# Chapter 1

# Positive Borel Measure

**Exercise 1.1.** Let $\{f_n\}$ be a sequence of real non-negative functions on $\mathbb{R}^1$, and consider the following four statements:

1. If $f_1$, $f_2$ are upper semicontinuous so is $f_1 + f_2$.
2. If $f_1, f_2$ are lower semicontinuous, so is $f_1 + f_2$.
3. If each $f_n$ is upper semicontinuous, so is $\sum^\infty f_n$.
4. If each $f_n$ is lower semicontinuous, so is $\sum^\infty f_n$.

**Solution.** Observe

$$\{f_1 + f_2 < a\} = \bigcup_{x \in \mathbb{R}} \{f_1 < x\} \cap \{f_2 < a - x\}$$

is open. To see the left is included in the right, for any $f_1(y) + f_2(y) < a$, take $f_1(y) < x < a - f_2(y)$ and the inclusion holds. Therefore 1 is verified. Similar argument goes with 2 if the above $<$ are replaced with $>$.

Notice 4 holds, fix any $x$ such that $\sum f(x) > a$. Since $f_n$'s are non-negative, there is $N$ such that $\sum^N f_n(x) > a$. Therefore there exists $\delta$, $\sum^N f_n(y) > a$ for any $y \in B_\delta(x)$ since finite sums of lower semicontinuous functions are lower semicontinuous. The proof is complete by observing

$$\sum f_n(y) \geq \sum^N f_n(y) > a$$

To give 3 a counterexample, consider $\sum f_n = \sum \mathcal{X}_{[-n,-1/n] \cup [1/n,n]}$. Obviously, every point but 0 is greater than or equal to 1. Hence

$$\{\sum f_n < 1\} = \{0\}$$

is closed.

**Exercise 1.2.** Let $f$ be an arbitrary complex function on $\mathbb{R}^1$, and define

$$\phi(x, \delta) = \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\},$$
$$\phi(x) = \inf\{\phi(x, \delta) : \delta > 0\}.$$

Prove that $\phi$ is upper semicontinuous, that $f$ is continuous at a point $x$ iff $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a $G_\delta$.

Formulate and prove an analogous statement for general topological spaces in place of $\mathbb{R}^1$.

**Solution.** Only give solution in the general case. Redefine

$$\phi(x) = \inf_{B \ni x} \mathrm{diam} f(B)$$

where the diameter is defined as $\mathrm{diam} A = \sup_{x,y \in A} |x - y|$. Take any $x \in \{\phi(x) < a\}$, there is $B \ni x$, $\mathrm{diam} f(B) < a$. Take any $y \in B$, then $\phi(y) \leq \mathrm{diam} f(B) < a$. This says $\{\phi(x) < a\}$ is open and $\phi$ is upper semicontinuous.

The relation between $\phi$ and continuity of $f$ is trivial. Since

$$\{\phi = 0\} = \bigcap_{q \in \mathbb{Q}^+} \{\phi < q\}$$

the set is a $G_\delta$.

**Exercise 1.3.** Let $X$ be a metric space, with metric $\rho$. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf_{y \in E} \rho(x, y)$$

Show that $\rho_E$ is uniformly continuous function on $X$. If $A$ and $B$ are disjoint nonempty closed subsets of $X$, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

**Solution.** Notice

$$\rho(a, x) + \rho(a, b) \geq \rho(x, b).$$

Taking infimum on $E$ on both sides gives

$$\rho_E(b) - \rho_E(a) \leq \rho(a, b)$$

By symmetry,

$$|\rho_E(b) - \rho_E(a)| \leq \rho(a, b)$$

showing the uniform continuity. Notice $0 \leq f \leq 1$ and $f = 1$ on $B$. Its support lies in $A^c$. Therefore if $B$ is compact, $B \prec f \prec A^c$.

**Exercise 1.4.** Examine the proof of Riesz theorem and prove the following two statments:

1. If $E_1 \subset V_1$ and $E_2 \subset V_2$, where $V_1$ and $V_2$ are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if $E_1, E_2$ are not in $\mathcal{M}$.

2. If $E \in \mathcal{M}_F$, then $E = N \cup K_1 \cup K_2 \ldots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

**Solution.**

1. Take any open set $U$ that covers $E_1 \cup E_2$. Observe

$$\mu(U) \geq \mu(U \cap V_1) + \mu(U \cap V_2) \geq \mu(E_1) + \mu(E_2).$$

   Taking infimum on both sides together with the subadditivity of $\mu$ gives the result.

2. Since $E \in \mathcal{M}_F, \mu(E) < \infty$. Take $K_1$

$$\mu(E) < \mu(K_1) + 1.$$

   Having chosen $K_1, ..., K_n$, denote $G = E - \cup^{n-1} K_i \in \mathcal{M}_F$. Pick $K_n$ such that

$$\mu(G) < \mu(K_n) + 1/n.$$

   Obviously $\mu(E - \bigcup K_n) = 0$ and this completes the proof.

