# Solution Manual to Real and Complex Analysis

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# Chapter 1

# Abstract Integration

**Exercise 1.1.** Does there exist an infinte $\sigma$-algebra which has only countably many members?

**Solution.** If $\mathcal{F}$ is a $\sigma$-algebra and $A$ a set, define $\mathcal{F} \cap A := \{A \cap F : F \in \mathcal{F}\}$. Hence if $A \in \mathcal{F}$, then $\mathcal{F} \cap A$ is a $\sigma$-subalgebra of $\mathcal{F}$. Further, if $\mathcal{F}$ is infinte, either $A \cap \mathcal{F}$ or $A^c \cap \mathcal{F}$ is infinte.

In other words, if $\mathcal{F}_n$ is infinte, then there is $A_{n+1} \in \mathcal{F}_n$ such that $\mathcal{F}_{n+1} = \mathcal{F}_n \cap A_n^c$ is infinte. Take $\mathcal{F}_0 = \mathcal{M}$, by induction we get a disjoint sequence of sets $A_n \in \mathcal{M}$. Since $\mathcal{M}$ must also contain any union of sets of $A_n$, and each union is different, this is to say $\mathcal{M}$ contains a embedding of all subsets of $\mathbb{N}$, therefore uncountable.

**Exercise 1.2.** Prove an analogue of Theorem 1.8 for $n$ functions.

**Solution.** It suffices to prove

$$f = f_1 \times f_2 \times ... \times f_n$$

is measurable is each $f_i$ is measurable. Similarly as in the theorem, take $R$ be any rectangles in $\mathbb{R}^n$. Notice

$$f^{-1}(R) = \bigcap f_i(I_i)$$

is measurable, if $R = I_1 \times I_2 \times ... \times I_n$. The rest of the proof is a repeat of Theorem 1.8.

**Exercise 1.3.** Prove that if $f$ is a real function on a measurable space $X$ such that $\{x : f(x) < r\}$ is measurable for every rational $r$, then $f$ is measurable.

**Solution.** Take $\Omega$ as all $E \subset \mathbb{R}$ such that $f^{-1}(E)$ is measurable. By Theorem 1.2, $\Omega$ is a $\sigma$-algebra. Notice

$$\{f > a\} = \bigcup_{q \in \mathbb{Q} \cap (-\infty, a)} \{f > q\}$$

for all $a \in \mathbb{R}$. Therefore, $\{f > a\}$ is measurable. By Theorem 1.2 again, $f$ is measurable.

**Exercise 1.4.** Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

1. $$\limsup -a_n = -\liminf a_n.$$

2. $$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

   provided none of the sums is of the form $\infty - \infty$.

3. If $a_n \leq b_n$ for all $n$, then

$$\liminf a_n \leq \liminf b_n$$

Show by example that strict inequality can hold in b).

**Solution.**

1. First prove $\sup -A = -\inf A$, where $A \in \bar{\mathbb{R}}$. Notice $-a \leq -\inf a$ implies $\sup -A \leq -\inf A$. Similarly $-\inf A \leq \sup -A$.

$$\limsup(-a_n) = \inf_{n\geq 1} \sup_{k\geq n}(-a_k) = -\sup\inf a_k = -\limsup a_n$$

2. Notice if $\limsup a_n + \limsup b_n$ is defined, $\sup_{k\geq n} a_k + \sup_{k\geq n} b_k$ is defined for sufficiently large $n$. Notice

$$a_n + b_n \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k$$

implies

$$\sup_{k\geq n} a_n + b_n \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k$$

for sufficiently large $n$. Taking the limits, whose existence indicated by the question, on both sides gives the result (any decreasing sequence has its limit the same as its infimum).

