

# Optimization

Department of Mathematics, Hamburg University, Bundesstrasse 55 , 20146, Hamburg, Germany

## Abstract:

**Keywords:** Optimization • Convexity

## Introduction

### 0.1. Definitions

#### Definition 0.1.

We say a functional  $J$  is proper if  $\text{dom } J \neq \emptyset$  and  $J > -\infty$ .

### 0.2. Useful lemmas and Theorems.

#### Lemma 0.1.

Let  $(x_n)_{n \in \mathbb{N}}$  be a bounded sequence in a Hilbert Space  $H$ . Then  $(x_n)_{n \in \mathbb{N}}$  possesses a weakly convergent subsequence.

#### Lemma 0.2.

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in a Hilbert Space  $H$ . Then  $(x_n)_{n \in \mathbb{N}}$  converges if and only if it is bounded and possesses at most one weak sequential cluster point.

#### Theorem 0.1.

Let  $f : H \rightarrow (\infty, \infty]$  be a convex functional on a Hilbert space. Then the following are equivalent:

- (i)  $f$  is weakly sequentially lower semicontinuous.
- (ii)  $f$  is sequentially lower semicontinuous.
- (iii)  $f$  is lower semicontinuous.
- (iv)  $f$  is weakly lower semicontinuous.

#### Lemma 0.3.

A convex set is closed if and only if it is weakly closed.

#### Lemma 0.4.

Every bounded linear operator over a Banach Space is weakly continuous.

**Lemma 0.5 (Parallelogram law).**

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

**Lemma 0.6.**

Let  $\mathcal{X}$  be a Hausdorff space and let  $(f_i)_{i \in I}$  be a family of lower semicontinuous functions from  $\mathcal{X}$  to  $[-\infty, \infty]$ . Then  $\sup_{i \in I} f_i$  is lower semi-continuous. If  $I$  is finite, then  $\min_{i \in I} f_i$  is lower-semicontinuous.

**Definition 0.2.**

Let  $\mathcal{X}$  be a Hausdorff space. The lower semicontinuous envelope of  $f : \mathcal{X} \rightarrow [-\infty, \infty]$  is

$$\bar{f} = \sup \{g : \mathcal{X} \rightarrow [-\infty, \infty] \mid g \leq f \text{ and } g \text{ is lower semicontinuous}\}.$$

## 1. Lecture 1

**Fact 1.1.**

Let  $x \in H$ , let  $U$  be a neighborhood of  $x$ , let  $G$  be a real Banach space, let  $T : U \rightarrow G$ , let  $V$  be a neighborhood of  $Tx$ , and let  $R : V \rightarrow K$ . Suppose that  $T$  is Frchet differentiable at  $x$  and that  $R$  is Gateaux differentiable at  $Tx$ . Then  $R \circ T$  is Gateaux differentiable at  $x$  and  $D(R \circ T)(x) = (DR(Tx)) \circ DT(x)$ . If  $R$  is Fréchet differentiable at  $x$ , then so is  $R \circ T$ .

**Fact 1.2.**

Let  $x \in H$ , let  $U$  be a neighborhood of  $x$ , let  $K$  be a real Banach space, and let  $T : U \rightarrow K$ . Suppose that  $T$  is twice Fréchet differentiable at  $x$ . Then  $\forall (y, z) \in H \times H$ ,  $(D^2T(x)y)z = (D^2T(x)z)y$ .

**Definition 1.1.**

Let  $x \in H$ , let  $C \in \mathcal{V}(x)$ , and let  $T : C \rightarrow K$ . Then  $T$  is Fréchet differentiable at  $x$  if there exists an operator  $DT(x) \in B(H, K)$ , called the Frchet derivative of  $T$  at  $x$ , such that

$$\lim_{0 \neq \|y\| \rightarrow 0} \frac{\|T(x+y) - Tx - DT(x)y\|}{\|y\|} = 0$$

## 2. Lecture 2

### 2.1. Convexity

**Definition 2.1.**

Let  $U$  be linear space. A functional  $J : U \rightarrow \bar{\mathbb{R}}$  is called convex, if for  $t \in [0, 1]$  and  $u_1, u_2 \in U$ .

$$J(tu_1 + (1-t)u_2) \leq tJ(u_1) + (1-t)J(u_2) \quad (1)$$

holds such that the right hand side is well defined.

- $J$  is strictly convex if (1) holds strictly for  $\forall u_1, u_2 \in U$ ,  $u_1 \neq u_2$  and  $t \in (0, 1)$  with  $J(u_1) < \infty$  and  $J(u_2) < \infty$ .
- An optimization problem,

$$\min_{u \in C} J(u)$$

is called convex if both  $C$  and  $J$  are convex.

**Lemma 2.1.**

If  $C$  and  $V$  are convex in  $U$ , then

- $\alpha V = \{w = \alpha v, v \in C\}$  is convex.
- $C + V$  is convex.

*Proof.* □

**Lemma 2.2.**

Let  $V$  be a collection of convex sets in  $U$ , then  $C = \bigcap_{K \in V} K$  is convex.

*Proof.* If  $C = \emptyset$ , then  $C$  the statement is vacuously true. Consider  $C \neq \emptyset$  and  $u_1, u_2 \in C$  then  $u_1, u_2 \in K$  for all  $K \in V$

$$\implies tu_1 + (1-t)u_2 \in K, \quad \forall K \in V \implies tu_1 + (1-t)u_2 \in \bigcap_{K \in V} K$$

□

**Lemma 2.3.**

Let  $C \in U$  convex and  $J : C \rightarrow \mathbb{R}$ . Define  $\alpha = \inf_{u \in C} J(u)$ . Then the set  $\Psi = \{u \mid J(u) = \alpha\}$  is convex, i.e. the solution of

$$\min_{u \in C} J(u)$$

is a convex set.

*Proof.* Let  $u_1, u_2 \in \Psi$  and  $u_t = tu_1 + (1-t)u_2$ . Since  $J$  is convex, it holds that  $J(u_t) \leq tJ(u_1) + (1-t)J(u_2) = \alpha$ . Thus  $J(u_t) = \alpha, \forall t \in [0, 1]$ . Implying  $u_t \in \Psi$ . Hence  $\Psi$  is convex. □

**Lemma 2.4.**

Let  $U$  be linear normed space, and  $C \subset U$  a convex set and  $J : U \rightarrow \overline{\mathbb{R}}$  convex functional. Let  $\bar{u} \in C$  such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in B_\epsilon(\bar{u}) \cap C,$$

for some ball  $B_\epsilon(\bar{u})$  in  $U$  with center in  $\bar{u}$ . Then  $J(\bar{u}) = \inf_{u \in C} J(u)$ . In other words, the local minimizer of a convex optimization problem is also a global minimizer.

*Proof.* Let  $B_\epsilon(\bar{u})$  be an open neighborhood of  $\bar{u}$  with  $J(\bar{u}) \leq J(u)$  for all  $u \in B_\epsilon(\bar{u}) \cap C$ . Take an arbitrary  $u^* \in C$  and consider  $u_t = t\bar{u} + (1-t)u^*$ . Since  $C$  is convex  $u_t \in C$ .

For some  $t \in (0, 1)$ ,  $u_t \in B_\epsilon(\bar{u})$ .

Thus,

$$J(\bar{u}) \leq J(u_t) \leq tJ(\bar{u}) + (1-t)J(u^*).$$

We have  $\forall t \in [0, 1]$  that  $(1-t) \leq 0$ , then

$$(1-t)J(\bar{u}) \leq (1-t)J(u^*) \quad \forall u^* \in C$$

Therefore,  $\bar{u}$  is a local minimizer for  $C$ . □

**Theorem 2.1.**

Let  $U$  is Banach Space,  $C \subset U$  convex and  $J : C \rightarrow \mathbb{R}$  Gateaux differentiable. Consider the minimization problem.

$$\min_{u \in C} J(u)$$

1. Let  $\bar{u}$  be a local solution. Then  $J'(\bar{u}; u - \bar{u}) \geq 0, \forall u \in C$ .

2. If  $J$  is convex on  $C$ , then  $J'(\bar{u}; u - \bar{u}) \geq 0, \forall u \in C$  is necessary and sufficient for global optimality of  $\bar{u}$
3. If  $J$  is strictly convex on  $C$ , then the minimization problem admits at most one solution.
4. If  $C$  is closed, and  $J$  is convex and continuous with

$$\lim_{\substack{u \in C \\ \|u\| \rightarrow \infty}} J(u) = \infty.$$

Then a global solution  $\bar{u} \in C$  exists.

