A study of Eulerian Walker Model

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Self-organization:
How Nature Works

- Spontaneous emergence of global order from local interactions.
- Systems able to survive/self repair substantial damage.
- Abundant in Nature.

Self-organized Criticality ("SOC")

Seen in *Slowly driven, Dynamic, Non-equilibrium* systems

- **Key Features:**
  - Critical state as attractor.
  - No external Fine tuning is needed.
  - Onset of Long Range Spatio-Temporal correlation,spontaneously.
- Plausible as a source of natural complexity:
  - Earthquakes
  - Forest Fires.
  - Epidemics. etc.

Motivation and objective

- Eulerian walk was the first problem which paved the way for graph theory.
- Eulerian Walker Model (EWM), as defined in the following context, serves as a model of self-organized criticality.
- Similarity between Abelian Sandpile Model (ASM) and EWM.
- Difference between ASM and EWM.

System:

- A graph G of N nodes
- j^{th} node having τ_j outgoing bonds
- j^{th} node having an outgoing *direction* n_j associated, where $1 \leqslant \; n_j \; \leqslant \; \tau_j \; \forall \; j$
- A Walker is added at any randomly chosen site i
- The configuration is completely specified by the set of values $\{n_i\}$ and the position of the walker

Dynamics:

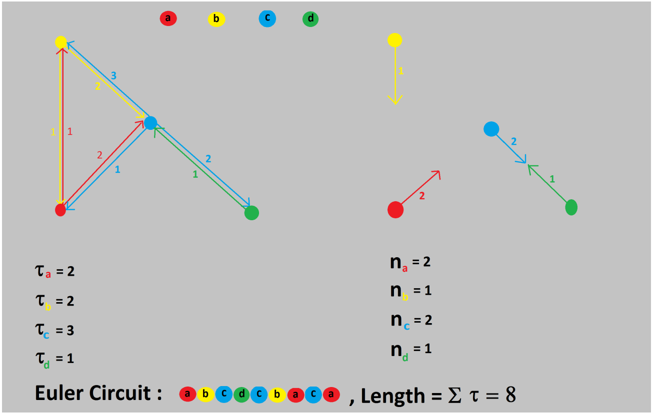
- After arriving at the site j, the walker changes the direction of arrow with the rule:

$$n_j \longrightarrow mod(n_j + 1, \tau_j)$$

- Walker moves along the new direction n_j.
- For an open graph the walker finally leaves the system.
- For a closed system the walker continues to walk forever, finally settling into a *limit cycle* of length $\sum_{i=1}^{i=N} \tau_i$. In this cycle every outgoing bond is visited exactly once. These walks are known as Eulerian circuits.

Eulerian circuit:

- An Eulerian walk on a closed graph of 4 nodes, where the walker finally settles into the *Eulerian Circuit: abcdcbaca* .

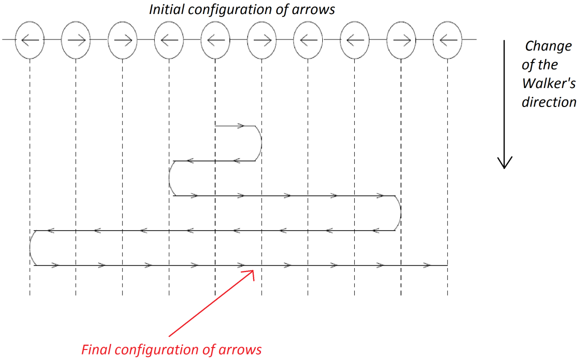


Description:

- **System:** 1D lattice of L sites.
- $\tau_i = 2 \; \forall \; i$.
- The outgoing direction, n_i has only 2 possible states, say Right and Left.
- The walker reverses the arrow-direction of the present site, and moves towards the new direction.
- The arrows in the region already visited by the walker get aligned in the same direction, i.e self-organization occurs.

