a point P is described by coordinates. The goal is to mathematically determine the coordinates of the point P in the tool system U.
The scalar product of the two coordinate vectors results in 0, since the two vectors are perpendicular to each other.
As already described in point 2.2, points can be written with the help of unit vectors. Appendix: Coordinate Transformation 1
Formula:

$$[P]_U = [O]_U + a \cdot \vec{x}_U + b \cdot \vec{y}_U = \begin{pmatrix} a \\ b \end{pmatrix}$$

If the displacement vector (i.e., the vector from O_U to O_T) is known, the formula can be rewritten as follows:

$$\vec{u} = \Delta x \cdot \vec{x}_U + \Delta y \cdot \vec{y}_U = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$[P]_T$ is represented by $\begin{pmatrix} x_T \\ y_T \end{pmatrix}$. P is described in U as follows:

$$[P]_U = \left[O_T + x_T \cdot \vec{a}_T + y_T \cdot \vec{b}_T \right]_U$$
$$= [O_T]_U + x_T \cdot [\vec{a}_T]_U + y_T \cdot [\vec{b}_T]_U$$

To determine the coordinates of the point $[P]_U$, it is sufficient to determine $[O_T]_U$, $[\vec{a}_T]_U$ and $[\vec{b}_T]_U$, as can be seen from the equation above.

3.4 Application

Given are $[P]_T = \begin{pmatrix} x_T \\ y_T \end{pmatrix}$, $[O_T]_U = \begin{pmatrix} x_{OTU} \\ y_{OTU} \end{pmatrix}$ and the angle of rotation α.
The unit vectors of the tool system T can be calculated with the help of sine and cosine.
Calculate:

$$[\vec{a}_T]_U = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$[\vec{b}_T]_U = \begin{pmatrix} \cos(\alpha + 90^\circ) \\ \sin(\alpha + 90^\circ) \end{pmatrix}$$

By the scalar product $\vec{a}_T \cdot \vec{b}_T$ the obtained result can be checked for correctness. The result of the scalar product must be 0, since both vectors are perpendicular to each other despite the rotation. The length of the individual vectors must also be 1.

$$\begin{aligned}
[P]_U &= [0_T]_U + x_T - [\vec{a}_T]_U + y_T - [\vec{b}_T]_U \\
&= \begin{pmatrix} x_{OTU} \\ y_{OTU} \end{pmatrix} + x_T - \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + y_T - \begin{pmatrix} \cos(\alpha + 90^\circ) \\ \sin(\alpha + 90^\circ) \end{pmatrix} \\
&= \begin{pmatrix} x_{OTU} + x_T - \cos\alpha + y_T - \cos(\alpha + 90^\circ) \\ y_{OTU} + x_T - \sin\alpha + y_T - \sin(\alpha + 90^\circ) \end{pmatrix}
\end{aligned}$$

This equation already gives the correct result. However, there is a way to simplify the equation. For this purpose, the so-called matrix calculation is used.

3.5 Matrices

A matrix is a rectangular number scheme. $M = \begin{pmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}$ is a matrix with two rows and two columns.

The above equation can be simplified by matzah as follows.

Since $\cos(\alpha + 90^\circ) = -\sin\alpha$ and $\sin(\alpha + 90^\circ) = \cos\alpha$ holds, the matrix can be defined as follows:

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

x_T and y_T are then multiplied by the matrix:

$$[P]_T - M = \begin{pmatrix} x_T \\ y_T \end{pmatrix} - \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

For the multiplication of 2 - 2 matrices, the number of columns of the first matrix must be equal to the number of rows of the second matrix. The number of rows of the first matrix is equal to the number of rows of the result matrix and the number of columns of the second matrix is equal to the number of columns of the result matrix.

The first row of the first matrix is multiplied by the first column of the second matrix. The second row of the first matrix is multiplied by the second column of the second matrix.

