Chapter 1

L^p Spaces

Exercise 1.1. Prove that the supremum of any collection of convex funtions on (a, b) is convex on (a, b) (if it is finite) and that pointwise limits of sequences of convex functions are convex. What can you say about the upper and lower limits of sequences of convex functions.

Solution. If f_n is a sequence of convex functions, and f is its supremum. For any $x, y \in (a, b)$, $n \in \mathbb{N}$,

$$\lambda f(x) + (1-\lambda)f(y) \geq \lambda f_n(x) + (1-\lambda)f_n(y) \geq f_n(\lambda x + (1-\lambda)y)$$

Taking supremum on the right side gives the result.

Now take $f = \lim f_n$, if that exists and $\{\infty, -\infty\} \not\subset f[(a, b)]$. Then f is convex by observing

$$f_n(\lambda x + (1-\lambda)y) \to f(\lambda x + (1-\lambda y)),$$
$$\lambda f_n(x) + (1-\lambda)f_n(y) \to \lambda f(x) + (1-\lambda)f(y).$$

Notice $\limsup f_n = \lim_{n=1} \sup_{k \geq n} f_n$ is convex by the above arguments.

However, $\liminf f_n$ may not be convex. Take $(a, b) = (0, 1)$. Define

$$f_n(x) = \begin{cases} x, n \text{ is odd} \\ 1-x, n \text{ is even} \end{cases}.$$

Clearly $\liminf f(x) = \begin{cases} x, 0 < x \leq 1/2 \\ 1-x, 1/2 \leq x < 1 \end{cases}$. That f is not convex is by verifying $f(1/2) > 1/2f(1/4) + 1/2f(3/4)$

Exercise 1.4. Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\phi(p) = \int_X |f^p| d\mu = ||f||_p^p, (0 < p < \infty).$$

Let $E = \{p : \phi(p) < \infty\}$. Assume $||f||_\infty > 0$.

1. If $r < p < s, r \in E$ and $s \in E$, prove that $p \in E$.
2. Prove that $\log \phi$ is convex in the interior of E and that ϕ is continuous on E.
3. By a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

4. if $r < p < s$, prove that $||f||_p \leq \max(||f||_r, ||f||_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

5. Assume that $||f||_r < \infty$ for some $r < \infty$ and prove that

$$||f||_p \to ||f||_\infty, \text{as } p \to \infty.$$

Exercise 1.5. Assume, in addition to the hypothesis of Exercise 4, that

$$\mu(X) = 1.$$

1. Prove that $||f||_r \leq ||f||_s$, if $0 < r < s \leq \infty$.

2. Under what conditions does it happen that $0 < r < s \leq \infty$ and $||f||_r = ||f||_s < \infty$?

3. Prove that $L^r(\mu) \subset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

4. Assume that $||f||_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to 0} ||f||_p = \exp\left\{\int_X \log|f| d\mu\right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

Solution.

1.

2.

3.

4. By definition of Lebesgue integral,

$$\int \log|f| d\mu = \int_{|f| \geq 1} \log|f| d\mu - \int_{|f| \leq 1} - \log|f| d\mu$$

since the above is essentially decomposing a integral of a function into that of its positive and negative parts. Note the integral of the positive part cannot be ∞, because otherwise by the inequality $\log t \leq t - 1$,

$$\infty = \int_{|f| \geq 1} \log|f| d\mu = \int_{|f| \geq 1} \frac{1}{p}(|f|^p - 1) d\mu \leq \frac{1}{p} \int_{|f| \geq 1} (|f|^p - 1)$$

showing $||f||_p = \infty$ for any $p > 0$, a contradition to the hypothesis in the question. Therefore $\int \log|f| = -\infty$ or it is finite.

Frist consider the finite case. Take any $p_n \to 0$, define $f_n = \frac{|f|^{p_n} - 1}{p_n}$. Notice $\lim f_n = \frac{d}{d|f|}|f|^p|_{|f|=0} = \log|f|$. Further for any $0 < p_n < r$, if $|f| \geq 1$,

$$|f_n| = \left|\frac{|f|^{p_n} - 1}{p_n}\right| = \int_1^{|f|} s^{p_n - 1} ds \leq \int_1^{|f|} s^{r-1} ds \leq \frac{|f|^r - 1}{r},$$

and if $|f| \leq 1$,

$$|f_n| = \left| \frac{|f|^{p_n} - 1}{p_n} \right| = \int_{|f|}^{1} s^{p_n - 1} ds \leq \int_{1}^{|f|} s^{-1} ds \leq -\log|f|.$$

Therefore $|f_n| \leq \frac{|f|^r - 1}{r} \mathcal{X}_{|f| \geq 1} - \log|f| \mathcal{X}_{|f| \leq 1}$. By Dominated Convergence,

$$\int f_n \to \int \log|f|.$$

Observe, by the inequality $\log t \leq t - 1$ again,

$$\frac{1}{p} \log \int |f|^p \leq \int \frac{|f|^p - 1}{p} \to \int \log|f|,$$

as $p \to 0$. By Jensen,

$$\frac{1}{p} \log \int |f|^p \geq \int \log|f|,$$

for any $p \leq r$, since log is concave over $\mathbb{R}$. Finally the result is obtained by sandwich theorem.
Now if $\int \log|f| d\mu = -\infty$, take $f_n = |f| \mathcal{X}_{|f| > 1/n} + \mathcal{X}_{|f| \leq 1/n}$, where n starts at 1. Clearly,

(a) $||f||_p \leq ||f_n||_p$.
(b) $\log|f_n| = \log|f| \mathcal{X}_{|f| > 1/n}$.
(c) $-\log|f_n| < \log n$

Notice the positive and negative part of $\log|f_n|$, by b), both converges increasingly to those of $\log|f|$. By Monotone Convergence, $\lim_{n\to\infty} \exp\{\int \log|f_n|\} \to \exp\{\int \log|f|\} = 0$. By c) $\int \log|f_n| \neq -\infty$. Obviously $f_n \in L^r(\mu)$, therefore $\log|f_n|$ can only have finite integral by the same proof as in $\log|f|$. Now together with previous arguments, $\lim_{p\to 0} ||f_n||_p \to \exp\{\int \log|f_n|\}$. Therefore by a),

$$\limsup_{p\to 0} ||f||_p = \lim_{n\to\infty} \limsup_{p\to 0} ||f||_p \leq \lim_{n\to\infty} \lim_{p\to 0} ||f_n||_p = \lim_{n\to\infty} \exp\left\{ \int \log|f_n| \right\} = 0.$$

Since $||f||_p \geq 0$, $||f||_p \to 0$, as $p \to 0$.