
Linear Algebra 1

Dictionary

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Chapter 1: Euclidean Vectors

Definition 1.1 (Euclidean vector or tuple).

A Euclidean vector is a list of n real numbers, also called an n -tuple. We write this list in parentheses, for example $(1, 3, -2, \dots, 0)$, and we say that this object belongs to \mathbb{R}^n . An arbitrary tuple can be written $\mathbf{v} = (v_1, v_2, \dots, v_n)$ where the components $v_i \in \mathbb{R}$ for any index i .

Definition 1.2 (Tuple addition).

Euclidean vectors are added to each other component by component. In symbols

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

Note: this means you can only add two tuples together of the same size. It makes no sense to add a 3-tuple to a 5-tuple.

Definition 1.3 (Scalar multiplication).

Let $c \in \mathbb{R}$, called a scalar quantity, and $\mathbf{v} \in \mathbb{R}^n$ with components v_i . Then the scalar multiplication $c\mathbf{v}$ gives a vector \mathbf{w} with components $w_i = cv_i$ for every index i . In tuple form

$$c(v_1, v_2, \dots, v_n) = (cv_1, cv_2, \dots, cv_n).$$

Definition 1.4 (Canonical Euclidean unit vectors).

The canonical Euclidean vectors in \mathbb{R}^n are the n vectors of the form

$$\begin{aligned}\mathbf{e}_1 &= (1, 0, \dots, 0) \\ \mathbf{e}_2 &= (0, 1, \dots, 0) \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \dots, 1).\end{aligned}$$

More compactly

$$\mathbf{e}_k = (\alpha_1, \alpha_2, \dots, \alpha_n) \quad \text{where} \quad \alpha_j = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases}$$

Definition 1.5 (Dot product).

For two n -tuples \mathbf{a} and \mathbf{b} , their dot product, also called scalar product and Euclidean inner product, is the real number given by the addition of component by component multiplication

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \sum_{i=1}^n a_i b_i.$$

Definition 5.7 (Surjectivity).

Let $f : V \rightarrow W$ be a linear map. We say that f is surjective if every vector in the output space has a corresponding input vector. In symbols

$$\forall \mathbf{w} \in W \quad \exists \mathbf{v} \in V \text{ such that } f(\mathbf{v}) = \mathbf{w}.$$

Definition 5.8 (Categories of linear maps).

Let $f : V \rightarrow W$ be a linear map.

- If $W = V$ we call f an endomorphism.
 - If f is both injective and surjective then we say it is bijective and we call it an isomorphism.
 - If f is both an isomorphism and an endomorphism we call it an automorphism.
-

Definition 5.9 (Composition of linear maps).

Composition of linear maps works exactly as you would expect if you remember the composition of regular functions. We must have a coherence between the output of one linear map and the input of another. So, two linear maps $f : A \rightarrow B$ and $g : U \rightarrow V$ can be composed as a well defined linear map $g \circ f$ ("g of f") if and only if the output space of f is the input space of g : $U = B$. For any $\mathbf{u} \in A$ the composition is written

$$g \circ f : A \rightarrow V \quad \text{and} \quad (g \circ f)(\mathbf{u}) = g(f(\mathbf{u})).$$

Chapter 5: Linear Maps

Definition 5.1 (Linear map).

A mapping, f , from a vector space V to a vector space W , denoted $f : V \rightarrow W$, is called a linear map if it satisfies the following property:

$$\forall \mathbf{u}, \mathbf{v} \in V, \forall \alpha, \beta \in \mathbb{R}$$

$$f(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}).$$

We say that a linear map preserves linear combinations.

Definition 5.2 (Image).

The image of a linear map $f : V \rightarrow W$, denoted $\text{im}(f)$, is the set of all possible "output" vectors of the map:

$$\text{im}(f) = \{ \mathbf{w} \in W \mid \exists \mathbf{v} \in V \, f(\mathbf{v}) = \mathbf{w} \} \subseteq W.$$

Definition 5.3 (Rank).

The rank of a linear map is the dimension of its image: $\text{rank}(f) = \dim(\text{im}(f))$.

Definition 5.4 (Kernel).

The kernel of a linear map $f : V \rightarrow W$, denoted $\ker(f)$, is the set of vectors that f maps to the zero vector, $\mathbf{0}_W$, of W . That is,

$$\ker(f) = \{ \mathbf{v} \in V \mid f(\mathbf{v}) = \mathbf{0}_W \}.$$

Definition 5.5 (Nullity).