*Proof.*

1. Let  $\bar{u}$  be a local solution  $J(\bar{u}) \leq J(u), \forall u \in B_\epsilon(\bar{u}) \cap C$ , let  $t \in [0, 1]$ ,  $u_t = \bar{u} + t(u - \bar{u})$ , then  $u_t \in C$ , since  $C$  is convex.

For small  $t > 0$ ,

$$0 \leq \frac{1}{t} [J(u_t) - J(u)] \leq \frac{1}{t} [J(\bar{u} + t(u - \bar{u})) - J(u)] \xrightarrow{t \downarrow 0} J'(\bar{u}; u - \bar{u})$$

2. Since  $J$  is convex we have for  $u \in C$ ,  $J(\bar{u} + t(u - \bar{u})) \leq J(\bar{u}) + t[J(u) - J(\bar{u})]$ , for  $t > 0$

$$\implies J(u) - J(\bar{u}) \geq \frac{1}{t} [J(\bar{u} + t(u - \bar{u})) - J(\bar{u})] \xrightarrow{t \downarrow 0} J'(\bar{u}; u - \bar{u}) \geq 0.$$

Therefore  $\bar{u}$  is a global minimizer.

3. Assume, that there are two solution for the minimization problem,  $\bar{u}, u^* \in C$ , such that  $\bar{u} \neq u^*$  and  $J(\bar{u}) = J(u^*) = \inf_{u \in C} J(u)$ . Since  $J$  is strictly convex  $J(u_t) = J(t\bar{u} + (1-t)u^*) < tJ(\bar{u}) + (1-t)J(u^*) = \alpha$  for all  $t \in [0, 1]$ . Contradicting our assumption that  $u^*$  and  $\bar{u}$  are solutions.
4.  $\alpha = \inf_{u \in C} J(u) \in \mathbb{R} \cup \{-\infty\}$ , choose a minimizing sequence  $(u_k)_k \subset C$  with  $J(u_k) \xrightarrow{k \rightarrow \infty} \alpha$

$\implies (u_k)_k$  is bounded, because  $J \rightarrow \infty$  as  $\|u\| \rightarrow \infty$ .

$\implies (u_k)_k$  contains a weakly convergent subsequence  $u_{k_e} \xrightarrow{e \rightarrow \infty} \bar{u} \in C$ . Since  $C$  is closed and convex.

$\implies J$  is weakly-lower semicontinuous because it is convex and continuous.

□

### 3. Lecture 3

Now consider Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  with the norm defined as  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ .

Let be  $J : H \rightarrow \mathbb{R}$  a functional over a Hilbert space  $H$ , we define the set,

$$\arg \min_{v \in C \subseteq H} J(x) := \{x \mid x \in C \wedge \forall v \in C : J(x) \leq J(v)\}.$$

By Riesz-Fréchet representation formula, exists a unique vector  $\nabla J(x) \in H$  such that,

$$(\forall y \in H) \quad J'(x; y) = \langle y, \nabla J(x) \rangle$$

namely Gateaux gradient of  $J$  at  $x$ .

**Lemma 3.1.**

Let  $H$  Hilbert space and  $C \subset H$  closed and convex. Define  $P_C : H \rightarrow C$ ,

$$P_C(x) = \arg \min_{v \in C} [\|v - x\|].$$

Then,

1.  $P_C$  is well defined, i.e.  $\forall x \in H, \exists! u \in C$  such that  $P_C(x) = \{u\}$ .
2.  $\forall u, x \in H$ , we have  $u = P_C(x) \iff u \in C$  and  $\langle x - u, v - u \rangle \leq 0 \ \forall v \in C$ .
3.  $\|P_C(u) - P_C(\bar{u})\| \leq \|u - \bar{u}\| \ \forall u, \bar{u} \in H$ , i.e. The projection  $P_C$  is non expansive.
4.  $\langle P_C(u) - P_C(\bar{u}), u - \bar{u} \rangle \leq 0, \ \forall u, \bar{u} \in H$
5. Let be  $t > 0$  a real number, then  $\forall u \in C$ , and  $\forall v \in H$ ,  $\phi(t) = \frac{1}{t} \|P_C(u + tv) - u\|$  is non-increasing.

*Proof.*

1. First we prove existence, let be  $(v_k)_k$  a minimizing sequence in  $C$ , such that

$$\|x - v_k\| \rightarrow \alpha = \inf_{v \in C} \|x - v\|,$$

By the parallelogram law,

$$\begin{aligned} 2\|v_j - x\|^2 + 2\|v_i - x\|^2 &= \|v_j - v_i\|^2 + \|v_j + v_i - 2x\|^2 \\ 2\|v_j - x\|^2 + 2\|v_i - x\|^2 &= \|v_j - v_i\|^2 + 4\left\|\frac{v_j + v_i}{2} - x\right\|^2 \\ \implies 2\|v_j - x\|^2 + 2\|v_i - x\|^2 - 4\left\|\frac{v_j + v_i}{2} - x\right\|^2 &= \|v_j - v_i\|^2 \end{aligned}$$

Since  $C$  is convex  $\frac{v_i + v_j}{2} \in C$ , then by definition of  $\alpha$ ,

$$0 \leq \alpha \leq \left\|\frac{v_j + v_i}{2} - x\right\|$$

Therefore the above equations become in the following inequality,

$$2\|v_j - x\|^2 + 2\|v_i - x\|^2 - 4\alpha^2 \geq \|v_j - v_i\|^2$$

Since  $\|v_i - x\| \rightarrow \alpha$  and  $\|v_j - x\| \rightarrow \alpha$ , we have that  $\|v_j - v_i\| \rightarrow 0$ , therefore the series is Cauchy and then converges. Since  $C$  is closed the series converges to a point  $v \in C$ .

Second we prove uniqueness, we proceed by contradiction, take  $v, v' \in C$  such  $v \neq v'$ , and both of them minimizing the distant with respect the point  $x$ , i.e.

$$\|x - v\| = \|x - v'\| = \alpha = \min_{u \in C} \|u - x\|$$

By the parallelogram law,

$$2\|x - v\|^2 + 2\|x - v'\|^2 = \|2x - v - v'\|^2 + \|v - v'\|^2$$

Since  $C$  is convex,  $\left\| \frac{v+v'}{2} - x \right\| \geq \alpha$

$$\begin{aligned} \|v - v'\|^2 &= 2\|x - v\|^2 + 2\|x - v'\|^2 - \|2x - v - v'\|^2 \\ \|v - v'\|^2 &= 2\|x - v\|^2 + 2\|x - v'\|^2 - 4\left\|x - \frac{v+v'}{2}\right\|^2 \\ \|v - v'\|^2 &= 2\alpha^2 + 2\alpha^2 - 4\left\|x - \frac{v+v'}{2}\right\|^2 \leq 0 \end{aligned}$$

Therefore  $\|v - v'\| = 0$ , and  $v = v'$ .

By the uniqueness and existence  $\arg \min_{u \in C} [\|u - x\|]$  is not empty set and has only one element for each  $x \in H$ .

Thus,  $P_C$  is well defined. □

### Theorem 3.1.

Let  $H$  be Hilbert space,  $C \subset H$  closed and convex,  $J : C \rightarrow \mathbb{R}$ , Gateaux differentiable at the local solution  $\bar{u}$  of  $\min_{u \in C} J(u)$ . Thus,  $J'(\bar{u}; u - \bar{u}) \geq 0$ ,  $\forall u \in C$  and it is equivalent to  $\bar{u} = P_C(\bar{u} - \delta \nabla J(\bar{u}))$ ,  $\forall \delta > 0$ .

*Proof.* Since every Hilbert Space is a Banach space, and  $C$  is closed and Convex subset of  $H$ , and  $\bar{u}$  is a solution of minimization problem; we can apply 2.1.

Thus  $J'(\bar{u}; u - \bar{u}) \geq 0 \iff \langle u - \bar{u}, \nabla J(\bar{u}) \rangle \geq 0 \forall u \in C$ .