3. Clearly $\inf_{k\geq n} a_k \leq \inf_{k\geq n} b_k$ for any $n$. This completes the proof.

Define

$$a_n = 1, -1, 1, -1, 1, ...$$
$$b_n = -1, 1, -1, 1, -1, ...$$

Obviously, $\limsup a_n = \limsup b_n = 1$, while $\limsup a_n + b_n = 0$.

**Exercise 1.5.** 1. Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$$

are measurable.

2. Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

**Solution.**

1. Notice

$$\{f < g\} = \bigcup_{r\in\mathbb{Q}} \{f < r\} \cap \{r < g\}$$

therefore measurable. By symmetry, $\{f > g\}$ is measurable as well. Now $\{f = g\}$ is the complement of $\{f > g\} \cup \{f < g\}$ thus must be measurable.

2. For any sequence of functions $f_n$, the set of points the sequence converges to a finite limit is

$$\{\limsup f_n = \liminf f_n\} \cap \{\limsup f_n < \infty\}.$$

Since the upper and lower limits of a sequence of functions are measurable functions, the above set is measurable.

**Exercise 1.6.** Let $X$ be an uncountable set, let $\mathcal{M}$ be the collection of all sets $E \subset X$ such that either $E$ or $E^c$ is at most countable, and define $\mu(E) = 0$ in the first case, $mu(E) = 1$ in the second. Prove that $\mathcal{M}$ is a $\sigma$-algebra in $X$ and that $\mu$ is a measure on $\mathcal{M}$. Describe the corresponding measurable functions and their integrals.

**Solution.** It suffices to prove $\mathcal{M}$ is closed under countable union and $\mu$ is a measure. Notice for any sequence of sets $E_n$ in $\mathcal{M}$, if all of them are at most countable, so must be their union. Therefore $\bigcup E_n \in \mathcal{M}$, and $\mu(\bigcup E_n) = 0 = \sum \mu(E_n)$. If $E_i$ have its complement at most countable, $(\bigcup E_n)^c = \bigcap E_n^c \subset E_i^c$ is at most countable. Hence $\bigcup E_n \in \mathcal{M}$. Further, if $E_n$ are disjoint, for any set $E_j, j \neq i$, $X \subset E_i^c \cup E_j^c$, showing $E_j^c$ must be uncountable, therefore $E_j$ must be at most countable and $\sum \mu(E_n) = 1 = \mu\left(\bigcup E_n\right)$.

Only describe integrals of non-negative functions, the general case follows is of no interest. Since any singalton is Borel, for any $r \in [0, \infty]$, a measurable function $f : X \to [0, \infty]$ would have $f^{-1}(r)$ either at most countable or have a at most countable complement. Take $A$ as the set of $r$ such that $f^{-1}(r)$ has an at most countable complement. Then if $A$ is countable, it is easy to check $\int f = \sum_{r \in A} r$. If $A$ is uncountable, then $\int f = \infty$. 21

# Chapter 2

# Positive Borel Measure

**Exercise 2.1.** Let $\{f_n\}$ be a sequence of real non-negative functions on $\mathbb{R}^1$, and consider the following four statements:

1. If $f_1$, $f_2$ are upper semicontinuous so is $f_1 + f_2$.
2. If $f_1, f_2$ are lower semicontinuous, so is $f_1 + f_2$.
3. If each $f_n$ is upper semicontinuous, so is $\sum^\infty f_n$.
4. If each $f_n$ is lower semicontinuous, so is $\sum^\infty f_n$.

**Solution.** Observe

$$\{f_1 + f_2 < a\} = \bigcup_{x\in\mathbb{R}} \{f_1 < x\} \cap \{f_2 < a - x\}$$

is open. To see the left is included in the right, for any $f_1(y) + f_2(y) < a$, take $f_1(y) < x < a - f_2(y)$ and the inclusion holds. Therefore 1 is verified. Similar argument goes with 2 if the above $<$ are replaced with $>$.

