Linear Algebra

Linear Algebra - Vector Spaces

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Linear Dependence and Independence

Definition - Linear dependence

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ from a vector space V is said to be *linearly dependent* if there exists a set of constants $\{\alpha_1, \ldots, \alpha_n\}$ *not all zero* such that

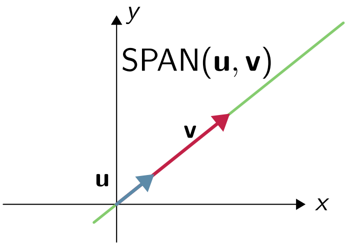
$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}_V.$$

Note: the right hand side of the equation is the *zero vector*, not the real number 0.

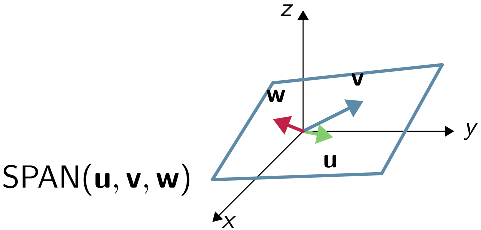
Example - Linear dependence of two Euclidean vectors

Properties

For Euclidean vectors, when we have two vectors that are linearly dependent we say that they are *colinear*.



When we have three 3d Euclidean vectors that are linearly dependent we say that they are *coplanar*. Any two of the vectors give a plane as their span and then adding the 3rd dependent vector gives linear combinations that remain in that plane.



Definition - Linear independence

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ from V is said to be *linearly independent* if they are not linearly dependent. That is, the equation

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}_V.$$

implies that the constants $\alpha_1, \ldots, \alpha_n$ *are all zero*.

Example - Linear independence of three Euclidean vectors

Methodology

Take note of the simple logic or methodology of the previous examples. We always start with the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}_V$$

and show either that by necessity all of the coefficients are zero (then the vectors are linearly independent), or that we can find at least one non-zero coefficient (then the vectors are linearly dependent). The exact method by which we determine these coefficients depends on which type of vectors we are considering.

Exercise - Linear dependence/independence

Determine whether the following vectors are linearly dependent or independent

$$\mathbf{u} = (1, 2, 0), \quad \mathbf{v} = (0, 1, -1) \quad \mathbf{w} = (1, 0, 3).$$

Properties

Given a set of linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, if a new set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{w}\}$ is linearly dependent, then $\mathbf{w}$ can be given as a linear combination of the other vectors. That is, there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n$$

Additionally, given a set of independent vectors, any linear combination of a subset of these vectors is also linearly independent.

Theorem - The span of a dependent set of vectors can be reduced

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors of V. If $\mathcal{B}$ is a set of linearly dependent vectors, then we can always remove one of the vectors to form a new set, $\mathcal{B}' = \mathcal{B} \backslash \{\mathbf{v}_k\}$ for some k, without changing the span: $\text{SPAN}(\mathcal{B}') = \text{SPAN}(\mathcal{B})$.

Basis

Definition - Spanning set

A set of vectors $\mathcal{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is said to *span* a vector space V if and only if

$$\text{SPAN}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$$

We also call $\mathcal{A}$ a spanning set of V. Every vector of V can be written as a linear combination vectors from a spanning set.

Examples - Spanning sets

Definition - Basis

A *basis of a vector space* V is a minimal set of vectors which spans the vector space. Formally, the set of vectors $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in a vector space V is a basis of V if it is a set of linearly independent vectors and $\mathrm{SPAN}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$.

Note: bases are not unique, but they always contain the same number of vectors.

Theorem - Bases of two dimensional Euclidean space

Any two linearly independent vectors in $\mathbb{R}^2$ forms a basis of $\mathbb{R}^2$.

Example - Basis of $\mathbb{R}^3$

Is the set $\mathcal{B} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \{(1, -2, 2), (2, 1, 0), (1, 1, 2)\}$ a basis of $\mathbb{R}^3$?

Dimension

Definition - Dimension

A vector space is said to be **finite-dimensional** if it has a basis with a finite number of vectors. This *number of vectors* (or cardinality) of the basis is called the dimension of the vector space, denoted $\dim(V)$. We may also speak of an *n-dimensional vector space* for a finite-dimensional vector space with dimension $n \in \mathbb{Z}$.

Dimension and Canonical Basis of Euclidean vector spaces

The Euclidean vector space $\mathbb{R}^n$ has dimension n.

