

Optimization

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Introduction

0.1. Definitions

0.2. Useful lemmas and Theorems.

Lemma 0.1.

Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in a Hilbert Space H . Then $(x_n)_{n \in \mathbb{N}}$ possesses a weakly convergent subsequence.

Lemma 0.2.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a Hilbert Space H . Then $(x_n)_{n \in \mathbb{N}}$ converges if and only if it is bounded and possesses at most one weak sequential cluster point.

Theorem 0.1.

Let $f : H \rightarrow (\infty, \infty]$ be a convex functional on a Hilbert space. Then the following are equivalent:

- (i) f is weakly sequentially lower semicontinuous.
- (ii) f is sequentially lower semicontinuous.
- (iii) f is lower semicontinuous.
- (iv) f is weakly lower semicontinuous.

Lemma 0.3.

A convex set is closed if and only if it is weakly closed.

1. Lecture 1

2. Lecture 2

2.1. Convexity

Definition 2.1.

Let U be linear space. A functional $J : U \rightarrow \overline{\mathbb{R}}$ is called convex, if for $t \in [0, 1]$ and $u_1, u_2 \in U$.

$$J(tu_1 + (1-t)u_2) \leq tJ(u_1) + (1-t)J(u_2) \quad (1)$$

holds such that the right hand side is well defined.

- J is strictly convex if (1) holds strictly for $\forall u_1, u_2 \in U, u_1 \neq u_2$ and $t \in (0, 1)$ with $J(u_1) < \infty$ and $J(u_2) < \infty$.
- An optimization problem,

$$\min_{u \in C} J(u)$$

is called convex if both C and J are convex.

Lemma 2.1.

If C and V are convex in U , then

- $\alpha V = \{w = \alpha v, v \in C\}$ is convex.
- $C + V$ is convex.

Proof.

□

Lemma 2.2.

Let V be a collection of convex sets in U , then $C = \bigcap_{K \in V} K$ is convex.

Proof. If $C = \emptyset$, then the statement is vacuously true. Consider $C \neq \emptyset$ and $u_1, u_2 \in C$ then $u_1, u_2 \in K$ for all $K \in V$

$$\implies tu_1 + (1-t)u_2 \in K, \quad \forall K \in V \implies tu_1 + (1-t)u_2 \in \bigcap_{K \in V} K$$

□

Lemma 2.3.

Let $C \in U$ convex and $J : C \rightarrow \mathbb{R}$. Define $\alpha = \inf_{u \in C} J(u)$. Then the set $\Psi = \{u \mid J(u) = \alpha\}$ is convex, i.e. the solution of

$$\min_{u \in C} J(u)$$

is a convex set.

Proof. Let $u_1, u_2 \in \Psi$ and $u_t = tu_1 + (1-t)u_2$. Since J is convex, it holds that $J(u_t) \leq tJ(u_1) + (1-t)J(u_2) = \alpha$. Thus $J(u_t) = \alpha, \forall t \in [0, 1]$. Implying $u_t \in \Psi$. Hence Ψ is convex. □

Lemma 2.4.

Let U be linear normed space, and $C \subset U$ a convex set and $J : U \rightarrow \overline{\mathbb{R}}$ convex functional. Let $\bar{u} \in C$ such that

$$J(\bar{u}) \leq J(u) \quad \forall u \in B_\epsilon(\bar{u}) \cap C,$$

for some ball $B_\epsilon(\bar{u})$ in U with center in \bar{u} . Then $J(\bar{u}) = \inf_{u \in C} J(u)$. In other words, the local minimizer of a convex optimization problem is also a global minimizer.

Proof. Let $B_\epsilon(\bar{u})$ be an open neighborhood of \bar{u} with $J(\bar{u}) \leq J(u)$ for all $u \in B_\epsilon(\bar{u}) \cap C$. Take an arbitrary $u^* \in C$ and consider $u_t = t\bar{u} + (1-t)u^*$. Since C is convex $u_t \in C$. For some $t \in (0, 1)$, $u_t \in B_\epsilon(\bar{u})$.

Thus,

$$J(\bar{u}) \leq J(u_t) \leq tJ(\bar{u}) + (1-t)J(u^*).$$

We have $\forall t \in [0, 1]$ that $(1-t) \leq 0$, then

$$(1-t)J(\bar{u}) \leq (1-t)J(u^*) \quad \forall u^* \in C$$

Therefore, \bar{u} is a local minimizer for C . □

Theorem 2.1.

Let U is Banach Space, $C \subset U$ convex and $J : C \rightarrow \mathbb{R}$ Gateaux differentiable. Consider the minimization problem.

$$\min_{u \in C} J(u)$$

1. Let \bar{u} be a local solution. Then $J'(\bar{u}, u - \bar{u}) \geq 0$, $\forall u \in C$.
2. If J is convex on C , then $J'(\bar{u}, u - \bar{u}) \geq 0$, $\forall u \in C$ is necessary and sufficient for global optimality of \bar{u}
3. If J is strictly convex on C , then the minimization problem admits at most one solution.
4. If C is closed, and J is convex and continuous with

$$\lim_{\substack{u \in C \\ \|u\| \rightarrow \infty}} J(u) = \infty.$$

Then a global solution $\bar{u} \in C$ exists.

Proof.

1. Let \bar{u} be a local solution $J(\bar{u}) \leq J(u)$, $\forall u \in B_\epsilon(\bar{u}) \cap C$, let $t \in [0, 1]$, $u_t = \bar{u} + t(u - \bar{u})$, then $u_t \in C$, since C is convex.

For small $t > 0$,

$$0 \leq \frac{1}{t} [J(u_t) - J(u)] \leq \frac{1}{t} [J(\bar{u} + t(u - \bar{u})) - J(u)] \xrightarrow{t \downarrow 0} J'(\bar{u}, u - \bar{u})$$

2. Since J is convex we have for $u \in C$, $J(\bar{u} + t(u - \bar{u})) \leq J(\bar{u}) + t[J(u) - J(\bar{u})]$, for $t > 0$

$$\implies J(u) - J(\bar{u}) \geq \frac{1}{t} [J(\bar{u} + t(u - \bar{u})) - J(\bar{u})] \xrightarrow{t \downarrow 0} J'(\bar{u}, u - \bar{u}) \geq 0.$$

Therefore \bar{u} is a global minimizer.

3. Assume, that there are two solution for the minimization problem, $\bar{u}, u^* \in C$, such that $\bar{u} \neq u^*$ and $J(\bar{u}) = J(u^*) = \inf_{u \in C} J(u)$. Since J is strictly convex $J(u_t) = J(t\bar{u} + (1-t)u^*) < tJ(\bar{u}) + (1-t)J(u^*) = \alpha$ for all $t \in [0, 1]$. Contradicting our assumption that u^* and \bar{u} are solutions.

4. $\alpha = \inf_{u \in C} J(u) \in \mathbb{R} \cup \{-\infty\}$, choose a minimizing sequence $(u_k)_k \subset C$ with $J(u_k) \xrightarrow{k \rightarrow \infty} \alpha$

$\implies (u_k)_k$ is bounded, because $J \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

$\implies (u_k)_k$ contains a weakly convergent subsequence $u_{k_e} \xrightarrow{e \rightarrow \infty} \bar{u} \in C$. Since C is closed and convex.

$\implies J$ is weakly-lower semicontinuous because it is convex and continuous.

□

3. Lecture 3

4. Lecture 4

4.1. Lecture 5

References