Notice 4 holds, fix any $x$ such that $\sum f(x) > a$. Since $f_n$'s are non-negative, there is $N$ such that $\sum^N f_n(x) > a$. Therefore there exists $\delta$, $\sum^N f_n(y) > a$ for any $y \in B_\delta(x)$ since finite sums of lower semicontinuous functions are lower semicontinuous. The proof is complete by observing

$$\sum f_n(y) \geq \sum^N f_n(y) > a$$

To give 3 a counterexample, consider $\sum f_n = \sum \mathcal{X}_{[-n,-1/n]\cup[1/n,n]}$. Obviously, every point but 0 is greater than or equal to 1. Hence

$$\{\sum f_n < 1\} = \{0\}$$

is closed.

**Exercise 2.2.** Let $f$ be an arbitrary complex function on $\mathbb{R}^1$, and define

$$\phi(x,\delta) = \sup\{|f(s) - f(t)| : s,t \in (x-\delta, x+\delta)\},$$
$$\phi(x) = \inf\{\phi(x,\delta) : \delta > 0\}.$$

Prove that $\phi$ is upper semicontinuous, that $f$ is continuous at a point $x$ iff $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a $G_\delta$.

Formulate and prove an analogous statement for general topological spaces in place of $\mathbb{R}^1$.

**Solution.** Only give solution in the general case. Redefine

$$\phi(x) = \inf_{B \ni x} \operatorname{diam} f(B)$$

where the diameter is defined as $\operatorname{diam} A = \sup_{x,y \in A} |x - y|$. Take any $x \in \{\phi(x) < a\}$, there is $B \ni x$, $\operatorname{diam} f(B) < a$. Take any $y \in B$, then $\phi(y) \leq \operatorname{diam} f(B) < a$. This says $\{\phi(x) < a\}$ is open and $\phi$ is upper semicontinuous.

The relation between $\phi$ and continuity of $f$ is trivial. Since

$$\{\phi = 0\} = \bigcap_{q \in \mathbb{Q}^+} \{\phi < q\}$$

the set is a $G_\delta$.

**Exercise 2.3.** Let $X$ be a metric space, with metric $\rho$. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf_{y \in E} \rho(x, y)$$

Show that $\rho_E$ is uniformly continuous function on $X$. If $A$ and $B$ are disjoint nonempty closed subsets of $X$, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

**Solution.** Notice

$$\rho(a, x) + \rho(a, b) \geq \rho(x, b).$$

Taking infimum on $E$ on both sides gives

$$\rho_E(b) - \rho_E(a) \leq \rho(a, b)$$

By symmetry,

$$|\rho_E(b) - \rho_E(a)| \leq \rho(a, b)$$

showing the uniform continuity. Notice $0 \leq f \leq 1$ and $f = 1$ on $B$. Its support lies in $A^c$. Therefore if $B$ is compact, $B \prec f \prec A^c$.

**Exercise 2.4.** Examine the proof of Riesz theorem and prove the following two statments:

1. If $E_1 \subset V_1$ and $E_2 \subset V_2$, where $V_1$ and $V_2$ are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if $E_1, E_2$ are not in $\mathcal{M}$.

2. If $E \in \mathcal{M}_F$, then $E = N \cup K_1 \cup K_2 \ldots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

**Solution.**

1. Take any open set $U$ that covers $E_1 \cup E_2$. Observe
$$\mu(U) \geq \mu(U \cap V_1) + \mu(U \cap V_2) \geq \mu(E_1) + \mu(E_2).$$
Taking infimum on both sides together with the subadditivity of $\mu$ gives the result.

2. Since $E \in \mathcal{M}_F, \mu(E) < \infty$. Take $K_1$
$$\mu(E) < \mu(K_1) + 1.$$
Having chosen $K_1, ..., K_n$, denote $G = E - \cup^{n-1} K_i \in \mathcal{M}_F$. Pick $K_n$ such that
$$\mu(G) < \mu(K_n) + 1/n.$$
Obviously $\mu(E - \bigcup K_n) = 0$ and this completes the proof.

