Taylor Series
$\frac{1}{x} = \sum_{n=0}^{\infty} x^n$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Permutations and Combinations
$P(n,k) = \frac{n!}{(n-k)!}$
$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$

Laplace Transforms
$F^*(s) = \int_0^{\infty} f(t)e^{-st}dt$
$f(t) = \int_0^{\infty} F^*(s)e^{st}ds$

Convolution Property
$f(t) * g(t) = \int_0^t f(t-x)g(x)dx \leftrightarrow F^*(s)G^*(s)$

Z-Transform
Mapping of discrete function f_n into complex fuction with variable z.
$F(z) = \sum_{n=0}^{\infty} f_n z^n$

Probability and Conditional
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A|B) = \frac{P(A \cap B)}{P(B)}$
A, B are independent if $P(A,B) = P(A)P(B)$

Total Probability
$P(B) = \sum_i P(A_i)P(B|A_i)$

Bayes' Rule
$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$

PMF (Probability Mass Function)
$p_X(x) = p(\{s \in \Omega \text{ s.t. } X(s) = x\})$
$\sum_x p_X(x) = 1$

Bernoulli Random Variable
$X = 1$ on success, $X = 0$ on failure.
$p(X = x) = p$, if $x = 1$
$p(X = x) = 1 - p$, if $x = 0$

Geometric Random Variable
Counts #trials until first success.
$p_X(x) = (1-p)^{x-1}p,\ x = 1, 2, \cdots$
$p(X \geq s+1 | X \geq t) = p(X \geq s)$

Binomial Random Variable
Counts #success in n identical independent experiments.
$p_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$, when $0 \leq x \leq n$
$p_X(x) = 0$, otherwise

Poisson Random Variable
Model occurrence of event over time interval assuming event happens at rate λ
$p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, when $x = 0, 1, \cdots$

PDF (Probability Density Function)
$\int_{-\infty}^{\infty} f_X(x)dx = 1$

CDF (Cumulative Distribution Function)
$F_X(x) = P(X \leq x)$
$\lim_{x \to -\infty} F_X(x) = 0$
$\lim_{x \to \infty} F_X(x) = 1$
$P(a < X \leq b) = F_X(b) - F_X(a)$

Uniform Distribution
$f_X(x) = \frac{1}{b-a}$, when $a \leq x \leq b$
$f_X(x) = 0$, otherwise

Exponential Distribution
Memoryless continuous distribution.
$f_X(x) = \lambda e^{-\lambda x}$, when $x \geq 0$
$F_X(x) = 1 - e^{-\lambda x}$, when $x \geq 0$
$F_X(x) = 0$, otherwise
$P(X > x) = e^{-\lambda x}$

Expectation
$E[X] = \sum_x xp(x)$
$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx$
If $Y = g(X)$, $E[Y] = \sum_x g(x)p(x)$,
$E[Y] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$
$E[X+Y] = E[X] + E[Y]$
$E[aX] = aE[X]$
$E[XY] = E[X]E[Y]$, if X,Y are independent.
For X, Y with joint PMF $p(x,y)$ or PDF $f_{X,Y}(x,y)$, $E[XY] = \sum_{(x,y)} xyp(x,y)$,
$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy$

Conditional Expectation
X,Y are random variables,
$E[Y|X] = \sum_y yP(Y = y|X = x) = \sum_y yp_{Y|X}(y|x)$,
$E[Y|X] = \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy$

Unconditional Expectation
$E[Y] = \sum_x E[Y|X]p_X(x)$
$E[Y] = \int_{\infty}^{\infty} E[Y|X]f_X(x)dx$

Variance
$Var[X] = E[(X - E[X])^2] = \sum_x (x - E[X])^2 p(x) = \int_{\infty}^{\infty} (x - E[X])^2 f_X(x)dx$
$Var[X] = E[X^2] - E[X]^2$
$Var[X+Y] = Var[X] + Var[Y] + 2Cov(X,Y)$
$Var[XY] = E[X^2Y^2] - E^2[XY] = E[X^2]E[Y^2] + Cov(X^2,Y^2) - (E[X]E[Y] + Cov(X,Y))^2$

Expectations and Variances
Binomial: np, $np(1-p)$
Geometric: $\frac{1}{p}$, $\frac{1-p}{p^2}$
Uniform: $\frac{a+b}{2}$, $\frac{(b-a)^2}{12}$
Exponential: $\frac{1}{\lambda}$, $\frac{1}{\lambda^2}$
Poisson: λ, λ

