

Taylor Series

$\frac{1}{x} = \sum_{n=0}^{\infty} x^n$
 $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Permutations and Combinations

$P(n, k) = \frac{n!}{(n-k)!}$

$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$

Laplace Transforms

$F^*(s) = \int_0^{\infty} f(t)e^{-st}dt$

$f(t) = \int_0^{\infty} F^*(s)e^{st}ds$

Convolution Property

$f(t) * g(t) = \int_0^t f(t-x)g(x)dx \leftrightarrow F^*(s)G^*(s)$

Z-Transform

Mapping of discrete function f_n into complex fuction with variable z .

$F(z) = \sum_{n=0}^{\infty} f_n z^n$

Probability and Conditional

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

A, B are independent if $P(A, B) = P(A)P(B)$

Total Probability

$P(B) = \sum_i P(A_i)P(B|A_i)$

Bayes' Rule

$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$

PMF (Probability Mass Function)

$p_X(x) = P(\{s \in \Omega \text{ s.t. } X(s) = x\})$

$\sum_x p_X(x) = 1$

Bernoulli Random Variable

$X = 1$ on success, $X = 0$ on failure.

$p(X = x) = p$, if $x = 1$

$p(X = x) = 1 - p$, if $x = 0$

Geometric Random Variable

Counts #trials until first success.

$p_X(x) = (1-p)^{x-1}p$, $x = 1, 2, \dots$

$p(X \geq s+1|X \geq t) = p(X \geq s)$

Binomial Random Variable

Counts #success in n identical independent experiments.

$p_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$, when $0 \leq x \leq n$

$p_X(x) = 0$, otherwise

Poisson Random Variable

Model occurrence of event over time interval assuming event happens at rate λ

$p_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$, when $x = 0, 1, \dots$

PDF (Probability Density Function)

$\int_{-\infty}^{\infty} f_X(x)dx = 1$

CDF (Cumulative Distribution Function)

$F_X(x) = P(X \leq x)$

$\lim_{x \rightarrow -\infty} F_X(x) = 0$

$\lim_{x \rightarrow \infty} F_X(x) = 1$

$P(a < X \leq b) = F_X(b) - F_X(a)$

Uniform Distribution

$f_X(x) = \frac{1}{b-a}$, when $a \leq x \leq b$

$f_X(x) = 0$, otherwise

Exponential Distribution

Memoryless continuous distribution.

$f_X(x) = \lambda e^{-\lambda x}$, when $x \geq 0$

$F_X(x) = 1 - e^{-\lambda x}$, when $x \geq 0$

$F_X(x) = 0$, otherwise

$P(X > x) = e^{-\lambda x}$

Expectation

$E[X] = \sum_x xp(x)$

$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx$

If $Y = g(X)$, $E[Y] = \sum_x g(x)p(x)$,

$E[Y] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$

$E[X + Y] = E[X] + E[Y]$

$E[aX] = aE[X]$

$E[XY] = E[X]E[Y]$, if X, Y are independent.

