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Optimization

Notes

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Abstract:

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Introduction

0.1. Definitions

Definition 0.1.

We say a functional J is proper if $\operatorname{dom} J \neq \emptyset$ and $J > -\infty$.

0.2. Useful lemmas and Theorems.

Lemma 0.1.

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Hilbert Space H. Then $(x_n)_{n\in\mathbb{N}}$ possesses a weakly convergent subsequence.

Lemma 0.2.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert Space H. Then $(x_n)_{n\in\mathbb{N}}$ converges if and only if it is bounded and possesses at most one weak sequential cluster point.

Theorem 0.1.

Let $f : H \to (\infty, \infty]$ be a convex functional on a Hilbert space. Then the following are equivalent:

(i) f is weakly sequentially lower semicontinuous.

(ii) f is sequentially lower semicontinuous.

(iii) f is lower semicontinuous.

(iv) f is weakly lower semicontinuous.

Lemma 0.3.

A convex set is closed if and only if it is weakly closed.

Lemma 0.4.

Every bounded linear operator over a Banach Space is weakly continuos.

Lemma 0.5 (Parallelogram law).

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Lemma 0.6.
Let $\mathcal{X}$ be a Hausdorff space and let $(f_i)_{i\in I}$ be a family of lower semicontinuous functions from $\mathcal{X}$ to $[-\infty,\infty]$. Then $\sup_{i\in I} f_i$ is lower semi-continuous. If I is finite, then $\min_{i\in I} f_i$ is lower-semicontinuous.

Definition 0.2.
Let $\mathcal{X}$ be a Hausdorff space. The lower semicontinuous envelope of $f : \mathcal{X} \to [-\infty,\infty]$ is

$$\overline{f} = \sup\left\{g : \mathcal{X} \to [-\infty,\infty] \mid \quad g \leq f \text{ and } g \text{ is lower semicontinuous}\right\}.$$

Proposition 0.1.
If C is a compact set in a normed space U, and G is a closed subset of C. Then G is compact.

Proof. Let $\{g_n\}$ a sequence contained in G. Since $G \subset C$ and C compact. $\exists\{g_n\}_k$ subsequence of $\{g_n\}$, contained in G such that $\{g_n\}_k \to g$, as $k \to \infty$, and then since G is closed $g \in G$. Therefore G is compact. $\square$

Clarify what this interval means $[x, x+\alpha y]$

1. Lecture 1

1.1. Infinite-Dimensional Optimization

Existence of solutions. Let (U,d) be a metric space and $J : U \to \overline{\mathbb{R}}$

Let (U,d) a metric space and $C \subset U$

$$\min_{u\in C} J(u)$$

Definition 1.1.
A point $u \in U$ is called:

- **Local Minimizer.** If there is a neighborhood $V \in U$ such that $J(u) \leq J(v)$, $\forall v \in V$.
- **Global Minimizer.** If $J(u) \leq J(v)$, $\forall v \in U$.

Definition 1.2.
Let be $\{u_k\} \in U$, a convergent sequence in U, such that converges to $u \in U$. The functional J is called lower semicontinuous at $u \in U$ if

$$J(u) \leq \liminf_{k\to\infty} J(u_k).$$

In general if J is lower semicontinuous at u, for all the $u \in U$. J is lower semicontinuous (l.s.c).

Theorem 1.1.
Let $J : U \to \mathbb{R}$ lower semicontinuous functional and $\exists\xi \in \mathbb{R}$, such that the level set $\mu_\xi = \{u \in U \mid J(u) \leq \xi\}$ be non-empty and compact set of U. Then there exists a global minimum.

Proof. Let $\alpha := \inf_{u\in U} J(u)$. Then $\exists\{u_n\} \in U$ such that $J(u_n) \to \alpha$. Then $\exists N \in \mathbb{N}$, such that $\forall k \geq N$, $J(u_k) \leq r$ (otherwise $r = \alpha$), then we have since μ_ξ is not empty, $u_k \in \mu_\xi$. Since μ_ξ is compact, $\exists\{u_k\}_l$ a subsequence of $\{u_k\}$ that converges in μ_ξ, i.e. $\{u_k\}_l \to \overline{u} \in \mu_\xi$, as $l \to \infty$. Since α is the infimimum and J is lower semicontinuous and,

$$\alpha \leq J(\overline{u}) \leq \liminf_{l\to\infty} J(u_{k_l})$$

On the other hand, since $J(u_k) \to \alpha$,

$$\liminf_{l\to\infty} J(u_k) \leq \alpha$$

Therefore $J(\overline{u}) = \alpha$, and hence $\overline{u}$ exists and it is a global minimizer. □

Corollary 1.1.
Let U be a Banach space. If the following conditions hold:

- $\exists \mu_\epsilon \in U$ *(level set) non-empty and compact.*
- $J : U \to \overline{\mathbb{R}}$ *is lower semicontinuous.*

Then set of global minimizers G is compact.

Proof. The theorem 1.1 implies that all minimizers are in the set μ_ξ. Therefore by proposition 0.1, G is precompact. Since J is lower semicontinuous, for any convergent sequence $(u_k) \in G$, we have

$$\alpha \leq J(u) \leq \liminf_{k\to\infty} J(u_k) = \alpha$$

Implying that the limit is also a global minimizer. Hence G is closed. □

1.2. Derivatives

Let U and V Banach spaces and $F : U \to V$ a mapping from U to V (that could be non linear).

Definition 1.3.
Let C be a subset of U, let $F : C \to V$, and let $x \in C$ be such that, for all $y \in U$, $\exists\alpha > 0$ and the set $[x, x + \alpha y] \subset C$. Then F is Gâteaux differentiable at x if there exists an operator $\mathsf{D}F(x) \in \mathcal{B}(U, V)$, called the Gâteaux derivative of F at x, such that,

$$\forall(y \in U) \quad \mathsf{D}F(x) = \lim_{\alpha\downarrow 0} \frac{F(x + \alpha y) - F(x)}{\alpha}$$

Thus, the second Gâteaux derivative of F at x is the operator $\mathsf{D}^2F(x) \in \mathcal{B}(U, \mathcal{B}(U, K))$ that satisfies

$$(\forall y \in U) \quad \mathsf{D}^2F(x)y = \lim_{\alpha\downarrow 0} \frac{\mathsf{D}F(x + \alpha y) - \mathsf{D}F(x)}{\alpha}$$

Remark 1.1.
The Gâteaux derivative $\mathsf{D}F(x)$ is unique whenever it exists.

Definition 1.4.
Let $x \in U$, let C a set contained in a neighborhood $\mathcal{V}(x)$ de x, and let $F : C \to V$. Then F is Fréchet differentiable at x if there exists an operator $\mathsf{D}F(x) \in \mathcal{B}(U, V)$, called the Fréchet derivative of F at x, such that

$$\lim_{0\neq\|y\|\to 0} \frac{\|F(x + y) - F(x) - \mathsf{D}F(x)y\|}{\|y\|} = 0.$$

Higher-order Frćhet derivatives are defined inductively. Thus, the second Fréchet derivative of F at x is the operator $\mathsf{D}^2F(x) \in \mathcal{B}(U, \mathcal{B}(U,V))$ that satisfies,

$$\lim_{0\neq\|y\|\to 0} \frac{\left\|\mathsf{D}F(x+y) - \mathsf{D}F(x) - \mathsf{D}^2F(x)y\right\|}{\|y\|} = 0.$$

Lemma 1.1.

Let $x \in U$, let be C a set $\mathcal{V}(x)$ contained in a neighborhood of x, and let $F : C \to V$. Suppose that F is Fréchet differentiable at x. Then the following hold:

- *F is Gâteaux differentiable at x and the two derivatives coincide.*
- *F is continuous at x.*

Proof. Denote the Fréchet derivative of F at x by L_x.

- Let $\alpha > 0$ and $y \in U\backslash\{0\}$. Then

  $$\left\|\frac{F(x+\alpha y) - Fx}{\alpha} - L_x y\right\| = \|y\| \frac{\|F(x+\alpha y) - Fx - L_x(\alpha y)\|}{\|\alpha y\|}$$

  converges to 0 as $\alpha \downarrow 0$, since F is Fréchet differentiable.

- Fix $\epsilon > 0$. By definition 1.4, we can find $\delta \in (0, \frac{\epsilon}{\epsilon + \|L_x\|}]$, such that for all y in the open ball of radius δ and center in zero, (i.e. $\forall y \in B_\delta(0)$),

  $$\|F(x+y) - Fx - L_x y\| \leq \epsilon \|y\|$$

  Thus, $\forall y \in B_\delta(0)$, by triangle inequality,

  $$\begin{aligned} \|F(x+y) - Fx\| &\leq \|F(x+y) - Fx - L_x y\| + \|L_x y\| \\ &\leq \epsilon \|y\| + \|L_x\| \|y\| \\ &\leq \delta(\epsilon + \|L_x\|) \\ &\leq \epsilon. \end{aligned}$$

  It follows that F is continuous at x.

□

Fact 1.1.

Let $x \in U$, let $\mathcal{U}$ be a neighborhood of x, and let G be a real Banach space, let $F : \mathcal{U} \to G$ a mapping from $\mathcal{U}$ to G, let $\mathcal{V}$ be a neighborhood of Tx, and let $R : V \to K$. Suppose that T is Fréchet differentiable at x and that R is Gâteaux differentiable at Tx. Then $R \circ T$ is Gateâux differentiable at x and $\mathsf{D}(R \circ T)(x) = (\mathsf{D}R(Tx)) \circ \mathsf{D}T(x)$. If R is Fréchet differentiable at x, then so is $R \circ T$.