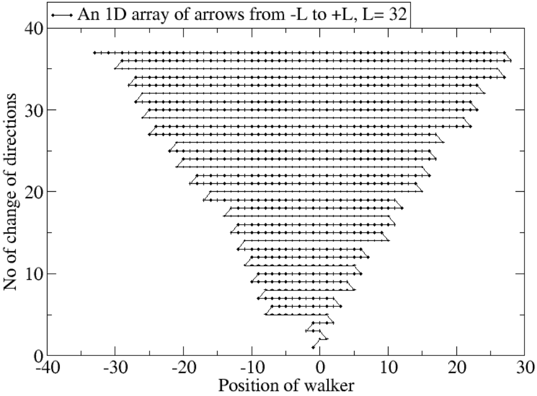
self-organization:

- The onset of alignment of arrows from a random initial environment.



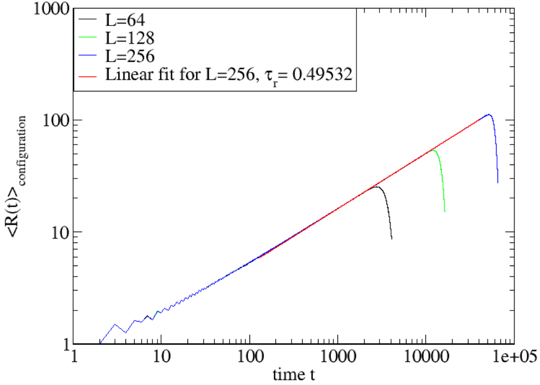
Results:

- Actual simulation on a lattice shows the trajectory with respect to change of direction of motion.



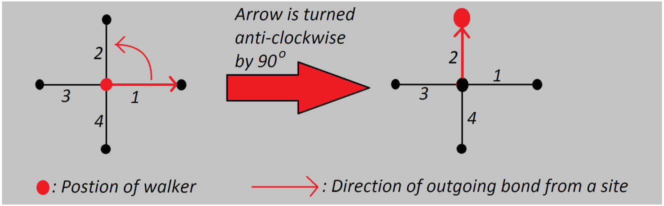
$\langle R(t)\rangle$ *vs* t plot:

- $t \equiv$ No of steps taken by the walker
  $\langle R(t)\rangle = \frac{\sum_i^N R_i(t)}{N}$, $N =$ No of configurations



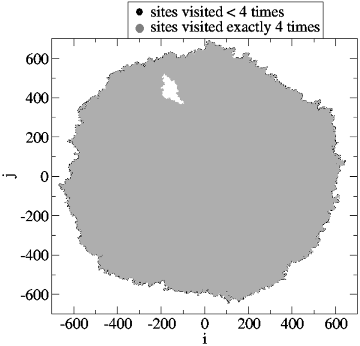
Description:

- **System:** 2D square lattice of LxL sites.
- $\tau_i = 4 \; \forall \; i$.
- The outgoing direction, n_i has only 4 possible states.
- The walker changes the arrow-direction of the present site anti-clockwise by 90^o, and moves along the new direction.



self-organization:

- If $S(T) \equiv$ No of distinct sites visited after T steps, then in the time window $T - 4S(T)$ to T, almost all the sites are visited exactly 4 times.



The white region within the grey cluster shows the sites which are visited more than 4 times in the given time window.

$T = 10^8$

$S(T) = 1237751$

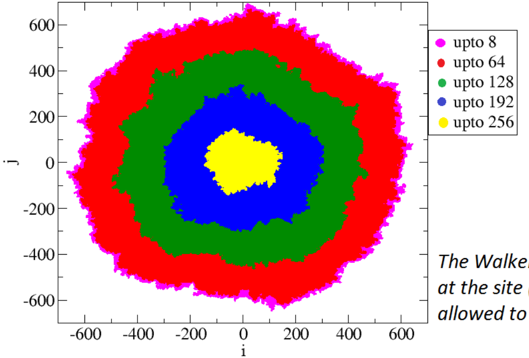
$S_{visit=4} = 1211424$

$S_{visit<4} = 18152$

$S_{visit>4} = 8175$

Shape of clusters:

- As T grows, the cluster of sites visited, approaches the shape of a circle. In fact, the sites with the equal number of visits form a circular perpheri.