For all  $\delta > 0$ , we multiply the Gateaux gradient  $(-\delta)$  and we have,

$$\langle u - \bar{u}, -\delta \nabla J(\bar{u}) \rangle \leq 0 \quad \forall u \in C,$$

adding zero to the gradient,  $\langle u - \bar{u}, \bar{u} - \delta \nabla J(\bar{u}) - \bar{u} \rangle \leq 0$ . Then we set  $w \in H$  as  $w := \bar{u} - \delta \nabla J(\bar{u})$ , and applying lemma 3.1 we have,

$$\bar{u} = P_C(w) \iff \langle u - \bar{u}, w - \bar{u} \rangle$$

Thus,

$$\bar{u} = P_C(\bar{u} - \delta \nabla J(\bar{u}))$$

□

## 3.1. Application

Consider  $U, Y, Z$  Hilbert spaces. Let be  $J : Y \times U \rightarrow \mathbb{R}$  a functional. Consider the minimization problem,

$$\begin{cases} \bar{u} = \min_{y, u} J(y, u) \\ Ay = Bu \quad u \in U_{ad} \subset U \end{cases}$$

For some set  $U_{ad}$  closed, convex and bounded. And  $A \in \mathcal{B}(Y, Z)$  bounded and invertible with  $A^{-1} \in \mathcal{B}(Z, Y)$  and  $B \in \mathcal{B}(U, Z)$ .

Then we can write  $y \in Y$  as a function of  $u \in U$ ,

$$y = y(u) = A^{-1}Bu$$

Consider the reduced cost functional  $F(u) := J(y(u), u)$ , then our problem is equivalent to

$$\bar{u}_{ad} = \min_{u \in U_{ad}} F(u)$$

Let  $(u_k)_k \in U_{ad}$  denote a minimizing sequence, i.e.  $F(u_k) \rightarrow \inf_{u \in U_{ad}} F(u)$ , since  $u_k \in U_{ad}$  the sequence is bounded. Therefore we can find a convergent subsequence  $u_{k_l} \xrightarrow{l \rightarrow \infty} \bar{u}$ , moreover since  $U_{ad}$  is closed and convex  $U_{ad}$  is weakly closed, implying  $\bar{u} \in U_{ad}$

### Proposition 3.1.

If  $J$  is continuous and weakly lower semicontinuous, then  $\bar{u} = \arg \min_{u \in U_{ad}} [F(u)]$ .

*Proof.* If  $J$  is weakly lower semicontinuous

$$J(y(\bar{u}), \bar{u}) \leq \liminf_{l \rightarrow \infty} J(y(u_k), u_k)$$

That is,

$$F(\bar{u}) \leq \liminf_{l \rightarrow \infty} F(u_k) = \alpha$$

Since  $u_{k_l} \xrightarrow{l \rightarrow \infty} \bar{u}$ ,  $\implies y(u_k) \rightharpoonup y(\bar{u})$  and  $A^{-1}Bu_k \rightharpoonup A^{-1}B\bar{u}$  □

$J$  is Gateaux differentiable, applying the chain rule to  $F$  and valuating in  $u$  we have

$$F_u(u; h) = J_y(y; \alpha(u; h), u) + J_u(y, u; h)$$

$$\begin{aligned} 0 &\leq \langle u - \bar{u}, \nabla_u F(\bar{u}) \rangle \quad \forall u \in U_{ad} \\ &= \langle A^{-1}B(u - \bar{u}), \nabla_y J(\bar{y}, \bar{u}) \rangle_{Y^*Y} + \langle u - \bar{u}, \nabla_u J(\bar{y}, \bar{u}) \rangle_{U^*U} \\ &= \langle A^{-1}B(u - \bar{u}), \nabla_y J(\bar{y}, \bar{u}) \rangle_{Y^*Y} + \langle u - \bar{u}, \nabla_u J(\bar{y}, \bar{u}) \rangle_{U^*U} \\ &= \langle u - \bar{u}, (A^{-1}B)^* \nabla_y J(\bar{y}, \bar{u}) \rangle_{U^*U} + \langle u - \bar{u}, \nabla_u J(\bar{y}, \bar{u}) \rangle_{U^*U} \\ &= \langle u - \bar{u}, (A^{-1}B)^* \nabla_y J(\bar{y}, \bar{u}) + \nabla_u J(\bar{y}, \bar{u}) \rangle_{U^*U} \end{aligned}$$

Setting  $p^* = (A^{-1}B)^* \nabla_y J(\bar{y}, \bar{u})$ . We have that  $\bar{u} = P_{U_{ad}}(\bar{u} - \delta(p^* + \nabla_u J(\bar{y}, \bar{u})))$

## 4. Lecture 4

### Lemma 4.1.

Let  $U$  be linear space and  $J : U \rightarrow \bar{\mathbb{R}}$ . Then

1. If  $J$  is convex, then the effective domain  $\text{dom}(J) = \{u \in U \mid J(u) < \infty\}$  is convex.
2.  $J$  is convex  $\iff \text{epi}(J) = \{(u, \alpha) \in U \times \mathbb{R} \mid J(u) \leq \alpha\}$  is convex.

*Proof.* Since  $U$  and  $\mathbb{R}$  are linear spaces, is easy to see that scalar multiplications and sums are well defined over  $U \times \mathbb{R}$  and so over  $\text{epi}(J)$ .

1. Assume  $J$  convex. If  $u_1 \in \text{dom}(J)$  and  $u_2$  are elements of  $\text{dom}(J)$ . Therefore,  $J(u_1) < \infty$ , and  $J(u_2) < \infty$ , therefore for  $t \in [0, 1]$ , we have  $tJ(u_1) < \infty$  and  $(1-t)J(u_2) < \infty$ . Since  $J$  is convex,

$$J(tu_1 + (1-t)u_2) \leq tJ(u_1) + (1-t)J(u_2) < \infty$$

,

Therefore,  $tu_1 + (1-t)u_2 \in \text{dom}(J)$ . Hence  $\text{dom } J$  is convex.

2. First consider  $J$  a convex functional, then we have for all  $u_1, u_2 \in U$ ,

$$J(tu_1 + (1-t)u_2) \leq tJ(u_1) + (1-t)J(u_2) \quad \forall t \in [0, 1]$$

Let  $(u_1, \alpha_1), (u_2, \alpha_2)$  elements of  $\text{epi}(J)$ , then  $J(u_1) < \alpha_1$  and  $J(u_2) < \alpha_2$ . Since  $J$  is convex.

$$J(tu_1 + (1-t)u_2) \leq t\alpha_1 + (1-t)\alpha_2$$

Then  $(tu_1 + (1-t)u_2, t\alpha_1 + (1-t)\alpha_2) \in \text{epi}(J)$ . Therefore, if  $J$  is convex, and  $(u_1, \alpha_1), (u_2, \alpha_2)$  are elements of  $\text{epi}(J)$  then,

$$t(u_1, \alpha_1) + (1-t)(u_2, \alpha_2) = (tu_1 + (1-t)u_2, t\alpha_1 + (1-t)\alpha_2) \in \text{epi}(J)$$

Hence  $\text{epi}(J)$  is convex.

Now assume  $\text{epi}(J)$  convex. Let  $(u_1, \alpha_1), (u_2, \alpha_2)$  elements of  $\text{epi}(J)$  then  $(tu_1 + (1-t)u_2, t\alpha_1 + (1-t)\alpha_2)$ , then

$$J(tu_1 + (1-t)u_2) \leq t\alpha_1 + (1-t)\alpha_2 \quad \forall t \in [0, 1]$$

By definition of  $\text{epi}(J)$ , if  $u_1, u_2 \in \text{dom } J$ , then  $(u_1, J(u_1))$  and  $(u_2, J(u_2))$ , are elements of  $\text{epi}(J)$ , therefore

$$J(tu_1 + (1-t)u_2) \leq tJ(u_1) + (1-t)J(u_2) \quad \forall t \in [0, 1]$$

Implying that  $J$  is convex.

□

#### Definition 4.1.

Let  $U$  a Banach space. Then the function  $J : U \rightarrow \overline{\mathbb{R}}$  is called lower semi-continuous at  $u_0 \in U$  if the following conditions holds:

- If  $\forall \epsilon > 0$  there is a neighborhood  $B_\delta(u_0)$  of  $u_0$ , such that  $J(u_0) - \epsilon \leq J(u) \quad \forall u \in B_\delta(u_0)$ .
- If  $J(u_0) \leq \liminf_{n \rightarrow \infty} J(u_n)$  holds for each sequence  $u_n \in U$ .

#### Remark 4.1.

If the second condition holds,  $J$  is called sometimes sequentially semi-continuous. If  $J$  is continuous it is also lower semi-continuous.

