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Introduction to Numerical Analysis: Exercise Booklet

Chapter 1

Introduction: square root methods

Exercise 1: Calculation of $\sqrt{7}$

a) Define the function $f(x)$ so that the solution of $f(x) = 0$ is $x = \sqrt{7}$. Is there a root of this equation in the interval $[1, 2]$? What about $[2, 3]$?

b) Use 3 iterations of the bisection method, starting with the interval $[2, 3]$, to estimate the value of $x = \sqrt{7}$.

c) How many iterations of the bisection method are required to achieve an accuracy better than 10^{-5}?

d) Use 3 iterations of Heron's method, starting at $x_0 = 3$, to estimate the value of $x = \sqrt{7}$.

Exercise 2: Calculation of $\sqrt{5}$

a) Give the terms of the Taylor expansion of $(1 + x)^p$ for $x = 4$ and $p = 1/2$ up to the third derivative. Will this series converge on $\sqrt{5}$?

b) Use 3 iterations of Heron's method, starting at $x_0 = 2$, to estimate the value of $\sqrt{5}$.

Exercise 3: Algorithm analysis

a) For Theon of Smyrna's method, by considering $p_{k+1}^2 - 2q_{k+1}^2$ and $p_0 = q_0 = 1$, prove this method converges to $x_0 = 2$.

b) Prove that Heron's method converges for any square root.

Chapter 2

Rootfinding

Exercise 4: Finding the intersection of $2\sin x = x$

a) Sketch a graph of $y = 2\sin x$ and $y = x$ in the domain $[-2\pi, 2\pi]$.

b) Write a function $f(x)$ that has roots corresponding to the solutions of $2\sin x = x$.

c) Use the Intermediate Value Theorem and the graph to find *the number* of roots of $f(x)$, and give intervals surrounding them.

d) Use the bisection algorithm to solve $2\sin x = x$ for as many iterations as needed until the solution stops changing its first 4 decimal places. Do this for each solution of $2\sin x = x$ that you found in part c.

e) Define a function $g(x)$ so that we have an iterative scheme: $x_{k+1} = g(x_k)$ with fixed points at the roots of $f(x)$.

f) Make a rough iteration figure exploring initial guesses x_0 and their subsequent iterates using the iteration scheme of part e. Are there any unstable fixed-points? What value does x_k converge to for initial x_0 where $\sin(x_0) > 0$? What about initial x_0 where $\sin(x_0) < 0$?

Exercise 5: Finding the intersection of $\sin x = \cos x$

a) Sketch a graph of $y = \sin x$ and $y = \cos x$ in the domain $[0, 2\pi]$.

b) Write a function $f(x)$ that has roots corresponding to the solutions of $\sin x = \cos x$.

c) Use the Intermediate Value Theorem and the graph to find *the number* of roots of $f(x)$, and give intervals surrounding them.

d) Use the bisection algorithm to solve $\sin x = x$ for as many iterations as needed until the solution stops changing its first 4 decimal places. Do this for each solution of $\sin x = \cos x$ that you found in part c.

Exercise 6: Fixed-point analysis $g(x) = x(x^2 - 1)$

a) How many fixed points, ξ_k, of $g(x)$ are there?

b) Sketch a graph of $y = g(x)$ and $y = x$. Make rough iteration figures at different initial guesses x_0 on each side of the fixed points. Which fixed point(s) do you expect to be stable?

c) Use the stability criterion on $|g(\xi_k)|$ to make a stability analysis of the fixed points.

Exercise 7: Fixed-point analysis $f(x) = 2e^{-x} + x - 2$

a) Find 2 functions $g(x)$ for which the $x = g(x)$ has the same solutions as zeros of $f(x)$.

b) Sketch separate graphs of the previous functions, as well as $y = 2e^x$ and $y = 2 - x$ to get an intuition on the location of the roots or fixed points.

c) Can you use Brouwer's theorem to guarantee the existence of any fixed points for either of the functions defined in part a?

d) Make a stability analysis of the fixed points for both functions.

e) For any stable fixed points, use the iterative scheme $x_{k+1} = g(x_k)$ for 5 iterations.