In Exercise 5 to 8, $m$ stands for Lebesgue measure on $\mathbb{R}^1$.

**Exercise 2.5.** Let $E$ be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even through $E$ and $\mathbb{R}^1$ have the same cardinality.

**Solution.** Denote $R$ as the set removed from $[0, 1]$ in construction of the Cantor set. Notice it is comprised of the union of $2^{n-1}$ open intervals of length $3^{-n}$ where $n$ ranges in $\mathbb{N}$. Therefore
$$\mu(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$
and $\mu(E) = \mu(I - R) = 0$.

To see $E$ has the same cardinality as $\mathbb{R}$. Notice each element in $E$ is a decimal of base 3 that has exactly one 1 at the end or no 1 at all. Therefore there is a surjection from $\mathbb{R}$ to $E$, and one from $E$ to decimals of base 2. This completes the proof.

**Exercise 2.6.** Construct a totally disconnected compact set $K \subset \mathbb{R}^1$ such that $m(K) > 0$. If $v$ is lower semicontinuous and $v \leq \mathcal{X}_K$, show that actually $v \leq 0$. Hence $\mathcal{X}_K$ cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali-Carathéodory theorem.

**Solution.** Construct $K$ similarly as the Cantor set in the previous exercise only we remove the middle fourths in place of thirds. Let $R$ be the union of removed intervals. $K$ is compact since each removal left a closed and bounded thus compact subset of $[0, 1]$ and $K$ is the intersection of all these compact sets. Similarly as above,
$$\mu(R) = \sum_{n-1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \sum \frac{1}{2^{n+1}} = \frac{1}{2}.$$
Therefore $\mu(K) = 1/2$. Notice $\{v > 0\}$ is open by definition. But it cannot be a subset of $K$ except for the empty set since $K$ is totally disconnected.

**Exercise 2.7.** If $0 < \epsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $m(E) = \epsilon$. (To say that $A$ is dense in $B$ means that the closure of $A$ contains $B$.)

**Solution.** The construction is similar as before. Notice if one takes out one middle $x$-th every time with $x > 2$, the removed set $R$ has measure

$$m(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{x^n} = \frac{1}{x-2}$$

Take $x = 1/\epsilon + 2$ and $m(R) = \epsilon$. $R$ is open since its complement in $[0,1]$ is compact (as proven before), therefore closed. To see $R$ is dense, notice every removal divides each remaining interval into its halves. Therefore to each point $x$ in $[0,1]$, there must be a point that has been removed after the $n$-th removal and lies in $B_{1/2^n}(x)$.

**Exercise 2.8.** Construct a Borel set $E \subset \mathbb{R}^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment $I$. Is it possible to have $m(E) < \infty$ for such a set?

**Solution.** Take $E = \mathbb{Q}$. Obviously $0 = m(E \cap I)$. Since $I$ is an nonempty segment, there is an open interval in it, i.e., $m(I) > 0$. Notice $\mathbb{Q}$ is Borel because it is the countable union of singaltons, and singaltons are Borel.

**Exercise 2.9.** Construct a sequence of continuous function $f_n$ on $[0,1]$ such that $0 \le f_n \le 1$, such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

**Solution.** Define

$$\begin{aligned}
&K_1 = [0,1], \\
&K_2 = [0,1/2], K_3 = [1/2,1], \\
&K_4 = [0,1/4], K_5 = [1/4,1/2], K_6 = [1/2,3/4], K_7 = [3/4,1]
\end{aligned}$$

Obviously $m(K_n) \to 0$. To each $K_n = [a,b]$, pick $V_n = (a - 1/2n, b + 1/2n)$ and $f_n$ that $K_n \prec f_n \prec V_n$. Finally,

$$\int f_n dm \le m(V_n) = m(K_n) + 1/n \to 0$$

But $f_n$ does not converge for any $x \in [0,1]$, since $f_n(x) = 1$ and $f_n(x) = 0$ both infinitely often.