Covariance: measure of joint probability
$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$
$Cov(X,Y) = E[XY] - E[X]E[Y]$
If X,Y are independent, $Cov(X,Y) = 0$

Correlation: scaled version of covariance
$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$, range $[-1, 1]$

Stationary Process: $F_X(\boldsymbol{x};\boldsymbol{t}) = F_X(\boldsymbol{x};\boldsymbol{t}+\tau)$

Independent Processs:
$F_X(\boldsymbol{x};\boldsymbol{t}) = F_{X_1}(x_1,t_1)F_{X_2}(x_2,t_2)\cdots F_{X_n}(x_n,t_n)$
$f_X(\boldsymbol{x},\boldsymbol{t}) = \prod_{i=1}^n f_{X_i}(x_i,x_t)$ **(Continuous State)**
$p_X(\boldsymbol{x},\boldsymbol{t}) = \prod_{i=1}^n p_{X_i}(x_i,t_i)$ **(Discrete State)**

Markovian Property: $P[X(t_{n+1}) \leq x_{n+1}|X(t_n) = x_n, X(t_{n+2}) = x_{n+2}, \cdots, X(t_0) = x_0] = P[X(t_{n+1}) \leq x_{n+1}|X(t_n) = x_n]$

Discrete Time Markov Chains (DTMC):
$p_{ij} = P[X_n = j|X_{n-1} = i]$ **(Homogenenous)**
$$\boldsymbol{P} = (p_{ij}) = \begin{bmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$
$\sum_j p_{ij} = 1$ for each row.

Initial State Probabilities:
$\boldsymbol{\pi}^{(0)} = (\pi_0^{(0)}, \pi_1^{(0)}, \cdots)$, where $\pi_j^{(0)} = P[X_0 = j]$

n-Step Transition Probabilities:
$p_{ij}^{(n)} = P[X_n = j|X_0 = i] = P[X_{n+k} = j|X_k = i]$

Chapman-Kolmogorov: $p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)}p_{kj}$

Limiting Distribution: $\pi = \lim_{n\to\infty} \pi^{(0)}P^n$
$\pi = \pi\boldsymbol{P}$ and $\sum_j \pi_j = 1$

Continuous Time Markov Chains (CTMC):
State transitions permitted at arbitrary time instances. Time spent in a state is exponentially distributed.

State Transition Probability:
$p_{ij}(t) = p(X(\tau + t) = j|X(\tau) = i)$

Chapman-Kolmogorov Equation:
$p_{ij}(s+t) = \sum_k p_{ik}(s)p_{kj}(t)$

Transition Probability: $\boldsymbol{H}(t) = \{p_{ij}(t)\}$
$\boldsymbol{H}(s+t) = \boldsymbol{H}(s)\boldsymbol{H}(t)$
$\boldsymbol{H}(t+\Delta t) = \boldsymbol{H}(t)\boldsymbol{H}(\Delta t)$
$\boldsymbol{H}(t+\Delta t) = \boldsymbol{H}(t)[\boldsymbol{H}(\Delta t) - I]$
$\frac{d\boldsymbol{H}(t)}{dt} = \boldsymbol{H}(t)\lim_{\Delta t\to 0}[\frac{\boldsymbol{H}(\Delta t) - I}{\Delta t}]$
$\boldsymbol{Q} = \lim_{\Delta t\to 0}[\frac{\boldsymbol{H}(\Delta t) - I}{\Delta t}]$
$\frac{d\boldsymbol{H}(t)}{dt} = \boldsymbol{H}(t)\boldsymbol{Q}$

Transition Rate Matrix: $\boldsymbol{Q}$, infinitesimal generator of $\boldsymbol{H}(t)$. $\boldsymbol{H}(t) = e^{\boldsymbol{Q}t}$.

Diagonal Elements ≤ 0.
$q_{ii} = \lim_{\Delta t\to 0}[\frac{p_{ii}(\Delta t) - 1}{\Delta t}]$.

