3 Perturbation methods

The perturbation approach computes the derivatives of the equilibrium variables w.r.t. the state variables at the deterministic steady state. It also computes the derivatives w.r.t. the standard deviation of the exogenous shock. It therefore allows to compute the effect of uncertainty on behavior.

Perturbation methods

- are *local* solutions: they provide high accuracy in the neighborhood of the deterministic steady state (i.e., in the model with small shocks); their accuracy for big shocks is unknown.
- give polynomial approximations; they cannot handle kinks such as the zero-lower-bound on the interest rate.

The method requires taking higher-order derivatives, which become very long and tedious to compute in real applications, this can only be done automatically on the computer. Dynare gives higher order perturbations, depending on the "order=n" command.

3.1 Simple deterministic growth model

The following gives example calculations for the simplest model, a one-equation deterministic growth model.

Euler equation:

$$u'(c_t) = \beta(1 + f'(k_{t+1}))u'(c_{t+1}) \tag{1}$$

Resource constraint:

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1} = f(k_t) + (1-\delta)k_t - S(k_t) \tag{2}$$

Assuming $k_{t+1} = S(k_t)$ where the function $S(k)$ has to be determined, we can write (1) as function of k:

$$u'\left(c(k)\right) = \beta R(k) u'\left(d(k)\right) \tag{3}$$

where

$$\begin{aligned} c(k) &= f(k) + (1-\delta)k - S(k) \\ d(k) &= f(S(k)) + (1-\delta)S(k) - S(S(k)) \\ R(k) &= (1 + f'(S(k))) \end{aligned} \tag{4}$$

First and second derivatives of these functions are

$$\begin{aligned}
c'(k) &= f'(k) + (1-\delta) - S'(k) \\
c''(k) &= f''(k) - S''(k) \\
d'(k) &= f'(S(k))S'(k) + (1-\delta)S'(k) - S'(S(k))S'(k) \\
d''(k) &= f''(S(k))\left(S'(k)\right)^2 + f'(S(k))S''(k) + (1-\delta)S''(k) \\
&\quad - S''(S(k))\left(S'(k)\right)^2 - S'(S(k))S''(k) \\
R'(k) &= f''(S(k))S'(k) \\
R''(k) &= f'''(S(k))\left(S'(k)\right)^2 + f''(S(k))S''(k)
\end{aligned} \tag{5}$$

Evaluating the derivatives at the steady state gives

$$\begin{aligned}
c'(k^*) &= f'(k^*) + (1-\delta) - S'(k^*) \\
c''(k^*) &= f''(k^*) - S''(k^*) \\
d'(k^*) &= f'(k^*)S'(k^*) + (1-\delta)S'(k^*) - S'(k^*)S'(k^*) \\
d''(k^*) &= f''(k^*)\left(S'(k^*)\right)^2 + f'(k^*)S''(k^*) + (1-\delta)S''(k^*) \\
&\quad - S''(k^*)\left(S'(k^*)\right)^2 - S'(k^*)S''(k^*) \\
R'(k^*) &= f''(k^*)S'(k^*) \\
R''(k^*) &= f'''(k^*)\left(S'(k^*)\right)^2 + f''(k^*)S''(k^*)
\end{aligned} \tag{6}$$

The aim is to learn about the properties of k in the neighborhood of the steady state $k^*S(K^*)$.

The first derivative of $S(k)$

Since (3) holds for all k, we can differentiate it and get

$$u''(c(k))c'(k) = \beta\left[R'(k)u'(d(k)) + R(k)u''(d(k))d'(k)\right] \tag{7}$$

Evaluating (7) at the steady state gives gives

$$\begin{aligned}
&u''(c^*)\left(f'(k^*) + (1-\delta) - S'(k^*)\right) = \\
&\qquad \beta\left[f''(k^*)S'(k^*)u'(c^*) + (1+f'(k^*))u''(c^*)f'(k^*)S'(k^*) + (1-\delta)S'(k^*) - \left(S'(k^*)\right)^2\right]
\end{aligned} \tag{8}$$

Notice that this is a quadratic equation in $S'(k^*)$, which has two solutions. Only one gives a stable solution; we find the right one with the usual methods for linearized equation systems. The policy function obtained from the linearized solution gives the first derivative of the policy function of the deterministic model at the deterministic steady state.

The second derivative of $S(k)$

Differentiating (7) again we get

$$\begin{aligned} u'''(c(k))\left(c'(k)\right)^2 + u''(c(k))c''(k) = \beta\Big[&R''(k)u'(d(k)) + R'(k)u''(d(k))d'(k) \\ &+ R'(k)u''(d(k))d'(k) + R(k)u'''(d(k))\left(d'(k)\right)^2 + R(k)u''(d(k))d''(k)\Big] \end{aligned} \tag{9}$$

Evaluating this at steady state gives

$$\begin{aligned} u'''(c^*)\left(c'(k^*)\right)^2 + u''(c^*)c''(k^*) = \beta\Big[&R''(k^*)u'(c^*) + R'(k^*)u''(c^*)d'(k^*) \\ &+ R'(k^*)u''(c^*)d'(k^*) + R(k^*)u'''(c^*)\left(d'(k^*)\right)^2 + R(k^*)u''(c^*)d''(k^*)\Big] \end{aligned} \tag{10}$$

Using the second derivatives in (6), we get a **linear** equation in $S''(k^*)$.

Higher order derivatives

We can differentiate (9) to obtain higher order derivatives. In each case, the next order derivative can be obtained by solving a linear equation.