Fact 1.2.

Let $x \in U$, let $\mathcal{U}$ be a neighborhood of x, let G be a real Banach space, and let $F : U \to U$. Suppose that F is twice Fréchet differentiable at x. Then $\forall (y,z) \in U \times U$, $(\mathsf{D}^2F(x)y)z = (\mathsf{D}^2F(x)z)y$.

2. Lecture 2

2.1. Convexity

Definition 2.1.
Let U be linear space. A functional $J : U \to \overline{\mathbb{R}}$ is called convex, if for $t \in [0, 1]$ and $u_1, u_2 \in U$.

$$J(tu_1 + (1-t)u_2) \le tJ(u_1) + (1-t)J(u_2) \tag{1}$$

holds such that the right hand side is well defined.

- J is strictly convex if (1) holds strictly for $\forall u_1, u_2 \in U$, $u_1 \neq u_2$ and $t \in (0, 1)$ with $J(u_1) < \infty$ and $J(u_2) < \infty$.
- An optimization problem,
$$\min_{u \in C} J(u)$$
is called convex if both C and J are convex.

Lemma 2.1.
If C and V are convex in U, then

- $\alpha V = \{w = \alpha v, v \in C\}$ *is convex.*
- $C + V$ *is convex.*

Proof. □

Lemma 2.2.
Let V be a collection of convex sets in U, then $C = \bigcap_{K \in V} K$ is convex.

Proof. If $C = \emptyset$, then C the statement is vacuously true. Consider $C \neq \emptyset$ and $u_1, u_2 \in C$ then $u_1, u_2 \in K$ for all $K \in V$

$$\implies tu_1 + (1-t)u_2 \in K, \quad \forall K \in V \implies tu_1 + (1-t)u_2 \in \bigcap_{K \in V} K$$

□

Lemma 2.3.
Let $C \in U$ convex and $J : C \to \mathbb{R}$. Define $\alpha = \inf_{u \in C} J(u)$. Then the set $\Psi = u | J(u) = \alpha$ is convex, i.e. the solution of

$$\min_{u \in C} J(u)$$

is a convex set.

Proof. Let $u_1, u_2 \in \Psi$ and $u_t = tu_1 + (1-t)u_2$.Sinc J is convex, it holds that $J(u_t) \le tJ(u_t) + (1-t)J(u_t) = \alpha$. Thus $J(u_t) = \alpha$, $\forall t \in [0, 1]$. Implying $u_t \in \Psi$ Hence Ψ is convex. □

Lemma 2.4.
Let U be linear normed space, and $C \subset U$ a convex set and $J : U \to \overline{\mathbb{R}}$ convex functional. Let $\overline{u} \in C$ such that

$$J(\overline{u}) \le J(u) \quad \forall u \in B_\epsilon(\overline{u}) \cap C,$$

for some ball $B_\epsilon(\overline{u})$ in U with center in $\overline{u}$. Then $J(\overline{u}) = \inf_{u \in C} J(u)$. In other words, the local minimizer of a convex optimization problem is also a global minimizer.

Proof. Let $B_\epsilon(\overline{u})$ be an open neighborhood of $\overline{u}$ with $J(\overline{u}) \le J(u)$ for all $u \in B_\epsilon(\overline{u}) \cap C$. Take an arbitrary $u^* \in C$ and consider $u_t = t\overline{u} + (1-t)u^*$. Since C is convex $u_t \in C$.
For some $t \in (0,1)$, $u_t \in B_\epsilon(\overline{u})$.
Thus,

$$J(\overline{u}) \le J(u_t) \le tJ(\overline{u}) + (1-t)J(u^*).$$

We have $\forall t \in [0,1]$ that $(1-t) \le 0$, then

$$(1-t)J(\overline{u}) \le (1-t)J(u^*) \qquad \forall u^* \in C$$

Therefore, $\overline{u}$ is a local minimizer for C. □

Theorem 2.1.

Let U is Banach Space, $C \subset U$ convex and $J : C \to \mathbb{R}$ Gateâux differentiable. Consider the minimization problem.

$$\min_{u \in C} J(u)$$

1. *Let $\overline{u}$ be a local solution. Then $J'(\overline{u}; u - \overline{u}) \ge 0$, $\forall u \in C$.*
2. *If J is convex on C, then $J'(\overline{u}; u - \overline{u}) \ge 0$, $\forall u \in C$ is necessary and sufficient for global optimality of $\overline{u}$*
3. *If J is strictly convex on C, then the minimization problem admits at most one solution.*
4. *If C is closed, and J is convex and continuous with*

$$\lim_{\substack{u \in C \\ \|u\| \to \infty}} J(u) = \infty.$$

   Then a global solution $\overline{u} \in C$ exists.

Proof.

1. Let $\overline{u}$ be a local solution $J(\overline{u}) \le J(u)$, $\forall u \in B_\epsilon(\overline{u}) \cap C$, let $t \in [0,1]$, $u_t = \overline{u} + t(u - \overline{u})$, then $u_t \in C$, since C is convex.

   For small $t > 0$,

$$0 \le \frac{1}{t}\left[J(u_t) - J(u)\right] \le \frac{1}{t}\left[J(\overline{u} + t(u - \overline{u})) - J(u)\right] \xrightarrow{t \downarrow 0} J'(\overline{u}; u - \overline{u})$$

2. Since J is convex we have for $u \in C$, $J(\overline{u} + t(u - \overline{u})) \le J(\overline{u}) + t\left[J(u) - J(\overline{u})\right]$, for $t > 0$

$$\implies J(u) - J(\overline{u}) \ge \frac{1}{t}\left[J(\overline{u} + t(u - \overline{u})) - J(\overline{u})\right] \xrightarrow{t \downarrow 0} J'(\overline{u}; u - \overline{u}) \ge 0.$$

   Therefore $\overline{u}$ is a global minimizer.

3. Assume, that there are two solution for the minimization problem, $\overline{u}, u^* \in C$, such that $\overline{u} \ne u^*$ and $J(\overline{u}) = J(u^*) = \inf_{u \in C} J(u)$. Since J is strictly convex $J(u_t) = J(t\overline{u} + (1-t)u^*) < tJ(\overline{u}) + (1-t)J(u^*) = \alpha$ for all $t \in [0,1]$. Contradicting our assumption that u^* and $\overline{u}$ are solutions.

4. $\alpha = \inf_{u \in C} J(u) \in \mathbb{R} \cup \{-\infty\}$, choose a minimizing sequence $(u_k)_k \subset C$ with $J(u_k) \xrightarrow{k \to \infty} \alpha$

   $\implies (u_k)_k$ is bounded, because $J \to \infty$ as $\|u\| \to \infty$.
   $\implies (u_k)_k$ contains a weakly convergent subsequence $u_{k_e} \xrightarrow[e \to \infty]{} \overline{u} \in C$. Since C is closed and convex.
   $\implies J$ is weakly-lower semicontinuos because it is convex and continuos.

□

3. Lecture 3

Now consider Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the norm defined as $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

Let be $J : H \to \mathbb{R}$ a functional over a Hilbert space H, we define the set,

$$\underset{v \in C \subseteq H}{\arg\min}\, J(x) := \{x \mid x \in H \wedge \forall v \in C :\ J(x) \leq J(v)\}.$$

By Riesz-Fréchet representation formula, exists a unique vector $\nabla J(x) \in H$ such that,

$$(\forall y \in H) \quad J'(x; y) = \langle y, \nabla J(x) \rangle$$

namely Gateâux gradient of J at x.

Lemma 3.1.
Let H Hilbert space and $C \subset H$ closed and convex. Define $P_C : H \to C$,

$$P_C(x) = \underset{v \in C}{\arg\min}\, [\|v - x\|].$$

Then,

1. *P_C is well defined, i.e. $\forall x \in H$, $\exists! u \in C$ such that $P_C(x) = \{u\}$.*
2. *$\forall u, x \in H$, we have $u = P_C(x) \iff u \in C$ and $\langle x - u, v - u \rangle \leq 0$ $\forall v \in C$.*
3. *$\|P_C(u) - P_C(\overline{u})\| \leq \|u - \overline{u}\|$ $\forall u, \overline{u} \in H$, i.e. The projection P_C is non expansive.*
4. *$\langle P_C(u) - P_C(\overline{u}), u - \overline{u} \rangle \leq 0, \quad \forall u, \overline{u} \in H$*
5. *Let be $t > 0$ a real number, then $\forall u \in C$, and $\forall v \in H$, $\phi(t) = \frac{1}{t}\|P_C(u + tv) - u\|$ is non-increasing.*

Proof. 1. First we prove existence, let be $(v_k)_k$ a minimizing sequence in C, such that

$$\|x - v_k\| \to \alpha = \inf_{v \in C} \|x - v\|,$$

By the parallelogram law,

$$\begin{aligned}
2\|v_j - x\|^2 + 2\|v_i - x\|^2 &= \|v_j - v_i\|^2 + \|v_j + v_i - 2x\|^2 \\
2\|v_j - x\|^2 + 2\|v_i - x\|^2 &= \|v_j - v_i\|^2 + 4\left\|\frac{v_j + v_i}{2} - x\right\|^2 \\
\Longrightarrow 2\|v_j - x\|^2 + 2\|v_i - x\|^2 - 4\left\|\frac{v_j + v_i}{2} - x\right\|^2 &= \|v_j - v_i\|^2
\end{aligned}$$

Since C is convex $\frac{v_i + v_j}{2} \in C$, then by definition of α,

$$0 \leq \alpha \leq \left\|\frac{v_j + v_i}{2} - x\right\|$$

Therefore the above equations become in the following inequality,

$$2\|v_j - x\|^2 + 2\|v_i - x\|^2 - 4\alpha^2 \geq \|v_j - v_i\|^2$$

Since $\|v_i - x\| \to \alpha$ and $\|v_j - x\| \to \alpha$, we have that $\|v_j - v_i\| \to 0$, therefore the series is Cauchy and then converges. Since C is closed the series converges to a point $v \in C$.