Notice the construction of $f_n$ need not follow that of Urysohn's. Define

$$l((x_1,y_1),(x_2,y_2))(x) = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) + y_1.$$

If $V_n = (a,b)$, $K_n = [c,d]$, and $a < c < d < b$. Pick $s,t$ that, $a < s < c < d < t < b$. Take

$$f_n = \begin{cases} 0, x \le s \\ l((s,0),(c,1)), s \le x \le c \\ 1, c \le x \le d \\ l((d,1),(t,0)), d \le x \le t \\ 0, x \ge t \end{cases}$$

# Chapter 3

# $L^p$ Spaces

**Exercise 3.1.** Prove that the supremum of any collection of convex funtions on $(a,b)$ is convex on $(a,b)$ (if it is finite) and that pointwise limits of sequences of convex functions are convex. What can you say about the upper and lower limits of sequences of convex functions.

**Solution.** If $f_n$ is a sequence of convex functions, and $f$ is its supremum. For any $x, y \in (a,b)$, $n \in \mathbb{N}$,

$$\lambda f(x) + (1-\lambda)f(y) \geq \lambda f_n(x) + (1-\lambda)f_n(y) \geq f_n(\lambda x + (1-\lambda)y)$$

Taking supremum on the right side gives the result.

Now take $f = \lim f_n$, if that exists and $\{\infty, -\infty\} \not\subset f[(a,b)]$. Then $f$ is convex by observing

$$f_n(\lambda x + (1-\lambda)y) \to f(\lambda x + (1-\lambda y)),$$
$$\lambda f_n(x) + (1-\lambda)f_n(y) \to \lambda f(x) + (1-\lambda)f(y).$$

Notice $\limsup f_n = \lim_{n=1} \sup_{k \geq n} f_n$ is convex by the above arguments.

However, $\liminf f_n$ may not be convex. Take $(a,b) = (0,1)$. Define

$$f_n(x) = \begin{cases} x, n \text{ is odd} \\ 1-x, n \text{ is even} \end{cases}.$$

Clearly $\liminf f(x) = \begin{cases} x, 0 < x \leq 1/2 \\ 1-x, 1/2 \leq x < 1 \end{cases}$. That $f$ is not convex is by verifying $f(1/2) > 1/2 f(1/4) + 1/2 f(3/4)$

**Exercise 3.4.** Suppose $f$ is a complex measurable function on $X$, $\mu$ is a positive measure on $X$, and

$$\phi(p) = \int_X |f^p| d\mu = ||f||_p^p, (0 < p < \infty).$$

Let $E = \{p : \phi(p) < \infty\}$. Assume $||f||_\infty > 0$.

1. If $r < p < s, r \in E$ and $s \in E$, prove that $p \in E$.
2. Prove that $\log \phi$ is convex in the interior of $E$ and that $\phi$ is continuous on $E$.
3. By a), $E$ is connected. Is $E$ necessarily open? Closed? Can $E$ consist of a single point? Can $E$ be any connected subset of $(0, \infty)$?

4. if $r < p < s$, prove that $||f||_p \leq \max(||f||_r, ||f||_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

5. Assume that $||f||_r < \infty$ for some $r < \infty$ and prove that

$$||f||_p \to ||f||_\infty, \text{as } p \to \infty.$$

**Exercise 3.5.** Assume, in addition to the hypothesis of Exercise 4, that

$$\mu(X) = 1.$$

1. Prove that $||f||_r \leq ||f||_s$, if $0 < r < s \leq \infty$.

2. Under what conditions does it happen that $0 < r < s \leq \infty$ and $||f||_r = ||f||_s < \infty$?

3. Prove that $L^r(\mu) \subset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

4. Assume that $||f||_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to 0} ||f||_p = \exp\left\{\int_X \log|f| d\mu\right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

**Solution.**

1.

