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Optimization

Notes

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Abstract:

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Introduction

0.1. Definitions

0.2. Useful lemmas and Theorems.

Lemma 0.1.
Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Hilbert Space H. Then $(x_n)_{n\in\mathbb{N}}$ possesses a weakly convergent subsequence.

Lemma 0.2.
Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert Space H. Then $(x_n)_{n\in\mathbb{N}}$ converges if and only if it is bounded and possesses at most one weak sequential cluster point.

Theorem 0.1.
Let $f : H \to (\infty, \infty]$ be a convex functional on a Hilbert space. Then the following are equivalent:

(i) f is weakly sequentially lower semicontinuous.

(ii) f is sequentially lower semicontinuous.

(iii) f is lower semicontinuous.

(iv) f is weakly lower semicontinuous.

Lemma 0.3.
A convex set is closed if and only if it is weakly closed.

1. Lecture 1

2. Lecture 2

2.1. Convexity

Definition 2.1.
Let U be linear space. A functional $J : U \to \overline{\mathbb{R}}$ is called convex, if for $t \in [0,1]$ and $u_1, u_2 \in U$.

$$J(tu_1 + (1-t)u_2) \leq tJ(u_1) + (1-t)J(u_2) \tag{1}$$

holds such that the right hand sid is well defined.

- J is strictly convex if (1) holds strictly for $\forall u_1, u_2 \in U$, $u_1 \neq u_2$ and $t \in (0,1)$ with $J(u_1) < \infty$ and $J(u_2) < \infty$.
- An optimization problem,
$$\min_{u \in C} J(u)$$
is called convex if both C and J are convex.

Lemma 2.1.
If C and V are convex in U, then

- $\alpha V = \{w = \alpha v, v \in C\}$ *is convex.*
- $C + V$ *is convex.*

Proof. □

Lemma 2.2.
Let V be a collection of convex sets in U, then $C = \bigcap_{K \in V} K$ is convex.

Proof. If $C = \emptyset$, then C the statement is vacuously true. Consider $C \neq \emptyset$ and $u_1, u_2 \in C$ then $u_1, u_2 \in K$ for all $K \in V$

$$\implies tu_1 + (1-t)u_2 \in K, \quad \forall K \in V \implies tu_1 + (1-t)u_2 \in \bigcap_{K \in V} K$$

□

Lemma 2.3.
Let $C \in U$ convex and $J : C \to \mathbb{R}$. Define $\alpha = \inf_{u \in C} J(u)$. Then the set $\Psi = u | J(u) = \alpha$ is convex, i.e. the solution of

$$\min_{u \in C} J(u)$$

is a convex set.

Proof. Let $u_1, u_2 \in \Psi$ and $u_t = tu_1 + (1-t)u_2$.Sinc J is convex, it holds that $J(u_t) \leq tJ(u_t) + (1-t)J(u_t) = \alpha$. Thus $J(u_t) = \alpha$, $\forall t \in [0,1]$. Implying $u_t \in \Psi$ Hence Ψ is convex. □

Lemma 2.4.
Let U be linear normed space, and $C \subset U$ a convex set and $J : U \to \overline{\mathbb{R}}$ convex functional. Let $\overline{u} \in C$ such that

$$J(\overline{u}) \leq J(u) \quad \forall u \in B_\epsilon(\overline{u}) \cap C,$$

for some ball $B_\epsilon(\overline{u})$ in U with center in $\overline{u}$. Then $J(\overline{u}) = \inf_{u \in C} J(u)$. In other words, the local minimizer of a convex optimization problem is also a global minimizer.

Proof. Let $B_\epsilon(\overline{u})$ be an open neighborhood of $\overline{u}$ with $J(\overline{u}) \leq J(u)$ for all $u \in B_\epsilon(\overline{u}) \cap C$. Take an arbitrary $u^* \in C$ and consider $u_t = t\overline{u} + (1-t)u^*$. Since C is convex $u_t \in C$.
For some $t \in (0,1)$, $u_t \in B_\epsilon(\overline{u})$.
Thus,

$$J(\overline{u}) \leq J(u_t) \leq tJ(\overline{u}) + (1-t)J(u^*).$$

We have $\forall t \in [0,1]$ that $(1-t) \leq 0$, then

$$(1-t)J(\overline{u}) \leq (1-t)J(u^*) \qquad \forall u^* \in C$$

Therefore, $\overline{u}$ is a local minimizer for C. □

Theorem 2.1.