Second we prove uniqueness, we proceed by contradiction, take $v, v' \in C$ such $v \neq v'$, and both of them minimizing the distant with respect the point x, i.e.

$$\|x - v\| = \|x - v'\| = \alpha = \min_{u \in C} \|u - x\|$$

By the parallelogram law,

$$2\|x - v\|^2 + 2\|x - v'\|^2 = \|2x - v - v'\|^2 + \|v - v'\|^2$$

Since C is convex, $\left\|\frac{v+v'}{2} - x\right\| \geq \alpha$

$$\|v - v'\|^2 = 2\|x - v\|^2 + 2\|x - v'\|^2 - \|2x - v - v'\|^2$$
$$\|v - v'\|^2 = 2\|x - v\|^2 + 2\|x - v'\|^2 - 4\left\|x - \frac{v - v'}{2}\right\|^2$$
$$\|v - v'\|^2 = 2\alpha^2 + 2\alpha^2 - 4\left\|x - \frac{v - v'}{2}\right\|^2 \leq 0$$

Therefore $\|v - v'\| = 0$, and $v = v'$.

By the uniqueness and existence $\arg\min_{u \in C} [\|u - x\|]$ is not empty set and has only one element for each $x \in H$.

Thus, P_C is well defined.

□

Theorem 3.1.
Let H be Hilbert space, $C \subset H$ closed and convex, $J : C \to \mathbb{R}$, Gateâux differentiable at the local solution $\overline{u}$ of $\min_{u \in C} J(u)$. Thus, $J'(\overline{u}; u - \overline{u}) \geq 0$, $\forall u \in C$ and it is equivalent to $\overline{u} = P_C(\overline{u} - \delta \nabla J(\overline{u}))$, $\forall \delta > 0$.

Proof. Since every Hilbert Space is a Banach space, and C is closed and Convex subset of H, and $\overline{u}$ is a solution of minimization problem; we can apply 2.1.
Thus $J'(\overline{u}; u - \overline{u}) \geq 0 \iff \langle u - \overline{u}, \nabla J(\overline{u})\rangle \geq 0 \; \forall u \in C$.
For all $\delta > 0$, we multiply the Gateâux gradient $(-\delta)$ and we have,

$$\langle u - \overline{u}, -\delta \nabla J(\overline{u})\rangle \leq 0 \; \forall u \in C,$$

adding zero to the gradient, $\langle u - \overline{u}, \overline{u} - \delta \nabla J(\overline{u}) - \overline{u}\rangle \leq 0$. Then we set $w \in H$ as $w := \overline{u} - \delta \nabla J(\overline{u})$, and applying lemma 3.1 we have,

$$\overline{u} = P_C(w) \iff \langle u - \overline{u}, w - \overline{u}\rangle$$

Thus,

$$\overline{u} = P_C(\overline{u} - \delta J(\overline{u}))$$

□

3.1. Application

Consider U, Y, Z Hilbert spaces. Let be $J : Y \times U \to \mathbb{R}$ a functional. Consider the minimization problem,

$$\begin{cases} \overline{u} = \min\limits_{y,u} J(y,u) \\ Ay = Bu \quad u \in U_{ad} \subset U \end{cases}$$

For some set U_{ad} closed, convex and bounded. And $A \in \mathcal{B}(Y, Z)$ bounded and invertible with $A^{-1} \in \mathcal{B}(Z, Y)$ and $B \in \mathcal{B}(U, Z)$.

Then we can write $y \in Y$ as a function of $u \in U$,

$$y = y(u) = A^{-1}Bu$$

Consider the reduced cost functional $F(u) := J(y(u), u)$, then our problem is equivalent to

$$\overline{u}_{ad} = \min_{u \in U_{ad}} F(u)$$

Let $(u_k)_k \in U_{ad}$ denote a minimizing sequence, i.e. $F(u_k) \to \inf_{u \in U_{ad}} F(u)$, since $u_k \in U_{ad}$ the sequence is bounded. Therefore we can find a convergent subsequence $u_{k_l} \xrightharpoonup[l\to\infty]{} \overline{u}$, moreover since U_{ad} is closed and convex U_{ad} is weakly closed, implying $\overline{u} \in U_{ad}$

Proposition 3.1.
If J is continuous and weakly lower semicontinuous, then $\overline{u} = \underset{u \in U_{ad}}{\arg\min}\,[F(u)]$.

Proof. If J is weakly lower semicontinuos

$$J(y(\overline{u}), \overline{u}) \le \liminf_{l\to\infty} J(y(u_k), u_k)$$

That is,

$$F(\overline{u}) \le \liminf_{l\to\infty} F(u_k) = \alpha$$

Since $u_{k_l} \xrightharpoonup[l\to\infty]{} \overline{u}$, $\implies$ $y(u_k) \rightharpoonup y(\overline{u})$ and $A^{-1}Bu_k \rightharpoonup A^{-1}B\overline{u}$ ☐

J is Gateâux differentiable, applying the chain rule to F and valuating in u we have

$$F_u(u; h) = J_y(y; \alpha(u; h), u) + J_u(y, u; h)$$

$$\begin{aligned} 0 &\le \langle u - \overline{u}, \nabla_u F(\overline{u})\rangle \quad \forall u \in U_{ad} \\ &= \langle A^{-1}B(u - \overline{u}), \nabla_y J(\overline{y}, \overline{u})\rangle_{Y^*Y} + \langle u - \overline{u}, \nabla_u J(\overline{y}, \overline{u})\rangle_{U^*U} \\ &= \langle A^{-1}B(u - \overline{u}), \nabla_y J(\overline{y}, \overline{u})\rangle_{Y^*Y} + \langle u - \overline{u}, \nabla_u J(\overline{y}, \overline{u})\rangle_{U^*U} \\ &= \langle u - \overline{u}, (A^{-1}B)^* \nabla_y J(y, \overline{u})\rangle_{U^*U} + \langle u - \overline{u}, \nabla_u J(y, \overline{u})\rangle_{U^*U} \\ &= \langle u - \overline{u}, (A^{-1}B)^* \nabla_y J(\overline{y}, \overline{u}) + \nabla_u J(\overline{y}, \overline{u})\rangle_{U^*U} \end{aligned}$$

Setting $p^* = (A^{-1}B)^* \nabla_y J(\overline{y}, \overline{u})$. We have that $\overline{u} = P_{U_{ad}}(\overline{u} - \delta(p^* + \nabla_u J(\overline{y}, \overline{u})))$

4. Lecture 4

Lemma 4.1.
Let U be linear space and $J : U \to \overline{\mathbb{R}}$. Then

1. *If J is convex, then the effective domain $\mathrm{dom}(J) = \{u \in U|\ J(u) < \infty\}$ is convex.*
2. *J is convex $\iff$ $\mathrm{epi}(J) = \{(u, \alpha) \in U \times \mathbb{R} | J(u) \leq \alpha\}$ is convex.*

Proof. Since U and $\mathbb{R}$ are linear spaces, is easy to see that scalar multiplications and sums are well defined over $U \times \mathbb{R}$ and so over $\mathrm{epi}\,(J)$.

1. Assume J convex. If $u_1 \in \mathrm{dom}\,(J)$ and u_2 are elements of $\mathrm{dom}\,(J)$. Therefore, $J(u_1) < \infty$, and $J(u_2) < \infty$, therefore for $t \in [0, 1]$, we have $tJ(u_1) < \infty$ and $(1 - t)J(u_2) < \infty$. Since J is convex,

$$J(tu_1 + (1 - t)u_2) \leq tJ(u_1) + (1 - t)J(u_2) < \infty$$

,

Therefore, $tu_1 + (1 - t)u_2 \in \mathrm{dom}\,(J)$. Hence $\mathrm{dom}\,J$ is convex.

2. First consider J a convex functional, then we have for all $u_1, u_2 \in U$,

$$J(tu_1 + (1 - t)u_2) \leq tJ(u_1) + (1 - t)J(u_2) \quad \forall t \in [0, 1]$$

Let (u_1, α_1), (u_2, α_2) elements of $\mathrm{epi}\,(J)$, then $J(u_1) < \alpha_1$ and $J(u_2) < \alpha_2$. Since J is convex.

$$J(tu_1 + (1 - t)u_2) \leq t\alpha_1 + (1 - t)\alpha_2$$

Then $(tu_1 + (1-t)u_2, t\alpha_1 + (1-t)\alpha_2) \in \mathrm{epi}\,(J)$. Therefore, if J is convex, and (u_1, α_1), (u_2, α_2) are elements of $\mathrm{epi}\,(J)$ then,

$$t(u_1, \alpha_1) + (1 - t)(u_2, \alpha_2) = (tu_1 + (1 - t)u_2, t\alpha_1 + (1 - t)\alpha_2) \in \mathrm{epi}\,(J)$$

Hence $\mathrm{epi}\,(J)$ is convex.

