École d'ingénieurs du numérique

Linear Algebra

Matrix Algebra

Andrew Lehmann

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Basic Concepts

Definition - Matrix

A matrix is a collection of numbers usually represented by a rectangular array. For example, an $m \times n$ (said m by n) matrix A with coefficients, a_{ij}, from a field $\mathbb{F}$ (e.g. rational numbers) would be represented by an array with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{12} & a_{13} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

By convention we denote matrices with capital letters and their coefficients in lowercase. We might sometimes use $a_{ij} = (A)_{ij}$ when it's clear.

Definition - Set of all $n \times m$ matrices

We write the **set of all $m \times n$ matrices** with coefficients in $\mathbb{F}$ as

$$\mathcal{M}_{m,n}(\mathbb{F})$$

Reminder of fields:

Natural numbers	$\mathbb{N}$	$\{1, 2, 3, 4, \dots\}$ and sometimes 0
Integers	$\mathbb{Z}$	$\{\dots, -3, -2, 0, 1, 2, 3, \dots\}$
Rational numbers	$\mathbb{Q}$	$\{\dots, -1/2, -1/3, 0, 1, 2, 5/2, 11/2, \dots\}$
Real numbers	$\mathbb{R}$	$\{\dots, -1, -1/2, 0, 1, 22/3, \pi, e, \sqrt{2}, \dots\}$
Complex numbers	$\mathbb{C}$	$\{\dots, 1+i, i, 0, \pi, e, \sqrt{2}, -1/2, \dots\}$

For the rest of this course we will almost exclusively use real numbers, and write $\mathcal{M}_{m,n} = \mathcal{M}_{m,n}(\mathbb{R})$.

Definition - Columns of a matrix

We denote the **columns** of a matrix $A \in \mathcal{M}_{m,n}$ as $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, etc where the j$^{\text{th}}$ column is

$$A^{(j)} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Definition - Rows of a matrix

We denote the **rows** of a matrix $A \in \mathcal{M}_{m,n}$ as $A_{(1)}$, $A_{(2)}$, $A_{(3)}$, etc where the i$^{\text{th}}$ row is

$$A_{(i)} = \begin{pmatrix} a_{i1} & a_{12} & a_{13} & \cdots & a_{ij} & \cdots & a_{in} \end{pmatrix}$$

Example matrices

Triangular matrices

For square matrices, an *upper* triangular matrix only has elements in the upper triangle, e.g.

$$\begin{pmatrix} 1.5 & 3 & \pi \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

while a *lower* triangular matrix only has elements in the lower triangle, e.g.

$$\begin{pmatrix} 1.5 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0.5 & 3 & 1 & 2 \end{pmatrix}$$

Canonical matrices

Canonical matrices have a single 1 amongst zeros, e.g. canonical columns

$$E^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad E^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots E^{(n)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Or we can write this compactly (but maybe harder to understand) $\left(E^{(j)}\right)_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

We could similarly define canonical rows.

Definition - Transpose

The **transpose** of a matrix $A \in \mathcal{M}_{m,n}$ is another matrix $B \in \mathcal{M}_{m,n}$ with coefficients defined by

$$b_{ij} = a_{ji} \quad \text{for} \quad 1 \leq i \leq m,\, 1 \leq j \leq n$$

In essence, the rows of B are the columns of A. Similarly, the columns of B are the rows of A. We denote the transpose $B = A^T$. Visually:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{12} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{i1} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{i2} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1j} & a_{2j} & \cdots & a_{ij} & \cdots & a_{mj} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{in} & \cdots & a_{mn} \end{pmatrix}$$

Examples - Transpose

Definition - Symmetric Matrices

We call a *square* matrix $A \in \mathcal{M}_{n,n}$ **symmetric** if it is equal to its transpose. That is, if

$$a_{ij} = a_{ji} \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq n$$

Examples - Symmetric matrices

Algebra of matrices

Definition - Matrix addition

Given two matrices *of the same shape*, $A, B \in \mathcal{M}_{m,n}$, we define the **addition**, $A + B$, to be a third matrix $C \in \mathcal{M}_{m,n}$ with coefficients given by:

$$c_{ij} = a_{ij} + b_{ij},$$

that is, we add matrices *coefficient by coefficient*.

