Solution Manual to Real and Complex Analysis

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Chapter 1

Abstract Integration

Exercise 1.1. Does there exist an infinte σ-algebra which has only countably many members?

Solution. If $\mathcal{F}$ is a σ-algebra and A a set, define $\mathcal{F} \cap A := \{A \cap F : F \in \mathcal{F}\}$. Hence if $A \in \mathcal{F}$, then $\mathcal{F} \cap A$ is a σ-subalgebra of $\mathcal{F}$. Further, if $\mathcal{F}$ is infinte, either $A \cap \mathcal{F}$ or $A^c \cap \mathcal{F}$ is infinte.

In other words, if $\mathcal{F}_n$ is infinte, then there is $A_{n+1} \in \mathcal{F}_n$ such that $\mathcal{F}_{n+1} = \mathcal{F}_n \cap A_n^c$ is infinte. Take $\mathcal{F}_0 = \mathcal{M}$, by induction we get a disjoint sequence of sets $A_n \in \mathcal{M}$. Since $\mathcal{M}$ must also contain any union of sets of A_n, and each union is different, this is to say $\mathcal{M}$ contains a embedding of all subsets of $\mathbb{N}$, therefore uncountable.

Exercise 1.2. Prove an analogue of Theorem 1.8 for n functions.

Solution. It suffices to prove

$$f = f_1 \times f_2 \times ... \times f_n$$

is measurable is each f_i is measurable. Similarly as in the theorem, take R be any rectangles in $\mathbb{R}^n$. Notice

$$f^{-1}(R) = \bigcap f_i(I_i)$$

is measurable, if $R = I_1 \times I_2 \times ... \times I_n$. The rest of the proof is a repeat of Theorem 1.8.

Exercise 1.3. Prove that if f is a real function on a measurable space X such that $\{x : f(x) < r\}$ is measurable for every rational r, then f is measurable.

Solution. Take Ω as all $E \subset \mathbb{R}$ such that $f^{-1}(E)$ is measurable. By Theorem 1.2, Ω is a σ-algebra. Notice

$$\{f > a\} = \bigcup_{q \in \mathbb{Q} \cap (-\infty, a)} \{f > q\}$$

for all $a \in \mathbb{R}$. Therefore, $\{f > a\}$ is measurable. By Theorem 1.2 again, f is measurable.

Chapter 2

Positive Borel Measure

Exercise 2.1. Let $\{f_n\}$ be a sequence of real non-negative functions on $\mathbb{R}^1$, and consider the following four statements:

1. If f_1, f_2 are upper semicontinuous so is $f_1 + f_2$.
2. If f_1, f_2 are lower semicontinuous, so is $f_1 + f_2$.
3. If each f_n is upper semicontinuous, so is $\sum^\infty f_n$.
4. If each f_n is lower semicontinuous, so is $\sum^\infty f_n$.

Solution. Observe

$$\{f_1 + f_2 < a\} = \bigcup_{x\in\mathbb{R}} \{f_1 < x\} \cap \{f_2 < a - x\}$$

is open. To see the left is included in the right, for any $f_1(y) + f_2(y) < a$, take $f_1(y) < x < a - f_2(y)$ and the inclusion holds. Therefore 1 is verified. Similar argument goes with 2 if the above $<$ are replaced with $>$.

Notice 4 holds, fix any x such that $\sum f(x) > a$. Since f_n's are non-negative, there is N such that $\sum^N f_n(x) > a$. Therefore there exists δ, $\sum^N f_n(y) > a$ for any $y \in B_\delta(x)$ since finite sums of lower semicontinuous functions are lower semicontinuous. The proof is complete by observing

$$\sum f_n(y) \geq \sum^N f_n(y) > a$$

To give 3 a counterexample, consider $\sum f_n = \sum \mathcal{X}_{[-n,-1/n]\cup[1/n,n]}$. Obviously, every point but 0 is greater than or equal to 1. Hence

$$\{\sum f_n < 1\} = \{0\}$$

is closed.

Exercise 2.2. Let f be an arbitrary complex function on $\mathbb{R}^1$, and define

$$\phi(x, \delta) = \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\},$$
$$\phi(x) = \inf\{\phi(x, \delta) : \delta > 0\}.$$

Prove that ϕ is upper semicontinuous, that f is continuous at a point x iff $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ.

Formulate and prove an analogous statement for general topological spaces in place of $\mathbb{R}^1$.

Solution. Only give solution in the general case. Redefine

$$\phi(x) = \inf_{B \ni x} \text{diam} f(B)$$

where the diameter is defined as $\text{diam} A = \sup_{x,y \in A} |x - y|$. Take any $x \in \{\phi(x) < a\}$, there is $B \ni x$, $\text{diam} f(B) < a$. Take any $y \in B$, then $\phi(y) \leq \text{diam} f(B) < a$. This says $\{\phi(x) < a\}$ is open and ϕ is upper semicontinuous.

The relation between ϕ and continuity of f is trivial. Since

$$\{\phi = 0\} = \bigcap_{q \in \mathbb{Q}^+} \{\phi < q\}$$

the set is a G_δ.

Exercise 2.3. Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf_{y \in E} \rho(x, y)$$

Show that ρ_E is uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Solution. Notice

$$\rho(a, x) + \rho(a, b) \geq \rho(x, b).$$

Taking infimum on E on both sides gives

$$\rho_E(b) - \rho_E(a) \leq \rho(a, b)$$

By symmetry,

$$|\rho_E(b) - \rho_E(a)| \leq \rho(a, b)$$

showing the uniform continuity. Notice $0 \leq f \leq 1$ and $f = 1$ on B. Its support lies in A^c. Therefore if B is compact, $B \prec f \prec A^c$.