Now assume $\mathrm{epi}\,(J)$ convex. Let (u_1, α_1), (u_2, α_2) elements of $\mathrm{epi}\,(J)$ then $(tu_1 + (1 - t)t\alpha_1 + (1 - t)\alpha_2)$, then

$$J(tu_1 + (1 - t)u_2) \leq t\alpha_1 + (1 - t)\alpha_2 \quad \forall t \in [0, 1]$$

By definition of $\mathrm{epi}\,(J)$, if $u_1, u_2 \in \mathrm{dom}\,J$, then $(u_1, J(u_1))$ and $(u_2, J(u_2))$, are elements of $\mathrm{epi}\,(J)$, therefore

$$J(tu_1 + (1 - t)u_2) \leq tJ(u_1) + (1 - t)J(u_2) \quad \forall t \in [0, 1]$$

Implying that J is convex.

□

Definition 4.1.
Let U a Banach space. Then the function $J : U \to \overline{\mathbb{R}}$ is called lower semi-continuous at $u_0 \in U$ if the following conditions holds:

- If $\forall \epsilon > 0$ there is a neighborhood $B_\delta(u_0)$ of u_0, such that $J(u_0) - \epsilon \leq J(u)\ \forall u \in B_\delta(u_0)$.
- If $J(u_0) \leq \liminf_{n\to\infty} J(u_n)$ holds for each sequence $u_n \in U$.

Remark 4.1.
If the second condition holds, J is called sometimes sequentially semi-continuous. If J is continuous it is also lower semi-continuous.

Theorem 4.1.
Let U be a Banach space and $J : U \to \overline{\mathbb{R}}$. Then sthe following conditions are equivalent.

1. *J is lower semi-continuous, i.e., J is lower semi-continuous at every point in U.*
2. *The* $\operatorname{epi}(J)$ *is closed.*
3. *The level sets $\mu_\xi = \{u \in U | J(u) \leq \xi\}$ is a closed set. Note that the sets μ_ξ are closed if and only if the sets $\gamma_\xi = \{u \in U | J(u) > \xi\}$ are open. (Since $\mu_\xi^c = \gamma_\xi$).*

Proof.

- (1) $\implies$ (2) Let (u_n, ξ_n), be a sequence in $\operatorname{epi}(J)$, such that converges to (u, ξ) in $U \times \mathbb{R}$. Then

  $$J(u) \leq \liminf_{n\to\infty} J(u_n) \leq \liminf_{n\to\infty} \xi_n = \xi.$$

  Hence $(u, \xi) \in \operatorname{epi}(J)$.

- (2) $\implies$ (3)Let $\xi \in \mathbb{R}$ and assume that $(u_n)_{n\in\mathbb{N}}$ is a sequence in μ_ξ that converges to u. Then the set $(u_n, \xi)_{n\in\mathbb{N}}$ is in $\operatorname{epi}(J)$. Since $\operatorname{epi}(J)$ is closed, we conclude that $(u, \xi) \in \operatorname{epi}(J)$, and hence $u \in \mu_k$.

- (3) $\implies$ (1) Let bet $u \in U$ an arbitrary member of the Banach space U, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence that converges to u. And we set the number $\eta = \liminf_{n\to\infty} J(u_n)$. Then we have to prove that $J(u) \leq \eta$. When $\eta = \infty$, the inequality is clear. Therefore we assume that $\eta < +\infty$. Since every sequence in $\mathbb{R}$ has a subsequence that converges to the lim inf, the sequence $(u_n)_n$ has a subsequence $(u_k)_k$, such that $J(u_k) \xrightarrow{k\to\infty} \eta$. Now, we can fix $\xi \in (\eta, \infty)$. By convergence we can find c such that $k \geq c$ implies that $(J(u_k))$ belongs to $(-\infty, \xi)$, therefore the set

  $$\{u_k |\ k \geq c \in \mathbb{N}\} \subset \mu_\xi.$$

  Since the sequence $u_n \to u$, the subsequence $u_k \to u$. And μ_ξ closed implies $u \in \mu_\xi$. Since this holds for all $\eta < \infty$, we take $\xi \downarrow \eta$. Implying $J(u) \leq \eta$.

$\square$

Example 4.1.
The indicator function of a set $C \subset U$, i.e. the function $I_C : U \to [-\infty, \infty]$

$$I_C(u) = \begin{cases} 0, & \text{if } u \in C \\ +\infty, & \textit{otherwise}, \end{cases}$$

is lower semi-continuous if and only if C is closed.

Proof. Take $\xi \in \mathbb{R}$. If $\xi < 0$, the set $\mu_\xi = \emptyset$. If $\xi > 0$, the set $\mu_\xi = C$. Therefore the sets m_ξ, for all $\xi \in \mathbb{R}$ is closed if and only if C is closed. By the theorem 4.1 I_C is lower semi-continuous if and only if C is closed. $\square$

The Dual Systems of Linear Spaces

Two linear spaces X and Y over the same scalar field Γ define a dual system if a fixed bilinear functional on their product is given:

$$(\cdot,\cdot): X \times Y \to \Gamma$$

.

The dual system is called separated if the following two properties hold:

1. $\forall x \in X \backslash \{0\}$ there is $y \in Y$ such that $(x, y) \neq 0$.

2. $\forall y \in Y \backslash \{0\}$ there is $x \in X$ such that $(x, y) \neq 0$.

In other words, X separates points in Y and Y separates points in X. We consider only separated dual systems. For each $x \in X$, we define the application $f_x : Y \to \Gamma$ by

$$f_x(y) = (x, y) \quad \forall y \in Y$$

We observe that f_x is a linear functional on Y and the mapping $x \to f_x, \quad \forall x \in X$, is linear and injective, as can be seen from condition (1). Hence, the correspondence is an embedding. Thus, the elements of X can be identified with the linear functionals on Y. In a similar way, the elements of Y can be considered as linear functionals of X, identifying an element $y \in Y$ with $g_y : X \to \Gamma$, defined by

$$g_y(x) = (x, y), \quad \forall x \in X.$$

Therefore, each dual system of linear spaces defines a mapping from either of the two linear spaces into the space of linear functionals on the other. We set,

$$p_y(x) = |(x, y)| = |g_y(x)|, \quad \forall x \in X$$

$$q_x(y) = |(x, y)| = |f_x(y)|, \quad \forall y \in Y$$

and we observe that $\mathcal{P} = \{p_y| \ y \in Y\}$ is a family of seminorms on X and $\mathcal{Q} = \{q_x| \ y \in X\}$ is a family of seminorms on Y.

Definition 4.2.

If U is a normed space, the the dual space $U^* = \mathcal{B}(U, \mathbb{R})$. Consists of all linear and bonded functionals mapping from U to $\mathbb{R}$.

Theorem 4.2.

Let be U a Banach space, then the dual U^* is also a Banach space relative to the norm of the functionals defined by

$$\|u^*\| = \sup_{\|u\|_U \leq 1} |u^*(u)|$$

Example 4.2.

Let $\Omega \subset \mathbb{R}$ be a measurable set. Let $f \in L^p(\Omega)$. Consider the functional $\phi_g : L^p(\Omega) :\to \mathbb{R}$ defined by,

$$\phi_g(f) = \int_\Omega fg d\mu$$

characterized for some g mapping Ω to the real line. This is a linear functional with respect $L^p(\Omega)$. We want an estimate of the norm of this functional. For this purpose we apply Hölder inequality, with $\frac{1}{p} + \frac{1}{q} = 1$, and $p, q > 1$,

$$\begin{aligned}
\|\phi_g\| &= \sup_{1 \geq \|f\|_{L^p(\Omega)}} \left| \int_\Omega gf d\mu \right| \\
&\leq \sup_{1 \geq \|f\|_{L^p(\Omega)}} \int_\Omega |gf| \, d\mu \\
&\text{By Hölder inequality} \\
&\leq \sup_{1 \geq \|f\|_{L^p(\Omega)}} \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_\Omega |g|^q \, d\mu \right)^{\frac{1}{q}} \\
&= \left(\int_\Omega |g|^q \, d\mu \right)^{\frac{1}{q}} \sup_{1 \geq \|f\|_{L^p(\Omega)}} \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega |g|^q \, d\mu \right)^{\frac{1}{q}} = \|g\|_{L^q(\Omega)}
\end{aligned}$$

This result implies that if $g \in L^q(\Omega)$, then ϕ_g is bounded, hence for all $g \in L^q(\Omega)$ we have that the functionals characterized by g, $\phi_g \in (L^p(\Omega))^*$. It's possible to demonstrate that all $\phi \in (L^p(\Omega))^*$ can be characterized by some g in $L^q(\Omega)$. Thus,

$$L^q(\Omega) = (L^p(\Omega))^*$$

Remark 4.2.