Off-Diagonal Elements ≥ 0.
$q_{ij} = \lim_{\Delta t\to 0}[\frac{p_{ij}(\Delta t) - 0}{\Delta t}]$, for $i \neq j$.
In each row, sum of off-diagonal = magnitude of diagonal: $q_{ii} = -\sum_{i\neq j} q_{ij}$

State Probabilities:
$\pi(t) = \pi(0)H(t),\ \pi(t) = \pi(0)e^{Qt}$

Stationary Distribution: $\pi Q = 0,\ \sum_j \pi_j = 1$

Birth-Death Process: At state k, λ_k, μ_k are birth and death rates. **Transition Matrix**: $Q =$
$$\begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Equilibrium Solution: $\pi\boldsymbol{Q} = 0$ and $\sum_j \pi_j = 1$

Differential Difference Equations: $\frac{\partial \pi_k(t)}{\partial t} = \lambda_{k-1}\pi_{k-1}(t) + \mu_{k+1}\pi_{k+1}(t) - (\lambda_k + \mu_k)\pi_k(t)$
$\frac{\partial \pi_0(t)}{\partial t} = \mu_1\pi_1(t) - \lambda_0\pi_0(t)$
$\frac{\partial \pi_k(t)}{\partial t} =$ flow in - flow out
flow in $= \lambda_{k-1}\pi_{k-1} + \mu_{k+1}\pi_{k+1}$
flow out $= (\lambda_k + \mu_k)\pi_k$

Queueing Theory: Analyze different systems. Poisson distribution model arrival process, exponential distribution model service times.

Poisson Process: **ArrivalRate of** λ:
$p(n$ arrivals in interval $T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}$

Exponential Inter-Arrival Time:
$p(\text{inter-arrival} \leq T) = 1 - e^{-\lambda T}$

Merging Property: Let A_1, A_2, ..., A_k be independent Poisson processes with rates λ_1, λ_2, ..., λ_k, then $A = \sum_i A_i$ is also Poisson process with rate $\lambda = \sum_i \lambda_i$

Splitting Property: Suppose every arrival is randomly routed with probability p to stream 1 and $(1-p)$ to stream 2. Stream 1 and 2 are Poisson with rates $p\lambda$ and $(1-p)\lambda$.

Kendall's Notation: A/B/C/D
A: inter-arrival time distribution
B: service time distribution
C: number of servers
D: maximum number of jobs possible
M:exponential, D: deterministic, G: general
$\alpha(t)$: number of arrivals in $(0,\ t)$
$\beta(t)$: number of departures in $(0,\ t)$
$N(t)$: number of customers in system at t
$T(t)$: average time in system for customer up to t
N: average number of customers in system.
$N = N_q + N_s$
T: average waiting time in system. $T = T_q + T_s$

Little's Law: $N = \lambda T$

Utilization Factor (Traffic Intensity Factor):
ρ = Work Arrival Rate / Server Capacity
System is **unstable** if $\rho > 1$

Single Server System
ρ is fraction of time server is busy

Multi-Server System
ρ is fraction of busy servers

M/M/1 Model: Assume state independent arrival λ, service rate μ. At equilibrium, flow in = flow out, $\lambda\pi_0 = \mu\pi_1 \to \pi_1 = \frac{\lambda}{\mu}\pi_0$, $\pi_1 = \rho\pi_0$.
Similarly $\lambda\pi_2 = \rho\pi_1 = \rho^2\pi_0$. In general $\pi_n = \rho^n\pi_0$.
$\sum_i \pi_i = 1 \to \pi_0 = 1 - \rho$, $\pi_n = (1-\rho)\rho^n$

Average Number Customers in System N:
$\sum_k k\pi_k = \sum_k k(1-\rho)\rho^k = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

Average Time in System T:
$T = N/\lambda$ (with Little's Law), $T = \frac{1}{\mu - \lambda}$

Average Time in Queue T_q:
$T_q = T - T_s = \frac{1}{\mu - \lambda} - \frac{1}{\mu}$

Average Number of Customers in Queue:
$N_q = \lambda T_q = N - \rho$ (with Little's Law)

Discouraged Arrivals: Arrival rates decrease with customers in system, $\lambda_k = \frac{\alpha}{k+1}$, $\mu_k = \mu$
Equilibrium state, $p_k = p_0(\frac{\alpha}{\mu})^k\frac{1}{k!}$. Since