The nullity of a linear map is the dimension of its kernel: $\text{nullity}(f) = \dim(\ker(f))$.

Definition 5.6 (Injectivity).

Let $f : V \rightarrow W$ be a linear map. We say f is injective if no two vectors of V are mapped to the same vector of W . In symbols we have two equivalent expressions

$$\forall \mathbf{x}, \mathbf{y} \in V, \quad (f(\mathbf{x}) = f(\mathbf{y}) \implies \mathbf{x} = \mathbf{y})$$

or

$$\forall \mathbf{x}, \mathbf{y} \in V, \quad (\mathbf{x} \neq \mathbf{y} \implies f(\mathbf{x}) \neq f(\mathbf{y}))$$

Chapter 2: Matrix Algebra

Definition 2.1 (Matrix).

A matrix is a collection of numbers from a field \mathbb{F} (e.g. rational numbers) usually represented by a rectangular array. For example, an $m \times n$ (said m by n) matrix A with coefficients $a_{ij} \in \mathbb{F}$ would be represented by an array with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix} = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}.$$

Sometimes it is convenient to refer to the coefficients in the array like so: $a_{ij} = (A)_{ij}$.

Definition 2.2 (Set of all $m \times n$ matrices).

We write the set of all $m \times n$ matrices with coefficients in \mathbb{F} as

$$\mathcal{M}_{m,n}(\mathbb{F})$$

Definition 2.3 (Matrix columns and rows).

For a matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$ we denote its j^{th} column and i^{th} row

$$A^{(j)} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix}, \quad A_{(i)} = (a_{i1} \quad a_{i2} \quad a_{i3} \quad \cdots \quad a_{ij} \quad \cdots \quad a_{in})$$

Definition 2.4 (Transpose of a matrix).

The transpose of an $m \times n$ matrix, A , is an $n \times m$ matrix, denoted A^T , with rows equal to the columns of A . That is, $(A^T)_{ij} = (A)_{ji}$ for all combinations of i and j .

Definition 4.10 (Linear independence).

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ from V is said to be linearly independent if they are not linearly dependent. That is, the equation

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}_V.$$

implies that the constants $\alpha_1, \dots, \alpha_n$ are all zero.

Definition 4.11 (Basis).

A basis of a vector space V is a minimal set of vectors which spans the vector space. Formally, the set of vectors $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in a vector space V is a basis of V if it is a set of linearly independent vectors and $\text{SPAN}(\mathbf{v}_1, \dots, \mathbf{v}_n) = V$. Note: bases are not unique, but they always contain the same number of vectors.

Definition 4.12 (Dimension).

The dimension of a vector space is the number of elements in a basis for that vector space.

Definition 4.13 (Canonical basis of \mathbb{R}^n).

The canonical basis of the vector space of real n -tuples, \mathbb{R}^n , is the ordered set of n n -tuples with k^{th} element, $\mathbf{c}_k = (\alpha_1, \dots, \alpha_n)$ such that

$$\alpha_j = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases}$$

That is, as a set the canonical basis is

$$\mathcal{C}_n = \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 0, \underbrace{1}_{k^{\text{th}} \text{ place}}, 0, \dots, 0), \dots, (0, 0, \dots, 1)\}.$$

Definition 4.14 (Canonical basis of \mathcal{P}_n).

The canonical basis of the vector space of polynomials with degree up to n , \mathcal{P}_n , is the ordered set of n polynomials with k^{th} element, $\mathbf{c}_k = x^k$. That is, as a set the canonical basis is

$$\mathcal{C}_n = \{1, x, x^2, \dots, x^n\}.$$

Definition 4.15 (Coordinates of a vector).

Let \mathbf{v} be a vector in a vector space V . The coordinates of \mathbf{v} with respect to a given basis \mathcal{B} , denoted $[\mathbf{v}]_{\mathcal{B}}$, is a column of the unique set of coefficients in the linear combination of \mathbf{v} in terms of the basis vectors.

Definition 4.5 (Cartesian form of Euclidean vector sub spaces).

Euclidean vector sub spaces can always be written as a set with some defining equations:

$$\{(x_1,\ldots,x_n)\in\mathbb{R}^n\mid\text{equations relating the }x_k\}.$$

For example, the general form of planar vector subspaces of \mathbb{R}^3 is

$$V_p=\left\{(x,y,z)\in\mathbb{R}^3\mid ax+by+cz=0\right\}$$

where a , b and c are some given constants. This set is read aloud as “all the triples (x,y,z) such that

$$ax+by+cz=0”.$$

Definition 4.6 (Sum of subspaces (sum space)).