The *canonical basis* of the vector space of real n-tuples, $\mathbb{R}^n$, is the ordered set of n n-tuples with k^{th} element, $\mathbf{c}_k = (\alpha_1, \ldots, \alpha_n)$ such that

$$\alpha_j = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases}$$

That is, as a set the canonical basis is

$$\mathcal{C}_n = \{(1, 0, \ldots, 0),\ (0, 1, \ldots, 0),\ \ldots,\ \underbrace{(0, 0, \ldots, 0, \overbrace{\ 1\ }^{k^{th} \text{ place}}, 0, \ldots, 0)}_{k^{th} \text{ tuple}},\ \ldots,\ (0, 0, \ldots, 1)\}.$$

Examples - Dimension

$\mathbb{R}^3$ has a basis $\mathcal{A} = \{(1, -2, 2), (2, 1, 0), (1, 1, 2)\}$ and therefore $\dim(\mathbb{R}^3) =$.

The set $\mathcal{B} = \{(1, 0, 0), (0, -1, 0), (0, 0, 2), (1, 1, -1)\}$ is not a basis of $\mathbb{R}^3$ because

...

The vector space of polynomials of degree up to 3, $\mathcal{P}_3$ has a basis $\mathcal{C} = \{1 - x, 1 + x^2, x^3, 2\}$ and therefore $\dim(\mathcal{P}_3) =$.

The vector space of 2x2 diagonal matrices, $\mathcal{M}$ has a basis $\mathcal{D} = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ and therefore $\dim(\mathcal{M}) =$.

Theorem - Dimension of subspaces

Let V be a finite-dimensional vector space of dimension n. Then every vector subspace, S, of V is also finite-dimensional and

$$\dim(S) \leq \dim(V).$$

Furthermore, if $\dim(S) = \dim(V)$ then $S = V$.

Theorem - Basis demonstration with dimension

Let V be a vector space and $\mathcal{B}$ be set of vectors in V. Then $\mathcal{B}$ forms a basis of V if its vectors are linearly independent and the number of vectors in $\mathcal{B}$ equals the dimension of V.

Note: this theorem merely results from the definition of dimension. It seems circular, but if you can use your geometrical intuition to find the dimension of a space (for example we can know that a planar equation will define a 2 dimensional vector space) without finding the basis first, then it cuts the work in half.

Example - Dimension of an intersection of Euclidean subspaces

Let $A = \text{SPAN}((1,1,0),\ (1,2,-1))$ and $B = \text{SPAN}((0,2,-1),\ (-1,1,-2))$ be two vector subspaces of $\mathbb{R}^3$. What is the dimension of $A \cap B$?

Example - Dimension of an intersection of Euclidean subspaces

Dimension of sum of subspaces

Let A and B be two finite-dimensional subspaces of a vector space V (which is either finite or infinite dimensional). Recall: the sum of these subspaces is the vector space given by
$A + B = \{\mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A, \, \mathbf{b} \in B\}$.

The sum space $A + B$ is also a finite-dimensional vector space with

$$\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B).$$

Particular cases:

- If $A + B = A \oplus B$ (a direct sum), then $A \cap B = \{\mathbf{0}_V\}$. So we have

$$\dim(A \oplus B) = \dim(A) + \dim(B)$$

- If A and B are *complementary*, then $A \cap B = \{\mathbf{0}_V\}$ and $A + B = V$. Hence

$$\dim(A \oplus B) = \dim(A) + \dim(B) = \dim(V)$$

Example - Dimension of sum space

For $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y - 3z = 0\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$: 1) Find a basis for B. 2) Show that $A + B = \mathbb{R}^3$. 3) Is $A + B$ a direct sum? 4) Does $\dim(A + B) = \dim(A) + \dim(B)$?

Coordinates

Coordinates in a basis

Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of a vector space V. As $\mathrm{SPAN}(\mathcal{B}) = V$, every vector of V can be expressed as a linear combination of the basis vectors

$$\forall \mathbf{v} \in V, \quad \exists\, \alpha_1, \ldots, \alpha_n, \quad \text{such that} \quad \mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n.$$

The *unique* coefficients in this linear combination are called the *coordinates* of the vector $\mathbf{v}$ *in the basis* $\mathcal{B}$. We denote these coordinates with a column

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Example - Coordinates in $\mathbb{R}^2$

Example - Coordinates in $\mathbb{R}^3$