Taylor Series
$\frac{1}{x} = \sum_{n=0}^{\infty} x^n$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Permutations and Combinations
$P(n,k) = \frac{n!}{(n-k)!}$
$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$

Laplace Transforms
$F^*(s) = \int_0^{\infty} f(t)e^{-st}dt$
$f(t) = \int_0^{\infty} F^*(s)e^{st}ds$

Convolution Property
$f(t) * g(t) = \int_0^t f(t-x)g(x)dx \leftrightarrow F^*(s)G^*(s)$

Z-Transform
Mapping of discrete function f_n into complex fuction with variable z.
$F(z) = \sum_{n=0}^{\infty} f_n z^n$

Probability and Conditional
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A|B) = \frac{P(A \cap B)}{P(B)}$
A, B are independent if $P(A,B) = P(A)P(B)$

Total Probability
$P(B) = \sum_i P(A_i)P(B|A_i)$

Bayes' Rule
$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$

PMF (Probability Mass Function)
$p_X(x) = p(\{s \in \Omega \text{ s.t. } X(s) = x\})$
$\sum_x p_X(x) = 1$

Bernoulli Random Variable
$X = 1$ on success, $X = 0$ on failure.
$p(X = x) = p$, if $x = 1$
$p(X = x) = 1 - p$, if $x = 0$

Geometric Random Variable
Counts #trials until first success.
$p_X(x) = (1-p)^{x-1}p$, $x = 1, 2, \cdots$
$p(X \geq s+1 | X \geq t) = p(X \geq s)$

Binomial Random Variable
Counts #success in n identical independent experiments.
$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, when $0 \leq x \leq n$
$p_X(x) = 0$, otherwise

Poisson Random Variable
Model occurrence of event over time interval assuming event happens at rate λ
$p_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$, when $x = 0, 1, \cdots$

PDF (Probability Density Function)
$\int_{-\infty}^{\infty} f_X(x)dx = 1$

CDF (Cumulative Distribution Function)
$F_X(x) = P(X \leq x)$
$\lim_{x \to -\infty} F_X(x) = 0$
$\lim_{x \to \infty} F_X(x) = 1$
$P(a < X \leq b) = F_X(b) - F_X(a)$

Uniform Distribution
$f_X(x) = \frac{1}{b-a}$, when $a \leq x \leq b$
$f_X(x) = 0$, otherwise

Exponential Distribution
Memoryless continuous distribution.
$f_X(x) = \lambda e^{-\lambda x}$, when $x \geq 0$
$F_X(x) = 1 - e^{-\lambda x}$, when $x \geq 0$
$F_X(x) = 0$, otherwise
$P(X > x) = e^{-\lambda x}$

Expectation
$E[X] = \sum_x xp(x)$
$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
If $Y = g(X)$, $E[Y] = \sum_x g(x)p(x)$,
$E[Y] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$
$E[X+Y] = E[X] + E[Y]$
$E[aX] = aE[X]$
$E[XY] = E[X]E[Y]$, if X,Y are independent.
For X, Y with joint PMF $p(x,y)$ or PDF $f_{X,Y}(x,y)$, $E[XY] = \sum_{(x,y)} xyp(x,y)$,
$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy$

Conditional Expectation
X,Y are random variables,
$E[Y|X] = \sum_y yP(Y = y|X = x) = \sum_y yp_{Y|X}(y|x)$,
$E[Y|X] = \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy$

Unconditional Expectation
$E[Y] = \sum_x E[Y|X]p_X(x)$
$E[Y] = \int_{\infty}^{\infty} E[Y|X]f_X(x)dx$

Variance
$Var[X] = E[(X - E[X])^2] = \sum_x (x - E[X])^2 p(x) = \int_{\infty}^{\infty} (x - E[X])^2 f_X(x)dx$
$Var[X] = E[X^2] - E[X]^2$
$Var[X+Y] = Var[X] + Var[Y]$, if X,Y are independent.

Expectations and Variances
Binomial: np, $np(1-p)$
Geometric: $\frac{1}{p}$, $\frac{1-p}{p^2}$
Uniform: $\frac{a+b}{2}$, $\frac{(b-a)^2}{12}$
Exponential: $\frac{1}{\lambda}$, $\frac{1}{\lambda^2}$
Poisson: λ, λ

Covariance: measure of joint probability
$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$
$Cov(X,Y) = E[XY] - E[X]E[Y]$
If X,Y are independent, $Cov(X,Y) = 0$

Correlation: scaled version of covariance
$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$, range $[-1, 1]$

CTMC: State transitions are permitted at arbitrary time instances. The amount of time spent in a state is exponentially distributed.