Exercise 8: The intersection of $e^x = \cos x + 1$

a) Sketch a graph of $y = e^x$ and $y = \cos x + 1$ in the domain $[-4\pi, 4\pi]$.

b) Write a function $f(x)$ that has roots corresponding to the solutions of $e^x = \cos x+1$.

c) From the graph, how many positive roots and how many negative roots of $f(x)$ will there be?

d) Use the Intermediate Value Theorem and the graph to prove there is a root of $f(x)$, for $x > 0$.

e) Use Netwon's method to propose an iterative scheme to find the roots of $f(x)$. Use this iterative scheme starting with $x_0 = 0$ to make a table of estimates x_k for $k = 0, 1, \ldots, 4$.

f) Show that the function $g(x) = \log(\cos x + 1)$ gives an iterative scheme $x_{k+1} = g(x_k)$ with fixed points equal to the roots of $f(x)$. Determine a second function $h(x)$ which also satisfies these properties.

g) Sketch a graph of $y = g(x)$ and $y = x$ in the domain $[-\pi, \pi]$.

h) Make a rough iteration figure exploring initial guesses x0 and their subsequent iterates using the iteration scheme with $g(x)$ of part f. Which fixed point(s) do you expect to be stable?

i) Use the stability criterion on $|g(\xi_k)|$ to make a stability analysis of the fixed points $\xi_k \in [-\pi, \pi]$.

Exercise 9: Fixed-point analysis $\log(x + 2) = x^2$

a) Sketch a graph of $y = \log(x + 2)$ and $y = x^2$.

b) Write a function $f(x)$ that has roots corresponding to the solutions of $\log(x+2) = x^2$.

c) From the graph, how many roots of $f(x)$ will there be?

d) Use the Intermediate Value Theorem and the graph to locate the roots of $f(x)$.

e) Use the Secant method to propose an iterative scheme to find the roots of $f(x)$. Use this iterative scheme twice, starting with $x_0 = 1$ and $x_1 = 2$ and then again with $x'_0 = -1$ and $x_1 = -1.5$ to make a table of estimates of x_k and x'_k until $k = 4$.

f) Show that the functions $g_1(x) = \exp(x^2) - 2$ and $g_2(x) = (\log(x + 2))1/2$ give iterative schemes $x_{k+1} = g_i(x_k)$ with fixed points equal to the roots of $f(x)$.

g) Sketch a graph of $y = g_1(x)$ and $y = x$. Sketch another graph of $y = g_2(x)$ and $y = x$. Can you use Brouwer's theorem to guarantee the existance of any fixed points for either $g_1(x)$ or $g_2(x)$?

h) Make a rough iteration figure exploring initial guesses x_0 and their subsequent iterates using the iteration scheme with $g(x)$ of part f. Which fixed point(s) do you expect to be stable?

i) Use the stability criterion on $|g_1(\xi_k)|$ and $|g_2(\xi_k)|$ to make a stability analysis of the fixed points ξ_k. For a stable fixed point of your choice, use the iterative scheme $x_{k+1} = g_i(x_k)$ to estimate the location of an intersection of $y = \log(x+2)$and $y = x_2$.

Chapter 3

Polynomial interpolation

Exercise 10: Given data $\{(1,2),(2,5),(3,3),(4,2)\}$

a) Make a piece-wise linear interpolation of the given data.

b) Find the unique polynomial of order less than 4 passing through each data point using the Lagrangian interpolation method.

c) Find the unique polynomial of order less than 4 passing through each data point using the Newtonian interpolation method.

Exercise 11: Given data $\{(1,2),(1.5,1.5),(3,2.5),(6,3)\}$

a) Make a piece-wise linear interpolation of the given data.

b) Find the unique polynomial of order less than 4 passing through each data point using the Lagrangian interpolation method.

c) Find the unique polynomial of order less than 4 passing through each data point using the Newtonian interpolation method.