$\sum_{k=0}^{\infty} p_k = 1$ we find that $p_0 = e^{-\alpha/\mu}$. So $p_k = ((\frac{\alpha}{\mu})^k e^{-\frac{\alpha}{\mu}})/k!$. **Utilization** $\rho = 1 - p_0 = 1 - e^{-\frac{\alpha}{\mu}}$

Expected number of customers in system: $N = E[k] = \alpha/\mu$. Expected time in system $T = N/\lambda$. λ_{eff} is expected (effective) arrival rate, $\lambda_{\text{eff}} = \mu\rho = \mu(1 - e^{-\alpha/\mu})$, so $T = \alpha/\mu^2(1 - e^{-\alpha/\mu})$

M/M/∞ Model: Service rate (number of servers) scales linarly with number of customers in system. → No waiting time, served immediately. $\lambda_k = \lambda$, $\mu_k = k\mu$. Equilibrium state, $p_k = (\frac{\lambda}{\mu})^k e^{-\lambda/\mu}/k!$, $p_0 = e^{-\lambda/\mu}$ (same as discouraged arrival system). Expected number of customers $N = E[k] = \lambda/\mu$. Expected time in system $T = N/\lambda = 1/\mu$.

M/M/m Model: Unlimited waiting room, constant arrival rate λ, maximum of m servers. Each server has service rate μ. $\lambda_k = \lambda$, $\mu_k = k\mu$ $(k < m)$, $m\mu$ $(k \geq m)$. Equilibrium distribution, $p_k = \frac{(m\rho)^k}{k!} p_0$ $(k < m)$, $\frac{(m\rho)^k}{m!} \frac{1}{m^{k-m}} p_0$ $(k \geq m)$ where $\rho = \lambda/m\mu$. When $\rho < 1$, $p_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)}\right]^{-1}$

M/M/1/K Model: System has single server and maximum of K customers. Equilibrium state, $p_k = p_0(\frac{\lambda}{\mu})^k$ $(k \leq K)$, 0 $(k > K)$, where $p_0 = \left(1 - \frac{\lambda}{\mu}\right)/\left(1 - (\frac{\lambda}{\mu})^{K+1}\right)$

Finite Population Systems: World has finite users and each user is either in system or to arrive with exponential time with mean $1/\mu$. $\lambda_k = \lambda(M - k)$ $(0 \leq k < M)$, 0 (otherwise). $\mu_k = \mu$ $(1 \leq k \leq M)$, 0 (otherwise). Equilibrium state, $p_k = p_0(\frac{\lambda}{\mu})^k \frac{M!}{(M-k)!}$ $(0 \leq k \leq M)$, 0 (otherwise). $p_0 = \left[\sum_{k=0}^{M} (\frac{\lambda}{\mu})^k \frac{M!}{(M-k)!}\right]^{-1}$

===========================

Q0: Using Laplace Transform, solve differential equations $x(t) = \frac{dx(t)}{dt} + 2\frac{d^2x(t)}{dt^2}$, $x(0) = 1$, $x'(0) = 1/2$.

$L\{x'(t)\} = sX(s) - x(0)$
$L\{x''(t) = s^2X(x) - sx(0) - x'(0)\}$
$X(s) = sX(s) - x(0) + 2[s^2X(x) - sx(0) - x'(0)]$
$X(s) = 1/(s - \frac{1}{2})$, $x(t) = e^{1/2t}$

Q1: Program has execution time uniformly distributed between 10 & 20 seconds. Number of interrupts during execution is Poisson random variable with parameter λt where t is program execution time. Probability distribution of the number of interrupts is $P(N = k) = (\lambda t)ke^{\lambda t}$.

(a) What is $E[N|T = t]$, where N is the number of interrupts, and T is running time.

$E[N|T = t] = \lambda t$, since for fixed running time, the number of interrupts is a Poisson random variable with mean λt.

(b) Expected number interrupts during randomly selected execution.

$E[N] = \int_{10}^{20} E[N|T = t]f_T(t)dt = \int_{10}^{20} \frac{\lambda t}{10} dt = 15\lambda$

Q2: We have m types of visitors with equal probability. Find expected visitors to have one of each type. Hint: Let X denote number of visitors needed. Represent X by $X = \sum_{i=1}^{m} X_i$ where each X_i is a geometric random variable.

