

# Solution Manual to Real and Complex Analysis

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## Chapter 1

# Abstract Integration



## Chapter 2

# Positive Borel Measure

**Exercise 2.1.** Let  $\{f_n\}$  be a sequence of real non-negative functions on  $\mathbb{R}^1$ , and consider the following four statements:

1. If  $f_1, f_2$  are upper semicontinuous so is  $f_1 + f_2$ .
2. If  $f_1, f_2$  are lower semicontinuous, so is  $f_1 + f_2$ .
3. If each  $f_n$  is upper semicontinuous, so is  $\sum^\infty f_n$ .
4. If each  $f_n$  is lower semicontinuous, so is  $\sum^\infty f_n$ .

**Solution.** Observe

$$\{f_1 + f_2 < a\} = \bigcup_{x \in \mathbb{R}} \{f_1 < x\} \cap \{f_2 < a - x\}$$

is open. To see the left is included in the right, for any  $f_1(y) + f_2(y) < a$ , take  $f_1(y) < x < a - f_2(y)$  and the inclusion holds. Therefore 1 is verified. Similar argument goes with 2 if the above  $<$  are replaced with  $>$ .

Notice 4 holds, fix any  $x$  such that  $\sum f_n(x) > a$ . Since  $f_n$ 's are non-negative, there is  $N$  such that  $\sum^N f_n(x) > a$ . Therefore there exists  $\delta$ ,  $\sum^N f_n(y) > a$  for any  $y \in B_\delta(x)$  since finite sums of lower semicontinuous functions are lower semicontinuous. The proof is complete by observing

$$\sum f_n(y) \geq \sum^N f_n(y) > a$$

To give 3 a counterexample, consider  $\sum f_n = \sum \chi_{[-n, -1/n] \cup [1/n, n]}$ . Obviously, every point but 0 is greater than or equal to 1. Hence

$$\{\sum f_n < 1\} = \{0\}$$

is closed.

**Exercise 2.2.** Let  $f$  be an arbitrary complex function on  $\mathbb{R}^1$ , and define

$$\begin{aligned}\phi(x, \delta) &= \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\}, \\ \phi(x) &= \inf\{\phi(x, \delta) : \delta > 0\}.\end{aligned}$$

Prove that  $\phi$  is upper semicontinuous, that  $f$  is continuous at a point  $x$  iff  $\phi(x) = 0$ , and hence that the set of points of continuity of an arbitrary complex function is a  $G_\delta$ .

Formulate and prove an analogous statement for general topological spaces in place of  $\mathbb{R}^1$ .

**Solution.** Only give solution in the general case. Redefine

$$\phi(x) = \inf_{B \ni x} \text{diam} f(B)$$

where the diameter is defined as  $\text{diam} A = \sup_{x, y \in A} |x - y|$ . Take any  $x \in \{\phi(x) < a\}$ , there is  $B \ni x$ ,  $\text{diam} f(B) < a$ . Take any  $y \in B$ , then  $\phi(y) \leq \text{diam} f(B) < a$ . This says  $\{\phi(x) < a\}$  is open and  $\phi$  is upper semicontinuous.

The relation between  $\phi$  and continuity of  $f$  is trivial. Since

$$\{\phi = 0\} = \bigcap_{q \in \mathbb{Q}^+} \{\phi < q\}$$

the set is a  $G_\delta$ .

**Exercise 2.3.** Let  $X$  be a metric space, with metric  $\rho$ . For any nonempty  $E \subset X$ , define

$$\rho_E(x) = \inf_{y \in E} \rho(x, y)$$

Show that  $\rho_E$  is uniformly continuous function on  $X$ . If  $A$  and  $B$  are disjoint nonempty closed subsets of  $X$ , examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

**Solution.** Notice

$$\rho(a, x) + \rho(a, b) \geq \rho(x, b).$$

Taking infimum on  $E$  on both sides gives

$$\rho_E(b) - \rho_E(a) \leq \rho(a, b)$$

By symmetry,

$$|\rho_E(b) - \rho_E(a)| \leq \rho(a, b)$$

showing the uniform continuity. Notice  $0 \leq f \leq 1$  and  $f = 1$  on  $B$ . Its support lies in  $A^c$ . Therefore if  $B$  is compact,  $B \prec f \prec A^c$ .

**Exercise 2.4.** Examine the proof of Riesz theorem and prove the following two statements:

1. If  $E_1 \subset V_1$  and  $E_2 \subset V_2$ , where  $V_1$  and  $V_2$  are disjoint open sets, then  $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ , even if  $E_1, E_2$  are not in  $\mathcal{M}$ .
2. If  $E \in \mathcal{M}_F$ , then  $E = N \cup K_1 \cup K_2 \dots$ , where  $\{K_i\}$  is a disjoint countable collection of compact sets and  $\mu(N) = 0$ .



**Solution.**

1. Take any open set  $U$  that covers  $E_1 \cup E_2$ . Observe

$$\mu(U) \geq \mu(U \cap V_1) + \mu(U \cap V_2) \geq \mu(E_1) + \mu(E_2).$$

Taking infimum on both sides together with the subadditivity of  $\mu$  gives the result.