Exercise 12: Estimate $f(x) = \log(x+2)$

a) Make a piece-wise linear interpolation of $f(x)$ at positions $x \in [0,2,4,6]$.

b) Give the Lagrangian polynomial of order less than 4 passing through $f(x)$ at positions $x \in [0,2,4,6]$.

c) Use Newtonian interpolation to give the polynomial of order less than 4 passing through $f(x)$ at positions $x \in [0,1,5,6]$.

d) Find the area under $f(x)$ between 0 and 6 by using the polynomials found in parts b and c. Compare to the analytic solution by directly integrating $f(x)$.

e) Using your table of divided differences from part c, add 1 data point at $x = 3$ to increase the polynomial order by 1.

Exercise 13: Estimate $f(x) = e^x$

a) Make a piece-wise linear interpolation of $f(x)$ at positions $x \in [-1,1,3,5]$.

b) Give the Lagrangian polynomial of order less than 4 passing through $f(x)$ at positions $x \in [-1,0,4,5]$.

c) Use Newtonian interpolation to give the polynomial of order less than 4 passing through $f(x)$ at positions $x \in [-1, 1, 3, 5]$.

d) Find the area under $f(x)$ between 0 and 6 by using the polynomials found in parts b and c. Compare to the analytic solution by directly integrating $f(x)$.

e) Using your table of divided differences from part c, add 1 data point at $x = 2$ to increase the polynomial order by 1.

Chapter 4

Least-squares

Exercise 14: Given data $\{(1,5),(3,6),(5,8),(7,9)\}$
Make a least-squares linear fit to the given data (generated by $y = 0.7x + 4$).
Exercise 15: Given data $\{(1,7),(3,10),(4,12),(5,14),(7,18)\}$
Make a least-squares linear fit to the given data (generated by $y = 2.2x + 3$).
Exercise 16: Given data $\{(1,5),(2,11),(4,33),(5,59)\}$
Make a least-squares exponential fit to the given data (generated by $y = 3e^{0.6x}$).
Exercise 17: Given data $\{(1,1),(3,2),(5,5),(7,9),(8,12)\}$
Make a least-squares exponential fit to the given data (generated by $y = e^{0.3x}$).
Exercise 18: Given data $\{(1,1),(3,3),(5,8),(7,15)\}$
Make a least-squares quadratic fit to the given data (generated by $y = 3x^2$).
Exercise 19: Given data $\{(1,10),(2,40),(3,89),(4,159),(6,359)\}$
Make a least-squares quadratic fit to the given data (generated by y=10x^2).

Chapter 5

Integration

Exercise 20: Given the function $f(x) = 2\log(x+1)$

a) Use the Trapezium rule to estimate the integral of $f(x)$ from [0,6] using 2 subdivisions.

b) Use the Trapezium rule to estimate the integral of $f(x)$ from [0,6] using 3 subdivisions.

c) Use Simpson's rule to estimate the integral of $f(x)$ from [0,6] using 2 subdivisions.

d) Use Simpson's rule to estimate the integral of $f(x)$ from [0,6] using 3 subdivisions.

e) Integrate $f(x)$ analytically to compare to the previous results.

Exercise 21: Given the function $f(x) = 3e^{0.5x} - 3$

a) Use the Trapezium rule to estimate the integral of $f(x)$ from [0,6] using 2 subdivisions.

b) Use the Trapezium rule to estimate the integral of $f(x)$ from [0,6] using 3 subdivisions.

c) Use Simpson's rule to estimate the integral of $f(x)$ from [0,6] using 2 subdivisions.

d) Use Simpson's rule to estimate the integral of $f(x)$ from [0,6] using 3 subdivisions.

e) Integrate $f(x)$ analytically to compare to the previous results.

Exercise 22: Given the function $f(x) = e^{\cos x}$

a) Use the Trapezium rule to estimate the integral of $f(x)$ from $[0, 2\pi]$ using 2 subdivisions.

b) Use the Trapezium rule to estimate the integral of $f(x)$ from $[0, 2\pi]$ using 3 subdivisions.

c) Use Simpson's rule to estimate the integral of $f(x)$ from $[0, 2\pi]$ using 2 subdivisions.

d) Use Simpson's rule to estimate the integral of $f(x)$ from $[0, 2\pi]$ using 3 subdivisions.