Suppose we have a vector space V with vector subspaces F and G . We define the **sum of subspaces** (or
sum space) as a new set denoted

$$F+G=\{\boldsymbol{f}+\boldsymbol{g}\mid\boldsymbol{f}\in F,\boldsymbol{g}\in G\}$$

Note: The sum space is a subset of the parent vector space: $F+G\subset V$.

Definition 4.7 (Direct sum).

Let F and G be two vector subspaces of a vector space V and let $E=F+G$ be the sum space. We say E is
a **direct sum** of F and G if each element of E has a **unique** decomposition as a sum of vectors in F and

vectors in G . That is, for every $\boldsymbol{v}\in E$, there exists unique vectors $\boldsymbol{f}\in F$ and $\boldsymbol{g}\in G$ such that $\boldsymbol{v}=\boldsymbol{f}+\boldsymbol{g}$. We
denote this direct sum with a new symbol

$$E=F\oplus G$$

Definition 4.8 (Complementary vector subspaces).

Let F and G be two vector subspaces of V . F and G are called **complementary** if V is a direct sum of F
and G . That is, if and only if

$$\bullet\quad V=F+G,\text{ and}$$

$$\bullet\quad F\cap G=\{\boldsymbol{0}_V\}$$

Definition 4.9 (Linear dependence).

A set of vectors $\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n\}$ from a vector space V is said to be linearly dependent if there exists a set of
constants $\{\alpha_1,\ldots,\alpha_n\}$ not all zero such that

$$\alpha_1\boldsymbol{v}_1+\ldots+\alpha_n\boldsymbol{v}_n=\boldsymbol{0}_V.$$

Note: the right hand side of the equation is the zero vector, not the real number 0.

$$(V)^i=a_{i1}x_1+a_{i2}x_2+\ldots+a_{im}x_m=\sum_{m=1}^ka_{ik}x_k$$

$Y\in\mathcal{M}_{m,1}$ with coefficients

Consider a matrix $A\in\mathcal{M}_{m,n}$ and a column $X\in\mathcal{M}_{n,1}$. We define the product AX to result in the column
Definition 2.11 (Multiplication of a matrix by a column).

additive inverse of A as $-A$.

Given a matrix $A\in\mathcal{M}_{ij}$, its additive inverse is the same matrix multiplied by the scalar -1 . We denote the
Definition 2.10 (Additive inverse).

The zero matrix of any shape is a matrix $M_0\in\mathcal{M}_{m,n}$ consisting entirely of zeros as coefficients.

Definition 2.9 (Zero matrix).

that is, we multiply every coefficient by the scalar.

$$b_{ij}=ka_{ij},$$

to be a matrix $B\in\mathcal{M}_{m,n}$ with coefficients given by:

Given a number $k\in\mathbb{R}$ (called a scalar) and a matrix $A\in\mathcal{M}_{m,n}$, we define matrix scalar multiplication, kA ,

Definition 2.8 (Scalar multiplication).

$$(A+B)_{ij}=(A)_{ij}+(B)_{ij}.$$

of the addition as the addition of the i,j^{th} coefficients of each matrix:

Definition 2.7 (Matrix addition).

Matrix addition is done coefficient by coefficient, that is, for two matrices A and B we define the i,j^{th} coefficient

A matrix A is symmetric if it is equal to its transpose, $A=A^T$.

Definition 2.6 (Symmetric matrix).

$i\neq j$.

A square matrix A is said to be diagonal if all its non-diagonal elements are zero, e.g. $(A)_{ij}=0$ whenever

Definition 2.5 (Diagonal matrix).

Visually

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_m \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

$$\implies Y = x_1 A^{(1)} + x_2 A^{(2)} + \cdots + x_m A^{(m)}$$

Definition 2.12 (Rotation matrix - arbitrary angle anti-clockwise).

By using a column $X \in \mathcal{M}_{2,1}$ to represent a Euclidean vector, the following matrix allows the operation of rotation, anti-clockwise, of X by an angle θ :

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

where the rotated vector is represented by a column $X' \in \mathcal{M}_{2,1}$ obtained by matrix multiplication $X' = R_\theta X$.

Definition 2.13 (Multiplication of two matrices).

