Chapter 1

Abstract Integration

Exercise 1.1. Does there exist an infinte σ-algebra which has only countably many members?

Solution. If $\mathcal{F}$ is a σ-algebra and A a set, define $\mathcal{F} \cap A := \{A \cap F : F \in \mathcal{F}\}$. Hence if $A \in \mathcal{F}$, then $\mathcal{F} \cap A$ is a σ-subalgebra of $\mathcal{F}$. Further, if $\mathcal{F}$ is infinte, either $A \cap \mathcal{F}$ or $A^c \cap \mathcal{F}$ is infinte.

In other words, if $\mathcal{F}_n$ is infinte, then there is $A_{n+1} \in \mathcal{F}_n$ such that $\mathcal{F}_{n+1} = \mathcal{F}_n \cap A_n^c$ is infinte. Take $\mathcal{F}_0 = \mathcal{M}$, by induction we get a disjoint sequence of sets $A_n \in \mathcal{M}$. Since $\mathcal{M}$ must also contain any union of sets of A_n, and each union is different, this is to say $\mathcal{M}$ contains a embedding of all subsets of $\mathbb{N}$, therefore uncountable.

Exercise 1.2. Prove an analogue of Theorem 1.8 for n functions.

Solution. It suffices to prove

$$f = f_1 \times f_2 \times ... \times f_n$$

is measurable is each f_i is measurable. Similarly as in the theorem, take R be any rectangles in $\mathbb{R}^n$. Notice

$$f^{-1}(R) = \bigcap f_i(I_i)$$

is measurable, if $R = I_1 \times I_2 \times ... \times I_n$. The rest of the proof is a repeat of Theorem 1.8.

Exercise 1.3. Prove that if f is a real function on a measurable space X such that $\{x : f(x) < r\}$ is measurable for every rational r, then f is measurable.

Solution. Take Ω as all $E \subset \mathbb{R}$ such that $f^{-1}(E)$ is measurable. By Theorem 1.2, Ω is a σ-algebra. Notice

$$\{f > a\} = \bigcup_{q \in \mathbb{Q} \cap (-\infty, a)} \{f > q\}$$

for all $a \in \mathbb{R}$. Therefore, $\{f > a\}$ is measurable. By Theorem 1.2 again, f is measurable.

Exercise 1.4. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

1. $$\limsup -a_n = -\liminf a_n.$$

2. $$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

   provided none of the sums is of the form $\infty - \infty$.

3. If $a_n \le b_n$ for all n, then

$$\liminf a_n \le \liminf b_n$$

Show by example that strict inequality can hold in b).

Solution.

1. First prove $\sup -A = -\inf A$, where $A \in \bar{\mathbb{R}}$. Notice $-a \le -\inf a$ implies $\sup -A \le -\inf A$. Similarly $-\inf A \le \sup -A$.

$$\limsup(-a_n) = \inf_{n\ge 1} \sup_{k\ge n}(-a_k) = -\sup\inf a_k = -\limsup a_n$$

2. Notice if $\limsup a_n + \limsup b_n$ is defined, $\sup_{k\ge n} a_k + \sup_{k\ge n} b_k$ is defined for sufficiently large n. Notice

$$a_n + b_n \le \sup_{k\ge n} a_k + \sup_{k\ge n} b_k$$

implies

$$\sup_{k\ge n} a_n + b_n \le \sup_{k\ge n} a_k + \sup_{k\ge n} b_k$$

for sufficiently large n. Taking the limits, whose existence indicated by the question, on both sides gives the result (any decreasing sequence has its limit the same as its infimum).

3. Clearly $\inf_{k\ge n} a_k \le \inf_{k\ge n} b_k$ for any n. This completes the proof.

Define

$$a_n = 1, -1, 1, -1, 1, ...$$
$$b_n = -1, 1, -1, 1, -1, ...$$

Obviously, $\limsup a_n = \limsup b_n = 1$, while $\limsup a_n + b_n = 0$.

Exercise 1.5. 1. Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$$

are measurable.

2. Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Solution.

1. Notice

$$\{f < g\} = \bigcup_{r\in\mathbb{Q}} \{f < r\} \cap \{r < g\}$$

therefore measurable. By symmetry, $\{f > g\}$ is measurable as well. Now $\{f = g\}$ is the complement of $\{f > g\} \cup \{f < g\}$ thus must be measurable.

2. For any sequence of functions f_n, the set of points the sequence converges to a finite limit is

$$\{\limsup f_n = \liminf f_n\} \cap \{\limsup f_n < \infty\}.$$

Since the upper and lower limits of a sequence of functions are measurable functions, the above set is measurable.

Exercise 1.6. Let X be an uncountable set, let $\mathcal{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $mu(E) = 1$ in the second. Prove that $\mathcal{M}$ is a σ-algebra in X and that μ is a measure on $\mathcal{M}$. Describe the corresponding measurable functions and their integrals.

Solution. It suffices to prove $\mathcal{M}$ is closed under countable union and μ is a measure. Notice for any sequence of sets E_n in $\mathcal{M}$, if all of them are at most countable, so must be their union. Therefore $\bigcup E_n \in \mathcal{M}$, and $\mu(\bigcup E_n) = 0 = \sum \mu(E_n)$. If E_i have its complement at most countable, $(\bigcup E_n)^c = \bigcap E_n^c \subset E_i^c$ is at most countable. Hence $\bigcup E_n \in \mathcal{M}$. Further, if E_n are disjoint, for any set $E_j, j \neq i$, $X \subset E_i^c \cup E_j^c$, showing E_j^c must be uncountable, therefore E_j must be at most countable and $\sum \mu(E_n) = 1 = \mu\left(\bigcup E_n\right)$.

Only describe integrals of non-negative functions, the general case follows is of no interest. Since any singalton is Borel, for any $r \in [0, \infty]$, a measurable function $f : X \to [0, \infty]$ would have $f^{-1}(r)$ either at most countable or have a at most countable complement. Take A as the set of r such that $f^{-1}(r)$ has an at most countable complement. Then if A is countable, it is easy to check $\int f = \sum_{r \in A} r$. If A is uncountable, then $\int f = \infty$.