

# Optimization

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**Abstract:**

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## Introduction

### 0.1. Definitions

### 0.2. Useful lemmas and Theorems.

**Lemma 0.1.**

Let  $(x_n)_{n \in \mathbb{N}}$  be a bounded sequence in a Hilbert Space  $H$ . Then  $(x_n)_{n \in \mathbb{N}}$  possesses a weakly convergent subsequence.

**Lemma 0.2.**

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in a Hilbert Space  $H$ . Then  $(x_n)_{n \in \mathbb{N}}$  converges if and only if it is bounded and possesses at most one weak sequential cluster point.

**Theorem 0.1.**

Let  $f : H \rightarrow (\infty, \infty]$  be a convex functional on a Hilbert space. Then the following are equivalent:

- (i)  $f$  is weakly sequentially lower semicontinuous.
- (ii)  $f$  is sequentially lower semicontinuous.
- (iii)  $f$  is lower semicontinuous.
- (iv)  $f$  is weakly lower semicontinuous.

**Lemma 0.3.**

A convex set is closed if and only if it is weakly closed.

# 1. Lecture 1

## 2. Lecture 2

### 2.1. Convexity

#### Definition 2.1.

Let  $U$  be linear space. A functional  $J : U \rightarrow \overline{\mathbb{R}}$  is called convex, if for  $t \in [0, 1]$  and  $u_1, u_2 \in U$ .

$$J(tu_1 + (1-t)u_2) \leq tJ(u_1) + (1-t)J(u_2) \quad (1)$$

holds such that the right hand side is well defined.

- $J$  is strictly convex if (1) holds strictly for  $\forall u_1, u_2 \in U, u_1 \neq u_2$  and  $t \in (0, 1)$  with  $J(u_1) < \infty$  and  $J(u_2) \leq \infty$ .
- An optimization problem,

$$\min_{u \in C} J(u)$$

is called convex if both  $C$  and  $J$  are convex.

#### Lemma 2.1.

If  $C$  and  $V$  are convex in  $U$ , then

- $\alpha V = \{w = \alpha v, v \in C\}$  is convex.
- $C + V$  is convex.

*Proof.* □

#### Lemma 2.2.

Let  $V$  be a collection of convex sets in  $U$ , then  $C = \bigcap_{K \in V} K$  is convex.

*Proof.* If  $C = \emptyset$ , then  $C$  the statement is vacuously true. Consider  $C \neq \emptyset$  and  $u_1, u_2 \in C$  then  $u_1, u_2 \in K$  for all  $K \in V$

$$\implies tu_1 + (1-t)u_2 \in K, \quad \forall K \in V \implies tu_1 + (1-t)u_2 \in \bigcap_{K \in V} K$$

□

#### Lemma 2.3.

Let  $C \in U$  convex and  $J : C \rightarrow \mathbb{R}$ . Define  $\alpha = \inf_{u \in C} J(u)$ . Then the set  $\Psi = \{u \mid J(u) = \alpha\}$  is convex, i.e. the solution of

$$\min_{u \in C} J(u)$$

is a convex set.

*Proof.* Let  $u_1, u_2 \in \Psi$  and  $u_t = tu_1 + (1-t)u_2$ . Since  $J$  is convex, it holds that  $J(u_t) \leq tJ(u_1) + (1-t)J(u_2) = \alpha$ . Thus  $J(u_t) = \alpha, \forall t \in [0, 1]$ . Implying  $u_t \in \Psi$ . Hence  $\Psi$  is convex. □

#### Lemma 2.4.

Let  $U$  be linear normed space, and  $C \subset U$  a convex set and  $J : U \rightarrow \overline{\mathbb{R}}$  convex functional. Let  $\bar{u} \in C$  such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in B_\epsilon(\bar{u}) \cap C,$$

for some ball  $B_\epsilon(\bar{u})$  in  $U$  with center in  $\bar{u}$ . Then  $J(\bar{u}) = \inf_{u \in C} J(u)$ . In other words, the local minimizer of a convex optimization problem is also a global minimizer.