For example:

$$\begin{pmatrix} 1 & 3 \\ 7 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Properties - Matrix addition

Remember: to add two matrices they must be the same shape!

Given three matrices $A, B, C \in \mathcal{M}_{m,n}$:

- matrix addition is associative: $A + (B + C) = (A + B) + C$,
- matrix addition is commutative: $A + B = B + A$.

Definition - Scalar multiplication

Given a number $k \in \mathbb{R}$ (called a scalar) and a matrix $A \in \mathcal{M}_{m,n}$, we define the **product**, kA, to be a matrix $B \in \mathcal{M}_{m,n}$ with coefficients given by:

$$b_{ij} = ka_{ij},$$

that is, we multiply *every coefficient* by the scalar.

For example:

$$k \begin{pmatrix} 1 & 3 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Properties - Scalar multiplication

For any $\alpha, \beta \in \mathbb{R}$ and matrices $A, B \in \mathcal{M}_{m,n}$ we have:

- Scalar multiplication is distributive:

$$\alpha(A + B) = \alpha A + \alpha B$$
$$(\alpha + \beta)A = \alpha A + \beta A$$

- Scalar multiplication is associative:

$$\alpha(\beta A) = (\alpha\beta)A$$

Definition - Multiplication of matrix by column

Consider a matrix $A \in \mathcal{M}_{m,n}$ and a column $X \in \mathcal{M}_{m,1}$. We define the product $Y = AX$ to be the column in $\mathcal{M}_{m,1}$ with coefficients

$$y_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \sum_{k=1}^{n} a_{ik}x_k$$

Visually

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_m \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$\implies Y = x_1 A^{(1)} + x_2 A^{(2)} + \cdots + x_n A^{(n)}$$

Examples - Multiplication of matrix by column - Method 1

Examples - Multiplication of matrix by column - Method 2

Properties

To be able to multiply a matrix by a column, the matrix must have the same number of columns as the elements of the column.

$$\underbrace{A}_{(\boxed{m},\boxed{n})}\ \underbrace{X}_{(\boxed{n},1)} = \underbrace{Y}_{(\boxed{m},1)}$$

For any $k \in \mathbb{R}$, matrices $A, B \in \mathcal{M}_{m,n}$ and columns $X, X' \in \mathcal{M}_{n,1}$ we have the following:

Distributivity: $\begin{aligned} \alpha(A+B)X &= AX + BX \\ A(X+X') &= AX + AX' \end{aligned}$

Associativity: $k(AX) = (kA)X$

Exercise

Let $\quad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Calculate AX, AY and $AX + AY$.

Let $\quad A = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 0 \\ 2 & 8 \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Calculate AX, BX and $AX + BX$.

Multiplication of two matrices

Suppose we have two matrices A and B. We would like to define the matrix multiplication $AB = C$ so that the following associative law holds:

$$CX = (AB)X = A(BX)$$

for a column X of an appropriate size.

Let $B \in \mathcal{M}_{m,n}$. This forces the size of X: $X \in \mathcal{M}_{n,1}$ and we have a new vector $Y \in \mathcal{M}_{m,1}$.
Matrix A must have the same number of columns as the elements of the column it multiplies, so let $A \in \mathcal{M}_{q,m}$ which gives a new vector $Z \in \mathcal{M}_{q,1}$.
Finally we must have $CX = Z$. This forces $C \in \mathcal{M}_{q,n}$.

