Taylor Series
$\frac{1}{x} = \sum_{n=0}^{\infty} x^n$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$
Permutations and Combinations
$P(n,k) = \frac{n!}{(n-k)!}$
$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$
Laplace Transforms
$F^*(s) = \int_0^{\infty} f(t)e^{-st}dt$
$f(t) = \int_0^{\infty} F^*(s)e^{st}ds$
Convolution Property
$f(t) * g(t) = \int_0^t f(t-x)g(x)dx \leftrightarrow F^*(s)G^*(s)$
Z-Transform
Mapping of discrete function f_n into complex fuction with variable z.
$F(z) = \sum_{n=0}^{\infty} f_n z^n$
Probability and Conditional
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A|B) = \frac{P(A \cap B)}{P(B)}$
A, B are independent if $P(A,B) = P(A)P(B)$
Total Probability
$P(B) = \sum_i P(A_i)P(B|A_i)$
Bayes' Rule
$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$
PMF (Probability Mass Function)
$p_X(x) = p(\{s \in \Omega \text{ s.t. } X(s) = x\})$
$\sum_x p_X(x) = 1$
Bernoulli Random Variable
$X = 1$ on success, $X = 0$ on failure.
$p(X = x) = p$, if $x = 1$
$p(X = x) = 1 - p$, if $x = 0$
Geometric Random Variable
Counts #trials until first success.
$p_X(x) = (1-p)^{x-1}p$, $x = 1, 2, \cdots$
$p(X \geq s+1 | X \geq t) = p(X \geq s)$
Binomial Random Variable
Counts #success in n identical independent experiments.
$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, when $0 \leq x \leq n$
$p_X(x) = 0$, otherwise
Poisson Random Variable
Model occurrence of event over time interval assuming event happens at rate λ
$p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, when $x = 0, 1, \cdots$
PDF (Probability Density Function)
$\int_{-\infty}^{\infty} f_X(x)dx = 1$
CDF (Cumulative Distribution Function)
$F_X(x) = P(X \leq x)$
$\lim_{x \to -\infty} F_X(x) = 0$
$\lim_{x \to \infty} F_X(x) = 1$
$P(a < X \leq b) = F_X(b) - F_X(a)$
Uniform Distribution
$f_X(x) = \frac{1}{b-a}$, when $a \leq x \leq b$
$f_X(x) = 0$, otherwise
Exponential Distribution
Memoryless continuous distribution.
$f_X(x) = \lambda e^{-\lambda x}$, when $x \geq 0$
$F_X(x) = 1 - e^{-\lambda x}$, when $x \geq 0$
$F_X(x) = 0$, otherwise
$P(X > x) = e^{-\lambda x}$
Expectation
$E[X] = \sum_x xp(x)$
$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx$
If $Y = g(X)$, $E[Y] = \sum_x g(x)p(x)$,
$E[Y] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$
$E[X+Y] = E[X] + E[Y]$
$E[aX] = aE[X]$
$E[XY] = E[X]E[Y]$, if X,Y are independent.
For X, Y with joint PMF $p(x,y)$ or PDF $f_{X,Y}(x,y)$, $E[XY] = \sum_{(x,y)} xyp(x,y)$,
$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy$
Conditional Expectation
X,Y are random variables,
$E[Y|X] = \sum_y yP(Y = y|X = x) = \sum_y yp_{Y|X}(y|x)$,
$E[Y|X] = \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy$
Unconditional Expectation
$E[Y] = \sum_x E[Y|X]p_X(x)$
$E[Y] = \int_{\infty}^{\infty} E[Y|X]f_X(x)dx$
Variance
$Var[X] = E[(X - E[X])^2] = \sum_x (x - E[X])^2 p(x) = \int_{\infty}^{\infty} (x - E[X])^2 f_X(x)dx$
$Var[X] = E[X^2] - E[X]^2$
$Var[X+Y] = Var[X] + Var[Y] + 2Cov(X,Y)$
$Var[XY] = E[X^2Y^2] - E^2[XY] = E[X^2]E[Y^2] + Cov(X^2,Y^2) - (E[X]E[Y] + Cov(X,Y))^2$