State Transition Probability:
$p_{ij}(t) = p(X(\tau + t) = j | X(\tau) = i)$

Chapman-Kolmogorov Equation:
$p_{ij}(s+t) = \sum_k p_{ik}(s)p_{kj}(t)$

Transition Probability $H(t)$: $H(t) = \{p_{ij}(t)\}$
$H(s+t) = H(s)H(t)$
$H(t + \Delta t) = H(t)H(\Delta t)$
$H(t + \Delta t) = H(t)[H(\Delta t) - I]$
$\frac{dH(t)}{dt} = H(t) \lim_{\Delta t \to 0} [\frac{H(\Delta t) - I}{\Delta t}]$
$Q = \lim_{\Delta t \to 0} [\frac{H(\Delta t) - I}{\Delta t}]$
$\frac{dH(t)}{dt} = H(t)Q$

Transition Rate Matrix: Q, infinitesimal generator of $H(t)$. $H(t) = e^{Qt}$.
Diagonal elements ≤ 0. $q_{ii} = \lim_{\Delta t \to 0} [\frac{p_{ii}(\Delta t) - 1}{\Delta t}]$.
Off-diagonal elements ≥ 0.
$q_{ij} = \lim_{\Delta t \to 0} [\frac{p_{ij}(\Delta t) - 0}{\Delta t}]$, for $i \neq j$.
In each row, sum of off-diagonal = magnitude of diagonal: $q_{ii} = -\sum_{i \neq j} q_{ij}$

State Probabilities:
$\pi(t) = \pi(0)H(t)$, $\pi(t) = \pi(0)e^{Qt}$

Stationary Distribution: $\pi Q = 0$, $\sum_j \pi_j = 1$

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Q1: A given program has an execution time that is uniformly distributed between 10 and 20 seconds. The number of interrupts that occur during execution is a Poisson random variable with parameter t where t is the program execution time. The probability distribution of the number of interrupts is therefore $P(N = k) = (\lambda t)ke^{\lambda t}$.

(a) What is $E[N|T = t]$, where N is the number of interrupts the program experiences, and T is the running time of the program.

$E[N|T = t] = \lambda t$, since for fixed running time, the number of interrupts is a Poisson random variable with mean λt.

(b) Find the expected number of interrupts the program experiences during a randomly selected execution.

$E[N] = \int_{10}^{20} E[N|T = t]f_T(t)dt = \int_{10}^{20} \frac{\lambda t}{10} dt = 15\lambda$

Q2: Suppose that you made a webpage and you are collecting the statistics from the visitors. There are m types of visitors. Each visit is equally likely to be any of the m types. Find the expected number of visitors needed in order to have at least one of each type. Hint: Let X denote the number of visitors needed. It is useful to represent X by $X = \sum_{i=1}^m X_i$ where each X_i is a geometric random variable.

Suppose the current visitor pool contains i different types. Let X_i denote the number of additional visitors needed until it contains $i+1$ types. The X_i is are independent geometric random variables with parameter $(m-i)/m$, $i = 0, 1, \cdots, m-1$.
$E[X] = E[\sum_{i=1}^m X_i] = \sum_{i=1}^m E[X_i] = \sum_{i=1}^m \frac{m}{m-i}$

Q3: A Markov chain $\{X_n, n \geq 0\}$ with states 0, 1, 2, has the transition probability matrix
$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ If $P[X_0 = 0] = P[X_0 = 1] = 1$, find the state probability vector $P[X_3 = 2]$.

Cubing the transition probability matrix, we obtain
$$P^3 = \begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{bmatrix}$$
$P[X_3 = 2] = \frac{1}{4} \cdot \frac{47}{108} + \frac{1}{4} \cdot \frac{11}{27} + \frac{1}{2} \cdot \frac{13}{36}$

Q4: A workstation tries to transmit frames through Ethernet. Suppose that whether or not collision occurs in the current transmission depends on the result of the last two trans- missions the workstation had. That is, suppose that if collisions have occurred in both of the past two transmissions, then with probability 0.7 a collision will occur in the current transmission; if a collision occurs in last transmission but not the transmission before the last one, then a collision will occur in the current transmission with probability 0.5; if a collision occurred in the transmission before the last one but not the last one, then one will occur in the current transmission with probability 0.4; if there have been no collision in the past two transmissions, then a collision will occur in the current transmission with probability 0.2. (Hint: Note that the state description needs to include status of last two transmissions).

(b) Find the transition probability matrix.
$$P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

(c) What fraction of frames suffer a collision?
Solve $\pi = \pi P$ to obtain the stationary state probabilities. Then the fraction of frames suffering a collision is $\pi_0 + \pi_2$ (or $\pi_0 + \pi_1$).