There is a natural duality between U and U^* determined by the bilinear functional $(\cdot, \cdot) : U \times U^* \to \mathbb{R}$, defined as

$$(u, u^*) = u^*(u), \quad \forall u \in U, \forall u^* \in U^*$$

Definition 4.3.

A sequence $(u_n)_n$ in a Banach space is called weakly convergent to some $u \in U$ if for all linears continuous functionals $u^* \in U^*$ we have

$$\lim_{n \to \infty} u^*(u_n) = u^*(u)$$

u is also called the weak-limit and we write $u_n \xrightharpoonup[n \to \infty]{} u$.

Theorem 4.3.

A sequence $(u_n)_n$ in U converges to $u \in U$ if and only if $\sup_{n \in \mathbb{N}} \|u_n\| < \infty$ and $u_n \xrightharpoonup[n \to \infty]{} u$

Theorem 4.4 (Bourbaki-Alaoglu-Katulami).

The closed unit ball in a Banach space U is weakly compact if and only if U is reflexive. If U is in an addition separable, then it's weakly sequentially compact.

Definition 4.4.

Let U be a Banach space and $J : U \to \mathbb{R}$, J is called weakly (sequentially) lower semi-continuous at point u_0 if for every weakly convergent sequence $(u_n)_n$ converges to u_0, i.e. $u_n \rightharpoonup u_0$, it holds

$$J(u) \leq \liminf_{n \to \infty} J(u_n)$$

Definition 4.5.
A non empty set $C \subset U$ is called weakly closed if for every weakly convergent sequence $(u_n)_n$ in C follows that the weak limit belongs to C. i.e. $u_n \rightharpoonup u$, with $u_n \in C$, implies $u \in C$.

Definition 4.6.
A non empty set $C \subset U$ is called weakly sequentially compact if for every sequence in C contains a weakly convergent subsequence whose limit belongs to C.

Theorem 4.5.
Let U be a Banach space and $J : U \to \overline{\mathbb{R}}$ the the following conditions are equivalent:

- *J is weakly lower semi-continuous on U for all $u \in U$.*
- *The level sets $\mu_\xi = \{u \in U | J(u) \leq \xi\}$ is weakly closed for each $\xi \in \mathbb{R}$.*

Lemma 4.2.
Let be $J : U \to \overline{\mathbb{R}}$ a convex and lower semicontinuous functional. Assume there is $u_0 \in U$ such that $J(u_0) = -\infty$, then J is nowhere finite.

Proof. Assume that there is $v \in U$ such that $-infty < J(v) < \infty$. Then by convexity $J(\lambda u_0 + (1-\lambda)v) = -\infty$, $\forall \lambda \in [0, 1]$. Because J is lower semicontinuos it follows that in the limit $\lambda \to 0$,

$$(\lambda u_0 + (1-\lambda)v) \to v \implies J(v) \leq J(\lambda u_0 + (1-\lambda)v) = -\infty$$

□

Lemma 4.3.
Every lower semi-continuous and convex function on a linear space U is weakly lower semi-continuous.

Corollary 4.1.
Assume that U is a reflexive Banach space, then every bounded sequence $(u_n)_n \in U$ that is $\sup_{n \in \mathbb{N}} \|u_n\| < \infty$ has a subsequence $(u_k)_k$ which is weakly convergent to some $u \in U$.

Remark 4.3.
Since every Hilbert space is reflexive the corollary applies to this case.

Lemma 4.4.
A closed set C is weakly closed if and only if the set is convex.

Definition 4.7.
Let U be a real linear space and $J : U \to \overline{\mathbb{R}}$. We said that J is sublinear if:

$$\begin{aligned} J(\lambda u) &= \lambda J(u) \qquad \forall u \in U, \text{ and } \mathbb{R} \ni \lambda > 0 \\ J(u+v) &\leq J(u) + J(v) \qquad \forall u, v \in U \end{aligned}$$

Remark 4.4.
Every sublinear function is convex.

Theorem 4.6.
Let U be a real linear space $J : U \to \overline{\mathbb{R}}$ a sublinear functional. Then there is a linear functional f on U such that,

$$f(u) \leq J(u) \quad \forall u \in U$$

Definition 4.8.
Let $J : U \to \overline{\mathbb{R}}$, we said that J is locally bounded around u_0 if $\exists V \subset U$ neighborhood of u_0 such that for some $M \in \mathbb{R}$

$$|J(u)| < M \qquad \forall u \in V$$

Lemma 4.5.
Let $J : U \to \mathbb{R}$ convex and U is a Banach space. If J is locally bounded around u, then J is lower semi-continuous in u.

Proof. Let $u_k \to u$ as $k \to \infty$. For each $\epsilon > 0$ we can find a sequence α_k such that $\left\| \frac{u-u_k}{\alpha_k} \right\| < \epsilon$, and $\alpha_k \to 0$ as $k \to \infty$. (Please read Maximal Monotone Operators and Evolution Systems in Banach Spaces of Barbu. Details Still to be recovered).
Moreover, for k sufficiently large we have $\|u - u_k\| < \epsilon$. Choose ϵ such that J is bounded in $\overline{B_{2\epsilon}(u)}$ by M and define $v_k = u_k + \frac{u-u_k}{\alpha_k} \in \overline{B_{2\epsilon}(u)}$, since $\|v_k - u\| \leq \|u_k - u\| + \left\| \frac{u-u_k}{\alpha_k} \right\| \leq 2\epsilon$.
Since J is convex

$$J(u) \leq \alpha_k J(v_k) + (1 - \alpha_k)J(u_k) \leq \alpha_k M + J(u_k)$$

.
Since $\alpha_k \to 0$, then

$$J(u) \leq \liminf_{k\to\infty}(\alpha_k M + J(u_k)) = \liminf_{k\to\infty} J(u_k)$$

Thus is if J is convex and locally bounded around u, then is lower semi-continuous around u. □

Remark 4.5.
The result that convexity and local boundedness imply lower semi-continuity is similar to classical result for linear operators where local boundness implies continuity. In general convexity plays in optimization the same role as linearity in solving equations.

5. Lecture 5

5.1. Subgradients

Characterization for J'' still to be analized the professor's notes are flawed.

Definition 5.1.
Let U be a Banach space and let $J : U \to (-\infty, \infty]$ be a convex and proper function. The subdifferential at a point $u \in \operatorname{dom} J$ is a mapping,

$$\partial J : U \to 2^{U^*}, \qquad \partial J(u) := \{p^* \in U^* \mid J(v) \geq J(u) + p^*(v - u),\ \forall v \in U\}$$

The elements of $p^* \in \partial J(u)$ are called subgradients of J at u.

Example 5.1.
Consider $J : \mathbb{R} \to \mathbb{R}$, $u \to |u|$ which is not differentiable at $u = 0$. If $u > 0$, then $J(u) = u$ and we can find $0 < v < u < w$. Then $p^* \in \partial J(u)$ implies by definition of subdifferential

$$\begin{aligned} v - u \geq p^*(v - u) \equiv (1 - p^*)(u - v) \leq 0 \\ w - u \geq p^*(w - u) \equiv (1 - p^*)(w - u) \geq 0. \end{aligned}$$

which implies for $u > 0$, $p^* \leq 1 \leq p^*$, then $p^* = 1$.

In the same way we obtain for $u < 0$, $p^* \geq -1 \geq p^*$. In the case $u = 0$, we need to satisfy $|v| \geq p^* v$, which is fulfilled if and only if $|p^*| \leq 1$. Hence for $J(u) = |u|$,

$$\partial |u| = \begin{cases} \{1\}, & u > 0 \\ [-1, 1], & u = 0 \\ \{-1\}, & u < 0 \end{cases} .$$

Example 5.2.
A convex function which is not subdifferentiable everywhere $J : \mathbb{R} \to \mathbb{R}$,

$$J(u) = \begin{cases} -\sqrt{1 - |u|^2} & |u| \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

For $|u| \geq 1$, we have $\partial J(u) = \emptyset$.

Example 5.3.
Let C be a convex and closed subset of U and I_C function defined by

$$I_C(u) = \begin{cases} 0 & u \in C \\ \infty & \text{otherwise} \end{cases}$$

The subdifferentiable is the definition of normal cone at u

$$\partial I_C(u) = \{u^* \in U^* \mid u^*(u - v) \geq \forall v \in C\} = \mathcal{N}_C(u)$$

.

Theorem 5.1.
Let U be a Banach space. And $J : U \to \overline{\mathbb{R}}$ a subdifferentiable function. Then $\partial J(u)$ is convex and weakly closed.

Remark 5.1.
Most of the rules for derivates also hold for subdifferentials with some additional assumptions,

- $J : U \to \overline{\mathbb{R}}$, $\lambda > 0$, $\partial J(\lambda u) = \lambda J(u)$.
- $\partial(J + F)(u) \supseteq \partial J(u) + \partial F(u)$.