#### Theorem 4.1.

Let  $U$  be a Banach space and  $J : U \rightarrow \overline{\mathbb{R}}$ . Then the following conditions are equivalent.

1.  $J$  is lower semi-continuous, i.e.,  $J$  is lower semi-continuous at every point in  $U$ .
2. The  $\text{epi}(J)$  is closed.



3. The level sets  $\mu_\xi = \{u \in U \mid J(u) \leq \xi\}$  is a closed set. Note that the sets  $\mu_\xi$  are closed if and only if the sets  $\gamma_\xi = \{u \in U \mid J(u) > \xi\}$  are open. (Since  $\mu_\xi^c = \gamma_\xi$ ).

*Proof.*

- (1)  $\implies$  (2) Let  $(u_n, \xi_n)$ , be a sequence in  $\text{epi}(J)$ , such that converges to  $(u, \xi)$  in  $U \times \mathbb{R}$ . Then

$$J(u) \leq \liminf_{n \rightarrow \infty} J(u_n) \leq \liminf_{n \rightarrow \infty} \xi_n = \xi.$$

Hence  $(u, \xi) \in \text{epi}(J)$ .

- (2)  $\implies$  (3) Let  $\xi \in \mathbb{R}$  and assume that  $(u_n)_{n \in \mathbb{N}}$  is a sequence in  $\mu_\xi$  that converges to  $u$ . Then the set  $(u_n, \xi)_{n \in \mathbb{N}}$  is in  $\text{epi}(J)$ . Since  $\text{epi}(J)$  is closed, we conclude that  $(u, \xi) \in \text{epi}(J)$ , and hence  $u \in \mu_\xi$ .
- (3)  $\implies$  (1) Let bet  $u \in U$  an arbitrary member of the Banach space  $U$ , and let  $(u_n)_{n \in \mathbb{N}}$  be a sequence that converges to  $u$ . And we set the number  $\eta = \liminf_{n \rightarrow \infty} J(u_n)$ . Then we have to prove that  $J(u) \leq \eta$ . When  $\eta = \infty$ , the inequality is clear. Therefore we assume that  $\eta < +\infty$ . Since every sequence in  $\mathbb{R}$  has a subsequence that converges to the liminf, the sequence  $(u_n)_n$  has a subsequence  $(u_k)_k$ , such that  $J(u_k) \xrightarrow{k \rightarrow \infty} \eta$ . Now, we can fix  $\xi \in (\eta, \infty)$ . By convergence we can find  $c$  such that  $k \geq c$  implies that  $(J(u_k))$  belongs to  $(-\infty, \xi)$ , therefore the set

$$\{u_k \mid k \geq c \in \mathbb{N}\} \subset \mu_\xi.$$

Since the sequence  $u_n \rightarrow u$ , the subsequence  $u_k \rightarrow u$ . And  $\mu_\xi$  closed implies  $u \in \mu_\xi$ . Since this holds for all  $\eta < \infty$ , we take  $\xi \downarrow \eta$ . Impling  $J(u) \leq \eta$ .

□

#### Example 4.1.

The indicator function of a set  $C \subset U$ , i.e. the function  $I_C : U \rightarrow [-\infty, \infty]$

$$I_C(u) = \begin{cases} 0, & \text{if } u \in C \\ +\infty, & \text{otherwise,} \end{cases}$$

is lower semi-continuous if and only if  $C$  is closed.

*Proof.* Take  $\xi \in \mathbb{R}$ . If  $\xi < 0$ , the set  $\mu_\xi = \emptyset$ . If  $\xi > 0$ , the set  $\mu_\xi = C$ . Therefore the sets  $\mu_\xi$ , for all  $\xi \in \mathbb{R}$  is closed if and only if  $C$  is closed. By the theorem 4.1  $I_C$  is lower semi-continuous if and only if  $C$  is closed. □

#### The Dual Systems of Linear Spaces

Two linear spaces  $X$  and  $Y$  over the same scalar field  $\Gamma$  define a dual system if a fixed bilinear functional on their product is given:

$$(\cdot, \cdot) : X \times Y \rightarrow \Gamma$$

.

The bilinear functional is sometimes omitted.

The dual system is called separated if the following two properties hold:

1.  $\forall x \in X \setminus \{0\}$  there is  $y \in Y$  such that  $(x, y) \neq 0$ .
2.  $\forall y \in Y \setminus \{0\}$  there is  $x \in X$  such that  $(x, y) \neq 0$ .

In other words,  $X$  separates points in  $Y$  and  $Y$  separates points in  $X$ . We consider only separated dual systems. For each  $x \in X$ , we define the application  $f_x : Y \rightarrow \Gamma$  by

$$f_x(y) = (x, y) \quad \forall y \in Y$$

We observe that  $f_x$  is a linear functional on  $Y$  and the mapping  $x \rightarrow f_x$ ,  $\forall x \in X$ , is linear and injective, as can be seen from condition (1). Hence, the correspondence is an embedding. Thus, the elements of  $X$  can be identified with the linear functionals on  $Y$ . In a similar way, the elements of  $Y$  can be considered as linear functionals of  $X$ , identifying an element  $y \in Y$  with  $g_y : X \rightarrow \Gamma$ , defined by

$$g_y(x) = (x, y), \quad \forall x \in X.$$

Therefore, each dual system of linear spaces defines a mapping from either of the two linear spaces into the space of linear functionals on the other. We set,

$$p_y(x) = |(x, y)| = |g_y(x)|, \quad \forall x \in X$$

$$q_x(y) = |(x, y)| = |f_x(y)|, \quad \forall y \in Y$$

and we observe that  $\mathcal{P} = \{p_y \mid y \in Y\}$  is a family of seminorms on  $X$  and  $\mathcal{Q} = \{q_x \mid x \in X\}$  is a family of seminorms on  $Y$ .

#### Definition 4.2.

If  $U$  is a normed space, the dual space  $U^* = \mathcal{B}(U, \mathbb{R})$ . Consists of all linear and bounded functionals mapping from  $U$  to  $\mathbb{R}$ .

#### Theorem 4.2.

Let  $U$  be a Banach space, then the dual  $U^*$  is also a Banach space relative to the norm of the functionals defined by

$$\|u^*\| = \sup_{\|u\|_U \leq 1} |u^*(u)|$$

#### Example 4.2.

Let  $\Omega \subset \mathbb{R}$  be a measurable set. Let  $f \in L^p(\Omega)$ . Consider the functional  $\phi_g : L^p(\Omega) \rightarrow \mathbb{R}$  defined by,

$$\phi_g(f) = \int_{\Omega} fg d\mu$$

characterized for some  $g$  mapping  $\Omega$  to the real line. This is a linear functional with respect  $L^p(\Omega)$ . We want an

estimate of the norm of this functional. For this purpose we apply Hölder inequality, with  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $p, q > 1$ ,

$$\begin{aligned}
\|\phi_g\| &= \sup_{1 \geq \|f\|_{L^p(\Omega)}} \left| \int_{\Omega} g f d\mu \right| \\
&\leq \sup_{1 \geq \|f\|_{L^p(\Omega)}} \int_{\Omega} |g f| d\mu \\
&\text{By Hölder inequality} \\
&\leq \sup_{1 \geq \|f\|_{L^p(\Omega)}} \left( \int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} \left( \int_{\Omega} |g|^q d\mu \right)^{\frac{1}{q}} \\
&= \left( \int_{\Omega} |g|^q d\mu \right)^{\frac{1}{q}} \sup_{1 \geq \|f\|_{L^p(\Omega)}} \left( \int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} \\
&= \left( \int_{\Omega} |g|^q d\mu \right)^{\frac{1}{q}} = \|g\|_{L^q(\Omega)}
\end{aligned}$$

This result implies that if  $g \in L^q(\Omega)$ , then  $\phi_g$  is bounded, hence for all  $g \in L^q(\Omega)$  we have that the functionals characterized by  $g$ ,  $\phi_g \in (L^p(\Omega))^*$ . It's possible to demonstrate that all  $\phi \in (L^p(\Omega))^*$  can be characterized by some  $g$  in  $L^q(\Omega)$ . Thus,

$$L^q(\Omega) = (L^p(\Omega))^*$$

#### Remark 4.2.

There is a natural duality between  $U$  and  $U^*$  determined by the bilinear functional  $(\cdot, \cdot) : U \times U^* \rightarrow \mathbb{R}$ , defined as

$$(u, u^*) = u^*(u), \quad \forall u \in U, \forall u^* \in U^*$$

#### Definition 4.3.

A sequence  $(u_n)_n$  in a Banach space is called weakly convergent to some  $u \in U$  if for all linear continuous functionals  $u^* \in U^*$  we have

$$\lim_{n \rightarrow \infty} u^*(u_n) = u^*(u)$$

$u$  is also called the weak-limit and we write  $u_n \xrightarrow[n \rightarrow \infty]{} u$ .