Consider two matrices $A \in \mathcal{M}_{n,m}$ and $B \in \mathcal{M}_{m,q}$. We define the product AB to be the matrix $C \in \mathcal{M}_{n,q}$ with coefficients

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^m a_{ik}b_{kj}$$

$$\implies \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mq} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}}_{\text{first column}} + \cdots + b_{m1} \underbrace{\begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}}_{\text{first column}} & \cdots & b_{1q} \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}}_{\text{first column}} + \cdots + b_{mq} \underbrace{\begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}}_{\text{first column}} \end{pmatrix}$$

Additionally, for the product

$$\underbrace{\begin{pmatrix} \textcolor{red}{n} & \textcolor{red}{m} \end{pmatrix}}_A \underbrace{\begin{pmatrix} \textcolor{blue}{m} & \textcolor{blue}{q} \end{pmatrix}}_B$$

we will call the indices for the columns of A and rows of B the inner indices (blue), whereas the indices for the rows of A and columns of B will be called the outer indices (red).

Chapter 4: Vector Spaces

Definition 4.1 (Vector space).

A vector space over a field \mathbb{F} is a set, call it V , with elements called vectors supplied with definitions of two operations, vector addition (VA) and scalar multiplication (SM), that satisfy the following vector space axioms:

- $$\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \quad \text{and} \quad \forall k, l \in \mathbb{F}$$
- (VA1) $\mathbf{u} + \mathbf{v} \in V$ (closure under vector addition)
- (VA2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity of vector addition)
- (VA3) $\exists \mathbf{0} \in V$, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ (additive identity)
- (VA4) $\exists -\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (additive inverse)
- (VA5) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity of vector addition)
- (SM1) $k\mathbf{u} \in V$ (closure under scalar multiplication)
- (SM2) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (distributivity over vector addition)
- (SM3) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$ (distributivity over field addition)
- (SM4) $k(l\mathbf{u}) = (kl)\mathbf{u}$ (compatibility of scalar and field multiplication)
- (SM5) $1\mathbf{u} = \mathbf{u}$ (multiplicative identity)

Definition 4.2 (Linear Combination).

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of vectors in a vector space V . A linear combination of these vectors is a new vector, $\mathbf{w} \in V$, of the form

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

where the α_k are real numbers.

Definition 4.3 (Vector subspace).

Suppose that V is a vector space and W is a subset of V . We call W a vector subspace if it satisfies the vector space axioms for the same definition of vector addition and scalar multiplication defined for V .

Definition 4.4 (Span).

Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of vectors from a vector space V . The span of these vectors is the set of all linear combinations of those vectors:

$$\text{SPAN}(\mathcal{B}) = \text{SPAN}(\mathbf{v}_1, \dots, \mathbf{v}_n) = \{\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \mid \forall \alpha_1, \dots, \alpha_n \in \mathbb{R}^n\}.$$

This set forms a vector subspace of V . It is obviously non-empty because it at least contains the vectors of \mathcal{B} . It is also automatically closed under vector addition and scalar multiplication because those are exactly the operations we used to create all the vectors in the span! Therefore $\text{SPAN}(\mathcal{B})$ is a vector subspace of V .

We say that (S) is a Cramer system if $\det(A) \neq 0$.

$$\overbrace{\begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{21} \\ a_{22} \\ \vdots \\ a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}}^X \overbrace{\begin{pmatrix} a_{n2} & \dots & a_{nn} \\ \vdots & \ddots & \vdots \\ a_{2n} & \dots & a_{nn} \end{pmatrix}}^X \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}^X =$$

with the associated matrix form

$$(S) \quad \begin{cases} a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = y_n \\ \vdots \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = y_2 \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = y_1 \end{cases}$$

Suppose we have the following linear system of equations (with unknowns equal to equations)

Definition 3.7 (Cramer system).

An **underdetermined system** has less equations than unknowns. We say "there are not enough equations". Such a system has either no solutions, or infinitely many.

Definition 3.6 (Underdetermined system).

Definition 3.5 (Overdetermined system). An overdetermined system has more equations than unknowns. We say "there are too many equations". Such a system allows solutions only if certain conditions are met.

$$\begin{aligned} (S_1) \quad \begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \\ 15I_1 - 4I_2 - 2I_3 = 20 \end{cases} & \equiv (S_2) \quad \begin{cases} I_1 + I_2 - I_3 = 0 \\ I_1 + I_2 - I_3 = 0 \end{cases} \\ \text{Adding a multiple of one equation to another equation} & \\ (S_1) \quad \begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \\ I_1 - (6/13)I_2 = (20/13) \end{cases} & \equiv (S_2) \quad \begin{cases} I_1 + I_2 - I_3 = 0 \\ I_1 - (6/13)I_2 = (20/13) \end{cases} \\ \text{Multiplying one equation by a non-zero constant} & \end{aligned}$$

without being explicit.