Expectations and Variances
Binomial: np, $np(1-p)$
Geometric: $\frac{1}{p}$, $\frac{1-p}{p^2}$
Uniform: $\frac{a+b}{2}$, $\frac{(b-a)^2}{12}$
Exponential: $\frac{1}{\lambda}$, $\frac{1}{\lambda^2}$
Poisson: λ, λ
Covariance: measure of joint probability
$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$
$Cov(X,Y) = E[XY] - E[X]E[Y]$
If X,Y are independent, $Cov(X,Y) = 0$
Correlation: scaled version of covariance
$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$, range $[-1, 1]$
Stationary Process: $F_X(\boldsymbol{x};\boldsymbol{t}) = F_X(\boldsymbol{x};\boldsymbol{t}+\tau)$
Independent Processs:
$F_X(\boldsymbol{x};\boldsymbol{t}) = F_{X_1}(x_1,t_1)F_{X_2}(x_2,t_2)\cdots F_{X_n}(x_n,t_n)$
$f_X(\boldsymbol{x},\boldsymbol{t}) = \prod_{i=1}^n f_{X_i}(x_i,x_t)$ **(Continuous State)**
$p_X(\boldsymbol{x},\boldsymbol{t}) = \prod_{i=1}^n p_{X_i}(x_i,t_i)$ **(Discrete State)**
Markovian Property: $P[X(t_{n+1}) \leq x_{n+1}|X(t_n) = x_n, X(t_{n+2}) = x_{n+2}, \cdots, X(t_0) = x_0] = P[X(t_{n+1}) \leq x_{n+1}|X(t_n) = x_n]$
Discrete Time Markov Chains (DTMC):
$p_{ij} = P[X_n = j|X_{n-1} = i]$ **(Homogenenous)**
$$\boldsymbol{P} = (p_{ij}) = \begin{bmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$
$\sum_j p_{ij} = 1$ for each row.
Initial State Probabilities:
$\boldsymbol{\pi}^{(0)} = (\pi_0^{(0)}, \pi_1^{(0)}, \cdots)$, where $\pi_j^{(0)} = P[X_0 = j]$
n-Step Transition Probabilities:
$p_{ij}^{(n)} = P[X_n = j|X_0 = i] = P[X_{n+k} = j|X_k = i]$
Chapman-Kolmogorov: $p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)}p_{kj}$
Limiting Distribution: $\pi = \lim_{n\to\infty} \pi^{(0)}P^n$
$\pi = \pi\boldsymbol{P}$ and $\sum_j \pi_j = 1$
Continuous Time Markov Chains (CTMC): State transitions permitted at arbitrary time instances. Time spent in a state is exponentially distributed.
State Transition Probability:
$p_{ij}(t) = p(X(\tau + t) = j|X(\tau) = i)$
Chapman-Kolmogorov Equation:
$p_{ij}(s+t) = \sum_k p_{ik}(s)p_{kj}(t)$
Transition Probability: $\boldsymbol{H}(t) = \{p_{ij}(t)\}$
$\boldsymbol{H}(s+t) = \boldsymbol{H}(s)\boldsymbol{H}(t)$
$\boldsymbol{H}(t+\Delta t) = \boldsymbol{H}(t)\boldsymbol{H}(\Delta t)$
$\boldsymbol{H}(t+\Delta t) = \boldsymbol{H}(t)[\boldsymbol{H}(\Delta t) - I]$
$\frac{d\boldsymbol{H}(t)}{dt} = \boldsymbol{H}(t)\lim_{\Delta t\to 0}[\frac{\boldsymbol{H}(\Delta t) - I}{\Delta t}]$
$\boldsymbol{Q} = \lim_{\Delta t\to 0}[\frac{\boldsymbol{H}(\Delta t) - I}{\Delta t}]$
$\frac{d\boldsymbol{H}(t)}{dt} = \boldsymbol{H}(t)\boldsymbol{Q}$
Transition Rate Matrix: $\boldsymbol{Q}$, infinitesimal generator of $\boldsymbol{H}(t)$. $\boldsymbol{H}(t) = e^{\boldsymbol{Q}t}$.
Diagonal Elements ≤ 0.
$q_{ii} = \lim_{\Delta t\to 0}[\frac{p_{ii}(\Delta t) - 1}{\Delta t}]$.
Off-Diagonal Elements ≥ 0.
$q_{ij} = \lim_{\Delta t\to 0}[\frac{p_{ij}(\Delta t) - 0}{\Delta t}]$, for $i \neq j$.
In each row, sum of off-diagonal = magnitude of diagonal: $q_{ii} = -\sum_{i\neq j} q_{ij}$
State Probabilities:
$\pi(t) = \pi(0)H(t)$, $\pi(t) = \pi(0)e^{Qt}$
Stationary Distribution: $\pi Q = 0$, $\sum_j \pi_j = 1$