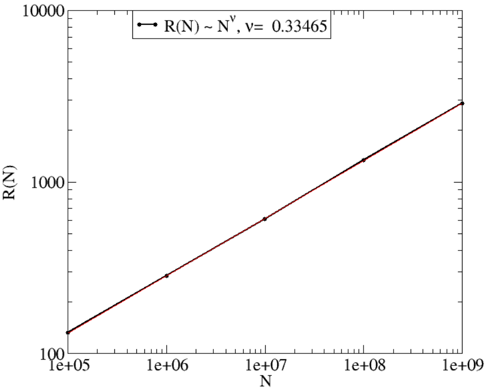


The Walker was dropped at the site (0,0) and was allowed to take 10^8 steps.

$\langle R(N) \rangle$ *vs* N plot:

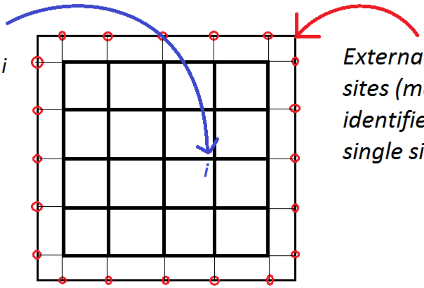
- $N \equiv$ No of steps taken by the walker

$\langle R(N) \rangle = \frac{\sum_i^{N_c} R_i(t)}{N_c}$, $N_c =$ No of configurations



Definition of the operator a_i

Randomly chosen site i out of N sites



External perimeter sites (marked red), identified with the single sink site i_0

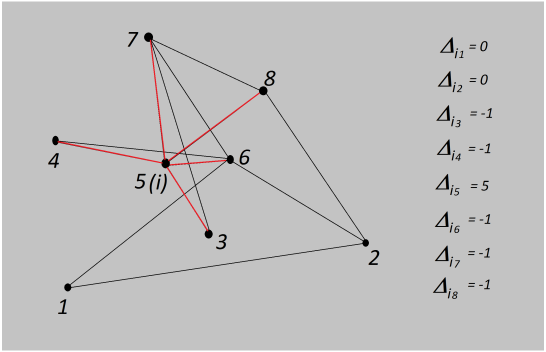
System at configuration C → *Walker added at i* → *Walker evolves till reaching i_0* → *System at configuration C'*

$C' = a_i C$

The Δ matrix

For a graph of N points, Δ_{NxN} is defined as:

$$\Delta_{ii} \equiv \text{No of outgoing bonds from site i}$$
$$-\Delta_{ij} \equiv \text{No of bonds from i to j}$$

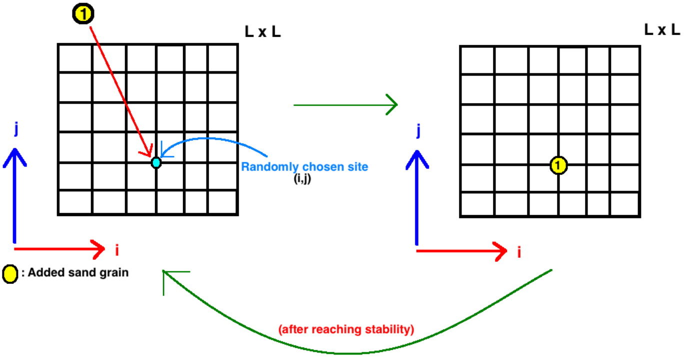


Relation between Δ *and* a_i

- a_i *and* a_j *commute*, $i.e$ $[a_i, a_j] = 0$
- $\prod_j a_i^{\Delta_{ij}} = I, \forall\ i$
- The particle addition operators in Abelian Sandpile Model obey the same algebra.