For X, Y with joint PMF $p(x, y)$ or PDF

$f_{X,Y}(x, y)$, $E[XY] = \sum_{(x,y)} xyp(x, y)$,

$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{X,Y}(x, y)dxdy$

Conditional Expectation

X, Y are random variables,

$E[Y|X] = \sum_y yP(Y = y|X = x) = \sum_y y p_{Y|X}(y|x)$,

$E[Y|X] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy$

Unconditional Expectation

$E[Y] = \sum_x E[Y|X]p_X(x)$

$E[Y] = \int_{-\infty}^{\infty} E[Y|X]f_X(x)dx$

Variance

$Var[X] = E[(X - E[X])^2] = \sum_x (x - E[X])^2 p(x) =$

$\int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)dx$

$Var[X] = E[X^2] - E[X]^2$

$Var[X + Y] = Var[X] + Var[Y] + 2Cov(X, Y)$

$Var[XY] = E[X^2Y^2] - E^2[XY] = E[X^2]E[Y^2] +$

$Cov(X^2, Y^2) - (E[X]E[Y] + Cov(X, Y))^2$

Expectations and Variances

Binomial: $np, np(1-p)$

Geometric: $\frac{1}{p}, \frac{1-p}{p^2}$

Uniform: $\frac{a+b}{2}, \frac{(b-a)^2}{12}$

Exponential: $\frac{1}{\lambda}, \frac{1}{\lambda^2}$

Poisson: λ, λ

Covariance: measure of joint probability

$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$

$Cov(X, Y) = E[XY] - E[X]E[Y]$

If X, Y are independent, $Cov(X, Y) = 0$

Correlation: scaled version of covariance

$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$, range $[-1, 1]$

Stationary Process: $F_X(\mathbf{x}; \mathbf{t}) = F_X(\mathbf{x}; \mathbf{t} + \tau)$

Independent Processes:

$F_X(\mathbf{x}; \mathbf{t}) = F_{X_1}(x_1, t_1)F_{X_2}(x_2, t_2) \dots F_{X_n}(x_n, t_n)$

$f_X(\mathbf{x}, \mathbf{t}) = \prod_{i=1}^n f_{X_i}(x_i, t_i)$ (**Continuous State**)

$p_X(\mathbf{x}, \mathbf{t}) = \prod_{i=1}^n p_{X_i}(x_i, t_i)$ (**Discrete State**)

Markovian Property: $P[X(t_{n+1}) \leq$

$x_{n+1}|X(t_n) = x_n, X(t_{n+2}) = x_{n+2}, \dots, X(t_0) =$

$x_0] = P[X(t_{n+1}) \leq x_{n+1}|X(t_n) = x_n]$

Discrete Time Markov Chains (DTMC):

$p_{ij} = P[X_n = j|X_{n-1} = i]$ (**Homogenous**)

$$\mathbf{P} = (p_{ij}) = \begin{bmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$\sum_j p_{ij} = 1$ for each row.

Initial State Probabilities:

$\boldsymbol{\pi}^{(0)} = (\pi_0^{(0)}, \pi_1^{(0)}, \dots)$, where $\pi_j^{(0)} = P[X_0 = j]$

n -Step Transition Probabilities:

$p_{ij}^{(n)} = P[X_n = j|X_0 = i] = P[X_{n+k} = j|X_k = i]$

Chapman-Kolmogorov: $p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)} p_{kj}$

Limiting Distribution: $\boldsymbol{\pi} = \lim_{n \rightarrow \infty} \boldsymbol{\pi}^{(0)} \mathbf{P}^n$

$\boldsymbol{\pi} = \boldsymbol{\pi} \mathbf{P}$ and $\sum_j \pi_j = 1$

Continuous Time Markov Chains (CTMC):

State transitions permitted at arbitrary time instances. Time spent in a state is exponentially distributed.

State Transition Probability:

$p_{ij}(t) = p(X(\tau + t) = j|X(\tau) = i)$

Chapman-Kolmogorov Equation:

$p_{ij}(s+t) = \sum_k p_{ik}(s)p_{kj}(t)$

Transition Probability: $\mathbf{H}(t) = \{p_{ij}(t)\}$

$\mathbf{H}(s+t) = \mathbf{H}(s)\mathbf{H}(t)$

$\mathbf{H}(t+\Delta t) = \mathbf{H}(t)\mathbf{H}(\Delta t)$

$\mathbf{H}(t+\Delta t) = \mathbf{H}(t)[\mathbf{H}(\Delta t) - \mathbf{I}]$

$\frac{d\mathbf{H}(t)}{dt} = \mathbf{H}(t) \lim_{\Delta t \rightarrow 0} [\frac{\mathbf{H}(\Delta t) - \mathbf{I}}{\Delta t}]$

$\mathbf{Q} = \lim_{\Delta t \rightarrow 0} [\frac{\mathbf{H}(\Delta t) - \mathbf{I}}{\Delta t}]$

$\frac{d\mathbf{H}(t)}{dt} = \mathbf{H}(t)\mathbf{Q}$

Transition Rate Matrix: \mathbf{Q} , infinitesimal generator of $\mathbf{H}(t)$. $\mathbf{H}(t) = e^{\mathbf{Q}t}$.