This results in the following equation:

$$\begin{aligned}
[P]_T - M &= \begin{pmatrix} x_T \\ y_T \end{pmatrix} - \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \\
&= x_T - \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + y_T - \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}
\end{aligned}$$

This part of the equation is now only extended by the position of the tool system T in the original system
U added.

The final equation is:

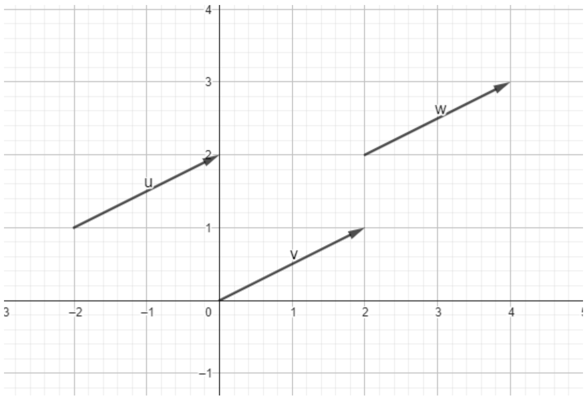
$$[P]_U = [0_T]_U + \begin{pmatrix} x_T \\ y_T \end{pmatrix} - \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

4 Closing words

The displacement of a point P in the tool system T into the original system U with rotated and displaced tool system was shown. It is also possible to carry out these displacements with several systems. For example a shift from the tool system T into an intermediate system Z and from there into the original system U. This possibility can be used to calculate the position of the tool system for a robot arm with several joints. For this purpose, the formula is optimized and intermediate systems are included at the joints, so that only one rotation or one displacement from one system to the other has to be performed. This approach can be found among others in the kinematics of robots and is called Denavit-Hartenberg notation.

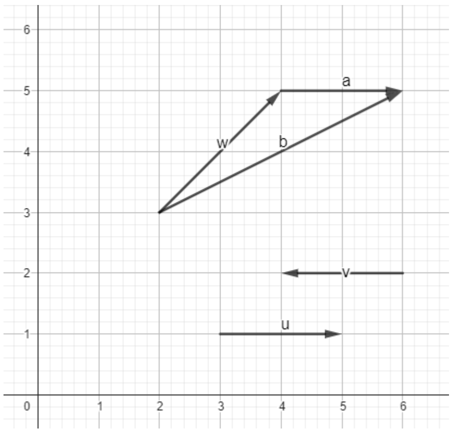
5 Appendix

5.1 Vectors 1



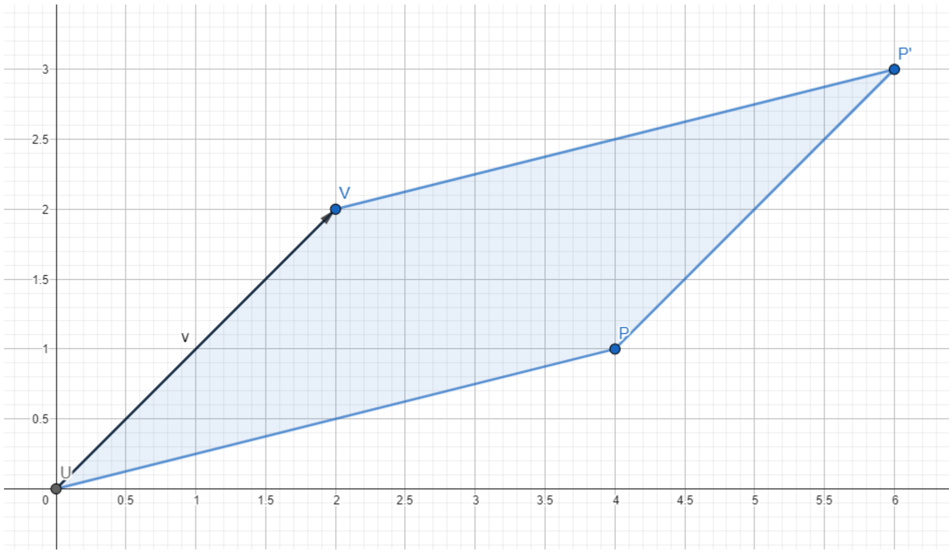
Coordinate system with the vector $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in three different positions.

