Taylor Series
$\frac{1}{x}=\sum_{n=0}^{\infty}x^n$
$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$
$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$
$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$
$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$
Permutations and Combinations
$P(n,k)=\frac{n!}{(n-k)!}$
$C(n,k)=\binom{n}{k}=\frac{n!}{(n-k)!k!}$
Laplace Transforms
$F^*(s)=\int_0^{\infty}f(t)e^{-st}dt$
$f(t)=\int_0^{\infty}F^*(s)e^{st}ds$
Convolution Property
$f(t)*g(t)=\int_0^t f(t-x)g(x)dx\leftrightarrow F^*(s)G^*(s)$
Z-Transform
Mapping of discrete function f_n into complex fuction with variable z.
$F(z)=\sum_{n=0}^{\infty}f_n z^n$
Probability and Conditional
$P(A\cup B)=P(A)+P(B)-P(A\cap B)$
$P(A|B)=\frac{P(A\cap B)}{P(B)}$
A, B are independent if $P(A,B)=P(A)P(B)$
Total Probability
$P(B)=\sum_i P(A_i)P(B|A_i)$
Bayes' Rule
$P(A_i|B)=\frac{P(A_i\cap B)}{P(B)}=\frac{P(A_i)P(B|A_i)}{P(B)}=\frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$
PMF (Probability Mass Function)
$p_X(x)=p(\{s\in\Omega \text{ s.t. } X(s)=x\})$
$\sum_x p_X(x)=1$
Bernoulli Random Variable
$X=1$ on success, $X=0$ on failure.
$p(X=x)=p$, if $x=1$
$p(X=x)=1-p$, if $x=0$
Geometric Random Variable
Counts #trials until first success.
$p_X(x)=(1-p)^{x-1}p$, $x=1,2,\cdots$
$p(X\geq s+1|X\geq t)=p(X\geq s)$
Binomial Random Variable
Counts #success in n identical independent experiments.
$p_X(x)=\binom{n}{x}p^x(1-p)^{n-x}$, when $0\leq x\leq n$
$p_X(x)=0$, otherwise
Poisson Random Variable
Model occurrence of event over time interval assuming event happens at rate λ
$p_X(x)=e^{-\lambda}\frac{\lambda^x}{x!}$, when $x=0,1,\cdots$
PDF (Probability Density Function)
$\int_{-\infty}^{\infty}f_X(x)dx=1$
CDF (Cumulative Distribution Function)
$F_X(x)=P(X\leq x)$
$\lim_{x\to-\infty}F_X(x)=0$
$\lim_{x\to\infty}F_X(x)=1$
$P(a<X\leq b)=F_X(b)-F_X(a)$
Uniform Distribution
$f_X(x)=\frac{1}{b-a}$, when $a\leq x\leq b$
$f_X(x)=0$, otherwise
Exponential Distribution
Memoryless continuous distribution.
$f_X(x)=\lambda e^{-\lambda x}$, when $x\geq 0$
$F_X(x)=1-e^{-\lambda x}$, when $x\geq 0$
$F_X(x)=0$, otherwise
$P(X>x)=e^{-\lambda x}$
Expectation
$E[X]=\sum_x xp(x)$
$E[X]=\int_{-\infty}^{\infty}xf_X(x)dx$
If $Y=g(X)$, $E[Y]=\sum_x g(x)p(x)$,
$E[Y]=\int_{-\infty}^{\infty}g(x)f_X(x)dx$
$E[X+Y]=E[X]+E[Y]$
$E[aX]=aE[X]$
$E[XY]=E[X]E[Y]$, if X,Y are independent.
For X, Y with joint PMF $p(x,y)$ or PDF $f_{X,Y}(x,y)$, $E[XY]=\sum_{(x,y)}xyp(x,y)$,
$E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf_{X,Y}(x,y)dxdy$
Conditional Expectation
X,Y are random variables,
$E[Y|X]=\sum_y yP(Y=y|X=x)=\sum_y yp_{Y|X}(y|x)$,
$E[Y|X]=\int_{-\infty}^{\infty}yf_{Y|X}(y|x)dy$
Unconditional Expectation
$E[Y]=\sum_x E[Y|X]p_X(x)$
$E[Y]=\int_{\infty}^{\infty}E[Y|X]f_X(x)dx$
Variance
$Var[X]=E[(X-E[X])^2]=\sum_x(x-E[X])^2p(x)=\int_{\infty}^{\infty}(x-E[X])^2f_X(x)dx$
$Var[X]=E[X^2]-E[X]^2$
$Var[X+Y]=Var[X]+Var[Y]+2Cov(X,Y)$
$Var[XY]=E[X^2Y^2]-E^2[XY]=E[X^2]E[Y^2]+Cov(X^2,Y^2)-(E[X]E[Y]+Cov(X,Y))^2$