The BTW sandpile model:
Description of system

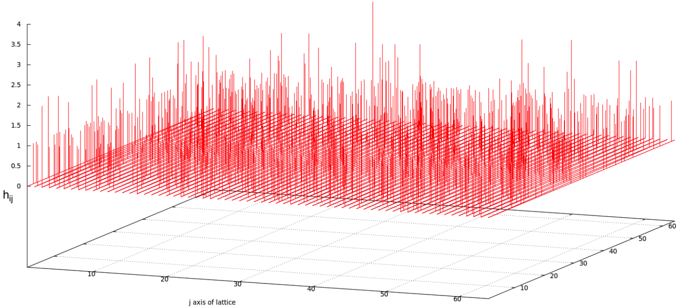
- **Addition of sand grain:**
  On an LxL square lattice with open boundary condition



The Process of Addition of Sandgrain

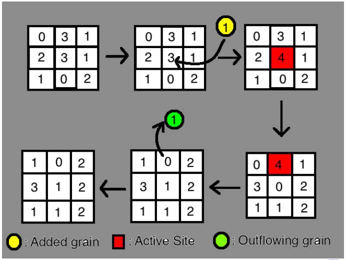
Storage of Sand grains:

- **In The form of Sand Column:**
  Number of grains at (i,j)
  = Height of Sand column at (i,j)
  $= h_{ij}$



Dynamics of System:
Toppling Mechanism and Outflow

- **Threshold Height:** $h_c = 4$
- If $h_{ij} < h_c \; \forall \; (i,j)$, (**Stable** Configuration)
  $h_{ij} \longrightarrow \; h_{ij}$
- If $h_{ij} \geq h_c$, for any (i,j) (**Unstable** Configuration)
  $h_{ij} \longrightarrow \; h_{ij} - 4$
  $h_{i'j'} \longrightarrow \; h_{i'j'} + 1, \; where, (i', j') \equiv (i \pm 1, j), (i, j \pm 1)$



The opeartor a_i

- If a particle is added at the site i on the lattice, which takes the system from a recurrent(stable) configuration C to another such configuration C', through a series of topplings at different sites, then

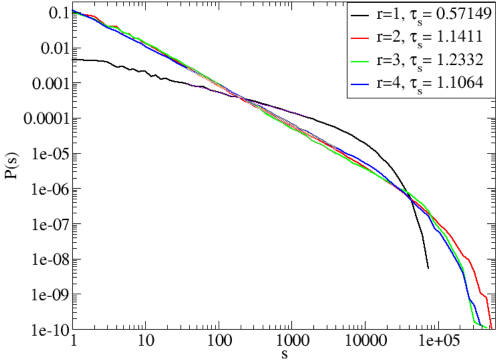
$$C' = a_i C$$
$$[a_i, a_j] = 0$$
$$\prod_j a_i^{\Delta_{ij}} = I, \ \forall i$$

Combining ASM and EWM

- Multiple walkers released at randomly chosen sites on the lattice.
- A walker arriving at the site j, waits until $h_j \geqslant r$
- Then these r particles take 1 step each in the directions $n_j + 1, n_j + 2, ..., n_j + r$ and the arrow is reset to mod $(n_j + r, \tau_j)$.
- $r = 1$ corresponds to EWM.
- $r = \tau_j$ corresponds to ASM.

Results:
P(s) vs s graph

- The distribution of avalanche sizes for r= 1, 2, 3, 4, on a 200x200 lattice, averaged over 10^6 configurations.



Conclusion:

- EWM serves as a model of self-organization, as seen in the 1D and 2D cases. The long-rage correlation in the system leads to criticality.
- The particle addition operator a_i in EWM satisfies the same alegbra as that in ASM.
- By introducing a parameter r, one can arrive at EWM and ASM as special cases.
- Although similar, EWM belongs a different universality class.

