Documentation

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1 Shallow water equations

1.1 Conservation form

The shallow water equations in conservation form are

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{1a}$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + \frac{\partial (huv)}{\partial y} = -\frac{g}{2}\frac{\partial h^2}{\partial x} - gh\frac{\partial z}{\partial x} - \frac{h}{\rho}\tau_{b,x} \tag{1b}$$

$$\frac{\partial (hv)}{\partial t} + \frac{\partial (huv)}{\partial x} + \frac{\partial (hv^2)}{\partial y} = -\frac{g}{2}\frac{\partial h^2}{\partial y} - gh\frac{\partial z}{\partial y} - \frac{h}{\rho}\tau_{b,y} \tag{1c}$$

We abbreviate this system as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} + \frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial y} = \begin{bmatrix} 0 \\ -gh\frac{\partial z}{\partial x} - \frac{h}{\rho}\tau_{b,x} \\ -gh\frac{\partial z}{\partial y} - \frac{h}{\rho}\tau_{b,y} \end{bmatrix} \tag{2a}$$

with the conservative variables

$$\boldsymbol{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix} \tag{2b}$$

and the fluxes

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix} = \begin{bmatrix} U_2 \\ \frac{U_2^2}{U_1} + \frac{1}{2}gU_1^2 \\ \frac{U_2\,U_3}{U_1} \end{bmatrix} \tag{2c}$$

and

$$\boldsymbol{G}(\boldsymbol{U}) = \begin{bmatrix} hu \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix} = \begin{bmatrix} U_3 \\ \frac{U_2\,U_3}{U_1} \\ \frac{U_3^2}{U_1} + \frac{1}{2}gU_1^2 \end{bmatrix}. \tag{2d}$$

1.2 Quasi-linear form

Using the chain rule of differentiation, we can rewrite the system as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}}\frac{\partial \boldsymbol{U}}{\partial x} + \frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial \boldsymbol{U}}\frac{\partial \boldsymbol{U}}{\partial y} = \begin{bmatrix} 0 \\ -gh\frac{\partial z}{\partial x} - \frac{h}{\rho}\tau_{b,x} \\ -gh\frac{\partial z}{\partial y} - \frac{h}{\rho}\tau_{b,y} \end{bmatrix} \tag{3a}$$

where the flux Jacobians result as

$$\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{U_2^2}{U_1^2} + gU_1 & 2\frac{U_2}{U_1} & 0 \\ -\frac{U_2\,U_3}{U_1^2} & \frac{U_3}{U_1} & \frac{U_2}{U_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ gh - u^2 & 2u & 0 \\ -uv & v & u \end{bmatrix} \tag{3b}$$

and

$$\frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial \boldsymbol{U}} = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{U_2\,U_3}{U_1^2} & \frac{U_3}{U_1} & \frac{U_2}{U_1} \\ -\frac{U_3^2}{U_1^2} + gU_1 & 0 & 2\frac{U_3}{U_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -uv & v & u \\ gh - v^2 & 0 & 2v \end{bmatrix} . \tag{3c}$$

We can find out about the speed and direction of information propagation by computing the eigenvalue decomposition of the flux Jacobians. The eigenvalue decompositions read with $c = \sqrt{gh}$:

$$\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}} = V^{-1} \begin{bmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{bmatrix} V \text{ with } V = \begin{bmatrix} 0 & 1 & 1 \\ 0 & u+c & u-c \\ 1 & v & v \end{bmatrix} \tag{4a}$$

and

$$\frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial \boldsymbol{U}} = W^{-1} \begin{bmatrix} v & 0 & 0 \\ 0 & v+c & 0 \\ 0 & 0 & v-c \end{bmatrix} W \text{ with } W = \begin{bmatrix} 0 & 1 & 1 \\ 1 & u & u \\ 0 & v+c & v-c \end{bmatrix} \tag{4b}$$