Diagonal Elements ≤ 0 .

$q_{ii} = \lim_{\Delta t \rightarrow 0} [\frac{p_{ii}(\Delta t) - 1}{\Delta t}]$.

Off-Diagonal Elements ≥ 0 .

$q_{ij} = \lim_{\Delta t \rightarrow 0} [\frac{p_{ij}(\Delta t) - 0}{\Delta t}]$, for $i \neq j$.

In each row, sum of off-diagonal = magnitude of diagonal: $q_{ii} = -\sum_{i \neq j} q_{ij}$

State Probabilities:

$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\mathbf{H}(t)$, $\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)e^{\mathbf{Q}t}$

Stationary Distribution: $\boldsymbol{\pi} \mathbf{Q} = 0$, $\sum_j \pi_j = 1$

Birth-Death Process: At state k , λ_k, μ_k

are birth and death rates. **Transition Matrix:** $\mathbf{Q} =$

$$\begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & \cdots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots & \cdots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Equilibrium Solution: $\boldsymbol{\pi} \mathbf{Q} = 0$ and $\sum_j \pi_j = 1$

Differential Difference Equations: $\frac{\partial \pi_k(t)}{\partial t} =$

$\lambda_{k-1}\pi_{k-1}(t) + \mu_{k+1}\pi_{k+1}(t) - (\lambda_k + \mu_k)\pi_k(t)$

$\frac{\partial \pi_0(t)}{\partial t} = \mu_1\pi_1(t) - \lambda_0\pi_0(t)$

$\frac{\partial \pi_k(t)}{\partial t} =$ flow in - flow out

flow in $= \lambda_{k-1}\pi_{k-1} + \mu_{k+1}\pi_{k+1}$

flow out $= (\lambda_k + \mu_k)\pi_k$

Queueing Theory: Analyze different systems.

Poisson distribution model arrival process, exponential distribution model service times.

Poisson Process: ArrivalRate of λ :

$p(n \text{ arrivals in interval } T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}$

Exponential Inter-Arrival Time:

$p(\text{inter-arrival} \leq T) = 1 - e^{-\lambda T}$

Merging Property: Let A_1, A_2, \dots, A_k be independent Poisson processes with rates $\lambda_1, \lambda_2, \dots, \lambda_k$, then $A = \sum_i A_i$ is also Poisson process with rate $\lambda = \sum_i \lambda_i$

Splitting Property: Suppose every arrival is randomly routed with probability p to stream 1 and $(1-p)$ to stream 2. Stream 1 and 2 are Poisson with rates $p\lambda$ and $(1-p)\lambda$.

Kendall's Notation: A/B/C/D

A: inter-arrival time distribution

B: service time distribution

C: number of servers

D: maximum number of jobs possible

M:exponential, D: deterministic, G: general

$\alpha(t)$: number of arrivals in $(0, t)$

$\beta(t)$: number of departures in $(0, t)$

$N(t)$: number of customers in system at t

$T(t)$: average time in system for customer up to t

N : average number of customers in system.

$N = N_q + N_s$

T : average waiting time in system. $T = T_q + T_s$

Little's Law: $N = \lambda T$

Utilization Factor (Traffic Intensity Factor):

$\rho = \text{Work Arrival Rate} / \text{Server Capacity}$

System is **unstable** if $\rho > 1$

Single Server System

ρ is fraction of time server is busy

Multi-Server System

ρ is fraction of busy servers

M/M/1 Model: Assume state independent arrival λ , service rate μ . At equilibrium, flow in = flow out, $\lambda\pi_0 = \mu\pi_1 \rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0, \pi_1 = \rho\pi_0$. Similarly $\lambda\pi_2 = \mu\pi_1 = \rho^2\pi_0$. In general $\pi_n = \rho^n\pi_0$. $\sum_i \pi_i = 1 \rightarrow \pi_0 = 1 - \rho, \pi_n = (1 - \rho)\rho^n$