3.2 Stochastic case

Now we assume the resource constraint

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t + \sigma\epsilon_{t+1} = S(k_t) + \sigma\epsilon_{t+1} \tag{11}$$

The Euler equation becomes

$$u'\left(c(k;\sigma)\right) = \beta\,\mathrm{E}_t\left[R(k;\sigma)u'\left(d(k;\sigma)\right)\right] \tag{12}$$

$$\begin{aligned} c(k;\sigma) &= f(k) + (1-\delta)k - S(k;\sigma) \\ d(k;\sigma) &= f(S(k;\sigma)+\sigma\epsilon) + (1-\delta)(S(k;\sigma)+\sigma\epsilon) - S(S(k;\sigma)+\sigma\epsilon;\sigma) \\ R(k;\sigma) &= (1 + f'(S(k;\sigma)+\sigma\epsilon)) \end{aligned} \tag{13}$$

Derivatives w.r.t. k, taken at $\sigma = 0$ (deterministic model) are as before. Differentiating w.r.t. σ gives

$$\begin{aligned} c_\sigma(k;\sigma) &= -S_\sigma(k;\sigma) \\ d_\sigma(k;\sigma) &= \left(f(k^*) + 1 - \delta\right)\left(S_\sigma(k;\sigma) + \epsilon\right) - S_k(k;\sigma)\left[S_\sigma(k;\sigma) + \epsilon\right] - S_\sigma(k;\sigma) \\ R_\sigma(k;\sigma) &= f''(k^*)\left(S_\sigma(k;\sigma) + \epsilon\right) \end{aligned} \tag{14}$$

Evaluating at $k = k^*$ and $\sigma = 0$ gives

$$\begin{aligned} c_\sigma(k^*, 0) &= -S_\sigma(k^*, 0) \\ d_\sigma(k^*, 0) &= (f(k^*) + 1 - \delta)\,(S_\sigma(k^*, 0) + \epsilon) - S_k(k^*; 0)\,[S_\sigma(k^*, 0) + \epsilon] - S_\sigma(k^*; 0) \\ R_\sigma(k^*, 0) &= f''(k^*)\,(S_\sigma(k^*, 0) + \epsilon) \end{aligned} \tag{15}$$

Differentiating (12) w.r.t. σ, evaluated at $k = k^*$ and $\sigma = 0$ gives

$$u''(c^*)c_\sigma(k^*; 0) = \beta\,\mathrm{E}_t\left[R_\sigma(k^*; 0)u'(c^*) + R(k^*; 0)u''(c^*)d_\sigma(k^*; 0)\right] \tag{16}$$

Using (15), we see that

$$\begin{aligned} -\,u''(c^*)S_\sigma(k^*, 0) = \beta f''(k^*)\,(S_\sigma(k^*, 0))\,u'(c^*)+ \\ R(k^*; 0)u''(c^*)d_\sigma(k^*; 0)\,[S_\sigma(k^*, 0) - S_k(k^*; 0)S_\sigma(k^*, 0) - S_\sigma(k^*; 0)] \end{aligned} \tag{17}$$

The whole equation is linear in $S_\sigma(k^*, 0)$ without a constant term, so that we conclude $S_\sigma(k^*, 0) = 0$.

Differentiating again w.r.t. σ, we obtain a linear equation in $S_{\sigma\sigma}(k^*, 0)$, the second derivative of saving w.r.t. the standard devation. Since the solution can be written as

$$S(k) = S(k^*, 0) + S_\sigma(k^*, 0)\sigma + S_{\sigma\sigma}(k^*, 0)\sigma^2 + \ldots = S(k^*, 0) + S_{\sigma\sigma}(k^*, 0)\sigma^2/2 + \ldots \tag{18}$$

we see that effect of uncertainty on saving is proportional to the variance σ^2, not the standard deviation σ.

3.3 Pruning

Assume that you have a dynamic stochastic model, and you have obtained a quadratic approximation to the solution by perturbation around the steady state $x^* = 0$:

$$x_t = ax_{t-1} + bx_{t-1}^2 + \sigma\epsilon_t + \ldots \tag{19}$$

with $|a| < 1$. (The quadratic approximation will generally have more terms, collected in the $\ldots$, but this is not essential for what follows.)

The quadratic term in (19) causes problems with stability. Notice that this equation has two deterministic steady states: $x^* = 0$ and $x^* = (1 - a)/b$. The second steady state is most likely an artefact of the quadratic approximation. It is not stable: whenever $x > x^* = (1 - a)/b$, x tends to grow towards infinity.

To avoid instability, researchers often use the "pruned" solution

$$\begin{aligned} x_t^{(1)} &= ax_{t-1}^{(1)} + \sigma\epsilon_t \\ x_t^{(2)} &= ax_{t-1}^{(2)} + b\left(x_{t-1}^{(1)}\right)^2 + c\sigma^2 + \sigma\epsilon_t \end{aligned} \tag{20}$$

- The solution $x_t^{(1)}$ is the linear solution. It is stable by the assumption on a. If ϵ has finite support (this is necessary to make sure the perturbation is properly defined), then $x_t^{(1)}$ is bounded.

- If $x_t^{(1)}$ is bounded, so is $\left(x_{t-1}^{(1)}\right)^2$.

- Then $x_t^{(2)}$ follows a stable linear process, with bounded exogenous input process $\left(x_{t-1}^{(1)}\right)^2 + \sigma\epsilon_t$

For an exact interpretation of the simulation obtained with pruning, cf. Lombardo and Uhlig (2014).

References

Lombardo, G. and H. Uhlig (2014, July). A theory of pruning. Working Paper Series 1696, European Central Bank.