Expectations and Variances
Binomial: np, $np(1-p)$
Geometric: $\frac{1}{p}$, $\frac{1-p}{p^2}$
Uniform: $\frac{a+b}{2}$, $\frac{(b-a)^2}{12}$
Exponential: $\frac{1}{\lambda}$, $\frac{1}{\lambda^2}$
Poisson: λ, λ
Covariance: measure of joint probability
$Cov(X,Y)=E[(X-E[X])(Y-E[Y])]$
$Cov(X,Y)=E[XY]-E[X]E[Y]$
If X,Y are independent, $Cov(X,Y)=0$
Correlation: scaled version of covariance
$\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$, range $[-1,1]$
Stationary Process: $F_X(\boldsymbol{x};\boldsymbol{t})=F_X(\boldsymbol{x};\boldsymbol{t}+\tau)$
Independent Processs:
$F_X(\boldsymbol{x};\boldsymbol{t})=F_{X_1}(x_1,t_1)F_{X_2}(x_2,t_2)\cdots F_{X_n}(x_n,t_n)$
$f_X(\boldsymbol{x},\boldsymbol{t})=\prod_{i=1}^n f_{X_i}(x_i,x_t)$ **(Continuous State)**
$p_X(\boldsymbol{x},\boldsymbol{t})=\prod_{i=1}^n p_{X_i}(x_i,t_i)$ **(Discrete State)**
Markovian Property: $P[X(t_{n+1})\leq x_{n+1}|X(t_n)=x_n,X(t_{n+2})=x_{n+2},\cdots,X(t_0)=x_0]=P[X(t_{n+1})\leq x_{n+1}|X(t_n)=x_n]$
Discrete Time Markov Chains (DTMC):
$p_{ij}=P[X_n=j|X_{n-1}=i]$ **(Homogenenous)**
$$\boldsymbol{P}=(p_{ij})=\begin{bmatrix}p_{00}&p_{01}&\cdots\\p_{10}&p_{11}&\cdots\\\vdots&\vdots&\vdots\end{bmatrix}$$
$\sum_j p_{ij}=1$ for each row.
Initial State Probabilities:
$\boldsymbol{\pi}^{(0)}=(\pi_0^{(0)},\pi_1^{(0)},\cdots)$, where $\pi_j^{(0)}=P[X_0=j]$
n-Step Transition Probabilities:
$p_{ij}^{(n)}=P[X_n=j|X_0=i]=P[X_{n+k}=j|X_k=i]$
Chapman-Kolmogorov: $p_{ij}^{(n)}=\sum_k p_{ik}^{(n-1)}p_{kj}$
Limiting Distribution: $\pi=\lim_{n\to\infty}\pi^{(0)}P^n$
$\pi=\pi\boldsymbol{P}$ and $\sum_j\pi_j=1$
Continuous Time Markov Chains (CTMC): State transitions permitted at arbitrary time instances. Time spent in a state is exponentially distributed.
State Transition Probability:
$p_{ij}(t)=p(X(\tau+t)=j|X(\tau)=i)$
Chapman-Kolmogorov Equation:
$p_{ij}(s+t)=\sum_k p_{ik}(s)p_{kj}(t)$
Transition Probability: $\boldsymbol{H}(t)=\{p_{ij}(t)\}$
$\boldsymbol{H}(s+t)=\boldsymbol{H}(s)\boldsymbol{H}(t)$
$\boldsymbol{H}(t+\Delta t)=\boldsymbol{H}(t)\boldsymbol{H}(\Delta t)$
$\boldsymbol{H}(t+\Delta t)=\boldsymbol{H}(t)[\boldsymbol{H}(\Delta t)-I]$
$\frac{d\boldsymbol{H}(t)}{dt}=\boldsymbol{H}(t)\lim_{\Delta t\to 0}[\frac{\boldsymbol{H}(\Delta t)-I}{\Delta t}]$
$\boldsymbol{Q}=\lim_{\Delta t\to 0}[\frac{\boldsymbol{H}(\Delta t)-I}{\Delta t}]$
$\frac{d\boldsymbol{H}(t)}{dt}=\boldsymbol{H}(t)\boldsymbol{Q}$
Transition Rate Matrix: $\boldsymbol{Q}$, infinitesimal generator of $\boldsymbol{H}(t)$. $\boldsymbol{H}(t)=e^{\boldsymbol{Q}t}$.
Diagonal Elements ≤ 0.
$q_{ii}=\lim_{\Delta t\to 0}[\frac{p_{ii}(\Delta t)-1}{\Delta t}]$.
Off-Diagonal Elements ≥ 0.
$q_{ij}=\lim_{\Delta t\to 0}[\frac{p_{ij}(\Delta t)-0}{\Delta t}]$, for $i\neq j$.
In each row, sum of off-diagonal = magnitude of diagonal: $q_{ii}=-\sum_{i\neq j}q_{ij}$
State Probabilities:
$\pi(t)=\pi(0)H(t)$, $\pi(t)=\pi(0)e^{Qt}$
Stationary Distribution: $\pi Q=0$, $\sum_j\pi_j=1$
Birth-Death Process: At state k, λ_k, μ_k are birth and death rates. **Transition Matrix**: $Q=$