Theorem 5.2 (Rockafeller).
Let U be a Banach space and $J : U \to \mathbb{R}$ proper and convex functions for $i = 1, \dots, n$. The sum-rule

$$\partial(J_1 + \cdots + J_n)(u) = \partial J_1(u) \dots \partial J_n(u), \qquad n \geq 2$$

holds if there exists $u_0 \in U$ such that all $J_i(u_0)$ are finite and all J_i except at most one J_k, $k \in \{1, 2, \dots n\}$ are continuous at u_0

6. Lecture 6

Theorem 6.1.
Let V, U, Hilbert Spaces. Let $J : V \to \overline{\mathbb{R}}$ convex. U and V Banach spaces, $A : U \to V$ linear and continuous with $A^* : U^* \to V^*$. Moreover, J is lower semi-continuous and let $A\overline{u}$ be a point where J is continuous and finite. Then te compose function $J \circ A : U \to \overline{\mathbb{R}}$ is subdifferentiable for all $u \in V$ and,

$$\partial(J \circ A)(u) = A^* \left(\partial J(Au)\right)$$

Proof. Let $p^* \in \partial J(Au)$,

$$J(p) \geq J(Au) + p^*(p - Au) \quad \forall p \in V$$

where $p = Av$ with $v \in U$,

$$(J \circ A)(v) \geq (J \circ A)(u) + p^*(A(v-u)) \quad \forall v \in U \tag{2}$$
$$= (J \circ A)(u) + A^* p^*(v-u) \quad \forall v \in U \tag{3}$$

i.e. $A^*p^* \in \partial(J \circ A)(u) \implies A^* \partial J(Au) \subseteq \partial(J \circ A)(u)$. Proof based again on the weak separation theorem of convex sets. (We have to check Bauschke) □

Theorem 6.2.
If $J : U \to \overline{\mathbb{R}}$ is convex and Frechét-differentiable at $u \in U$, then $\partial J(u) = \{J'(u)\}$

Proof. Let $p^* \in \partial J(u)$. Then for each $t > 0$, $J(u + tv) - J(u) \geq p^*(tv) = tp^*(v)$, diving by t and takin the limit $t \to 0$ we obtain,

$$J'(u)(v) \geq p^*(v) \quad \forall v \in U \tag{4}$$
$$\implies (J'(u) - p^*)(v) \geq 0 \quad \forall v \in U. \tag{5}$$

Since $J'(u)$ is Frechét differentiable the operator $J'(u)$ is linear with respect to v and $p^* \in U^*$ implies $(J'(u) - p^*)$ is linear, taking $-v \in U$, we obtain that $(J'(u) - p^*)(v) \leq 0$. Therefore $p^* = J'(u)$.
On the other hand, if J is differentiable, it follows that $J'(u) \in \partial J(u)$. For $v \in U$, we set $w = v - u$, $u \in U$ we have,

$$J(u + w) - J(u) \geq (J'(u))(w) \tag{6}$$
$$\implies \quad J(v) - J(u) \geq (J'(u))(v - u) \tag{7}$$

Since the above inequality holds for all $v \in U$ implies $J'(u) \in \partial J(u)$. □

Remark 6.1.
The subgradient can be used to obtain local optimality conditions that are necessary and sufficient for convex problem.

Theorem 6.3.
Let U be a Banach Space and $J : U \to \mathbb{R}$ convex and proper. Then each local minimum is global minimum. Moreover $\overline{u} \in U$ is a minimizer if and only if $0 \in \partial J(\overline{u})$.

Proof. If $0 \in \partial J(\overline{u})$: $J(v) \geq J(\overline{u}) + (0)(v - \overline{u}) = J(\overline{u})$, $\forall v \in U$, and hence $\overline{u}$ is a global minimizer.
Assume that $0 \notin J(\overline{u})$, then $\exists v \in U$, such that

$$J(v) < J(\overline{u}) + (0)(v - \overline{u}) = J(\overline{u}).$$

Therefore $\overline{u}$ cannot be a minimizer. □

Definition 6.1 (Duality).
Let $J : U \to \overline{\mathbb{R}}$, and U a Banach space. Then the convex conjugate function $J^* : U^* \to \mathbb{R}$ is defined by

$$J^*(p^*) = \sup_{u \in U} \{p^*(u) - J(u)\}$$

implies that $-\sup_{u\in U} \{p^*(u) - J(u)\} = -J^*(p^*) = \inf_{u\in U} \{J(u) - p^*(u)\}$.

Example 6.1.
Consider the indicator function of a convex set C, $I_C : U \to \overline{\mathbb{R}}$

$$I_C(u) = \begin{cases} 0 & u \in C \\ \infty & \text{otherwise.} \end{cases}$$

Then we have that the convex conjugate is given by

$$J^*(p^*) = \sup_{u\in U} \{p^*(u) - I_C(u)\} = \sup_{u\in C} \{p^*(u)\}.$$

Example 6.2.
$J : \mathbb{R}_+ \to \mathbb{R}$

Example 6.3.
Let $J : \mathbb{R} \to \mathbb{R}$, such that $J(u) = \exp u$, then $J^*(p^*) = \sup_{u\in\mathbb{R}}\{p^*u - \exp u\}$. Let $f(u) = p^*u - \exp(u)$, therefore $f'(u) = p^* - \exp u$, $\forall u \in \mathbb{R}$. Which is zero for $\overline{u} = \ln p^*$, if $p^* > 0$. Since $f''(u) < 0$, then $\overline{u}$ is indeed maximum. And we see that $\lim\limits_{u\to\pm\infty} f(u) = -\infty$. If $p^* = 0$, $f(u) = -\exp u < 0$ and therefore the $\sup_{u\in\mathbb{R}} f(u) = 0$ (Consider the limit when $u \to -\infty$). Then we have,

$$J^*(p^*) = \begin{cases} p^*(\ln p^* - 1) & p^* > 0 \\ 0 & p^* = 0 \end{cases}$$

Example 6.4.
Let U be a Hilbert space and $J(u) = \frac{1}{2}\|u\|^2$. Since U is Hilbert, by Riesz, for each linear and bounded functional $\phi_{p^*} \in H$, $\exists p^* \in H$ such that, $\phi_{p^*}(u) = \langle u, p^*\rangle$. Using the definition of conjugate function,

$$\begin{aligned} J^*(p^*) &= \sup_{u\in U} \left\{\langle u, p^*\rangle - \frac{1}{2}\|u\|^2\right\} \\ &= -\inf_{u\in U} \left\{\frac{1}{2}\|u\|^2 - \langle u, p^*\rangle\right\} \end{aligned}$$

Note that,

$$\frac{1}{2}\|u - p^*\|^2 = \frac{1}{2}\|u\|^2 - \langle u, p^*\rangle + \frac{1}{2}\|p^*\|^2$$

Therefore we can substitute in the above equation to find an equivalent form to the conjugate function,

$$\begin{aligned} J^*(p^*) &= -\inf_{u\in U} \left\{\frac{1}{2}\left(\|u - p^*\|^2 - \|p^*\|^2\right)\right\} \\ &= -\frac{1}{2}\inf_{u\in U} \left\{\|u - p^*\|^2\right\} + \frac{1}{2}\|p^*\|^2 \end{aligned}$$

We have $\|u - p^*\| \geq 0$, $\forall u \in H$, then,

$$\inf_{u\in U} \left\{\|u - p^*\|^2\right\} = 0,$$

since we can take $u = p^*$. Hence,

$$J^*(p^*) = \frac{1}{2}\|p^*\|^2 \tag{8}$$

Theorem 6.4.
Let U be a Banach space and $J : U \to \overline{\mathbb{R}}$. Then J^* is convex.

Proof. Let $p^*, q^* \in U^*$, and $\lambda \in [0,1]$,

$$\begin{aligned}
J^*(\lambda p^* + (1-\lambda)q^*) &= \sup_{u\in U}\{(\lambda p^* + (1-\lambda)q^*)(u) - J(u)\} && (9)\\
&= \sup_{u\in U}\{\lambda p^*(u) - \lambda J(u) + (1-\lambda)q^*(u) - (1-\lambda)J(u)\} && (10)\\
&\leq \sup_{v,u\in U}\{\lambda p^*(u) - \lambda J(u) + (1-\lambda)q^*(v) - (1-\lambda)J(v)\} && (11)\\
&= \sup_{u\in U}\{\lambda p^*(u) - \lambda J(u)\} + \sup_{v\in U}\{(1-\lambda)q^*(v) - (1-\lambda)J(v)\} && (12)\\
&= \lambda J^*(p^*) + (1-\lambda)J^*(q^*). && (13)
\end{aligned}$$

Hence J^* is convex. □

7. Lecture 7

Remark 7.1.
Some elementary properties of conjugate functions

- **Young inequality** $J(u) + J^*(p^*) \geq p^*(u) \quad \forall u \in U, \forall p^* \in U^*$
- $J^*(0) = \sup_{u\in U}((0,u) - J(u)) = \sup_{u\in U}(-J(u)) = \inf_{u\in U} J(u)$

In many applications in optimization, is used the equivalent formulation,

$$\inf_{u\in U} J(u) = -J^*(0).$$

$J \leq F \implies J^* \geq F^*$

Theorem 7.1.
Let U a Banach space and $J^* : U^* \to \overline{\mathbb{R}}$ be the conjugate of the $J : U \to \overline{\mathbb{R}}$. Then for all $u \in U$.

$$p^* \in \partial J(u) \iff J(u) + J^*(p^*) = p^*(u)$$

.

Proof. content... □

Corollary 7.1.
It follows from previous theorem that $\partial J(u) = \{p^* \in U^* | J(u) + J^*(p^*) = (p^*, u)\}$.