5.2 Vectors 2



Multiplying the vector $\vec{u}$ by -1 gives the vector $\vec{v}$. Adding the vectors $\vec{w}$ and $\vec{a}$ results in the vector $\vec{b}$.

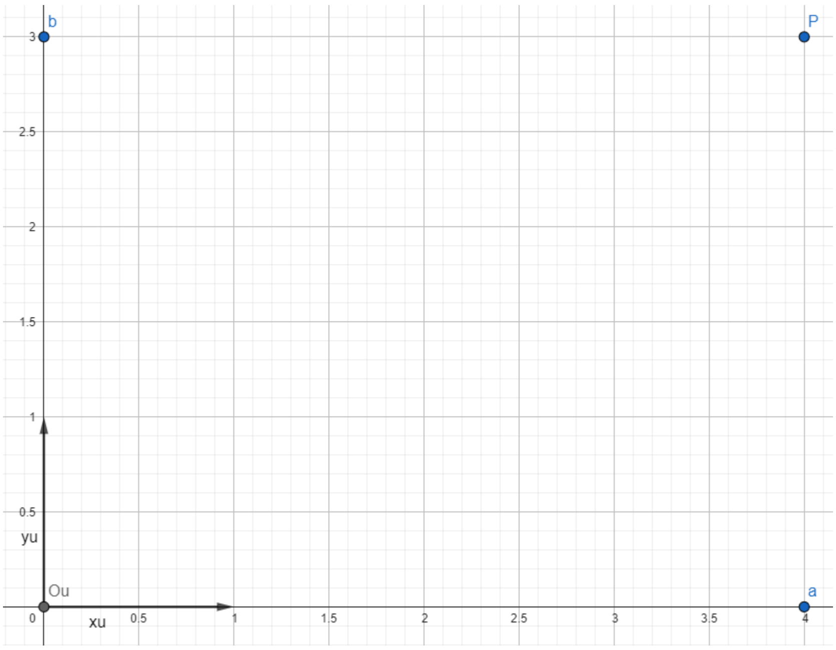
5.3 Vectors 3



The point P is moved with the vector $\vec{UV}$. The paralellogram results in the new point P′. The vector $\vec{UV}$ is the vector between the two points U and V and is described as follows:

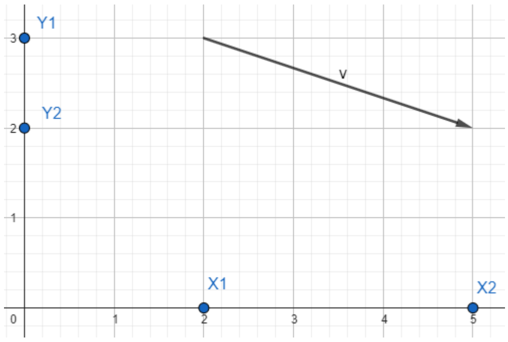
$$\vec{v} = \vec{UV} = V - U$$

5.4 Vectors 4



Unit vectors xu and yu in the coordinate system U with the point P and its descriptive sections a and b.

