1 The Basic RBC Model

1.1 Description of Model

Variables

y	output
z	level of technology (exogenous)
k	capital at end of period
L	labor input
c	consumption
I	investment
w	wage
r	interest rate

Households

The economy is populated by a "representative household". That means, all households are identical, both ex ante and ex post.

The household maximizes

$$\max \mathrm{E} \sum_{t=0}^{\infty} \beta^t U(c_t, L_t) \tag{1}$$

subject to the law of motion for end-of-period assets A_t:

$$A_t = (1 + r_t)A_{t-1} + w_t L_t - C_t \tag{2}$$

r_t is the stochastic return to the asset.

We assume that U is strictly concave in consumption and leisure. To rule out Ponzi schemes, we assume a suitable transversality condition.

Firms

Firms rent capital K and labor L from households. In each period, they maximize profits:

$$\max_{K,L} F(K, L, z) - wL - r^K K \tag{3}$$

where r^K is the rental rate of capital.

Notice that the firm solves a sequence of static optimization problems, not a dynamic problem.

Assets and capital

We assume that households have only one asset available. This is equal to physical capital. Then K_{t-1} (the capital available at the beginning of t for production) must be equal to the accumulated savings of households at the end of period $t-1$, A_t.

$$K_{t-1} = A_{t-1} \tag{4}$$

While each individual firm chooses its level of capital in period t, the price mechanism will make that, in general equilibrium, aggregate capital K_{t-1} will equal accumulated savings A_{t-1}. The equilibrium amount of capital used for production in period t is therefore already determined in period $t-1$. We will therefore use the notation k_{t-1} in the formulation of the model.

Allowing for only one asset is not a loss of generality, because

- Capital is the only *physical* asset in the economy, i.e., the only asset that can be in positive net supply.
- Financial assets are all in zero net supply (any long position in the asset must be compensated by a corresponding short position), because we assume a closed economy. Households are the only agents who can in principle hold such an asset. Since all agents are the same, and we only consider symmetric equilibria, each households holds zero assets in equilibrium. Then we can just drop the financial assets from the model. This does not affect the real allocation.

Investment

We assume that physical capital suffers from a constant rate of depreciation, δ. Then aggregate capital evolves according to

$$K_t = (1-\delta)K_{t-1} + I_t \tag{5}$$

where K_{t-1} is capital at the end of period $t-1$, i.e., at the beginning of period t. The rental rate of capital r_t^K is then related to households' net return on capital r_t by

$$r_t^K = r_t + \delta \tag{6}$$

Aggregate Resource Constraint

We assume that the production function $Y_t = F(K_{t-1}, L_t, z_t)$ is constant returns to scale in K and L. This implies that

$$Y_t = r^K K_{t-1} + w_t L_t \tag{7}$$

Plugging (4), (5), (6) and (7) into (2), we obtain the aggregate resource constraint

$$Y_t = C_t + I_t \tag{8}$$

1.2 Model Equations

1.3 Parameters

The model frequency is quarterly.

Discount factor: $\beta = 0.99$; depreciation rate for capital: $\delta = 0.025$; output share of capital: $\alpha = 0.4$; weight of leisure in utility: $\eta = 1.5$; autocorrelation of technology shock: $\rho = 0.95$;

1.4 Functions

Marginal utility of consumption:

$$U_c(c, l) = 1/c \tag{9}$$

Marginal utility of leisure:

$$U_L(c, l) = \eta/(1 - l) \tag{10}$$

production function:

$$F(z, k, l) = zk^{\alpha} l^{(1-\alpha)} \tag{11}$$

Marginal productivity of capital:

$$F_k(z, k, l) = \alpha z (l/k)^{1-\alpha} \tag{12}$$

Marginal productivity of labor:

$$F_L(z, k, l) = (1 - \alpha) z (k/l)^{\alpha} \tag{13}$$

1.5 Equations

Exogenous equation for productivity:

$$\log(z_t) = \rho \log(z_{t-1}) + \epsilon_t \tag{14}$$

Household Euler equation:

$$U_c(c_t, L_t) = \beta(1 + r_{t+1}) U_c(c_{t+1}, L_{t+1}) \tag{15}$$

Optimal capital input: gross interest rate = marginal productivity of capital:

$$F_k(z_t, k_{t-1}, L_t) = (r_t + \delta) \tag{16}$$

Production function

$$y_t = F(z_t, k_{t-1}, L_t) \tag{17}$$

Law of motion for capital:

$$I_t = k_t - (1 - \delta) k_{t-1} \tag{18}$$

Aggregate resource constraint:

$$c_t = y_t - I_t \tag{19}$$

Optimal labor input: wage = marginal productivity of labor:

$$w_t = F_L(z_t, k_{t-1}, L_t) \tag{20}$$

Labor supply:

$$0 = w_t U_c(c_t, L_t) - U_L(c_t, L_t) \tag{21}$$

2 Competitive equilibrium and Planner's solution

2.1 Competitive equilibrium

The equilibrium we consider is competitive, recursive and symmetric. For a definition of this equilibrium, see also Cooley and Prescott (1995).

- *Competititve* equilibrium.

  We assume perfect competition. When making their decisions, economic agents take the stochastic processes for all aggregate variables, in particular prices (interest rate, wage rate) as given.

- *Recursive* equilibrium.

  The decisions of all agents are functions of a fixed number of state variables (and the number of state variables does not change from one period to the next).