Theorem 7.2.
Let U be a Banach space and $J : U \to \mathbb{R}$ be proper function. If $p^* \in \partial J(u)$ then $u \in \partial J^*(p^*)$

Proof. Let $p^* \in \partial J(u)$. For any $g^* \in U^*$, it follows

$$J^*(g^*) = \sup_{v\in U}\{g^*(v) - J(v)\} \geq g^*(u) - J(u) \geq g^*(u) - J(u)$$

From theorem 7.1

$$J^*(g^*) \leq g^*(u) - p^*(u) + J^*(p^*) = (g^* - p^*)(u) + J^*(p^*) \implies u \in \partial J^*(p^*).$$

□

By iteration the definition, we obtain the bipolar function $(J^*)^* = J^{**} : U^{**} \to \overline{\mathbb{R}}$,

$$J^{**}(u) = \sup_{p^* \in U^*} \{p^*(u) - J^*(p^*)\}$$

Theorem 7.3 (Convex envelope theorem.).
Let U be a reflexive Banach space. The J^{**} is the maximum convex functional below J (also called convex envelope), i.e. $J^{**}(u) \leq J(u)$, $\forall u \in U$ and $F(u) \leq J^{**}(u)$, $\forall u \in U$ if F is also convex and $F(u) \leq J(u)$, $\forall u$. In particular $J^{**} = J$ if and only if J is convex.

Proof. Let $\phi_u \in U^{**}$,

$$\begin{aligned} J^{**}(u) &= \sup_{p^* \in U^*} \{p^*(u) - J^*(p^*)\} && (14)\\ &= \sup_{p^* \in U^*} \left\{ p^*(u) - \sup_{v \in U} \{p^*(v) - J(v)\} \right\} && (15)\\ &= \sup_{p^* \in U^*} \left\{ p^*(u) + \inf_{v \in U} \{J(v) - p^*(v)\} \right\} && (16)\\ &= \sup_{p^* \in U^*} \left\{ \inf_{v \in U} \{p^*(u) + J(v) - p^*(v)\} \right\} && (17)\\ &= \sup_{p^* \in U^*} \left\{ \inf_{v \in U} \{p^*(u-v) + J(v)\} \right\} && (18) \end{aligned}$$

Taking $v = u$ in the expression and comparing it with its infimum the inequality holds,

$$\begin{aligned} &\inf_{v \in U} \{p^*(u-v) + J(v)\} \leq p^*(u-u) + J(u)\\ &\inf_{v \in U} \{p^*(u-v) + J(v)\} \leq J(u) \end{aligned}$$

We have that $J^{**}(u) \leq J(u)$.

$$\begin{aligned} \sup_{p^* \in U^*} \inf_{v \in U} \{p^*(u-v) + J(v)\} &\leq J(u)\\ J^{**}(u) &\leq J(u) \end{aligned}$$

Now we assume that F is a convex functional and $g^* \in \partial F(u)$ for $u \in U$.

$$\begin{aligned} \implies F(v) &\geq F(v) + q^*(v-u) && (19)\\ F^{**}(u) &= \sup_{p^* \in U^*} \inf_{v \in U} \{p^*(u-v) + F(u) + q^*(v-u)\} && (20)\\ &\geq \sup_{p^* \in U^*} \inf_{v \in U} \{(p^* - q^*)(u-v) + F(u)\} && (21)\\ &\geq \inf_{v \in U} \{(q^* - q^*)(u-v) + F(u)\} && (22)\\ &= F(u) && (23) \end{aligned}$$

If F is convex,

$$\implies F(u) \leq F^{**}(u) \leq F(u) \implies F(u) = F^{**}(u), \quad (24)$$

$$F(u) = F^{**}(u) = \sup_{p^* \in U^*} \inf_{v \in U} \{p^*(u-v) + F(v)\} \leq J^{**}(u) \quad (25)$$

□

8. Lecture 8

Definition 8.1.

Let U and Y Banach spaces and $J : U \to \overline{\mathbb{R}}$ is a proper function. We consider an optimization problem (P) called primal problem

$$\inf_{u\in U} J(u) \tag{P}$$

Then the problem is said to be nontrivial if there is $\overline{u} \in U$ such that $J(\overline{u}) < \infty$. A function $\Phi : U \times Y \to \overline{\mathbb{R}}$ is said to be a perturbation function of J,

$$\inf_{u\in U} \Phi(u,p) \tag{Pp}$$

if $\Phi(u,0) = J(u)$ for all $u \in U$. For each $p \in Y$, the minimization problem (Pp) is called a perturbation problem. The variable p is called perturbation parameter. If we denote by Φ^* the convex conjugate function of Φ, the *dual problem*, with respect to Φ is defined by

$$\sup_{p^*\in Y^*} -\Phi^*(0,p^*) \tag{P*}$$

where $\Phi^* : (U \times Y)^* \cong U^* \times Y^* \to \overline{\mathbb{R}}$, a function defined as follows.

$$\Phi^*(u^*,p^*) = \sup_{\substack{u\in U\\ p\in Y}} \{u^*(u) + p^*(p) - \Phi(u,p)\}$$

Remark 8.1.

For $p = 0$, (P*) $\equiv$ (Pp). We denote the infimum for problem (P) by $\inf$ (P) and the supremum for problem (P*) by $\sup$ (P*)

Lemma 8.1 (Weak duality).

For the problem (P) *and* (P*) *it holds that*

$$-\infty \le \sup(\text{P*}) \le \inf(\text{P}) \le \infty.$$

Proof. Let $p^* \in Y^*$. It follows

$$\begin{aligned}
-\Phi^*(0,p^*) &= -\sup_{\substack{u\in U\\ p\in Y}} \{0(u) + p^*(p) - \Phi(u,p)\} && (26)\\
&= \inf_{\substack{u\in U\\ p\in Y}} \{\Phi(u,p) - p^*(p)\} && (27)\\
&\le \Phi(u,0) - p^*(0) \quad \forall u \in U, p^* \in Y^* && (28)\\
\implies \sup_{p^*\in Y^*} \{-\Phi(0,p^*)\} &\le \inf_{u\in U} \Phi(u,0) = \inf(P) && (29)
\end{aligned}$$

□

By iteration we can define, a bidual problem

$$-\sup_{u\in U} \{-\Phi^*(u,0)\} = \inf_{u\in U} \Phi^*(u,0) \tag{P**}$$

In case the space U is reflexive then $U^{**} = U$.

If the perturbation function $\Phi(u,p)$ is proper, convex and weakly lower semicontinuous. Then $\Phi^{**} = \Phi$. In this case $\Phi(u,0) = \Phi^{**}(u,0)$ i.e (P)$\equiv$ (P**)

Definition 8.2.
Consider the infimal value function

$$h(p) = \inf(\text{Pp}) = \inf_{u \in U} \Phi(u, p)$$

The problem (P) is called stable if $h(0)$ is finite and its sub-differentiable in zero is not empty.

Theorem 8.1.
The primal problem (P) *is stable if and only if the following conditions are simultaneously satisfied:*

- *The dual problem* (P*) *has a solution.*
- *There is no duality gap, i.e.*

$$\inf(\text{P}) = \sup(\text{P*}) \leq \infty$$

Theorem 8.2 (Extremal relation).
Let $\Phi : U \times Y \to \overline{\mathbb{R}}$*, be convex the the following statements are equivalent:*

1. (P) *and* (Pp) *have solutions* $\overline{u}$ *and* $\overline{p^*}$ *and* $\inf(P) = \sup(P^*)$
2. $\Phi(\overline{u}, 0) + \Phi^* \left(0, \overline{p^*}\right) = 0$
3. $\left(0, \overline{p^*}\right) \in \partial\Phi(u, 0)$ *and* $(\overline{u}, 0) \in \partial\Phi^*(0, p^*)$

Proof. We proceed by parts:

1. (1) $\implies$ (2): $\overline{u}$ solution of $\inf(\text{P})$ and $\overline{p^*}$ solution of $\sup(\text{P*})$ and $\inf(\text{P}) = \sup(\text{P*})$. This properties implies, $\Phi(\overline{u}, 0) = \inf(\text{P}) = \sup(\text{P*}) = -\Phi(0, \overline{p^*}) \implies \Phi(\overline{u}, 0) + \Phi^*(0, \overline{p^*}) = 0$.
2. (2) $\implies$ (1): $-\Phi^*(0, \overline{p^*}) = \sup(\text{P*}) \leq \inf(\text{P}) = \Phi(\overline{u}, 0) = -\Phi^*(0, \overline{p^*}) \implies \sup(\text{P*}) = \inf(\text{P})$
3. (2) $\iff$ (3): $\Phi(\overline{u}, 0) + \Phi^*(0, \overline{p^*}) = 0 = (0, \overline{u}) + (\overline{p^*}, 0) = \left((0, \overline{p^*}), (\overline{u}, 0)\right) \iff (0, \overline{p^*}) \in \partial\Phi(\overline{u}, 0)\ \forall u \in U, \forall p^* \in \partial J(u) \iff J(u) + J^*(p^*) = (p^*, u)$

□

Fencel duality.

Consider the functional $J : U \to \overline{\mathbb{R}}$,

$$J(u) = F(u) + G(Au)$$

with $F : U \to \overline{\mathbb{R}}$, G convex function $G : V \to \overline{\mathbb{R}}$ and $A : U \to V$ bounded and linear.