2 A Finite-Volume discretization

2.1 The discrete conservation law

For discretization, we choose an equidistant Cartesian grid with a collocated arrangement of variables. As primitive fields, we choose the conservative variables h, hu and hv. We integrate the shallow water equations (2) over a cell $\left[x_i - \frac{\Delta x}{2}; x_i + \frac{\Delta x}{2}\right] \times \left[y_j - \frac{\Delta y}{2}; y_j + \frac{\Delta y}{2}\right]$ and we obtain:

$$\begin{aligned} &\int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \frac{\partial \boldsymbol{U}}{\partial t}\,\mathrm{d}y\,\mathrm{d}x + \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \left[\boldsymbol{F}\left(\boldsymbol{U}\right)\right]_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}}\,\mathrm{d}y + \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \left[\boldsymbol{G}\left(\boldsymbol{U}\right)\right]_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}}\,\mathrm{d}x \\ &= \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \begin{bmatrix} 0 \\ -gh\frac{\partial z}{\partial x} - \frac{h}{\rho}\tau_{b,x} \\ -gh\frac{\partial z}{\partial y} - \frac{h}{\rho}\tau_{b,y} \end{bmatrix} \mathrm{d}y\,\mathrm{d}x\,. \end{aligned} \tag{5}$$

We now define the volume averaged quantities

$$\overline{\boldsymbol{U}}_{i,j} = \frac{1}{\Delta x\,\Delta y} \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \boldsymbol{U}\,\mathrm{d}y\,\mathrm{d}x, \tag{6}$$

for which we obtain a system of ordinary differential equations

$$\begin{aligned} &\frac{\mathrm{d}\overline{\boldsymbol{U}}_{i,j}}{\mathrm{d}t}\Delta x\,\Delta y + \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \left[\boldsymbol{F}\left(\boldsymbol{U}\right)\right]_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}}\,\mathrm{d}y + \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \left[\boldsymbol{G}\left(\boldsymbol{U}\right)\right]_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}}\,\mathrm{d}x \\ &= \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \begin{bmatrix} 0 \\ -gh\frac{\partial z}{\partial x} - \frac{h}{\rho}\tau_{b,x} \\ -gh\frac{\partial z}{\partial y} - \frac{h}{\rho}\tau_{b,y} \end{bmatrix} \mathrm{d}y\,\mathrm{d}x\,. \end{aligned} \tag{7}$$

Also, we define the numerical fluxes

$$\tilde{\boldsymbol{F}}_{i+\frac{1}{2},j} = \frac{1}{\Delta y} \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \left[\boldsymbol{F}\left(\boldsymbol{U}\right)\right]_{x_i+\frac{\Delta x}{2}}\,\mathrm{d}y \tag{8a}$$

$$\tilde{\boldsymbol{G}}_{i,j+\frac{1}{2}} = \frac{1}{\Delta x} \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \left[\boldsymbol{G}\left(\boldsymbol{U}\right)\right]_{y_j+\frac{\Delta y}{2}}\,\mathrm{d}x\,, \tag{8b}$$

so our system results as

$$\begin{aligned} &\frac{\mathrm{d}\overline{\boldsymbol{U}}_{i,j}}{\mathrm{d}t}\Delta x\,\Delta y + \left(\tilde{\boldsymbol{F}}_{i+\frac{1}{2},j} - \tilde{\boldsymbol{F}}_{i-\frac{1}{2},j}\right)\Delta y + \left(\tilde{\boldsymbol{G}}_{i,j+\frac{1}{2}} - \tilde{\boldsymbol{G}}_{i,j-\frac{1}{2}}\right)\Delta x \\ &= \int_{x_i-\frac{\Delta x}{2}}^{x_i+\frac{\Delta x}{2}} \int_{y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \begin{bmatrix} 0 \\ -gh\frac{\partial z}{\partial x} - \frac{h}{\rho}\tau_{b,x} \\ -gh\frac{\partial z}{\partial y} - \frac{h}{\rho}\tau_{b,y} \end{bmatrix} \mathrm{d}y\,\mathrm{d}x\,. \end{aligned} \tag{9}$$

This equation is a discrete conservation law for the quantities h, hu and hv on the cell (i, j). To obtain a numerical scheme, we need to approximate the source terms and the fluxes in terms of the cell-averaged quantities $\overline{\boldsymbol{U}}_{i,j}$ and choose a time integration scheme.

2.2 Roe's scheme

The idea of **?** is to approximate the non-linear conservation law with a