Chapter 1

Abstract Integration

Exercise 1.1. Does there exist an infinte σ-algebra which has only countably many members?

Solution. If $\mathcal{F}$ is a σ-algebra and A a set, define $\mathcal{F} \cap A := \{A \cap F : F \in \mathcal{F}\}$. Hence if $A \in \mathcal{F}$, then $\mathcal{F} \cap A$ is a σ-subalgebra of $\mathcal{F}$. Further, if $\mathcal{F}$ is infinte, either $A \cap \mathcal{F}$ or $A^c \cap \mathcal{F}$ is infinte.

In other words, if $\mathcal{F}_n$ is infinte, then there is $A_{n+1} \in \mathcal{F}_n$ such that $\mathcal{F}_{n+1} = \mathcal{F}_n \cap A_n^c$ is infinte. Take $\mathcal{F}_0 = \mathcal{M}$, by induction we get a disjoint sequence of sets $A_n \in \mathcal{M}$. Since $\mathcal{M}$ must also contain any union of sets of A_n, and each union is different, this is to say $\mathcal{M}$ contains a embedding of all subsets of $\mathbb{N}$, therefore uncountable.

Exercise 1.2. Prove an analogue of Theorem 1.8 for n functions.

Solution. It suffices to prove

$$f = f_1 \times f_2 \times ... \times f_n$$

is measurable is each f_i is measurable. Similarly as in the theorem, take R be any rectangles in $\mathbb{R}^n$. Notice

$$f^{-1}(R) = \bigcap f_i(I_i)$$

is measurable, if $R = I_1 \times I_2 \times ... \times I_n$. The rest of the proof is a repeat of Theorem 1.8.

Exercise 1.3. Prove that if f is a real function on a measurable space X such that $\{x : f(x) < r\}$ is measurable for every rational r, then f is measurable.

Solution. Take Ω as all $E \subset \mathbb{R}$ such that $f^{-1}(E)$ is measurable. By Theorem 1.2, Ω is a σ-algebra. Notice

$$\{f > a\} = \bigcup_{q \in \mathbb{Q} \cap (-\infty, a)} \{f > q\}$$

for all $a \in \mathbb{R}$. Therefore, $\{f > a\}$ is measurable. By Theorem 1.2 again, f is measurable.