We introduce the perturbation $\Phi(u, p) = F(u) + G(Au - p)$. The dual problem is obtained with,

$$\Phi^*(0, p^*) = \sup_{\substack{u \in U \\ p \in V}} \{p^*(p) - F(u) - G(Au - p)\}$$

For fixed u we set $q : Au - p$.

$$
\begin{aligned}
\Phi^*(0,p^*) &= \sup_{u\in U}\sup_{q\in V}\left\{p^*\left(Au-q\right)-F(u)-G(q)\right\}\\
&= \sup_{u\in U}\sup_{q\in V}\left\{p^*\left(Au\right)-p^*(q)-F(u)-G(q)\right\}\\
&= \sup_{u\in U}\left\{p^*\left(Au\right)-F(u)\right\}+\sup_{q\in V}\left\{(-p^*)(q)-G(q)\right\}\\
&= \sup_{u\in U}\left\{\left(A^*\circ p^*\right)(u)-F(u)\right\}+\sup_{q\in V}\left\{(-p^*)(q)-G(q)\right\}\\
&=F^*(A^*\circ p^*)+G^*(-p^*)
\end{aligned}
$$

Where $(A^*\circ p^*)\in U^*$, defined as $(A^*\circ p^*):U\to\overline{\mathbb{R}}$

$$(A^*\circ p^*)(u)=p^*(Au)$$

In case U is a Hilbert space A^* is the adjoint operator of A.

9. Lecture 9

We check the optimality conditions.

$$
\begin{aligned}
0 &= \Phi(\overline{u},0)+\Phi^*(u,\overline{p^*})\\
&= F(\overline{u})+G(A\overline{u})+F^*(A^*\overline{p^*})\\
&= [F(\overline{u})+F^*(A^*\overline{p^*})-A^*\circ p^*(u)]+[G(A\overline{u})+G^*(-\overline{p^*})-(-p^*)(A\overline{u})]
\end{aligned}
$$

Using Young inequality $J(u)+J^*(u^*)-u^*(u)\geq 0$, $\forall u\in U$, and $\forall u^*\in U^*$, we see that both square brackets are nonnegative; and the sum is zero. Then

$$
\begin{aligned}
F(\overline{u})+F^*(A^*\overline{p^*})=A^*\circ p^*(u) &\implies A^*p^*\in\partial F(\overline{u})\\
G(A\overline{u})+G^*(-\overline{p^*})=(-p^*)(A\overline{u}) &\implies -p^*\in\partial G(A\overleftarrow{u})
\end{aligned}
$$

F, G are convex and locally bounded, one can show that $\sup\left(\text{P*}\right)=\inf\left(\text{P}\right)$.

Example 9.1 (Denoising with bounded variation.).

Let be $u,v\in L^2(\Omega)$. And let be $g:\Omega\to\mathbb{R}^n$, such that, $g\in C_0^\infty(\Omega,\mathbb{R}^n)$. Consider the following functional $J:L^2(\Omega)\to\mathbb{R}$, defined as follows,

$$J(u)=\frac{1}{2}\int_\Omega|u(x)-v(x)|^2+\alpha\sup_{\|g\|\leq 1}\int_\Omega u\mathrm{div}\,(g)dx$$

Also consider the minimization problem

$$\min_{u \in BV(\Omega)} J(u),$$

restricted to the set of functions with bounded total variations,

$$BV(\Omega) = \left\{ u \in L^1(\Omega) \mid V(u, \Omega) < \infty \right\},$$

where a total bounded variation is defined as,

$$V(u, \Omega) = \sup \left\{ \int_\Omega u \text{ div } (g) dx; \text{ such that } g \in C_0^\infty(\Omega, \mathbb{R}^n), \ \|g\|_\infty \leq 1 \right\}$$

Remark 9.1.
For u smooth enough, it is possible to apply integration by parts, considering the contributions due g has compact support and $\Omega \subset \mathbb{R}^n$, $\int_\Omega u \text{ div } g dx = -\int_\Omega g \cdot \nabla u dx$.

Consider the norm defined on $BV(\Omega)$ as follows,

$$\|u\|_{BV} := \|u\|_{L^1(\Omega)} + V(u, \Omega).$$

If we consider $J(u) = F(u) + G(Au)$, we can set

$$F(u) = \frac{1}{2} \int_\Omega |u(x) - v(x)|^2 \, dx = \frac{1}{2} \|u - v\|^2_{L^2(\Omega)}$$
$$G(Au) = \alpha \int_\Omega |\nabla u| \, dx$$

Where $A := \alpha \nabla$, and $G(u) = \int_\Omega |u| \, dx$. We proceed to give

$$F^*(p^*) = \frac{1}{2} \int_\Omega |p^*(x) - v(x)|^2 - \frac{1}{2} v^2(x) dx$$
$$G^*(p^*) = \begin{cases} 0, & \|p^*\| \leq 1 \\ -\infty, & \text{otherwise} \end{cases}$$

$A^* = -\alpha(\nabla \cdot)$, therefore

$$-J(p^*) = -\frac{1}{2} \int_\Omega -\alpha \nabla \cdot p^* + v^2 + \frac{1}{2} v^2 dx$$

9.1. Lagrangians

Definition 9.1.
The function $L : U \times Y^* \to \overline{\mathbb{R}}$, $-L(u, p^*) = \sup_{p \in Y} \{p^*(p) - \Phi(u, p)\}$, is called Lagrangian or (P) relative to the perturbation Φ. If we denote by Φ_u for fixed $u \in U$ the function $p \to \Phi(u, p)$, then $-L(u, p^*) = \Phi_u^*(p^*)$

Lemma 9.1.
For all $u \in U$, the function $L_u : Y^* \to \overline{\mathbb{R}}$, $p^* \to L(u, p)$ is a concave function (i.e. $-L_u$ is convex) and weak upper semi-continuous. If Φ is convex the for all $p^* \in Y^*$ the function $L_{p^*} : U \to \overline{\mathbb{R}}$, $u \to L(u, p^*)$ is convex.

Proof. □

Without assuming anything about Φ, we obtain

$$\begin{aligned}\Phi^*(u^*,p^*) &= \sup_{u\in U, p\in U}\{u^*(u)+p^*(p)-\Phi(u,p)\}\\ &= \sup_{u\in U}\left\{u^*(u)+\sup_{p\in Y}[p^*(p)-\Phi(u,p)]\right\}\\ &= \sup_{u\in U}\{u^*(u)-L(u,p^*)\}\end{aligned}$$

This implies that,

$$(\mathrm{P}^*)\ \sup_{p^*\in Y^*}\{-\Phi^*(0,p^*)\} = \sup_{p^*\in Y^*}\inf_{u\in U} L(u,p^*)$$

Now we assume that Φ is convex and weak lower semi-continuous, then for $u\in U$, the function $\Phi_u : Y \to \overline{\mathbb{R}}$ is convex and weak lower semi-continuous and thus $\Phi_u^* * = \Phi_u$. Moreover

$$\begin{aligned}\Phi(u,p) &= \Phi_u^{**}(p)\\ &= \sup_{p^*\in Y^*}\{p^*(p)-\Phi_u^*(p)\}\\ &= \sup_{p^*\in Y^*}\{p^*(p)+L(u,p^*)\}\\ &= \sup_{p^*\in Y^*}\{L(u,p^*)\}\end{aligned}$$

Thus,

$$(\mathrm{P})\quad \inf_{u,p}\Phi(u,p) = \inf_{u\in U}\sup_{p^*\in Y^*} L(u,p^*) \tag{30}$$

Remark 9.2.

The problems (P) and (P*) are related to min-max problem we have that the weak duality means

$$\sup\inf L \le \inf\sup L$$

Definition 9.2.

An element $(\overline{u},\overline{p^*})\in U\times Y^*$ is called saddle point of L if

$$L(\overline{u},p^*) \le L(\overline{u},\overline{p^*}) \le L(u,p^*),\quad \forall u\in U, \forall p^*\in Y^*.$$

Theorem 9.1.

Assume that Φ convex and weak lower semicontinuous. Then $(u^*,\overline{p^*})$ is a saddle point of L if and only if $\overline{u}$ is solution of (P), $\overline{p^*}$ is solution of (P) and* $\inf(\mathrm{P}) = \sup(\mathrm{P}^*)$.

Proof. Let $(\overline{u},\overline{p^*})$ be a saddle point of L. We have that,

$$\left.\begin{array}{r} L(\overline{u},\overline{p^*}) = \inf_{u\in U} L(u,\overline{p^*}) = -\Phi^*(0,\overline{p^*}) \\ L(\overline{u},\overline{p^*}) = \sup_{p^*\in Y^*} L(\overline{u},\overline{p^*}) = -\Phi^*(\overline{u},0)\end{array}\right\} \implies \Phi(\overline{u},0)+\Phi^*(0,\overline{p}^*) = 0$$

Theorem about extremal conditions $\implies$ $\overline{u}$ is a solution of (P), $\overline{p^*}$ solution of (P*) and

$$\inf(\mathrm{P}) = \sup(\mathrm{P}^*)$$

"other direction" follows the same argumentation. $\square$

Theorem 9.2 (Saddle point theorem.).
Let $\Phi : U \times Y \to \overline{\mathbb{R}}$ be convex, weak lower semicontinuous and (P) is stable. Then $\overline{u} \in U$ is a solution of (P) if and only if then exist $\overline{p^*} \in Y^*$ such that $(\overline{u}, \overline{p^*})$, is a saddle point of L.

Proof. Out of the scope of the course. According to professor. $\square$

References