#### Theorem 4.3.

A sequence  $(u_n)_n$  in  $U$  converges to  $u \in U$  if and only if  $\sup_{n \in \mathbb{N}} \|u_n\| < \infty$  and  $u_n \xrightarrow[n \rightarrow \infty]{} u$

#### Theorem 4.4 (Bourbaki-Alaoglu-Katulami).

The closed unit ball in a Banach space  $U$  is weakly compact if and only if  $U$  is reflexive. If  $U$  is in addition separable, then it's weakly sequentially compact.

#### Definition 4.4.

Let  $U$  be a Banach space and  $J : U \rightarrow \mathbb{R}$ ,  $J$  is called weakly (sequentially) lower semi-continuous at point  $u_0$  if for every weakly convergent sequence  $(u_n)_n$  converges to  $u_0$ , i.e.  $u_n \rightharpoonup u_0$ , it holds

$$J(u) \leq \liminf_{n \rightarrow \infty} J(u_n)$$

#### Definition 4.5.

A non empty set  $C \subset U$  is called weakly closed if for every weakly convergent sequence  $(u_n)_n$  in  $C$  follows that the weak limit belongs to  $C$ . i.e.  $u_n \rightharpoonup u$ , with  $u_n \in C$ , implies  $u \in C$ .

**Definition 4.6.**

A non empty set  $C \subset U$  is called weakly sequentially compact if for every sequence in  $C$  contains a weakly convergent subsequence whose limit belongs to  $C$ .

**Theorem 4.5.**

Let  $U$  be a Banach space and  $J : U \rightarrow \overline{\mathbb{R}}$  the the following conditions are equivalent:

- $J$  is weakly lower semi-continuous on  $U$  for all  $u \in U$ .
- The level sets  $\mu_\xi = \{u \in U | J(u) \leq \xi\}$  is weakly closed for each  $\xi \in \mathbb{R}$ .

**Lemma 4.2.**

Let be  $J : U \rightarrow \overline{\mathbb{R}}$  a convex and lower semicontinuous functional. Assume there is  $u_0 \in U$  such that  $J(u_0) = -\infty$ , then  $J$  is nowhere finite.

*Proof.* Assume that there is  $v \in U$  such that  $-\infty < J(v) < \infty$ . Then by convexity  $J(\lambda u_0 + (1 - \lambda)v) = -\infty$ ,  $\forall \lambda \in [0, 1]$ . Because  $J$  is lower semicontinuous it follows that in the limit  $\lambda \rightarrow 0$ ,

$$(\lambda u_0 + (1 - \lambda)v) \rightarrow v \implies J(v) \leq J(\lambda u_0 + (1 - \lambda)v) = -\infty$$

□

**Lemma 4.3.**

Every lower semi-continuous and convex function on a linear space  $U$  is weakly lower semi-continuous.

**Corollary 4.1.**

Assume that  $U$  is a reflexive Banach space, then every bounded sequence  $(u_n)_n \in U$  that is  $\sup_{n \in \mathbb{N}} \|u_n\| < \infty$  has a subsequence  $(u_k)_k$  which is weakly convergent to some  $u \in U$ .

**Remark 4.3.**

Since every Hilbert space is reflexive the corollary applies to this case.

**Lemma 4.4.**

A closed set  $C$  is weakly closed if and only if the set is convex.

**Definition 4.7.**

Let  $U$  be a real linear space and  $J : U \rightarrow \overline{\mathbb{R}}$ . We said that  $J$  is sublinear if:

$$\begin{aligned} J(\lambda u) &= \lambda J(u) & \forall u \in U, \text{ and } \mathbb{R} \ni \lambda > 0 \\ J(u + v) &\leq J(u) + J(v) & \forall u, v \in U \end{aligned}$$

**Remark 4.4.**

Every sublinear function is convex.

**Theorem 4.6.**

Let  $U$  be a real linear space  $J : U \rightarrow \overline{\mathbb{R}}$  a sublinear functional. Then there is a linear functional  $f$  on  $U$  such that,

$$f(u) \leq J(u) \quad \forall u \in U$$

**Definition 4.8.**

Let  $J : U \rightarrow \overline{\mathbb{R}}$ , we said that  $J$  is locally bounded around  $u_0$  if  $\exists V \subset U$  neighborhood of  $u_0$  such that for some  $M \in \mathbb{R}$

$$|J(u)| < M \quad \forall u \in V$$

**Lemma 4.5.**

Let  $J : U \rightarrow \mathbb{R}$  convex and  $U$  is a Banach space. If  $J$  is locally bounded around  $u$ , then  $J$  is lower semi-continuous in  $u$ .

*Proof.* Let  $u_k \rightarrow u$  as  $k \rightarrow \infty$ . For each  $\epsilon > 0$  we can find a sequence  $\alpha_k$  such that  $\left\| \frac{u - u_k}{\alpha_k} \right\| < \epsilon$ , and  $\alpha_k \rightarrow 0$  as  $k \rightarrow \infty$ . (Please read Maximal Monotone Operators and Evolution Systems in Banach Spaces of Barbu. Details Still to be recovered).

Moreover, for  $k$  sufficiently large we have  $\|u - u_k\| < \epsilon$ . Choose  $\epsilon$  such that  $J$  is bounded in  $\overline{B_{2\epsilon}(u)}$  by  $M$  and define  $v_k = u_k + \frac{u - u_k}{\alpha_k} \in \overline{B_{2\epsilon}(u)}$ , since  $\|v_k - u\| \leq \|u_k - u\| + \left\| \frac{u - u_k}{\alpha_k} \right\| \leq 2\epsilon$ .

Since  $J$  is convex

$$J(u) \leq \alpha_k J(v_k) + (1 - \alpha_k) J(u_k) \leq \alpha_k M + J(u_k)$$

Since  $\alpha_k \rightarrow 0$ , then

$$J(u) \leq \liminf_{k \rightarrow \infty} (\alpha_k M + J(u_k)) = \liminf_{k \rightarrow \infty} J(u_k)$$

Thus is if  $J$  is convex and locally bounded around  $u$ , then is lower semi-continuous around  $u$ .  $\square$

**Remark 4.5.**

The result that convexity and local boundedness imply lower semi-continuity is similar to classical result for linear operators where local boundedness implies continuity. In general convexity plays in optimization the same role as linearity in solving equations.

## 5. Lecture 5

### 5.1. Subgradients

Characterization for  $J''$  still to be analyzed the professor's notes are flawed.

**Definition 5.1.**

Let  $U$  be a Banach space and let  $J : U \rightarrow (-\infty, \infty]$  be a convex and proper function. The subdifferential at a point  $u \in \text{dom } J$  is a mapping,

$$\partial J : U \rightarrow 2^{U^*}, \quad \partial J(u) := \{p^* \in U^* \mid J(v) \geq J(u) + p^*(v - u), \forall v \in U\}$$

The elements of  $p^* \in \partial J(u)$  are called subgradients of  $J$  at  $u$ .

**Example 5.1.**

Consider  $J : \mathbb{R} \rightarrow \mathbb{R}$ ,  $u \rightarrow |u|$  which is not differentiable at  $u = 0$ . If  $u > 0$ , then  $J(u) = u$  and we can find  $0 < v < u < w$ . Then  $p^* \in \partial J(u)$  implies by definition of subdifferential

$$\begin{aligned} v - u &\geq p^*(v - u) \equiv (1 - p^*)(u - v) \leq 0 \\ w - u &\geq p^*(w - u) \equiv (1 - p^*)(w - u) \geq 0. \end{aligned}$$

which implies for  $u > 0$ ,  $p^* \leq 1 \leq p^*$ , then  $p^* = 1$ .

In the same way we obtain for  $u < 0$ ,  $p^* \geq -1 \geq p^*$ . In the case  $u = 0$ , we need to satisfy  $|v| \geq p^*v$ , which is fulfilled if and only if  $|p^*| \leq 1$ . Hence for  $J(u) = |u|$ ,

$$\partial |u| = \begin{cases} \{1\}, & u > 0 \\ [-1, 1], & u = 0 \\ \{-1\}, & u < 0 \end{cases}.$$

**Example 5.2.**

A convex function which is not subdifferentiable everywhere  $J : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$J(u) = \begin{cases} -\sqrt{1 - |u|^2} & |u| \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

For  $|u| \geq 1$ , we have  $\partial J(u) = \emptyset$ .