$$\begin{bmatrix}-\lambda_0&\lambda_0&0&0&\cdots&\cdots\\\mu_1&-(\lambda_1+\mu_1)&\lambda_1&0&\cdots&\cdots\\0&\mu_2&-(\lambda_2+\mu_2)&\lambda_2&0&\cdots\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\end{bmatrix}$$
Equilibrium Solution: $\pi\boldsymbol{Q}=0$ and $\sum_j\pi_j=1$
Differential Difference Equations: $\frac{\partial\pi_k(t)}{\partial t}=\lambda_{k-1}\pi_{k-1}(t)+\mu_{k+1}\pi_{k+1}(t)-(\lambda_k+\mu_k)\pi_k(t)$
$\frac{\partial\pi_0(t)}{\partial t}=\mu_1\pi_1(t)-\lambda_0\pi_0(t)$
$\frac{\partial\pi_k(t)}{\partial t}=$ flow in - flow out
flow in $=\lambda_{k-1}\pi_{k-1}+\mu_{k+1}\pi_{k+1}$
flow out $=(\lambda_k+\mu_k)\pi_k$
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Q1: Program has execution time uniformly distributed between 10 & 20 seconds. Number of interrupts during execution is Poisson random variable with parameter λt where t is program execution time. Probability distribution of the number of interrupts is $P(N=k)=(\lambda t)ke^{\lambda t}$.
(a) What is $E[N|T=t]$, where N is the number of interrupts, and T is running time.
$E[N|T=t]=\lambda t$, since for fixed running time, the number of interrupts is a Poisson random variable with mean λt.
(b) Expected number interrupts during randomly selected execution.
$E[N]=\int_{10}^{20}E[N|T=t]f_T(t)dt=\int_{10}^{20}\frac{\lambda t}{10}dt=15\lambda$
Q2: We have m types of visitors with equal probability. Find expected visitors to have one of each type. Hint: Let X denote number of visitors needed. Represent X by $X=\sum_{i=1}^m X_i$ where each X_i is a geometric random variable.
Suppose the current visitor pool contains i different types. Let X_i denote the number of additional visitors needed until it contains $i+1$ types. The X_i is are independent geometric random variables with parameter $(m-i)/m$, $i=0,1,\cdots,m-1$.
$E[X]=E[\sum_{i=1}^m X_i]=\sum_{i=1}^m E[X_i]=\sum_{i=1}^m\frac{m}{m-i}$
Q3: A Markov chain $\{X_n,n\geq 0\}$ with states 0, 1, 2 has transition probability matrix
$\begin{bmatrix}1/2&1/3&1/6\\0&1/3&2/3\\1/2&0&1/2\end{bmatrix}$ If $P[X_0=0]=P[X_0=1]=\frac{1}{4}$, find state probability vector $P[X_3=2]$.
Cubing transition probability matrix
$$P^{(3)}=\begin{bmatrix}13/36&11/54&47/108\\4/9&4/27&11/27\\5/12&2/9&13/36\end{bmatrix}$$
$P[X_3=2]=\frac{1}{4}\cdot\frac{47}{108}+\frac{1}{4}\cdot\frac{11}{27}+\frac{1}{2}\cdot\frac{13}{36}$
Q4: Whether or not collision occurs depends on result of last two transmissions If collisions occurred in both of the past two, collision will occur with probability 0.7; collision occurs in last but not the transmission before the last one, then a collision will occur with probability 0.5; collision occurred in transmission before the last one but not the last one, collision with probability 0.4; no collision in the past two transmissions, collision with probability 0.2.
(b) Find the transition probability matrix.
$$P=\begin{bmatrix}0.7&0&0.3&0\\0.5&0&0.5&0\\0&0.4&0&0.6\\0&0.2&0&0.8\end{bmatrix}$$
(c) What fraction of frames suffer a collision?