Note: we generally have to specify in words that we create a submatrix. The notation A_{ij} is a little ambiguous

$$\text{The submatrix } A_{ij} = \begin{pmatrix} a_{m,1} & \dots & a_{m,j-1} & a_{m,j+1} & \dots & a_{m,n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{1,1} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1,n} \end{pmatrix}$$

$$\text{For } A = \begin{pmatrix} a_{m,1} & \dots & a_{m,j-1} & a_{m,j} & \dots & a_{m,n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j} & \dots & a_{i+1,n} \\ a_{i,1} & \dots & a_{i,j-1} & a_{i,j} & \dots & a_{i,n} \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j} & \dots & a_{i-1,n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_{1,1} & \dots & a_{1,j-1} & a_{1,j} & \dots & a_{1,n} \end{pmatrix}$$

From a matrix A we generate the submatrix A_{ij} by deleting the i th row and j th column:

Definition 2.17 (Submatrix).

$$\det((a)) = a$$

Definition 2.16 (Determinant of a 1 by 1 matrix). The determinant of any 1 by 1 matrix is given by its only coefficient.

This matrix B is called the inverse of A and is denoted A^{-1} . As we have commutative matrices, $AB = BA$, recall that this can only happen if A is square. So, only square matrices can have inverses.

$$AB = BA = I$$

Definition 2.15 (Invertible matrix). A matrix A is invertible if and only if there exists a matrix B such that

$$(I)_{ij} = \begin{cases} 1 & \text{whenever } i = j, \\ 0 & \text{whenever } i \neq j. \end{cases}$$

elsewhere, that is,

Definition 2.14 (Identity matrix). The n -dimensional identity matrix I is a square matrix of size $n \times n$ with 1s along the diagonal and 0s

Definition 2.18 (Determinant of an $n \times n$ matrix).

For any square matrix $A \in \mathcal{M}_{n,n}$, its determinant is given by

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

where the a_{ij} are coefficients of A , A_{ij} is the i, j^{th} submatrix of A and for any $1 \leq j \leq n$. We can also sum over the j index for any $1 \leq i \leq n$

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

and we will show that the answer is the same.

Definition 2.19 (Cramer system).

Suppose we have the following linear system of equations (with unknowns equal to equations)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases} \quad (S)$$

with the associated matrix form

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_X = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_Y$$

We say that (S) is a Cramer system if $\det(A) \neq 0$.

Definition 2.20 (Cofactor matrix).

From a matrix A we generate its cofactor matrix C_A which has entries given by determinants of submatrices of A with the same plus/minus pattern as in a determinant calculation. That is, the entries of C_A are $c_{ij} = (-1)^{i+j} \det(A_{ij})$:

$$C_A = \begin{pmatrix} |A_{11}| & -|A_{12}| & |A_{13}| & \cdots \\ -|A_{21}| & |A_{22}| & -|A_{23}| & \cdots \\ |A_{31}| & -|A_{32}| & |A_{33}| & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Chapter 3: Linear Systems

Definition 3.1 (Linear system of equations).

A system of m linear equations with n unknowns, denoted (S), has the general form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m \end{cases} \quad (S)$$

where the x_j are the unknowns we want to find, a_{ij} are the coefficients and the y_i are the constant terms.

Definition 3.2 (Homogeneous linear system).

For any system of linear equations, (S), given by $AX = Y$, we associate the **homogeneous system**, denoted (H):

$$AX = 0_m$$

for the column

$$0_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{m,1}$$

We will denote the solution set of (H) by \mathcal{H} .

Note: the homogeneous system always admits at least one solution, the trivial solution $X = 0_n$.

Definition 3.3 (Equivalent systems).

Two systems of linear equations are **equivalent** if they share the same set of solutions.

Definition 3.4 (Elementary operations).

There are **elementary operations** that we can do to systems of equations that give new systems that remain equivalent to the old.

$$(S_1) \begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \end{cases} \xrightarrow{\text{Exchanging two equations}} \equiv (S_2) \begin{cases} 13I_1 - 6I_2 = 20 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$