Suppose the current visitor pool contains i different types. Let X_i denote the number of additional visitors needed until it contains $i + 1$ types. The X_i is are independent geometric random variables with parameter $(m - i)/m$, $i = 0, 1, \cdots, m - 1$. $E[X] = E[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} E[X_i] = \sum_{i=1}^{m} \frac{m}{m-i}$

Q3: A Markov chain $\{X_n, n \geq 0\}$ with states 0, 1, 2 has transition probability matrix $\begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ If $P[X_0 = 0] = P[X_0 = 1] = \frac{1}{4}$, find state probability vector $P[X_3 = 2]$.

Cubing transition probability matrix

$$P^{(3)} = \begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$$

$P[X_3 = 2] = \frac{1}{4} \cdot \frac{47}{108} + \frac{1}{4} \cdot \frac{11}{27} + \frac{1}{2} \cdot \frac{13}{36}$

Q4: Whether or not collision occurs depends on result of last two transmissions If collisions occurred in both of the past two, collision will occur with probability 0.7; collision occurs in last but not the transmission before the last one, then a collision will occur with probability 0.5; collision occurred in transmission before the last one but not the last one, collision with probability 0.4; no collision in the past two transmissions, collision with probability 0.2.

(b) Find the transition probability matrix.

$$P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

(c) What fraction of frames suffer a collision?

Solve $\pi = \pi P$ to obtain the stationary state probabilities. Then the fraction of frames suffering a collision is $\pi_0 + \pi_2$ (or $\pi_0 + \pi_1$).

Q5: A shop has room for 2 customers. Customers arrive at Poisson rate 3/hour and service times are exponential random variables with mean 0.25 hours.

(a) Average number of customers in shop?

We have birth-death process with $\lambda = 3$ and $\mu = 4$. $E[k] = \sum_{k=0}^{2} k\pi_k$. Solve for π_k, $\pi_1 = \frac{\lambda}{\mu}\pi_0$, $\pi_2 = \frac{\lambda}{\mu}\pi_1$, $\sum_{j=0}^{2} \pi_j = 1$, so $\pi_0 = 16/37$, $E[k] = 30/37$.

(b) What is proportion of customers who get serviced? $\pi_0 + \pi_1 = 28/37$.

Q6: Packets arrive at router Poisson rate 3/ms and time to foward exponential with mean 0.2 ms. Fraction of time buffer empty?

State is num packets in router, $\lambda = 3$, $\mu = 5$. Solve $p_1 = \frac{\lambda}{\mu}p_0$, $p_2 = \frac{\lambda}{\mu}p_1$, ..., $p_{k+1} = \frac{\lambda}{\mu}p_k$, $\sum_{k=0}^{\infty} p_k = 1$, thus $p_0 = 2/5$.

Q7: 2 machines produce products at n products/hour. Lifetime of machine follows exponential with mean $1/x$ hours, time to fix machine folows exponential with mean $1/y$ hours. Expected long term producing rate?

State is num machines working. $\lambda_0 = \lambda_1 = y$, $\mu_1 = x$, $\mu_2 = 2x$ bc if either machine fails, num working machines reduce from 2 to 1. Solve $p_1 = \frac{y}{x}p_0$, $p_2 = \frac{y}{2x}p_1$, $p_0 + p_1 + p_2 = 1$, thus $p_0 = \frac{2x^2}{2x^2+2xy+y^2}$. Expected num products produced per hour: $p_0 \times 0 + p_1 \times n + p_2 \times 2n$

Q8: M/M/1 queueing system. Jobs arrive to be scheduled at rate λ jobs/ms. Single core serving jobs at rate of μ jobs/ms. Suppose that the we want the average time spent in waiting queue to be no more than 3 milliseconds. If $\lambda = 10$ jobs/ms, find minimum of μ.

Steady state probabilities are $\pi_0 = 1\rho$, $\pi_k = \rho^k(1\rho)$ where $\rho = \lambda/\mu$. Average customers $N = \frac{\rho}{1-\rho}$. By Little's Law, $N = \lambda \cdot T \to T = N$ where T is average time in system. Average waiting time $W = T - 1/\mu$, so we need $W = \frac{\rho}{\lambda(1-\rho)} - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} \leq 3$

Q9: M customers go to single-server. When customer arrives, enters service if the server is free or joins queue. Upon leaving, customer returns after exponential time with rate λ. Service time exponentially distributed μ.

(a) Define states and set up the balance equations.

(b) The average rate at which customers enter the station.

(c) The average time that a customer spends.