References I

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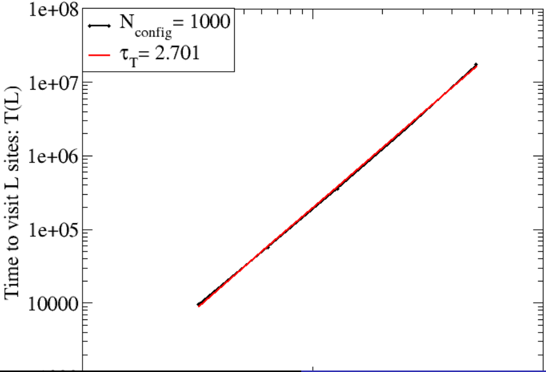
Eulerian Walkers as a Model of Self-Organized Criticality.
Physical Review Letters, 77, 25(1996), 5079.

S. S. Manna, D. Dhar, S. N. Majumdar
Spanning trees in two dimensions.
Physical Review A, 46, 8(1992), R4471.

R. Kapri, D. Dhar.
Asymptotic shape of the region visited by an Eulerian Walker.
arXiv: 0906.5506v2 [cond-mat.stat-mech] , 17 Sep 2009.

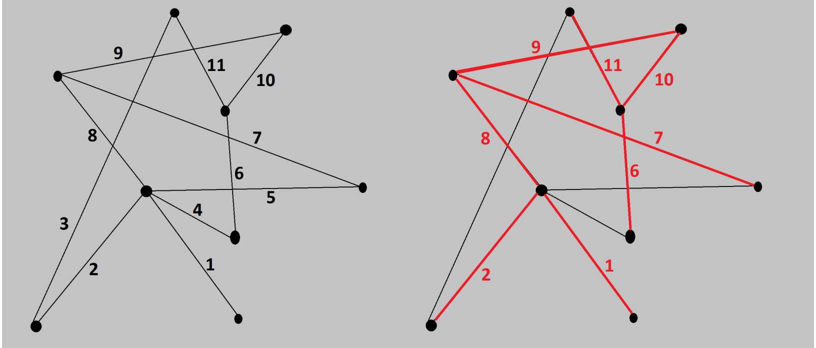
Sub-diffusive nature:

- The number of steps taken to visit the L^2 sites at least once also captures the sub-diffusive nature of Eulerian walk, the exponent of the power law depeb=ndance is about 2.70 .



What is a *spanning tree*?

- A collection of (N-1) bonds on a graph of N nodes, which form a single connected cluster.



A graph G of 9 nodes and 11 links

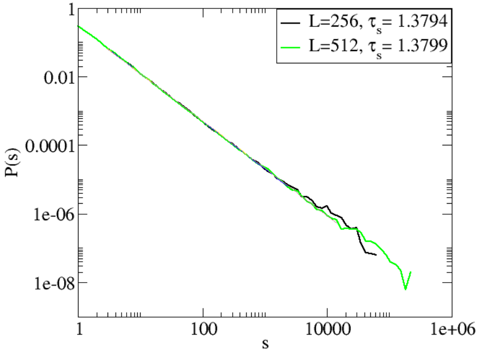
A spanning tree on the graph G
No of links = (9 - 1) = 8

Generating a Spanning tree
On a square lattice

- **Broder's algorithm:**
  A random walker is allowed to walk until it visits all the sites at least once.
  Only the last exit directions at each site is kept in memory. If the sites are now connected according to the "outgoing directions" at each site, it forms a spanning tree, rooted at the last site that was visited.

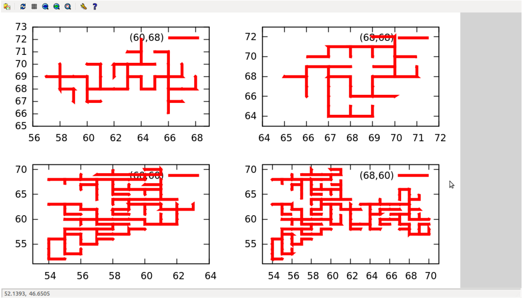
Loop perimeter distribution

- Adding a bond at random to a spanning tree closes a loop. The perimeter (s) of the loop is also power law distributed. $Prob(s) \; \alpha \; s^{-\tau_s}$



Spanning trees in EWM:

- A Eulerian walk ending at site j, generates a spanning tree rooted at j.
- The figure shows four such spanning trees generated by the "first Eulerian circuit".



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