Birth-Death Process: At state k, λ_k, μ_k are birth and death rates. **Transition Matrix**: $Q =$
$$\begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \cdots & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$
Equilibrium Solution: $\pi\boldsymbol{Q} = 0$ and $\sum_j \pi_j = 1$
Differential Difference Equations: $\frac{\partial \pi_k(t)}{\partial t} = \lambda_{k-1}\pi_{k-1}(t) + \mu_{k+1}\pi_{k+1}(t) - (\lambda_k + \mu_k)\pi_k(t)$
$\frac{\partial \pi_0(t)}{\partial t} = \mu_1\pi_1(t) - \lambda_0\pi_0(t)$
$\frac{\partial \pi_k(t)}{\partial t} =$ flow in - flow out
flow in $= \lambda_{k-1}\pi_{k-1} + \mu_{k+1}\pi_{k+1}$
flow out $= (\lambda_k + \mu_k)\pi_k$
Queueing Theory: Analyze different systems. Poisson distribution model arrival process, exponential distribution model service times.
Poisson Process: **ArrivalRate of** λ:
$p(n \text{ arrivals in interval } T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}$
Exponential Inter-Arrival Time:
$p(\text{inter-arrival} \leq T) = 1 - e^{-\lambda T}$
Merging Property: Let A_1, A_2, ..., A_k be independent Poisson processes with rates λ_1, λ_2, ..., λ_k, then $A = \sum_i A_i$ is also Poisson process with rate $\lambda = \sum_i \lambda_i$
Splitting Property: Suppose every arrival is randomly routed with probability p to stream 1 and $(1-p)$ to stream 2. Stream 1 and 2 are Poisson with rates $p\lambda$ and $(1-p)\lambda$.
Kendall's Notation: A/B/C/D
A: inter-arrival time distribution
B: service time distribution
C: number of servers
D: maximum number of jobs possible
M:exponential, D: deterministic, G: general
$\alpha(t)$: number of arrivals in $(0, t)$
$\beta(t)$: number of departures in $(0, t)$
$N(t)$: number of customers in system at t
$T(t)$: average time in system for customer up to t
N: average number of customers in system.
$N = N_q + N_s$
T: average waiting time in system. $T = T_q + T_s$
Little's Law: $N = \lambda T$
Utilization Factor (Traffic Intensity Factor):
ρ = Work Arrival Rate / Server Capacity
System is **unstable** if $\rho > 1$
Single Server System
ρ is fraction of time server is busy
Multi-Server System
ρ is fraction of busy servers
M/M/1 Model: Assume state independent arrival λ, service rate μ. At equilibrium, flow in = flow out, $\lambda\pi_0 = \mu\pi_1 \to \pi_1 = \frac{\lambda}{\mu}\pi_0$, $\pi_1 = \rho\pi_0$.
Similarly $\lambda\pi_2 = \rho\pi_1 = \rho^2\pi_0$. In general $\pi_n = \rho^n\pi_0$.
$\sum_i \pi_i = 1 \to \pi_0 = 1 - \rho$, $\pi_n = (1-\rho)\rho^n$
Average Number Customers in System N:
$\sum_k k\pi_k = \sum_k k(1-\rho)\rho^k = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$
Average Time in System T:
$T = N/\lambda$ (with Little's Law), $T = \frac{1}{\mu - \lambda}$
Average Time in Queue T_q:
$T_q = T - T_s = \frac{1}{\mu-\lambda} - \frac{1}{\mu}$
Average Number of Customers in Queue:
$N_q = \lambda T_q = N - \rho$ (with Little's Law)
Discouraged Arrivals: Arrival rates decrease with customers in system, $\lambda_k = \frac{\alpha}{k+1}$, $\mu_k = \mu$
Equilibrium state, $p_k = p_0(\frac{\alpha}{\mu})^k\frac{1}{k!}$. Since