  Denote by $d_t(i)$ the individual decision at time t of household i, $s_t(i)$ the n-vector of individual state variables of household i, S_t the n-vector of corresponding aggregate state variables, and z_t the m-vector of stochastic exogenous aggregate states (there are no idiosyncratic exogenous states: representative agent model). Then the individual decision can be written as

  $$d_t(i) = d_i(S_t, s_t(i), z_t) \tag{22}$$

  The corresponding aggregate variables satisfy

  $$D_t = D(S_t, z_t) \tag{23}$$

  Decisions (23) give rise to the aggregate law of motion (ALM)

  $$S_{t+1} = \mathcal{M}(S_t, z_t, z_{t+1}) \tag{24}$$

- *Symmetric* equilibrium.

  Decision functions are the same across agents:

  $$d_i(S, s, z) = d_j(S, s, z), \qquad \forall i, j \tag{25}$$

  Then, in particular,

  $$d_i(S, S, z) = D(S, z) \qquad \forall i \tag{26}$$

That means: if the individual state of an agent equals the aggregate state, then its decision equals the aggregate decision (where aggregates are understood as averages).

We assume that in period 0, all agents start with the same individual state variables. The symmetry of decision functions than imply that states will be the same across agents in all periods.

The concept of *equilibrium* implies

- Given the ALM (24), the decisions (22) are optimal.
- The aggregate decisions are compatible with individual decisions, i.e., D_t equals the integral of d_i over all agents. In a symmetric equilibrium, this simplifies to

$$D(S_t, z_t) = d_i(S_t, S_t, z_t) \tag{27}$$

We assume there is an aggregate transition function

$$S_{t+1} = \mathcal{T}(S_t, D_t, z_t, z_{t+1}) \tag{28}$$

such that a decision function of the form (23) implies an aggregate law of motion (24)

$$S_{t+1} = \mathcal{T}(S_t, D(S_t, z_t), z_t, z_{t+1}) = \mathcal{M}(S_t, z_t, z_{t+1}) \tag{29}$$

2.2 Planner's solution

The model of Section (1) has the following characteristics:

- Perfect competition in all markets.
- We can assume complete asset markets (as explained above, all the financial assets are not traded in equilibrium, so it is not necessary to model them explicitly).
- There are no externalities, taxes or other kind of frictions.

So we guess that the centralized equilibrium is equivalent to the planner's solution. One can show this formally by deriving the FOCs of the planner's problem and show that they are equivalent to the equations of the competitive equilibrium. (One also has to take care of the transversality conditions.)

3 Solving the model by linear approximation

For a start, we compute an approximate solution of the model, based on a linear approximation. This requires the following steps (more details will come later).

1. Find deterministic steady state ($\epsilon_t = 0$ always).
2. Linearize all equations around the steady state.
3. Get linearized solution:
$$x_t = Ax_{t-1} + B\epsilon_t \tag{30}$$
   where x is the vector of all variables in the model. The linearized solution is calculated in Dynare when setting order=1 in the shock_simul command:

   ```
   stoch_simul(order=1,irf=40,hp_filter=1600);
   ```

   The matrices A and B are computed by the Matlab routine "dynare2b.m" in the archive libm.tar.gz.

Dynare provides higher-order (perturbation) approximations by setting order¿1. More about this later.

4 Evaluating the Model

To evaluate the model, we compare the implications of the numerical solution of the model to the data. There are two main tools for this

1. Impulse response functions
2. Second moments of the data and the model solution

To compute them, we first have to find numerical values for the parameters.

4.1 Impulse response

1. Start from the steady state, so (in deviations from steady state) $x_0 = 0$.
2. Assume a shock hits in period 1, so either $\epsilon_1 = 0.01$ (interpret as 1 percent, if x is in logs) or $\epsilon_1 = \sigma_\epsilon$.
3. Assume no further shock uccurs, i.e., $\epsilon_t = 0, \qquad t = 2, 3, \ldots$.

Then simulate the model (means, compute (30) for $t = 1, 2, \ldots, T$) with this sequence of ϵ_t.

Interpretation: response of the economy to a one-time shock. Since the solution is linear(ized), any model simulation can be interpreted as a superposition of impulses responses. In this sense, the impulse response function gives a complete description of the dynamic properties of the linear(ized) system.

4.2 Second Moments

If we assume that the matrix A from the solution of the model (cf. Equ(30)) is asymptotically stable (means: all eigenvalues are strictly smaller than 1 in absolute value), then (30) generates a stationary processes for the vector of variables x, that is stationary in the sense that all first and second moments are constant over time.

The following statistics are often considered:

- Standard deviations of all series (considered typically relative to GDP)
- First-order autocorrelation of all series
- Contemporaneous correlation of all series with GDP
- Correlation of all series with GDP at lags up to ± 5 quarters.

Important: before computing second moments, the model simulations are subject to the same detrending procedure that has been applied to the data. The detrending can have a strong effect on second moments.

4.3 Parameter values: calibration

In econometrics, parameter values are estimated by methods such as LS or ML. One can say that parameters are chosen so as to optimize the fit between model and data. This has an important disadvantage: with many free parameters, it is hard to say whether a good fit is due to a good model, or just the suitable choice of many parameters.

In contrast, the traditional RBC literature uses "calibration". In my opinion, the main rationale for this approach is the separation of parameter determination and model evaluation:

1. Evaluation is done using second moments
2. Parameters are found by
   - first moments (choose parameters so as to get averages right, for example average capital/labor ratio in the data compared to steady state capital/labor ratio in the model)
   - evidence from other data sets, such as micro data, if available

Advantage: if the model fits the data, is not because the parameters where chosen to maximize the fit.

Disadvantage: there is some discretion which information to use (which steady state ratios, which micro data etc.).

References

Cooley, T. F. and E. C. Prescott (1995). Economic growth and business cycles. In T. F. Cooley (Ed.), *Frontiers of Business Cycle Research*. Princeton: Princeton University Press.