**Example 5.3.**

Let  $C$  be a convex and closed subset of  $U$  and  $I_C$  function defined by

$$I_C(u) = \begin{cases} 0 & u \in C \\ \infty & \text{otherwise} \end{cases}$$

The subdifferentiable is the definition of normal cone at  $u$

$$\partial I_C(u) = \{u^* \in U^* \mid u^*(u - v) \geq \forall v \in C\} = \mathcal{N}_C(u)$$

.

**Theorem 5.1.**

Let  $U$  be a Banach space. And  $J : U \rightarrow \overline{\mathbb{R}}$  a subdifferentiable function. Then  $\partial J(u)$  is convex and weakly closed.

**Remark 5.1.**

Most of the rules for derivatives also hold for subdifferentials with some additional assumptions,

- $J : U \rightarrow \overline{\mathbb{R}}, \lambda > 0, \partial J(\lambda u) = \lambda \partial J(u)$ .
- $\partial(J + F)(u) \supseteq \partial J(u) + \partial F(u)$ .

**Theorem 5.2 (Rockafeller).**

Let  $U$  be a Banach space and  $J : U \rightarrow \overline{\mathbb{R}}$  proper and convex functions for  $i = 1, \dots, n$ . The sum-rule

$$\partial(J_1 + \dots + J_n)(u) = \partial J_1(u) \dots \partial J_n(u), \quad n \geq 2$$

holds if there exists  $u_0 \in U$  such that all  $J_i(u_0)$  are finite and all  $J_i$  except at most one  $J_k, k \in \{1, 2, \dots, n\}$  are continuous at  $u_0$

## 6. Lecture 6

**Theorem 6.1.**

Let  $V, U$ , Hilbert Spaces. Let  $J : V \rightarrow \overline{\mathbb{R}}$  convex.  $U$  and  $V$  Banach spaces,  $A : U \rightarrow V$  linear and continuous with  $A^* : U^* \rightarrow V^*$ . Moreover,  $J$  is lower semi-continuous and let  $A\bar{u}$  be a point where  $J$  is continuous and finite. Then the composed function  $J \circ A : U \rightarrow \overline{\mathbb{R}}$  is subdifferentiable for all  $u \in V$  and,

$$\partial(J \circ A)(u) = A^* [\partial J(Au)]$$

*Proof.* Let  $p^* \in \partial J(Au)$ ,

$$J(p) \geq J(Au) + p^*(p - Au) \quad \forall p \in V$$

where  $p = Av$  with  $v \in U$ ,

$$(J \circ A)(v) \geq (J \circ A)(u) + p^*(A(v - u)) \quad \forall v \in U \quad (2)$$

$$= (J \circ A)(u) + A^*p^*(v - u) \quad \forall v \in U \quad (3)$$

i.e.  $A^*p^* \in \partial(J \circ A)(u) \implies A^*\partial J(Au) \subseteq \partial(J \circ A)(u)$ . Proof based again on the weak separation theorem of convex sets. (We have to check Bauschke)  $\square$

### Theorem 6.2.

If  $J : U \rightarrow \mathbb{R}$  is convex and Frechét-differentiable at  $u \in U$ , then  $\partial J(u) = \{J'(u)\}$

*Proof.* Let  $p^* \in \partial J(u)$ . Then for each  $t > 0$ ,  $J(u + tv) - J(u) \geq p^*(tv) = tp^*(v)$ , dividing by  $t$  and taking the limit  $t \rightarrow 0$  we obtain,

$$J'(u)(v) \geq p^*(v) \quad \forall v \in U \quad (4)$$

$$\implies (J'(u) - p^*)(v) \geq 0 \quad \forall v \in U. \quad (5)$$

Since  $J'(u)$  is Frechét differentiable the operator  $J'(u)$  is linear with respect to  $v$  and  $p^* \in U^*$  implies  $(J'(u) - p^*)$  is linear, taking  $-v \in U$ , we obtain that  $(J'(u) - p^*)(v) \leq 0$ . Therefore  $p^* = J'(u)$ .

On the other hand, if  $J$  is differentiable, it follows that  $J'(u) \in \partial J(u)$ . For  $v \in U$ , we set  $w = v - u$ ,  $u \in U$  we have,

$$J(u + w) - J(u) \geq (J'(u))(w) \quad (6)$$

$$\implies J(v) - J(u) \geq (J'(u))(v - u) \quad (7)$$

Since the above inequality holds for all  $v \in U$  implies  $J'(u) \in \partial J(u)$ .  $\square$

### Remark 6.1.

The subgradient can be used to obtain local optimality conditions that are necessary and sufficient for convex problem.

### Theorem 6.3.

Let  $U$  be a Banach Space and  $J : U \rightarrow \mathbb{R}$  convex and proper. Then each local minimum is global minimum. Moreover  $\bar{u} \in U$  is a minimizer if and only if  $0 \in \partial J(\bar{u})$ .

*Proof.* If  $0 \in \partial J(\bar{u})$ :  $J(v) \geq J(\bar{u}) + (0)(v - \bar{u}) = J(\bar{u})$ ,  $\forall v \in U$ , and hence  $\bar{u}$  is a global minimizer.

Assume that  $0 \notin \partial J(\bar{u})$ , then  $\exists v \in U$ , such that

$$J(v) < J(\bar{u}) + (0)(v - \bar{u}) = J(\bar{u}).$$

Therefore  $\bar{u}$  cannot be a minimizer.  $\square$

### Definition 6.1 (Duality).

Let  $J : U \rightarrow \mathbb{R}$ , and  $U$  a Banach space. Then the convex conjugate function  $J^* : U^* \rightarrow \mathbb{R}$  is defined by

$$J^*(p^*) = \sup_{u \in U} \{p^*(u) - J(u)\}$$

implies that  $-\sup_{u \in U} \{p^*(u) - J(u)\} = -J^*(p^*) = \inf_{u \in U} \{J(u) - p^*(u)\}$ .

**Example 6.1.**

Consider the indicator function of a convex set  $C$ ,  $I_C : U \rightarrow \overline{\mathbb{R}}$

$$I_C(u) = \begin{cases} 0 & u \in C \\ \infty & \text{otherwise.} \end{cases}$$

Then we have that the convex conjugate is given by

$$J^*(p^*) = \sup_{u \in U} \{p^*(u) - I_C(u)\} = \sup_{u \in C} \{p^*(u)\}.$$

**Example 6.2.**

$J : \mathbb{R}_+ \rightarrow \mathbb{R}$

**Example 6.3.**

Let  $J : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $J(u) = \exp u$ , then  $J^*(p^*) = \sup_{u \in \mathbb{R}} \{p^*u - \exp u\}$ . Let  $f(u) = p^*u - \exp(u)$ , therefore  $f'(u) = p^* - \exp u$ ,  $\forall u \in \mathbb{R}$ . Which is zero for  $\bar{u} = \ln p^*$ , if  $p^* > 0$ . Since  $f''(u) < 0$ , then  $\bar{u}$  is indeed maximum. And we see that  $\lim_{u \rightarrow \pm\infty} f(u) = -\infty$ . If  $p^* = 0$ ,  $f(u) = -\exp u < 0$  and therefore the  $\sup_{u \in \mathbb{R}} f(u) = 0$  (Consider the limit when  $u \rightarrow -\infty$ ). Then we have,

$$J^*(p^*) = \begin{cases} p^*(\ln p^* - 1) & p^* > 0 \\ 0 & p^* = 0 \end{cases}$$

**Example 6.4.**

Let  $U$  be a Hilbert space and  $J(u) = \frac{1}{2} \|u\|^2$ . Since  $U$  is Hilbert, by Riesz, for each linear and bounded functional  $\phi_{p^*} \in H$ ,  $\exists p^* \in H$  such that,  $\phi_{p^*}(u) = \langle u, p^* \rangle$ . Using the definition of conjugate function,

$$\begin{aligned} J^*(p^*) &= \sup_{u \in U} \left\{ \langle u, p^* \rangle - \frac{1}{2} \|u\|^2 \right\} \\ &= - \inf_{u \in U} \left\{ \frac{1}{2} \|u\|^2 - \langle u, p^* \rangle \right\} \end{aligned}$$

Note that,

$$\frac{1}{2} \|u - p^*\|^2 = \frac{1}{2} \|u\|^2 - \langle u, p^* \rangle + \frac{1}{2} \|p^*\|^2$$

Therefore we can substitute in the above equation to find an equivalent form to the conjugate function,

$$\begin{aligned} J^*(p^*) &= - \inf_{u \in U} \left\{ \frac{1}{2} (\|u - p^*\|^2 - \|p^*\|^2) \right\} \\ &= - \frac{1}{2} \inf_{u \in U} \left\{ \|u - p^*\|^2 \right\} + \frac{1}{2} \|p^*\|^2 \end{aligned}$$

We have  $\|u - p^*\| \geq 0$ ,  $\forall u \in H$ , then,

$$\inf_{u \in U} \left\{ \|u - p^*\|^2 \right\} = 0,$$

since we can take  $u = p^*$ . Hence,

$$J^*(p^*) = \frac{1}{2} \|p^*\|^2 \tag{8}$$

**Theorem 6.4.**

Let  $U$  be a Banach space and  $J : U \rightarrow \overline{\mathbb{R}}$ . Then  $J^*$  is convex.