Let U is Banach Space, $C \subset U$ convex and $J : C \to \mathbb{R}$ Gateâux differentiable. Consider the minimization problem.

$$\min_{u \in C} J(u)$$

1. *Let $\overline{u}$ be a local solution. Then $J'(\overline{u}, u - \overline{u}) \geq 0$, $\forall u \in C$.*
2. *If J is convex on C, then $J'(\overline{u}, u - \overline{u}) \geq 0$, $\forall u \in C$ is necessary and sufficient for global optimality of $\overline{u}$*
3. *If J is strictly convex on C, then the minimization problem admits at most one solution.*
4. *If C is closed, and J is convex and continuous with*

   $$\lim_{\substack{u \in C \\ \|u\| \to \infty}} J(u) = \infty.$$

   Then a global solution $\overline{u} \in C$ exists.

Proof.

1. Let $\overline{u}$ be a local solution $J(\overline{u}) \leq J(u)$, $\forall u \in B_\epsilon(\overline{u}) \cap C$, let $t \in [0,1]$, $u_t = \overline{u} + t(u - \overline{u})$, then $u_t \in C$, since C is convex.

   For small $t > 0$,

   $$0 \leq \frac{1}{t}\left[J(u_t) - J(u)\right] \leq \frac{1}{t}\left[J(\overline{u} + t(u - \overline{u})) - J(u)\right] \xrightarrow{t \downarrow 0} J'(\overline{u}, u - \overline{u})$$

2. Since J is convex we have for $u \in C$, $J(\overline{u} + t(u - \overline{u})) \leq J(\overline{u}) + t\left[J(u) - J(\overline{u})\right]$, for $t > 0$

   $$\implies J(u) - J(\overline{u}) \geq \frac{1}{t}\left[J(\overline{u} + t(u - \overline{u})) - J(\overline{u})\right] \xrightarrow{t \downarrow 0} J'(\overline{u}, u - \overline{u}) \geq 0.$$

   Therefore $\overline{u}$ is a global minimizer.

3. Assume, that there are two solution for the minimization problem, $\overline{u}, u^* \in C$, such that $\overline{u} \neq u^*$ and $J(\overline{u}) = J(u^*) = \inf_{u \in C} J(u)$. Since J is strictly convex $J(u_t) = J(t\overline{u} + (1-t)u^*) < tJ(\overline{u}) + (1-t)J(u^*) = \alpha$ for all $t \in [0,1]$. Contradicting our assumption that u^* and $\overline{u}$ are solutions.

4. $\alpha = \inf_{u \in C} J(u) \in \mathbb{R} \cup \{-\infty\}$, choose a minimizing sequence $(u_k)_k \subset C$ with $J(u_k) \xrightarrow{k \to \infty} \alpha$

   $\implies (u_k)_k$ is bounded, because $J \to \infty$ as $\|u\| \to \infty$.
   $\implies (u_k)_k$ contains a weakly convergent subsequence $u_{k_e} \xrightarrow[e \to \infty]{} \overline{u} \in C$. Since C is closed and convex.
   $\implies J$ is weakly-lower semicontinuos because it is convex and continuos.

□

3. Lecture 3

Now consider Hilbert space $(H, <.>)$ with $\|x\| = \sqrt{(x,x)}$.

4. Lecture 4

5. Lecture 5

6. Lecture 6

7. Lecture 7

8. Lecture 8

Definition 8.1.
A function $phi : U \times Y \to \overline{\mathbb{R}}$ is said to be a perturbation function of J (function of minimization problem in U), if $\phi(u, 0) = J(u)$ for all $u \in U$. For each $p \in Y$, the minimization problem.

$$\inf_{u \in U} \phi(u, p) \tag{Pp}$$

is called a perturbation problem to (Pp). The variable p is called perturbation parameter. If we denote by ϕ^* the convex conjugate function of ϕ, the *dual problem*, with respect to ϕ is defined by

$$\sup_{p^* \in Y^*} -\phi^*(0, p^*)$$

where $\phi^* : (U \times Y)^* \cong U^* \times Y^* \to \overline{\mathbb{R}}$.

$$\phi^*(u^*, p^*) = \sup_{u \in U, p \in Y} ((u^*, u)_{U^*U} + (p^*, p)_{Y^*Y} - \phi(u, p))$$

Remark: for $p = 0$, (8.1) $\equiv$ (Pp).

