

Solution Manual to Real and Complex Analysis

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Chapter 1

Abstract Integration

Chapter 2

Positive Borel Measure

Exercise 2.1. Let $\{f_n\}$ be a sequence of real non-negative functions on \mathbb{R}^1 , and consider the following four statements:

1. If f_1, f_2 are upper semicontinuous so is $f_1 + f_2$.
2. If f_1, f_2 are lower semicontinuous, so is $f_1 + f_2$.
3. If each f_n is upper semicontinuous, so is $\sum^\infty f_n$.
4. If each f_n is lower semicontinuous, so is $\sum^\infty f_n$.

Solution. Observe

$$\{f_1 + f_2 < a\} = \bigcup_{x \in \mathbb{R}} \{f_1 < x\} \cap \{f_2 < a - x\}$$

is open. To see the left is included in the right, for any $f_1(y) + f_2(y) < a$, take $f_1(y) < x < a - f_2(y)$ and the inclusion holds. Therefore 1 is verified. Similar argument goes with 2 if the above $<$ are replaced with $>$.

Notice 4 holds, fix any x such that $\sum f_n(x) > a$. Since f_n 's are non-negative, there is N such that $\sum^N f_n(x) > a$. Therefore there exists δ , $\sum^N f_n(y) > a$ for any $y \in B_\delta(x)$ since finite sums of lower semicontinuous functions are lower semicontinuous. The proof is complete by observing

$$\sum f_n(y) \geq \sum^N f_n(y) > a$$

To give 3 a counterexample, consider $\sum f_n = \sum \chi_{[-n, -1/n] \cup [1/n, n]}$. Obviously, every point but 0 is greater than or equal to 1. Hence

$$\{\sum f_n < 1\} = \{0\}$$

is closed.

Exercise 2.2. Let f be an arbitrary complex function on \mathbb{R}^1 , and define

$$\begin{aligned}\phi(x, \delta) &= \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\}, \\ \phi(x) &= \inf\{\phi(x, \delta) : \delta > 0\}.\end{aligned}$$

Prove that ϕ is upper semicontinuous, that f is continuous at a point x iff $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ .

Formulate and prove an analogous statement for general topological spaces in place of \mathbb{R}^1 .

Solution. Only give solution in the general case. Redefine

$$\phi(x) = \inf_{B \ni x} \text{diam} f(B)$$

where the diameter is defined as $\text{diam} A = \sup_{x, y \in A} |x - y|$. Take any $x \in \{\phi(x) < a\}$, there is $B \ni x$, $\text{diam} f(B) < a$. Take any $y \in B$, then $\phi(y) \leq \text{diam} f(B) < a$. This says $\{\phi(x) < a\}$ is open and ϕ is upper semicontinuous.

The relation between ϕ and continuity of f is trivial. Since

$$\{\phi = 0\} = \bigcap_{q \in \mathbb{Q}^+} \{\phi < q\}$$

the set is a G_δ .

Exercise 2.3. Let X be a metric space, with metric ρ . For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf_{y \in E} \rho(x, y)$$

Show that ρ_E is uniformly continuous function on X . If A and B are disjoint nonempty closed subsets of X , examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Solution. Notice

$$\rho(a, x) + \rho(a, b) \geq \rho(x, b).$$

Taking infimum on E on both sides gives

$$\rho_E(b) - \rho_E(a) \leq \rho(a, b)$$

By symmetry,

$$|\rho_E(b) - \rho_E(a)| \leq \rho(a, b)$$

showing the uniform continuity. Notice $0 \leq f \leq 1$ and $f = 1$ on B . Its support lies in A^c . Therefore if B is compact, $B \prec f \prec A^c$.

Exercise 2.4. Examine the proof of Riesz theorem and prove the following two statements:

1. If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1, E_2 are not in \mathcal{M} .
2. If $E \in \mathcal{M}_F$, then $E = N \cup K_1 \cup K_2 \dots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

Solution.

1. Take any open set U that covers $E_1 \cup E_2$. Observe

$$\mu(U) \geq \mu(U \cap V_1) + \mu(U \cap V_2) \geq \mu(E_1) + \mu(E_2).$$

Taking infimum on both sides together with the subadditivity of μ gives the result.

2. Since $E \in \mathcal{M}_F$, $\mu(E) < \infty$. Take K_1

$$\mu(E) < \mu(K_1) + 1.$$

Having chosen K_1, \dots, K_n , denote $G = E - \bigcup^{n-1} K_i \in \mathcal{M}_F$. Pick K_n such that

$$\mu(G) < \mu(K_n) + 1/n.$$

Obviously $\mu(E - \bigcup K_n) = 0$ and this completes the proof.

Exercise 2.5. Let E be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even though E and \mathbb{R}^1 have the same cardinality.

Solution. Denote R as the set removed from $[0, 1]$ in construction of the Cantor set. Notice it is comprised of the union of 2^{n-1} open intervals of length 3^{-n} where n ranges in \mathbb{N} . Therefore

$$\mu(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$

and $\mu(E) = \mu(I - R) = 0$.

To see E has the same cardinality as \mathbb{R} . Notice each element in E is a decimal of base 3 that has exactly one 1 at the end or no 1 at all. Therefore there is a surjection from \mathbb{R} to E , and one from E to decimals of base 2. This completes the proof.

Exercise 2.6. Construct a totally disconnected compact set $K \subset \mathbb{R}^1$ such that $m(K) > 0$. If v is lower semicontinuous and $v \leq \chi_K$, show that actually $v \leq 0$. Hence χ_K cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali-Carathéodory theorem.

Solution. Construct K similarly as the Cantor set in the previous exercise only we remove the middle fourths in place of thirds. Let R be the union of removed intervals. Similarly as above,

$$\mu(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \sum \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore $\mu(K) = 1/2$. Notice $\{v > 0\}$ is open by definition. But it cannot be a subset of K except for the empty set since K is totally disconnected.