Exercise 2.4. Examine the proof of Riesz theorem and prove the following two statments:

1. If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1, E_2 are not in $\mathcal{M}$.

2. If $E \in \mathcal{M}_F$, then $E = N \cup K_1 \cup K_2 \ldots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

Solution.

1. Take any open set U that covers $E_1 \cup E_2$. Observe
$$\mu(U) \geq \mu(U \cap V_1) + \mu(U \cap V_2) \geq \mu(E_1) + \mu(E_2).$$
Taking infimum on both sides together with the subadditivity of μ gives the result.

2. Since $E \in \mathcal{M}_F, \mu(E) < \infty$. Take K_1
$$\mu(E) < \mu(K_1) + 1.$$
Having chosen $K_1, ..., K_n$, denote $G = E - \cup^{n-1} K_i \in \mathcal{M}_F$. Pick K_n such that
$$\mu(G) < \mu(K_n) + 1/n.$$
Obviously $\mu(E - \bigcup K_n) = 0$ and this completes the proof.

In Exercise 5 to 8, m stands for Lebesgue measure on $\mathbb{R}^1$.

Exercise 2.5. Let E be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even through E and $\mathbb{R}^1$ have the same cardinality.

Solution. Denote R as the set removed from $[0, 1]$ in construction of the Cantor set. Notice it is comprised of the union of 2^{n-1} open intervals of length 3^{-n} where n ranges in $\mathbb{N}$. Therefore
$$\mu(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$
and $\mu(E) = \mu(I - R) = 0$.

To see E has the same cardinality as $\mathbb{R}$. Notice each element in E is a decimal of base 3 that has exactly one 1 at the end or no 1 at all. Therefore there is a surjection from $\mathbb{R}$ to E, and one from E to decimals of base 2. This completes the proof.

Exercise 2.6. Construct a totally disconnected compact set $K \subset \mathbb{R}^1$ such that $m(K) > 0$. If v is lower semicontinuous and $v \leq \mathcal{X}_K$, show that actually $v \leq 0$. Hence $\mathcal{X}_K$ cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali-Carathéodory theorem.

Solution. Construct K similarly as the Cantor set in the previous exercise only we remove the middle fourths in place of thirds. Let R be the union of removed intervals. K is compact since each removal left a closed and bounded thus compact subset of $[0, 1]$ and K is the intersection of all these compact sets. Similarly as above,
$$\mu(R) = \sum_{n-1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \sum \frac{1}{2^{n+1}} = \frac{1}{2}.$$
Therefore $\mu(K) = 1/2$. Notice $\{v > 0\}$ is open by definition. But it cannot be a subset of K except for the empty set since K is totally disconnected.

Exercise 2.7. If $0 < \epsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $m(E) = \epsilon$. (To say that A is dense in B means that the closure of A contains B.)

Solution. The construction is similar as before. Notice if one takes out one middle x-th every time with $x > 2$, the removed set R has measure

$$m(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{x^n} = \frac{1}{x-2}$$

Take $x = 1/\epsilon + 2$ and $m(R) = \epsilon$. R is open since its complement in $[0,1]$ is compact (as proven before), therefore closed. To see R is dense, notice every removal divides each remaining interval into its halves. Therefore to each point x in $[0,1]$, there must be a point that has been removed after the n-th removal and lies in $B_{1/2^n}(x)$.

Exercise 2.8. Construct a Borel set $E \subset \mathbb{R}^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment I. Is it possible to have $m(E) < \infty$ for such a set?

Solution. Take $E = \mathbb{Q}$. Obviously $0 = m(E \cap I)$. Since I is an nonempty segment, there is an open interval in it, i.e., $m(I) > 0$. Notice $\mathbb{Q}$ is Borel because it is the countable union of singaltons, and singaltons are Borel.

Exercise 2.9. Construct a sequence of continuous function f_n on $[0,1]$ such that $0 \leq f_n \leq 1$, such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

Solution. Define

$$\begin{aligned}
&K_1 = [0,1], \\
&K_2 = [0,1/2], K_3 = [1/2,1], \\
&K_4 = [0,1/4], K_5 = [1/4,1/2], K_6 = [1/2,3/4], K_7 = [3/4,1]
\end{aligned}$$

Obviously $m(K_n) \to 0$. To each $K_n = [a,b]$, pick $V_n = (a - 1/2n, b + 1/2n)$ and f_n that $K_n \prec f_n \prec V_n$. Finally,

$$\int f_n dm \leq m(V_n) = m(K_n) + 1/n \to 0$$

But f_n does not converge for any $x \in [0,1]$, since $f_n(x) = 1$ and $f_n(x) = 0$ both infinitely often.

Notice the construction of f_n need not follow that of Urysohn's. Define

$$l((x_1,y_1),(x_2,y_2))(x) = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) + y_1.$$

If $V_n = (a,b)$, $K_n = [c,d]$, and $a < c < d < b$. Pick s,t that, $a < s < c < d < t < b$. Take

$$f_n = \begin{cases} 0, x \leq s \\ l((s,0),(c,1)), s \leq x \leq c \\ 1, c \leq x \leq d \\ l((d,1),(t,0)), d \leq x \leq t \\ 0, x \geq t \end{cases}$$