$\sum_{k=0}^{\infty} p_k = 1$ we find that $p_0 = e^{-\alpha/\mu}$. So $p_k = ((\frac{\alpha}{\mu})^k e^{-\frac{\alpha}{\mu}})/k!$. **Utilization** $\rho = 1 - p_0 = 1 - e^{-\frac{\alpha}{\mu}}$

Expected number of customers in system: $N = E[k] = \alpha/\mu$. Expected time in system $T = N/\lambda$. λ_{eff} is expected (effective) arrival rate, $\lambda_{\text{eff}} = \mu\rho = \mu(1 - e^{-\alpha/\mu})$, so $T = \alpha/\mu^2(1 - e^{-\alpha/\mu})$

M/M/∞ Model: Service rate (number of servers) scales linarly with number of customers in system. → No waiting time, served immediately. $\lambda_k = \lambda$, $\mu_k = k\mu$. Equilibrium state, $p_k = (\frac{\lambda}{\mu})^k e^{-\lambda/\mu}/k!$, $p_0 = e^{-\lambda/\mu}$ (same as discouraged arrival system). Expected number of customers $N = E[k] = \lambda/\mu$. Expected time in system $T = N/\lambda = 1/\mu$.

M/M/m Model: Unlimited waiting room, constant arrival rate λ, maximum of m servers. Each server has service rate μ. $\lambda_k = \lambda$, $\mu_k = k\mu$ $(k < m)$, $m\mu$ $(k \geq m)$. Equilibrium distribution, $p_k = \frac{(m\rho)^k}{k!} p_0$ $(k < m)$, $\frac{(m\rho)^k}{m!} \frac{1}{m^{k-m}} p_0$ $(k \geq m)$ where $\rho = \lambda/m\mu$. When $\rho < 1$, $p_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)}\right]^{-1}$

M/M/1/K Model: System has single server and maximum of K customers. Equilibrium state, $p_k = p_0(\frac{\lambda}{\mu})^k$ $(k \leq K)$, 0 $(k > K)$, where $p_0 = \left(1 - \frac{\lambda}{\mu}\right)/\left(1 - (\frac{\lambda}{\mu})^{K+1}\right)$

Finite Population Systems: World has finite users and each user is either in system or to arrive with exponential time with mean $1/\mu$. $\lambda_k = \lambda(M - k)$ $(0 \leq k < M)$, 0 (otherwise). $\mu_k = \mu$ $(1 \leq k \leq M)$, 0 (otherwise). Equilibrium state, $p_k = p_0(\frac{\lambda}{\mu})^k \frac{M!}{(M-k)!}$ $(0 \leq k \leq M)$, 0 (otherwise). $p_0 = \left[\sum_{k=0}^{M} (\frac{\lambda}{\mu})^k \frac{M!}{(M-k)!}\right]^{-1}$

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Q0: Using Laplace Transform, solve differential equations $x(t) = \frac{dx(t)}{dt} + 2\frac{d^2x(t)}{dt^2}$, $x(0) = 1$, $x'(0) = 1/2$.

$L\{x'(t)\} = sX(s) - x(0)$
$L\{x''(t) = s^2X(x) - sx(0) - x'(0)\}$
$X(s) = sX(s) - x(0) + 2[s^2X(x) - sx(0) - x'(0)]$
$X(s) = 1/(s - \frac{1}{2})$, $x(t) = e^{1/2t}$

Q1: Program has execution time uniformly distributed between 10 & 20 seconds. Number of interrupts during execution is Poisson random variable with parameter λt where t is program execution time. Probability distribution of the number of interrupts is $P(N = k) = (\lambda t)ke^{\lambda t}$.

(a) What is $E[N|T = t]$, where N is the number of interrupts, and T is running time.

