Universität Hamburg
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Optimization

Notes

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Abstract:

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Introduction

0.1. Definitions

0.2. Useful lemmas and Theorems.

Lemma 0.1.

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Hilbert Space H. Then $(x_n)_{n\in\mathbb{N}}$ possesses a weakly convergent subsequence.

Lemma 0.2.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hilbert Space H. Then $(x_n)_{n\in\mathbb{N}}$ converges if and only if it is bounded and possesses at most one weak sequential cluster point.

Theorem 0.1.

Let $f : H \to (\infty, \infty]$ be a convex functional on a Hilbert space. Then the following are equivalent:

(i) *f is weakly sequentially lower semicontinuous.*

(ii) *f is sequentially lower semicontinuous.*

(iii) *f is lower semicontinuous.*

(iv) *f is weakly lower semicontinuous.*

1. Lecture 1

2. Lecture 2

2.1. Convexity

Definition 2.1.

Let U be linear space. A functional $J : U \to \overline{\mathbb{R}}$ is called convex, if for $t \in [0, 1]$ and $u_1, u_2 \in U$.

$$J(tu_1 + (1-t)u_2) \le tJ(u_1) + (1-t)J(u_2) \tag{1}$$

holds such that the right hand sid is well defined.

- J is strictly convex if (1) holds strictly for $\forall u_1, u_2 \in U$, $u_1 \neq u_2$ and $t \in (0,1)$ with $J(u_1) < \infty$ and $J(u_2) \leq \infty$.
- An optimization problem,
$$\min_{u \in C} J(u)$$
is called convex if both C and J are convex.

Lemma 2.1.
If C and V are convex in U, then

- $\alpha V = \{w = \alpha v, v \in C\}$ *is convex.*
- $C + V$ *is convex.*

Proof. □

Lemma 2.2.
Let V be a collection of convex sets in U, then $C = \bigcap_{K \in V} K$ is convex.

Proof. If $C = \emptyset$, then C the statement is vacuously true. Consider $C \neq \emptyset$ and $u_1, u_2 \in C$ then $u_1, u_2 \in K$ for all $K \in V$

$$\implies tu_1 + (1-t)u_2 \in K, \quad \forall K \in V \implies tu_1 + (1-t)u_2 \in \bigcap_{K \in V} K$$

□

Lemma 2.3.
Let $C \in U$ convex and $J : C \to \mathbb{R}$. Define $\alpha = \inf_{u \in C} J(u)$. Then the set $\Psi = u | J(u) = \alpha$ is convex, i.e. the solution of

$$\min_{u \in C} J(u)$$

is a convex set.

Proof. Let $u_1, u_2 \in \Psi$ and $u_t = tu_1 + (1-t)u_2$.Sinc J is convex, it holds that $J(u_t) \leq tJ(u_t) + (1-t)J(u_t) = \alpha$. Thus $J(u_t) = \alpha$, $\forall t \in [0,1]$. Implying $u_t \in \Psi$ Hence Ψ is convex. □

Lemma 2.4.
Let U be linear normed space, and $C \subset U$ a convex set and $J : U \to \overline{\mathbb{R}}$ convex functional. Let $\overline{u} \in C$ such that

$$J(\overline{u}) \leq J(u) \quad \forall u \in B_\epsilon(\overline{u}) \cap C,$$

for some ball $B_\epsilon(\overline{u})$ in U with center in $\overline{u}$. Then $J(\overline{u}) = \inf_{u \in C} J(u)$. In other words, the local minimizer of a convex optimization problem is also a global minimizer.

Proof. Let $B_\epsilon(\overline{u})$ be an open neighborhood of $\overline{u}$ with $J(\overline{u}) \leq J(u)$ for all $u \in B_\epsilon(\overline{u}) \cap C$. Take an arbitrary $u^* \in C$ and consider $u_t = t\overline{u} + (1-t)u^*$. Since C is convex $u_t \in C$.
For some $t \in (0,1)$, $u_t \in B_\epsilon(\overline{u})$.
Thus,

$$J(\overline{u}) \leq J(u_t) \leq tJ(\overline{u}) + (1-t)J(u^*).$$

We have $\forall t \in [0,1]$ that $(1-t) \leq 0$, then

$$(1-t)J(\overline{u}) \leq (1-t)J(u^*) \qquad \forall u^* \in C$$

Therefore, $\overline{u}$ is a local minimizer for C. □

Theorem 2.1.
Let U is Banach Space, $C \subset U$ convex and $J : C \to \mathbb{R}$ Gateâux differentiable. Consider the minimization problem.

$$\min_{u\in C} J(u)$$

1. *Let $\overline{u}$ be a local solution. Then $J'(\overline{u}, u-\overline{u}) \geq 0$, $\forall u \in C$.*
2. *If J is convex on C, then $J'(\overline{u}, u-\overline{u}) \geq 0$, $\forall u \in C$ is necessary and sufficient for global optimality of $\overline{u}$*
3. *If J is strictly convex on C, then the minimization problem admits at most one solution.*
4. *If C is closed, and J is convex and continuous with*

$$\lim_{\substack{u\in C \\ \|u\|\to\infty}} J(u) = \infty.$$

Then a global solution $\overline{u} \in C$ exists.

Proof.

1. Let $\overline{u}$ be a local solution $J(\overline{u}) \leq J(u)$, $\forall u \in B_\epsilon(\overline{u}) \cap C$, let $t \in [0,1]$, $u_t = \overline{u} + t(u-\overline{u})$, then $u_t \in C$, since C is convex.
   For small $t > 0$,
   $$0 \leq \frac{1}{t}\left[J(u_t) - J(u)\right] \leq \frac{1}{t}\left[J(\overline{u} + t(u-\overline{u})) - J(u)\right] \xrightarrow{t\downarrow 0} J'(\overline{u}, u-\overline{u})$$
2. Since J is convex we have for $u \in C$, $J(\overline{u} + t(u-\overline{u})) \leq J(\overline{u}) + t\left[J(u) - J(\overline{u})\right]$, for $t > 0$
   $$\implies J(u) - J(\overline{u}) \geq \frac{1}{t}\left[J(\overline{u} + t(u-\overline{u})) - J(\overline{u})\right] \xrightarrow{t\downarrow 0} J'(\overline{u}, u-\overline{u}) \geq 0.$$
   Therefore $\overline{u}$ is a global minimizer.
3. Assume, that there are two solution for the minimization problem, $\overline{u}, u^* \in C$, such that $\overline{u} \neq u^*$ and $J(\overline{u}) = J(u^*) = \inf_{u\in C} J(u)$. Since J is strictly convex $J(u_t) = J(t\overline{u} + (1-t)u^*) < tJ(\overline{u}) + (1-t)J(u^*) = \alpha$ for all $t \in [0,1]$. Contradicting our assumption that u^* and $\overline{u}$ are solutions.
4. $\alpha = \inf_{u\in C} J(u) \in \mathbb{R} \cup \{-\infty\}$, choose a minimizing sequence $(u_k)_k \subset C$ with $J(u_k) \xrightarrow{k\to\infty} \alpha$

   $\implies (u_k)_k$ is bounded, because $J \to \infty$ as $\|u\| \to \infty$.
   $\implies (u_k)_k$ contains a weakly convergent subsequence $u_{k_e} \xrightarrow[e\to\infty]{} \overline{u} \in C$. Since C is closed and convex.
   $\implies J$ is weakly-lower semicontinuos because it is convex and continuos.

□

3. Lecture 3

4. Lecture 4

4.1. Lecture 5

References