*Proof.* Let  $p^*, q^* \in U^*$ , and  $\lambda \in [0, 1]$ ,

$$J^*(\lambda p^* + (1 - \lambda)q^*) = \sup_{u \in U} \{(\lambda p^* + (1 - \lambda)q^*)(u) - J(u)\} \quad (9)$$

$$= \sup_{u \in U} \{\lambda p^*(u) - \lambda J(u) + (1 - \lambda)q^*(u) - (1 - \lambda)J(u)\} \quad (10)$$

$$\leq \sup_{v, u \in U} \{\lambda p^*(u) - \lambda J(u) + (1 - \lambda)q^*(v) - (1 - \lambda)J(v)\} \quad (11)$$

$$= \sup_{u \in U} \{\lambda p^*(u) - \lambda J(u)\} + \sup_{v \in U} \{(1 - \lambda)q^*(v) - (1 - \lambda)J(v)\} \quad (12)$$

$$= \lambda J^*(p^*) + (1 - \lambda)J^*(q^*). \quad (13)$$

Hence  $J^*$  is convex. □

## 7. Lecture 7

### Remark 7.1.

Some elementary properties of conjugate functions

- **Young inequality**  $J(u) + J^*(p^*) \geq p^*(u) \quad \forall u \in U, \forall p^* \in U^*$
- $J^*(0) = \sup_{u \in U} ((0, u) - J(u)) = \sup_{u \in U} (-J(u)) = \inf_{u \in U} J(u)$

In many applications in optimization, is used the equivalent formulation,

$$\inf_{u \in U} J(u) = -J^*(0).$$

$$J \leq F \implies J^* \geq F^*$$

### Theorem 7.1.

Let  $U$  a Banach space and  $J^* : U^* \rightarrow \overline{\mathbb{R}}$  be the conjugate of the  $J : U \rightarrow \overline{\mathbb{R}}$ . Then for all  $u \in U$ .

$$p^* \in \partial J(u) \iff J(u) + J^*(p^*) = p^*(u)$$

.

*Proof.* content... □

### Corollary 7.1.

It follows from previous theorem that  $\partial J(u) = \{p^* \in U^* | J(u) + J^*(p^*) = (p^*, u)\}$ .

### Theorem 7.2.

Let  $U$  be a Banach space and  $J : U \rightarrow \mathbb{R}$  be proper function. If  $p^* \in \partial J(u)$  then  $u \in \partial J^*(p^*)$

*Proof.* Let  $p^* \in \partial J(u)$ . For any  $g^* \in U^*$ , it follows

$$J^*(g^*) = \sup_{v \in U} \{g^*(v) - J(v)\} \geq g^*(u) - J(u) \geq g^*(u) - J(u)$$

From theorem 7.1

$$J^*(g^*) \leq g^*(u) - p^*(u) + J^*(p^*) = (g^* - p^*)(u) + J^*(p^*) \implies u \in \partial J^*(p^*).$$

□

By iteration the definition, we obtain the bipolar function  $(J^*)^* = J^{**} : U^{**} \rightarrow \overline{\mathbb{R}}$ ,

$$J^{**}(u) = \sup_{p^* \in U^*} \{p^*(u) - J^*(p^*)\}$$

### Theorem 7.3.

Let  $U$  be a reflexive Banach space. The  $J^{**}$  is the maximum convex functional below  $J$  (also called convex envelope), i.e.  $J^{**}(u) \leq J(u)$ ,  $\forall u \in U$  and  $F(u) \leq J^{**}(u)$ ,  $\forall u \in U$  if  $F$  is also convex and  $F(u) \leq J(u)$ ,  $\forall u$ . In particular  $J^{**} = J$  if and only if  $J$  is convex.

*Proof.*

$$J^{**}(u) = \sup_{p^* \in U^*} \{p^*(u) - J^*(p^*)\} \quad (14)$$

$$= \sup_{p^* \in U^*} \left\{ p^*(u) - \sup_{v \in U} \{p^*(v) - J(v)\} \right\} \quad (15)$$

$$= \sup_{p^* \in U^*} \left\{ p^*(u) + \inf_{v \in U} \{p^*(v) - J(v)\} \right\} \quad (16)$$

$$(17)$$

Since for any  $p^* \in U^*$ ,

$$\inf_{v \in U} \{p^*(u - v) + J(v)\} \leq p^*(u - u) + J(u)$$

We have that  $J^{**}(u) \leq J(u)$ .

Now we assume that  $F$  is a convex functional and  $g^* \in \partial F(u)$  for  $u \in U$ .

$$\implies F(v) \geq F(u) + g^*(v - u) \quad (18)$$

$$F^{**}(u) = \sup_{p^* \in U^*} \inf_{v \in U} \{p^*(u - v) + F(v)\} \quad (19)$$

$$\geq \sup_{p^* \in U^*} \inf_{v \in U} \{(p^* - g^*)(u - v) + F(u)\} \quad (20)$$

$$\geq \inf_{v \in U} \{(g^* - q^*)(u - v) + F(u)\} \quad (21)$$

$$= F(u) \quad (22)$$

If  $F$  is convex,

$$\implies F(u) \leq F^{**}(u) \leq F(u) \implies F(u) = F^{**}(u), \quad (23)$$

$$F(u) = F^{**}(u) = \sup_{p^* \in U^*} \inf_{v \in U} \{p^*(u - v) + F(v)\} \leq J^{**}(u) \quad (24)$$

□

## 8. Lecture 8

### Definition 8.1.

Let  $U$  and  $Y$  Banach spaces and  $J : U \rightarrow \overline{\mathbb{R}}$  is a proper function. We consider an optimization problem (P) called primal problem

$$\inf_{u \in U} J(u) \quad (P)$$

Then the problem is said to be nontrivial if there is  $\bar{u} \in U$  such that  $J(\bar{u}) < \infty$ . A function  $\Phi : U \times Y \rightarrow \overline{\mathbb{R}}$  is said to be a perturbation function of  $J$ ,

$$\inf_{u \in U} \Phi(u, p) \quad (Pp)$$

if  $\Phi(u, 0) = J(u)$  for all  $u \in U$ . For each  $p \in Y$ , the minimization problem (Pp) is called a perturbation problem. The variable  $p$  is called perturbation parameter. If we denote by  $\Phi^*$  the convex conjugate function of  $\Phi$ , the *dual problem*, with respect to  $\Phi$  is defined by

$$\sup_{p^* \in Y^*} -\Phi^*(0, p^*) \quad (\text{P}^*)$$

where  $\Phi^* : (U \times Y)^* \cong U^* \times Y^* \rightarrow \overline{\mathbb{R}}$ , a function defined as follows.

$$\Phi^*(u^*, p^*) = \sup_{\substack{u \in U \\ p \in Y}} \{u^*(u) + p^*(p) - \Phi(u, p)\}$$

**Remark 8.1.**

For  $p = 0$ ,  $(\text{P}^*) \equiv (\text{Pp})$ . We denote the infimum for problem (P) by  $\inf(\text{P})$  and the supremum for problem  $(\text{P}^*)$  by  $\sup(\text{P}^*)$

**Lemma 8.1 (Weak duality).**

For the problem (P) and  $(\text{P}^*)$  it holds that

$$-\infty \leq \sup(\text{P}^*) \leq \inf(\text{P}) \leq \infty.$$

*Proof.* Let  $p^* \in Y^*$ . It follows

$$-\Phi^*(0, p^*) = -\sup_{\substack{u \in U \\ p \in Y}} \{0(u) + p^*(p) - \Phi(u, p)\} \quad (25)$$

$$= \inf_{\substack{u \in U \\ p \in Y}} \{\Phi(u, p) - p^*(p)\} \quad (26)$$

$$\leq \Phi(u, 0) - p^*(0) \quad \forall u \in U, p^* \in Y^* \quad (27)$$

$$\implies \sup_{p^* \in Y^*} \{-\Phi^*(0, p^*)\} \leq \inf_{u \in U} \Phi(u, 0) = \inf(P) \quad (28)$$

□

By iteration we can define, a bidual problem

$$-\sup_{u \in U} \{-\Phi^*(u, 0)\} = \inf_{u \in U} \Phi^*(u, 0) \quad (\text{P}^{**})$$

In case the space  $U$  is reflexive then  $U^{**} = U$ .