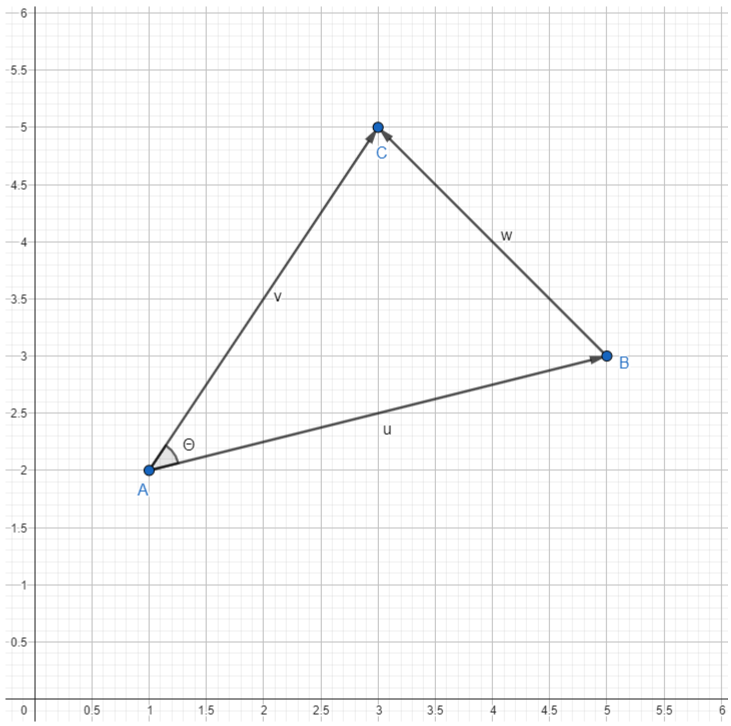
5.5 Vectors 5



The vector $\vec{v}$ is represented by the difference of the start and end points of the vector.

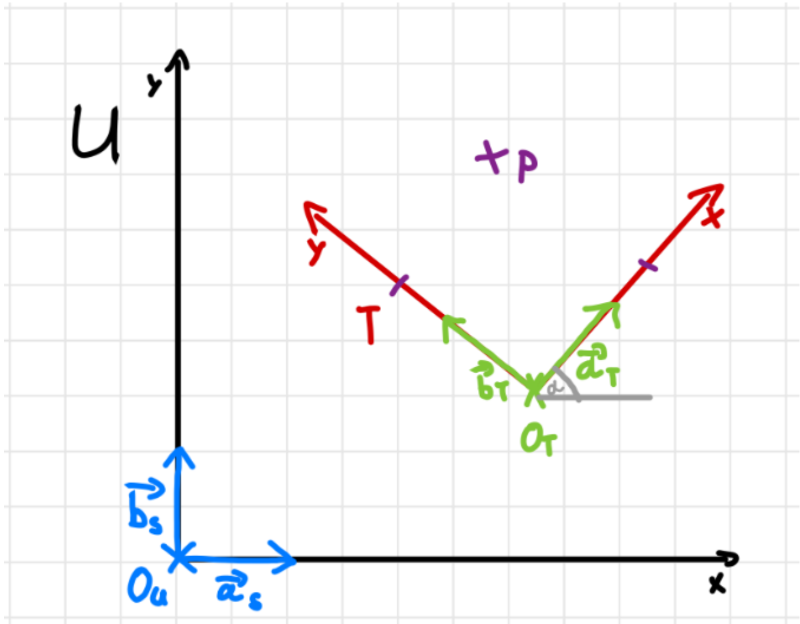
$$\vec{v} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} X2 - X1 \\ Y2 - Y1 \end{pmatrix} = \begin{pmatrix} 5 - 2 \\ 2 - 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

5.6 Scalar product 1



$\vec{w}$ is the difference vector of $\vec{v}$ and $\vec{u}$. It has the length $\|\vec{v} - \vec{u}\|$. The angle between $\vec{v}$ and $\vec{u}$ is θ.

5.7 Coordinate transformation 1



6 Sources

- own records
- https://mathe-lok.de/schuelervorlesung-kinematik-von-roboterarmen (Videos and slides from week 1 and 2) (Last call: 12/28/2022 @ 18:22:27)
- https://www.mathebibel.de/skalarprodukt (Last call: 12/26/2022 @ 17:23:01)

7 Information

The latest version of this document is available at: https://github.com/AstragoDEEdu/homework_robotics_12-2022/blob/main/homework.pdf. Source documents for appendices, as well as images and the source text of the document in LATEX are available at the following address: https://github.com/AstragoDEEdu/hausarbeit_robotik_12-2022.

8 License