Average Number Customers in System N :

$\sum_k k\pi_k = \sum_k k(1 - \rho)\rho^k = \frac{\rho}{1 - \rho} = \frac{\lambda}{\mu - \lambda}$

Average Time in System T :

$T = N/\lambda$ (with Little's Law), $T = \frac{1}{\mu - \lambda}$

Average Time in Queue T_q :

$T_q = T - T_s = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$

Average Number of Customers in Queue:

$N_q = \lambda T_q = N - \rho$ (with Little's Law)

Discouraged Arrivals: Arrival rates decrease

with customers in system, $\lambda_k = \frac{\alpha}{k+1}, \mu_k = \mu$

Equilibrium state, $p_k = p_0 (\frac{\alpha}{\mu})^k \frac{1}{k!}$. Since $\sum_{k=0}^{\infty} p_k = 1$ we find that $p_0 = e^{-\alpha/\mu}$. So $p_k = ((\frac{\alpha}{\mu})^k e^{-\frac{\alpha}{\mu}})/k!$. **Utilization**
 $\rho = 1 - p_0 = 1 - e^{-\frac{\alpha}{\mu}}$
 Expected number of customers in system:
 $N = E[k] = \alpha/\mu$. Expected time in system
 $T = N/\lambda$. λ_{eff} is expected (effective) arrival rate,
 $\lambda_{\text{eff}} = \mu\rho = \mu(1 - e^{-\alpha/\mu})$, so $T = \alpha/\mu^2(1 - e^{-\alpha/\mu})$

M/M/∞ Model: Service rate (number of servers) scales linearly with number of customers in system.
 → No waiting time, served immediately. $\lambda_k = \lambda$, $\mu_k = k\mu$. Equilibrium state, $p_k = (\frac{\lambda}{\mu})^k e^{-\lambda/\mu}/k!$, $p_0 = e^{-\lambda/\mu}$ (same as discouraged arrival system). Expected number of customers $N = E[k] = \lambda/\mu$. Expected time in system $T = N/\lambda = 1/\mu$.
M/M/m Model: Unlimited waiting room, constant arrival rate λ , maximum of m servers. Each server has service rate μ . $\lambda_k = \lambda$, $\mu_k = k\mu$ ($k < m$), $m\mu$ ($k \geq m$). Equilibrium distribution, $p_k = \frac{(m\rho)^k}{k!}p_0$ ($k < m$), $\frac{(m\rho)^k}{m!} \frac{1}{m^{k-m}}p_0$ ($k \geq m$) where $\rho = \lambda/m\mu$. When $\rho < 1$,
 $p_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)} \right]^{-1}$

M/M/1/K Model: System has single server and maximum of K customers. Equilibrium state, $p_k = p_0 (\frac{\lambda}{\mu})^k$ ($k \leq K$), 0 ($k > K$), where $p_0 = \left(1 - \frac{\lambda}{\mu}\right) / \left(1 - (\frac{\lambda}{\mu})^{K+1}\right)$
Finite Population Systems: World has finite users and each user is either in system or to arrive with exponential time with mean $1/\mu$. $\lambda_k = \lambda(M - k)$ ($0 \leq k < M$), 0 (otherwise). $\mu_k = \mu$ ($1 \leq k \leq M$), 0 (otherwise). Equilibrium state, $p_k = p_0 (\frac{\lambda}{\mu})^k \frac{M!}{(M-k)!}$ ($0 \leq k \leq M$), 0 (otherwise).
 $p_0 = \left[\sum_{k=0}^M (\frac{\lambda}{\mu})^k \frac{M!}{(M-k)!} \right]^{-1}$
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Q0: Using Laplace Transform, solve differential equations $x(t) = \frac{dx(t)}{dt} + 2\frac{d^2x(t)}{dt^2}$, $x(0) = 1$, $x'(0) = 1/2$.
 $L\{x'(t)\} = sX(s) - x(0)$
 $L\{x''(t)\} = s^2X(s) - sx(0) - x'(0)$
 $X(s) = sX(s) - x(0) + 2[s^2X(s) - sx(0) - x'(0)]$
 $X(s) = 1/(s - \frac{1}{2})$, $x(t) = e^{1/2t}$