If the perturbation function  $\Phi(u, p)$  is proper, convex and weakly lower semicontinuous. Then  $\Phi^{**} = \Phi$ . In this case  $\Phi(u, 0) = \Phi^{**}(u, 0)$  i.e.  $(\text{P}) \equiv (\text{P}^{**})$

**Definition 8.2.**

Consider the infimal value function

$$h(p) = \inf(\text{Pp}) = \inf_{u \in U} \Phi(u, p)$$

The problem (P) is called stable if  $h(0)$  is finite and its sub-differentiable in zero is not empty.

**Theorem 8.1.**

The primal problem (P) is stable if and only if the following conditions are simultaneously satisfied:

- The dual problem  $(\text{P}^*)$  has a solution.

- There is no duality gap, i.e.

$$\inf(\mathbf{P}) = \sup(\mathbf{P}^*) \leq \infty$$

### Theorem 8.2 (Extremal relation).

Let  $\Phi : U \times Y \rightarrow \overline{\mathbb{R}}$ , be convex the the following statements are equivalent:

1.  $(\mathbf{P})$  and  $(\mathbf{P}^*)$  have solutions  $\bar{u}$  and  $\bar{p}^*$  and  $\inf(\mathbf{P}) = \sup(\mathbf{P}^*)$
2.  $\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = 0$
3.  $(0, \bar{p}^*) \in \partial\Phi(u, 0)$  and  $(\bar{u}, 0) \in \partial\Phi^*(0, p^*)$

*Proof.* We proceed by parts:

1.  $(1) \implies (2)$ :  $\bar{u}$  solution of  $\inf(\mathbf{P})$  and  $\bar{p}^*$  solution of  $\sup(\mathbf{P}^*)$  and  $\inf(\mathbf{P}) = \sup(\mathbf{P}^*)$ . This properties implies,  $\Phi(\bar{u}, 0) = \inf(\mathbf{P}) = \sup(\mathbf{P}^*) = -\Phi^*(0, \bar{p}^*) \implies \Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = 0$ .
2.  $(2) \implies (1)$ :  $-\Phi^*(0, \bar{p}^*) = \sup(\mathbf{P}^*) \leq \inf(\mathbf{P}) = \Phi(\bar{u}, 0) = -\Phi^*(0, \bar{p}^*) \implies \sup(\mathbf{P}^*) = \inf(\mathbf{P})$
3.  $(2) \iff (3)$ :  $\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = 0 = (0, \bar{u}) + (\bar{p}^*, 0) = ((0, \bar{p}^*), (\bar{u}, 0)) \iff (0, \bar{p}^*) \in \partial\Phi(\bar{u}, 0) \forall u \in U, \forall p^* \in \partial J(u) \iff J(u) + J^*(p^*) = (p^*, u)$

□

### Fencel duality.

Consider the functional  $J : U \rightarrow \overline{\mathbb{R}}$ ,

$$J(u) = F(u) + G(Au)$$

with  $F : U \rightarrow \overline{\mathbb{R}}$ ,  $G$  convex function  $G : V \rightarrow \overline{\mathbb{R}}$  and  $A : U \rightarrow V$  bounded and linear.

We introduce the perturbation  $\Phi(u, p) = F(u) + G(Au - p)$ . The dual problem is obtained with,

$$\Phi^*(0, p^*) = \sup_{\substack{u \in U \\ p \in V}} \{p^*(p) - F(u) - G(Au - p)\}$$

For fixed  $u$  we set  $q : Au - p$ .

$$\begin{aligned} \Phi^*(0, p^*) &= \sup_{u \in U} \sup_{q \in V} \{p^*(Au - q) - F(u) - G(q)\} \\ &= \sup_{u \in U} \sup_{q \in V} \{p^*(Au) - p^*(q) - F(u) - G(q)\} \\ &= \sup_{u \in U} \{p^*(Au) - F(u)\} + \sup_{q \in V} \{(-p^*)(q) - G(q)\} \\ &= \sup_{u \in U} \{(A^* \circ p^*)(u) - F(u)\} + \sup_{q \in V} \{(-p^*)(q) - G(q)\} \\ &= F^*(A^* \circ p^*) + G^*(-p^*) \end{aligned}$$

Where  $(A^* \circ p^*) \in U^*$ , defined as  $(A^* \circ p^*) : U \rightarrow \overline{\mathbb{R}}$

$$(A^* \circ p^*)(u) = p^*(Au)$$

In case  $U$  is a Hilbert space  $A^*$  is the adjoint operator of  $A$ .

## 9. Lecture 9

We check the optimality conditions.

$$\begin{aligned} 0 &= \Phi(\bar{u}, 0) + \Phi^*(u, \bar{p}^*) \\ &= F(\bar{u}) + G(A\bar{u}) + F^*(A^*\bar{p}^*) = [F(\bar{u}) + F^*(A^*\bar{p}^*) - A^* \circ p^*(u)] + [G(A\bar{u}) + G^*(-\bar{p}^*) - (-p^*)(A\bar{u})] \end{aligned}$$

Using Young inequality  $J(u) + J^*(u^*) - u^*(u) \geq 0$ ,  $\forall u \in U$ , and  $\forall u^* \in U^*$ , we see that both square brackets are nonnegative; and the sum is zero. Then

$$\begin{aligned} F(\bar{u}) + F^*(A^*\bar{p}^*) &= A^* \circ p^*(u) \implies A^*p^* \in \partial F(\bar{u}) \\ G(A\bar{u}) + G^*(-\bar{p}^*) &= (-p^*)(A\bar{u}) \implies -p^* \in \partial G(A\bar{u}) \end{aligned}$$

$F, G$  are convex and locally bounded, one can show that  $\sup(\mathbf{P}^*) = \inf(\mathbf{P})$ .

**Example 9.1 (....Bounded variation).**

$$J(u) = \frac{1}{2} \int_{\Omega} |u(x) - v(x)|^2 + \alpha \sup_{g \in C_0^\infty, |g| \leq 1} \int_{\Omega} u dv g dx \quad (29)$$

$\min_{u \in B(g)} J(u)$ ,  $\mathcal{B} = \{u \in L^1(\Omega) \mid V(u, \Omega) < \infty\}$ , where

$$V(u, \Omega) = \left\{ \sup \int_{\Omega} u dv g dx, g \in C_0^\infty(\Omega, \mathbb{R}^n), \|g\|_\infty \leq 1 \right\} \quad (30)$$

$$\|u\|_{BV} := \|u\|_{C^1(\Omega)} + V(u, \Omega),$$

$$\implies F(u) = \frac{1}{2} \int_{\Omega} |u(x) - v(x)|^2 dx = \|u - v\|_{L^2(\Omega)}^2$$

$$G(u) = \int_{\Omega} |u| dx$$

where  $A : \alpha \nabla$ ,

$$F^*(p^*) = \frac{1}{2} \int_{\Omega} |p^*(x) - v(x)|^2 - \frac{1}{2} v^2(x) dx$$

and

$$G^*(p^*) = \begin{cases} 0, & |p^*| \leq 1 \\ -\infty, & \text{otherwise} \end{cases}$$

$A^* = -\alpha \div$ , therefore

$$J(p^*) = -\frac{1}{2} - \frac{1}{2} \int_{\Omega} -\alpha \div p^* + v^2 + \frac{1}{2} v^2 dx$$

### 9.1. Lagrangians

#### References