$E[N|T = t] = \lambda t$, since for fixed running time, the number of interrupts is a Poisson random variable with mean λt.

(b) Expected number interrupts during randomly selected execution.

$E[N] = \int_{10}^{20} E[N|T = t] f_T(t)dt = \int_{10}^{20} \frac{\lambda t}{10} dt = 15\lambda$

Q2: We have m types of visitors with equal probability. Find expected visitors to have one of each type. Hint: Let X denote number of visitors needed. Represent X by $X = \sum_{i=1}^{m} X_i$ where each X_i is a geometric random variable.

Suppose the current visitor pool contains i different types. Let X_i denote the number of additional visitors needed until it contains $i + 1$ types. The X_i is are independent geometric random variables with parameter $(m - i)/m$, $i = 0, 1, \cdots, m - 1$. $E[X] = E[\sum_{i=1}^{m} X_i] = \sum_{i=1}^{m} E[X_i] = \sum_{i=1}^{m} \frac{m}{m-i}$

Q3: A Markov chain $\{X_n, n \geq 0\}$ with states 0, 1, 2 has transition probability matrix $\begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ If $P[X_0 = 0] = P[X_0 = 1] = \frac{1}{4}$, find state probability vector $P[X_3 = 2]$.

Cubing transition probability matrix

$P^{(3)} = \begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$

$P[X_3 = 2] = \frac{1}{4} \cdot \frac{47}{108} + \frac{1}{4} \cdot \frac{11}{27} + \frac{1}{2} \cdot \frac{13}{36}$

Q4: Whether or not collision occurs depends on result of last two transmissions If collisions occurred in both of the past two, collision will occur with probability 0.7; collision occurs in last but not the transmission before the last one, then a collision will occur with probability 0.5; collision occurred in transmission before the last one but not the last one, collision with probability 0.4; no collision in the past two transmissions, collision with probability 0.2.

(b) Find the transition probability matrix.

$P = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$

(c) What fraction of frames suffer a collision?

Solve $\pi = \pi P$ to obtain the stationary state probabilities. Then the fraction of frames suffering a collision is $\pi_0 + \pi_2$ (or $\pi_0 + \pi_1$).

Q5: A shop has room for 2 customers. Customers arrive at Poisson rate 3/hour and service times are exponential random variables with mean 0.25 hours.

(a) Average number of customers in shop?

We have birth-death process with $\lambda = 3$ and $\mu = 4$. $E[k] = \sum_{k=0}^{2} k\pi_k$. Solve for π_k, $\pi_1 = \frac{\lambda}{\mu}\pi_0$, $\pi_2 = \frac{\lambda}{\mu}\pi_1$, $\sum_{j=0}^{2} \pi_j = 1$, so $\pi_0 = 16/37$, $E[k] = 30/37$.

(b) What is proportion of customers who get serviced? $\pi_0 + \pi_1 = 28/37$.

Q6: Packets arrive at router Poisson rate 3/ms and time to foward exponential with mean 0.2 ms. Fraction of time buffer empty?

State is num packets in router, $\lambda = 3$, $\mu = 5$. Solve $p_1 = \frac{\lambda}{\mu}p_0$, $p_2 = \frac{\lambda}{\mu}p_1$, ..., $p_{k+1} = \frac{\lambda}{\mu}p_k$, $\sum_{k=0}^{\infty} p_k = 1$, thus $p_0 = 2/5$.

Q7: 2 machines produce products at n products/hour. Lifetime of machine follows exponential with mean $1/x$ hours, time to fix machine folows exponential with mean $1/y$ hours. Expected long term producing rate?

State is num machines working. $\lambda_0 = \lambda_1 = y$, $\mu_1 = x$, $\mu_2 = 2x$ bc if either machine fails, num working machines reduce from 2 to 1. Solve $p_1 = \frac{y}{x}p_0$, $p_2 = \frac{y}{2x}p_1$, $p_0 + p_1 + p_2 = 1$, thus $p_0 = \frac{2x^2}{2x^2+2xy+y^2}$. Expected num products produced per hour: $p_0 \times 0 + p_1 \times n + p_2 \times 2n$