$$\underbrace{B}_{(m,n)}\underbrace{X}_{(n,1)} = \underbrace{Y}_{(m,1)}$$

$$\underbrace{A}_{(q,m)}\underbrace{Y}_{(m,1)} = \underbrace{Z}_{(q,1)}$$

$$\underbrace{C}_{(q,n)}\underbrace{X}_{(n,1)} = \underbrace{Z}_{(q,1)}$$

Multiplication of two matrices

So, *in order that the multiplication is well defined*, the number of columns of the left matrix must match the number rows of the right matrix.That is:

$$\underbrace{(q, m)}_{A}\ \underbrace{(m, n)}_{B} = \underbrace{(q, n)}_{C}$$

I say "the *inner* indices of A and B must match and their *outer* indices gives the shape of the result". For example

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 10 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{well defined}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{not defined}$$

Definition - Multiplication of two matrices

Consider two matrices $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{n,q}$. We define the product AB to be the matrix $C \in \mathcal{M}_{m,q}$ with coefficients

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$\Longrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix}$$

$$= \left(b_{11} \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + b_{n1} \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \quad \cdots \quad b_{1q} \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + b_{nq} \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right)$$

Example - Method 1

Example - Method 2

Example - Method 3

Properties - Matrix multiplication

Given matrices $A, A' \in \mathcal{M}_{m,n}$, $B, B' \in \mathcal{M}_{n,q}$ and constant $k \in \mathbb{R}$ we have the following properties

- $A(B + B') = AB + AB'$ and $(A + A')B = AB + A'B$
- $A(kB) = k(AB) = (kA)B$
- $(AB)^T = B^T A^T$

Definition - Identity matrix

The identity matrix, I_k, is a square matrix satisfying: $I_n A = A = A I_m$ for any $A \in \mathcal{M}_{m,n}$. I_k is a $k \times k$ square matrix with 1s on the diagonal and 0s everywhere else:

$$(I_k)_{ij} = \begin{cases} 1, & \text{when } i = j \\ 0, & \text{when } i \neq j. \end{cases}$$

For example

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Properties

The multiplication of two matrices can give the null matrix, even if both matrices are non-null. For example

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Properties

Given matrices $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{n,q}$:

- The product $C = AB$ is a matrix of size (m, q).
- If we want to define the product $C' = BA$, what must be true?
  - C' must be a square matrix of size (n, n)
  - C must be a square matrix of size $(m, m) = (q, q)$
  - If $n \neq m$, then C and C' have different sizes, and we say that C and C' are not compatible.
  - If $n = m$, the products AB and BA are the same size, but are not necessarily the same element by element. Matrix multiplication *is not generally commutative*.

Example - Non-commutative matrix multiplication

Multiplication by diagonal matrices

Given two square matrices $A, \Lambda \in \mathcal{M}_{m,m}$ where Λ is diagonal:

$$\Lambda A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} = \begin{pmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \cdots & \lambda_1 a_{1m} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \cdots & \lambda_2 a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_m a_{m1} & \lambda_m a_{m2} & \cdots & \lambda_m a_{mm} \end{pmatrix}$$

The rows of A are multiplied, respectively, by $\lambda_1, \lambda_2, \ldots, \lambda_m$.

$$A\Lambda = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} = \begin{pmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} & \cdots & \lambda_m a_{1m} \\ \lambda_1 a_{21} & \lambda_2 a_{22} & \cdots & \lambda_m a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 a_{m1} & \lambda_2 a_{m2} & \cdots & \lambda_m a_{mm} \end{pmatrix}$$

The columns of A are multiplied, respectively, by $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Definition - Invertible matrix

Consider a square matrix $A \in \mathcal{M}_{n,n}$. We say that A is **invertible** if there exists a matrix, denoted A^{-1}, which satisfies the following

$$A^{-1}A = I_n = AA^{-1}$$

Note:

- If one of the equalities can be shown, the other follows automatically.
- If a matrix is invertible, the inverse is unique.

We will develop methods to find inverses of matrices in future lessons.