Q1: Program has execution time uniformly distributed between 10 & 20 seconds. Number of interrupts during execution is Poisson random variable with parameter λt where t is program execution time. Probability distribution of the number of interrupts is $P(N = k) = (\lambda t)k e^{\lambda t}$.
 (a) What is $E[N|T = t]$, where N is the number of interrupts, and T is running time.
 $E[N|T = t] = \lambda t$, since for fixed running time, the number of interrupts is a Poisson random variable with mean λt .

(b) Expected number interrupts during randomly selected execution.
 $E[N] = \int_{10}^{20} E[N|T = t]f_T(t)dt = \int_{10}^{20} \frac{\lambda t}{10} dt = 15\lambda$
Q2: We have m types of visitors with equal probability. Find expected visitors to have one of each type. Hint: Let X denote number of visitors needed. Represent X by $X = \sum_{i=1}^m X_i$ where each X_i is a geometric random variable.
 Suppose the current visitor pool contains i different types. Let X_i denote the number of additional visitors needed until it contains $i + 1$ types. The X_i is are independent geometric random variables with parameter $(m - i)/m$, $i = 0, 1, \dots, m - 1$.
 $E[X] = E[\sum_{i=1}^m X_i] = \sum_{i=1}^m E[X_i] = \sum_{i=1}^m \frac{m}{m-i}$

Q3: A Markov chain $\{X_n, n \geq 0\}$ with states 0, 1, 2 has transition probability matrix $\begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ If $P[X_0 = 0] = P[X_0 = 1] = \frac{1}{4}$, find state probability vector $P[X_3 = 2]$.
 Cubing transition probability matrix

$P^{(3)} = \begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$
 $P[X_3 = 2] = \frac{1}{4} \cdot \frac{47}{108} + \frac{1}{4} \cdot \frac{11}{27} + \frac{1}{2} \cdot \frac{13}{36}$
Q4: Whether or not collision occurs depends on result of last two transmissions If collisions occurred in both of the past two, collision will occur with probability 0.7; collision occurs in last but not the transmission before the last one, then a collision will occur with probability 0.5; collision occurred in transmission before the last one but not the last one, collision with probability 0.4; no collision in the past two transmissions, collision with probability 0.2.
 (b) Find the transition probability matrix.
 $P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$
 (c) What fraction of frames suffer a collision?
 Solve $\pi = \pi P$ to obtain the stationary state probabilities. Then the fraction of frames suffering a collision is $\pi_0 + \pi_2$ (or $\pi_0 + \pi_1$).

Q5: A shop has room for 2 customers. Customers arrive at Poisson rate 3/hour and service times are exponential random variables with mean 0.25 hours.
 (a) Average number of customers in shop?
 We have birth-death process with $\lambda = 3$ and $\mu = 4$.
 $E[k] = \sum_{k=0}^2 k\pi_k$. Solve for π_k , $\pi_1 = \frac{\lambda}{\mu}\pi_0$,
 $\pi_2 = \frac{\lambda}{\mu}\pi_1$, $\sum_{j=0}^2 \pi_j = 1$, so $\pi_0 = 16/37$,
 $E[k] = 30/37$.
 (b) What is proportion of customers who get serviced? $\pi_0 + \pi_1 = 28/37$.
Q6: Packets arrive at router Poisson rate 3/ms and time to forward exponential with mean 0.2 ms. Fraction of time buffer empty?
 State is num packets in router, $\lambda = 3$, $\mu = 5$. Solve $p_1 = \frac{\lambda}{\mu}p_0$, $p_2 = \frac{\lambda}{\mu}p_1$, ..., $p_{k+1} = \frac{\lambda}{\mu}p_k$, $\sum_{k=0}^{\infty} p_k = 1$, thus $p_0 = 2/5$.