*Proof.* Let  $B_\epsilon(\bar{u})$  be an open neighborhood of  $\bar{u}$  with  $J(\bar{u}) \leq J(u)$  for all  $u \in B_\epsilon(\bar{u}) \cap C$ . Take an arbitrary  $u^* \in C$  and consider  $u_t = t\bar{u} + (1-t)u^*$ . Since  $C$  is convex  $u_t \in C$ . For some  $t \in (0, 1)$ ,  $u_t \in B_\epsilon(\bar{u})$ .

Thus,

$$J(\bar{u}) \leq J(u_t) \leq tJ(\bar{u}) + (1-t)J(u^*).$$

We have  $\forall t \in [0, 1]$  that  $(1-t) \leq 0$ , then

$$(1-t)J(\bar{u}) \leq (1-t)J(u^*) \quad \forall u^* \in C$$

Therefore,  $\bar{u}$  is a local minimizer for  $C$ . □

### Theorem 2.1.

Let  $U$  is Banach Space,  $C \subset U$  convex and  $J : C \rightarrow \mathbb{R}$  Gateaux differentiable. Consider the minimization problem.

$$\min_{u \in C} J(u)$$

1. Let  $\bar{u}$  be a local solution. Then  $J'(\bar{u}, u - \bar{u}) \geq 0$ ,  $\forall u \in C$ .
2. If  $J$  is convex on  $C$ , then  $J'(\bar{u}, u - \bar{u}) \geq 0$ ,  $\forall u \in C$  is necessary and sufficient for global optimality of  $\bar{u}$
3. If  $J$  is strictly convex on  $C$ , then the minimization problem admits at most one solution.
4. If  $C$  is closed, and  $J$  is convex and continuous with

$$\lim_{\substack{u \in C \\ \|u\| \rightarrow \infty}} J(u) = \infty.$$

Then a global solution  $\bar{u} \in C$  exists.

*Proof.*

1. Let  $\bar{u}$  be a local solution  $J(\bar{u}) \leq J(u)$ ,  $\forall u \in B_\epsilon(\bar{u}) \cap C$ , let  $t \in [0, 1]$ ,  $u_t = \bar{u} + t(u - \bar{u})$ , then  $u_t \in C$ , since  $C$  is convex.

For small  $t > 0$ ,

$$0 \leq \frac{1}{t} [J(u_t) - J(u)] \leq \frac{1}{t} [J(\bar{u} + t(u - \bar{u})) - J(u)] \xrightarrow{t \downarrow 0} J'(\bar{u}, u - \bar{u})$$

2. Since  $J$  is convex we have for  $u \in C$ ,  $J(\bar{u} + t(u - \bar{u})) \leq J(\bar{u}) + t[J(u) - J(\bar{u})]$ , for  $t > 0$

$$\implies J(u) - J(\bar{u}) \geq \frac{1}{t} [J(\bar{u} + t(u - \bar{u})) - J(\bar{u})] \xrightarrow{t \downarrow 0} J'(\bar{u}, u - \bar{u}) \geq 0.$$

Therefore  $\bar{u}$  is a global minimizer.

3. Assume, that there are two solution for the minimization problem,  $\bar{u}, u^* \in C$ , such that  $\bar{u} \neq u^*$  and  $J(\bar{u}) = J(u^*) = \inf_{u \in C} J(u)$ . Since  $J$  is strictly convex  $J(u_t) = J(t\bar{u} + (1-t)u^*) < tJ(\bar{u}) + (1-t)J(u^*) = \alpha$  for all  $t \in [0, 1]$ . Contradicting our assumption that  $u^*$  and  $\bar{u}$  are solutions.

4.  $\alpha = \inf_{u \in C} J(u) \in \mathbb{R} \cup \{-\infty\}$ , choose a minimizing sequence  $(u_k)_k \subset C$  with  $J(u_k) \xrightarrow{k \rightarrow \infty} \alpha$

$\implies (u_k)_k$  is bounded, because  $J \rightarrow \infty$  as  $\|u\| \rightarrow \infty$ .

$\implies (u_k)_k$  contains a weakly convergent subsequence  $u_{k_e} \xrightarrow{e \rightarrow \infty} \bar{u} \in C$ . Since  $C$  is closed and convex.

$\implies J$  is weakly-lower semicontinuous because it is convex and continuous.

□

### 3. Lecture 3

### 4. Lecture 4

#### 4.1. Lecture 5

References

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