2. Since  $E \in \mathcal{M}_F$ ,  $\mu(E) < \infty$ . Take  $K_1$

$$\mu(E) < \mu(K_1) + 1.$$

Having chosen  $K_1, \dots, K_n$ , denote  $G = E - \bigcup^{n-1} K_i \in \mathcal{M}_F$ . Pick  $K_n$  such that

$$\mu(G) < \mu(K_n) + 1/n.$$

Obviously  $\mu(E - \bigcup K_n) = 0$  and this completes the proof.

In Exercise 5 to 8,  $m$  stands for Lebesgue measure on  $\mathbb{R}^1$ .

**Exercise 2.5.** Let  $E$  be Cantor's familiar "middle thirds" set. Show that  $m(E) = 0$ , even though  $E$  and  $\mathbb{R}^1$  have the same cardinality.

**Solution.** Denote  $R$  as the set removed from  $[0, 1]$  in construction of the Cantor set. Notice it is comprised of the union of  $2^{n-1}$  open intervals of length  $3^{-n}$  where  $n$  ranges in  $\mathbb{N}$ . Therefore

$$\mu(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$

and  $\mu(E) = \mu(I - R) = 0$ .

To see  $E$  has the same cardinality as  $\mathbb{R}$ . Notice each element in  $E$  is a decimal of base 3 that has exactly one 1 at the end or no 1 at all. Therefore there is a surjection from  $\mathbb{R}$  to  $E$ , and one from  $E$  to decimals of base 2. This completes the proof.

**Exercise 2.6.** Construct a totally disconnected compact set  $K \subset \mathbb{R}^1$  such that  $m(K) > 0$ . If  $v$  is lower semicontinuous and  $v \leq \chi_K$ , show that actually  $v \leq 0$ . Hence  $\chi_K$  cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali-Carathéodory theorem.

**Solution.** Construct  $K$  similarly as the Cantor set in the previous exercise only we remove the middle fourths in place of thirds. Let  $R$  be the union of removed intervals.  $K$  is compact since each removal left a closed and bounded thus compact subset of  $[0, 1]$  and  $K$  is the intersection of all these compact sets. Similarly as above,

$$\mu(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \sum \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore  $\mu(K) = 1/2$ . Notice  $\{v > 0\}$  is open by definition. But it cannot be a subset of  $K$  except for the empty set since  $K$  is totally disconnected.

**Exercise 2.7.** If  $0 < \epsilon < 1$ , construct an open set  $E \subset [0, 1]$  which is dense in  $[0, 1]$ , such that  $m(E) = \epsilon$ . (To say that  $A$  is dense in  $B$  means that the closure of  $A$  contains  $B$ .)

**Solution.** The construction is similar as before. Notice if one takes out one middle  $x$ -th every time with  $x > 2$ , the removed set  $R$  has measure

$$m(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{x^n} = \frac{1}{x-2}$$

Take  $x = 1/\epsilon + 2$  and  $m(R) = \epsilon$ .  $R$  is open since its complement in  $[0, 1]$  is compact (as proven before), therefore closed. To see  $R$  is dense, notice every removal divides each remaining interval into its halves. Therefore to each point  $x$  in  $[0, 1]$ , there must be a point that has been removed after the  $n$ -th removal and lies in  $B_{1/2^n}(x)$ .

**Exercise 2.8.** Construct a Borel set  $E \subset \mathbb{R}^1$  such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment  $I$ . Is it possible to have  $m(E) < \infty$  for such a set?

**Solution.** Take  $E = \mathbb{Q}$ . Obviously  $0 = m(E \cap I)$ . Since  $I$  is an nonempty segment, there is an open interval in it, i.e.,  $m(I) > 0$ . Notice  $\mathbb{Q}$  is Borel because it is the countable union of singaltons, and singaltons are Borel.

**Exercise 2.9.** Construct a sequence of continuous function  $f_n$  on  $[0, 1]$  such that  $0 \leq f_n \leq 1$ , such that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0$$

but such that the sequence  $\{f_n(x)\}$  converges for no  $x \in [0, 1]$ .

**Solution.** Define

$$\begin{aligned} K_1 &= [0, 1], \\ K_2 &= [0, 1/2], K_3 = [1/2, 1], \\ K_4 &= [0, 1/4], K_5 = [1/4, 1/2], K_6 = [1/2, 3/4], K_7 = [3/4, 1] \end{aligned}$$

Obviously  $m(K_n) \rightarrow 0$ . To each  $K_n = [a, b]$ , pick  $V_n = (a - 1/2n, b + 1/2n)$  and  $f_n$  that  $K_n \prec f_n \prec V_n$ . Finally,

$$\int f_n dm \leq m(V_n) = m(K_n) + 1/n \rightarrow 0$$

But  $f_n$  does not converge for any  $x \in [0, 1]$ , since  $f_n(x) = 1$  and  $f_n(x) = 0$  both infinitely often.

Notice the construction of  $f_n$  need not follow that of Urysohn's. Define

$$l((x_1, y_1), (x_2, y_2))(x) = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) + y_1.$$

If  $V_n = (a, b)$ ,  $K_n = [c, d]$ , and  $a < c < d < b$ . Pick  $s, t$  that,  $a < s < c < d < t < b$ . Take

$$f_n = \begin{cases} 0, & x \leq s \\ l((s, 0), (c, 1)), & s \leq x \leq c \\ 1, & c \leq x \leq d \\ l((d, 1), (t, 0)), & d \leq x \leq t \\ 0, & x \geq t \end{cases}$$