Q7: 2 machines produce products at n products/hour. Lifetime of machine follows exponential with mean $1/x$ hours, time to fix machine follows exponential with mean $1/y$ hours. Expected long term producing rate?
 State is num machines working. $\lambda_0 = \lambda_1 = y$, $\mu_1 = x$, $\mu_2 = 2x$ bc if either machine fails, num working machines reduce from 2 to 1. Solve $p_1 = \frac{y}{x}p_0$, $p_2 = \frac{y}{2x}p_1$, $p_0 + p_1 + p_2 = 1$, thus $p_0 = \frac{2x^2}{2x^2 + 2xy + y^2}$. Expected num products produced per hour: $p_0 \times 0 + p_1 \times n + p_2 \times 2n$
Q8: M/M/1 queueing system. Jobs arrive to be scheduled at rate λ jobs/ms. Single core serving jobs at rate of μ jobs/ms. Suppose that the we want the average time spent in waiting queue to be no more than 3 milliseconds. If $\lambda = 10$ jobs/ms, find minimum of μ .

Steady state probabilities are $\pi_0 = 1\rho$, $\pi_k = \rho^k(1\rho)$ where $\rho = \lambda/\mu$. Average customers $N = \frac{\rho}{1-\rho}$. By Little's Law, $N = \lambda \cdot T \rightarrow T = N$ where T is average time in system. Average waiting time $W = T - 1/\mu$, so we need $W = \frac{\rho}{\lambda(1-\rho)} - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} \leq 3$
Q9: M customers go to single-server. When customer arrives, enters service if the server is free or joins queue. Upon leaving, customer returns after exponential time with rate λ . Service time exponentially distributed μ .

(a) Define states and set up the balance equations. States are # customers in server. Balance equations $M\lambda\pi_0 = \mu\pi_1$
 $((M - i)\lambda + \mu)\pi_i = (M0i + 1)\lambda\pi_{i-1} + \mu\pi_{i+1}$ for $i = 1, 2, \dots, M - 1$
 (b) Average rate customers enter the station.
 $\lambda_{\text{avg}} = \sum_{i=0}^M (M - i)\lambda\pi_i$
 (c) Average time customer spends. Avg # customers $N = \sum_{i=0}^M i\pi_i$. Avg time $T = N/\lambda_{\text{avg}}$

Q10: UCLA receives 10 average applications. Each has 0.7 probability getting accepted. Program takes 2 years. Assume Poisson arrival, exponential time for time length.

What is probability no students in program? What is average number of students in program each year?

Modeled as M/M/∞ system with $\lambda = 10 \cdot 0.7 = 7$, $\mu = 0.5$. No students in program: $p_0 = e^{-\lambda/\mu}$
 Average number of students $N = \lambda/\mu = 14$.

Q11: Two servers. 90 reqs/hour and placed in single buffer one server becomes free. Service time 1.2 minutes. Assuming inter-arrival times and service times exponential. Determine probability no requests in the system. Expected number of requests in the system. Expected time request waits in buffer.

Modeled as M/M/2 with $\lambda = 90$, $\mu = 50$.
 $\rho = \lambda/\mu = 1.8$. $\rho/m = 1.8/2 = 0.9 < 1$. Thus system is stable. $P[\text{No Requests}]$
 $= \left[\frac{\rho^m}{m!} \frac{m}{m-\rho} + \sum_{n=0}^{m-1} \frac{\rho^n}{n!} \right]^{-1}$. Expected number of reqs, $N_s = \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{m-1} np_n + \sum_{n=m}^{\infty} np_n$,
 $N_s = \sum_{n=0}^{m-1} n \frac{\rho^n}{m!} p_0 + \sum_{n=m}^{\infty} n \frac{\rho^n}{m!m^{n-m}p_0}$.
 Expected time in buffer, $T_q = T_s - 1.2(\text{min})$, where $T_s = N_s/\lambda$. $T_q = 5.117$ (min).