Solve $\pi = \pi P$ to obtain the stationary state probabilities. Then the fraction of frames suffering a collision is $\pi_0 + \pi_2$ (or $\pi_0 + \pi_1$).

Q5: A shop has room for 2 customers. Customers arrive at Poisson rate 3/hour and service times are exponential random variables with mean 0.25 hours.

(a) Average number of customers in shop?

We have birth-death process with $\lambda = 3$ and $\mu = 4$. $E[k] = \sum_{k=0}^{2} k\pi_k$. Solve for π_k, $\pi_1 = \frac{\lambda}{\mu}\pi_0$, $\pi_2 = \frac{\lambda}{\mu}\pi_1$, $\sum_{j=0}^{2} \pi_j = 1$, so $\pi_0 = 16/37$, $E[k] = 30/37$.

(b) What is proportion of customers who get serviced? $\pi_0 + \pi_1 = 28/37$.

Q6: Packets arrive at router Poisson rate 3/ms and time to foward exponential with mean 0.2 ms. Fraction of time buffer empty?

State is num packets in router, $\lambda = 3$, $\mu = 5$. Solve $p_1 = \frac{\lambda}{\mu}p_0$, $p_2 = \frac{\lambda}{\mu}p_1$, ..., $p_{k+1} = \frac{\lambda}{\mu}p_k$, $\sum_{k=0}^{\infty} p_k = 1$, thus $p_0 = 2/5$.

Q7: 2 machines produce products at n products/hour. Lifetime of machine follows exponential with mean $1/x$ hours, time to fix machine folows exponential with mean $1/y$ hours. Expected long term producing rate?

State is num machines working. $\lambda_0 = \lambda_1 = y$, $\mu_1 = x$, $\mu_2 = 2x$ bc if either machine fails, num working machines reduce from 2 to 1. Solve $p_1 = \frac{y}{x}p_0$, $p_2 = \frac{y}{2x}p_1$, $p_0 + p_1 + p_2 = 1$, thus $p_0 = \frac{2x^2}{2x^2+2xy+y^2}$. Expected num products produced per hour: $p_0 \times 0 + p_1 \times n + p_2 \times 2n$