We denote the infimum for problem (**??**) by $\inf(P)$ and the sup of (Pp) by $\sup P*$

Lemma 8.1 (Weak duality).
For the problem (P)and (P) it holds that*

$$-\infty \leq \sup(P*) \leq \inf(P) \leq \infty.$$

Proof. Let $p^* \in Y^*$. It follows

$$-\phi^*(0, p^*) = -\sup_{u \in U, p \in Y} ((0, u) + (p^*, p) - \phi(u, p)) \tag{2}$$
$$= \inf_{u \in U, p \in Y} (\phi(u, p) - (p^*, p)) \tag{3}$$
$$\leq (\phi(u, 0) - (p^*, 0)) \quad \forall u \in U, p^* \in Y^* \tag{4}$$
$$\implies \sup_{p^* \in Y^*} \leq \inf_{u \in U} \phi(u, 0) = \inf(P) \tag{5}$$

By iteration we can define, a bidual problem

$$-\sup_{u\in U}\left(-\phi^*(u,0)\right)=\inf_{u\in U}\phi^*(u,0)$$

If the perturbation function $\phi(u,p)$ is proper, convex and weakly lower semicontinuous. Then $\phi^{**}=\phi$. In this case $\phi(u,0)=\phi^{**}(u,p)$ i.e 8.1 $\equiv$ **??** $\square$

Definition 8.2.

Consider the infimal value function

$$h(p^*)=\inf Pp$$

The problem is called stable if h(0) is finite and subdifferentiable in zero is not empty.

Theorem 8.1.

The primal problem (P) is stable if and only if the following this condition are simultaneously satisfied:

- *The dual problem (P^*) has solution.*
- *There is no duality gap, i.e.*

$$\inf(P)=\sup(P^*)\leq\infty$$

Theorem 8.2 (Extremal relation).

Let $\phi:U\times Y\to\overline{\mathbb{R}}$, be convex the the following statements are equivalent:

1. *(P) and Pp have solutions $\overline{u}$ and $\overline{p^*}$ and* $\inf(P)=sup(P^*)$
2. $\phi(\overline{u},0)+\phi^*\left(0,\overline{p^*}\right)=0$
3. $\left(0,\overline{p^*}\right)\in\partial\phi(u,0)$ *and* $(\overline{u},0)\in\partial\phi^*(0,p^*)$

Proof. We proceed by parts:

1. (1) $\implies$ (2): $\overline{u}$ solution of $\inf(P)$ and $\overline{p^*}$ solution $\sup(p^*)$ and $\inf(P)=\sup(P*)$ $\implies$ $\phi(\overline{u},0)=\inf(P)=\sup(P*)=-\phi(0,\overline{p^*})$, then $\implies$ $\phi(\overline{u},0)+\phi^*(0,\overline{p^*})=0$
2. (2) $\implies$ (1): $-\phi^*(0,\overline{p^*})=\sup(P*)\leq\inf(P)=\phi(\overline{u},0)=-\phi^*(0,\overline{p^*})$ $\implies$ $\sup(P*)=\inf(P)$
3. (2) $\iff$ (3): $\phi(\overline{u},0)+\phi^*(0,\overline{p^*})=0=(0,\overline{u})+(\overline{p^*},0)=\left((0,\overline{p^*},(\overline{u},0))\right)$ $\iff$ $(0,\overline{p^*})\in\partial\phi(\overline{u},0)$ $\forall u\in U,p^*\in\partial J(u)$ $\iff$ $J(u)+J^*(p^*)=(p^*,u)$

$\square$

Functional duality.

$$J(u)=F(u)+G(Au)$$

with $F:U\to\overline{\mathbb{R}}$, G convex function $G:V\to\overline{\mathbb{R}}$ and $A:U\to V$ bounded and linear.

We introduce the perturbation $\phi(u,p)=F(u)+G(Au-p)$. The dual problem is obtained with,

$$\phi^*(0,p^*)=\sup_{\substack{u\in U\\p\in V}}\left((p^*,p)-F(u)-G(Au-p)\right)$$

For fixed u we set $q:Au-p$.

$$
\begin{aligned}
\phi^*(0,p^*) =& \sup_{u\in U}\sup_{p\in V}\left((p^*, Au-q) - F(u) - G(q)\right)\\
=& \sup_{u\in U}\sup_{p\in V}\left((A^*p^*, u) - (p^*, q) - F(u) - G(q)\right)\\
=& \sup_{u\in U}\left((p^*, Au) - F(u)\right) + \sup_{p\in V}\left((-p^*, q) - G(q)\right)\\
=& F^*(A^*p^*) + G^*(-p^*)
\end{aligned}
$$

References