2.

3.

4. By definition of Lebesgue integral,

$$\int \log|f| d\mu = \int_{|f|\geq 1} \log|f| d\mu - \int_{|f|\leq 1} - \log|f| d\mu$$

since the above is essentially decomposing a integral of a function into that of its positive and negative parts. Note the integral of the positive part cannot be $\infty$, because otherwise by the inequality $\log t \leq t-1$,

$$\infty = \int_{|f|\geq 1} \log|f| d\mu = \int_{|f|\geq 1} \frac{1}{p}(|f|^p - 1) d\mu \leq \frac{1}{p} \int_{|f|\geq 1} (|f|^p - 1)$$

showing $||f||_p = \infty$ for any $p > 0$, a contradition to the hypothesis in the question. Therefore $\int \log|f| = -\infty$ or it is finite.

Frist consider the finite case. Take any $p_n \to 0$, define $f_n = \frac{|f|^{p_n}-1}{p_n}$. Notice $\lim f_n = \frac{d}{d|f|}|f|^p|_{|f|=0} = \log|f|$. Further for any $0 < p_n < r$, if $|f| \geq 1$,

$$|f_n| = \left|\frac{|f|^{p_n} - 1}{p_n}\right| = \int_1^{|f|} s^{p_n - 1} ds \leq \int_1^{|f|} s^{r-1} ds \leq \frac{|f|^r - 1}{r},$$

and if $|f| \leq 1$,

$$|f_n| = \left|\frac{|f|^{p_n} - 1}{p_n}\right| = \int_{|f|}^{1} s^{p_n - 1} ds \leq \int_{1}^{|f|} s^{-1} ds \leq -\log|f|.$$

Therefore $|f_n| \leq \frac{|f|^r - 1}{r}\mathcal{X}_{|f|\geq 1} - \log|f|\mathcal{X}_{|f|\leq 1}$. By Dominated Convergence,

$$\int f_n \to \int \log|f|.$$

Observe, by the inequality $\log t \leq t - 1$ again,

$$\frac{1}{p}\log\int|f|^p \leq \int\frac{|f|^p - 1}{p} \to \int\log|f|,$$

as $p \to 0$. By Jensen,

$$\frac{1}{p}\log\int|f|^p \geq \int\log|f|,$$

for any $p \leq r$, since log is concave over $\mathbb{R}$. Finally the result is obtained by sandwich theorem.

Now if $\int\log|f|d\mu = -\infty$, take $f_n = |f|\mathcal{X}_{|f|>1/n} + \mathcal{X}_{|f|\leq 1/n}$, where $n$ starts at 1. Clearly,

(a) $||f||_p \leq ||f_n||_p$.

(b) $\log|f_n| = \log|f|\mathcal{X}_{|f|>1/n}$.

(c) $-\log|f_n| < \log n$

Notice the positive and negative part of $\log|f_n|$, by b), both converges increasingly to those of $\log|f|$. By Monotone Convergence, $\lim\limits_{n\to\infty}\exp\{\int\log|f_n|\} \to \exp\{\int\log|f|\} = 0$. By c) $\int\log|f_n| \neq -\infty$. Obviously $f_n \in L^r(\mu)$, therefore $\log|f_n|$ can only have finite integral by the same proof as in $\log|f|$. Now together with previous arguments, $\lim\limits_{p\to 0}||f_n||_p \to \exp\{\int\log|f_n|\}$. Therefore by a),

$$\limsup_{p\to 0}||f||_p = \lim_{n\to\infty}\limsup_{p\to 0}||f||_p \leq \lim_{n\to\infty}\lim_{p\to 0}||f_n||_p = \lim_{n\to\infty}\exp\left\{\int\log|f_n|\right\} = 0.$$

Since $||f||_p \geq 0$, $||f||_p \to 0$, as $p \to 0$.