In Exercise 5 to 8, $m$ stands for Lebesgue measure on $\mathbb{R}^1$.

**Exercise 1.5.** Let $E$ be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even through $E$ and $\mathbb{R}^1$ have the same cardinality.

**Solution.** Denote $R$ as the set removed from $[0, 1]$ in construction of the Cantor set. Notice it is comprised of the union of $2^{n-1}$ open intervals of length $3^{-n}$ where $n$ ranges in $\mathbb{N}$. Therefore

$$\mu(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$

and $\mu(E) = \mu(I - R) = 0$.

To see $E$ has the same cardinality as $\mathbb{R}$. Notice each element in $E$ is a decimal of base 3 that has exactly one 1 at the end or no 1 at all. Therefore there is a surjection from $\mathbb{R}$ to $E$, and one from $E$ to decimals of base 2. This completes the proof.

**Exercise 1.6.** Construct a totally disconnected compact set $K \subset \mathbb{R}^1$ such that $m(K) > 0$. If $v$ is lower semicontinuous and $v \leq \mathcal{X}_K$, show that actually $v \leq 0$. Hence $\mathcal{X}_K$ cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali-Carathéodory theorem.

**Solution.** Construct $K$ similarly as the Cantor set in the previous exercise only we remove the middle fourths in place of thirds. Let $R$ be the union of removed intervals. $K$ is compact since each removal left a closed and bounded thus compact subset of $[0, 1]$ and $K$ is the intersection of all these compact sets. Similarly as above,

$$\mu(R) = \sum_{n-1}^{\infty} \frac{2^{n-1}}{2^{2n}} = \sum \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore $\mu(K) = 1/2$. Notice $\{v > 0\}$ is open by definition. But it cannot be a subset of $K$ except for the empty set since $K$ is totally disconnected.

**Exercise 1.7.** If $0 < \epsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $m(E) = \epsilon$. (To say that $A$ is dense in $B$ means that the closure of $A$ contains $B$.)

**Solution.** The construction is similar as before. Notice if one takes out one middle $x$-th every time with $x > 2$, the removed set $R$ has measure

$$m(R) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{x^n} = \frac{1}{x-2}$$

Take $x = 1/\epsilon + 2$ and $m(R) = \epsilon$. $R$ is open since its complement in $[0, 1]$ is compact (as proven before), therefore closed. To see $R$ is dense, notice every removal divides each remaining interval into its halves. Therefore to each point $x$ in $[0, 1]$, there must be a point that has been removed after the $n$-th removal and lies in $B_{1/2^n}(x)$.

**Exercise 1.8.** Construct a Borel set $E \subset \mathbb{R}^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment $I$. Is it possible to have $m(E) < \infty$ for such a set?

**Solution.** Take $E = \mathbb{Q}$. Obviously $0 = m(E \cap I)$. Since $I$ is an nonempty segment, there is an open interval in it, i.e., $m(I) > 0$. Notice $\mathbb{Q}$ is Borel because it is the countable union of singaltons, and singaltons are Borel.

**Exercise 1.9.** Construct a sequence of continuous function $f_n$ on $[0, 1]$ such that $0 \le f_n \le 1$, such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0, 1]$.

**Solution.** Define

$$\begin{aligned}
&K_1 = [0, 1],\\
&K_2 = [0, 1/2], K_3 = [1/2, 1],\\
&K_4 = [0, 1/4], K_5 = [1/4, 1/2], K_6 = [1/2, 3/4], K_7 = [3/4, 1]
\end{aligned}$$

Obviously $m(K_n) \to 0$. To each $K_n = [a, b]$, pick $V_n = (a - 1/2n, b + 1/2n)$ and $f_n$ that $K_n \prec f_n \prec V_n$. Finally,

$$\int f_n dm \le m(V_n) = m(K_n) + 1/n \to 0$$

But $f_n$ does not converge for any $x \in [0, 1]$, since $f_n(x) = 1$ and $f_n(x) = 0$ both infinitely often.

Notice the construction of $f_n$ need not follow that of Urysohn's. Define

$$l((x_1, y_1), (x_2, y_2))(x) = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) + y_1.$$

If $V_n = (a, b)$, $K_n = [c, d]$, and $a < c < d < b$. Pick $s, t$ that, $a < s < c < d < t < b$. Take

$$f_n = \begin{cases} 0, x \le s \\ l((s, 0), (c, 1)), s \le x \le c \\ 1, c \le x \le d \\ l((d, 1), (t, 0)), d \le x \le t \\ 0, x \ge t \end{cases}$$