Note: $f(x)$ cannot be integrated analytically (into elementary functions) to compare to the previous results.

Chapter 6

Differentiation

Exercise 23: Given the data

x	2.0000	2.0025	2.0050	2.0075	2.0100
$f(x)$	1.202604	1.214698	1.226929	1.239299	1.251809

a) Use the forward finite difference scheme with h=0.005 to estimate $f'(2.005)$.

b) Use the forward finite difference scheme with h=0.0025 to estimate $f'(2.005)$.

c) Use the backward finite difference scheme with h=0.005 to estimate $f'(2.005)$.

d) Use the backward finite difference scheme with h=0.0025 to estimate $f'(2.005)$.

e) Use the centered finite difference scheme with h=0.005 to estimate $f'(2.005)$.

f) Use the centered finite difference scheme with h=0.0025 to estimate $f'(2.005)$.

g) The data were generated with $f(x) = \exp(x^2 + 10)/1000000$. What is the true derivative at $x = 2.005$?

Exercise 24: Given the data

x	7.5000	7.5025	7.5050	7.5075	7.5100
$f(x)$	384.375	384.785	385.194	385.604	386.015

a) Use the forward finite difference scheme with h=0.005 to estimate $f'(7.505)$.

b) Use the forward finite difference scheme with h=0.0025 to estimate $f'(7.505)$.

c) Use the backward finite difference scheme with h=0.005 to estimate $f'(7.505)$.

d) Use the backward finite difference scheme with h=0.0025 to estimate $f'(7.505)$.

e) Use the centered finite difference scheme with h=0.005 to estimate $f'(7.505)$.

f) Use the centered finite difference scheme with h=0.0025 to estimate $f'(7.505)$.

g) The data were generated with $f(x) = x^3 - 5x$. What is the true derivative at $x = 7.505$?

x	1.560	1.561	1.562	1.563	1.564
$f(x)$	92.6204	102.076	113.681	128.263	147.136

Exercise 25: Given the data

a) Use the forward finite difference scheme with h=0.002 to estimate $f'(1.562)$.

b) Use the forward finite difference scheme with h=0.001 to estimate $f'(1.562)$.

c) Use the backward finite difference scheme with h=0.002 to estimate $f'(1.562)$.

d) Use the backward finite difference scheme with h=0.001 to estimate $f'(1.562)$.

e) Use the centered finite difference scheme with h=0.002 to estimate $f'(1.562)$.

f) Use the centered finite difference scheme with h=0.001 to estimate $f'(1.562)$.

g) The data were generated with $f(x) = \tan x$. What is the true derivative at $x = 7.505$?

Chapter 7

Differential equations

Exercise 26: Consider $y'(x) = -2y$

a) Write the iteration scheme given by Euler's method.

b) Write the iteration scheme given by the implicit Trapezium method. Define a function $F(z)$ so that the root of the equation $F(z) = 0$ is equal to the next iteration of the solution in your iteration scheme.

c) From the Trapezium method, propose an explicit iteration scheme to solve the differential equation.

d) Write the iteration scheme given by the second order Runge-Kutta method.

e) Write the iteration scheme given by the fourth order Runge-Kutta method.

f) What is the analytic solution of the differential equation?

Exercise 27: Consider $y'(x) = xy$

a) Write the iteration scheme given by Euler's method.

b) Write the iteration scheme given by the implicit Trapezium method. Define a function $F(z)$ so that the root of the equation $F(z) = 0$ is equal to the next iteration of the solution in your iteration scheme.

c) From the Trapezium method, propose an explicit iteration scheme to solve the differential equation.

d) Write the iteration scheme given by the second order Runge-Kutta method.

e) Write the iteration scheme given by the fourth order Runge-Kutta method.

f) What is the analytic solution of the differential equation?

Exercise 28: Consider $y'(x) = -2y^2$

Keep 4 decimal places throughout your calculations in the following questions.

a) Estimate $y(0.1)$ using Euler's method, with $y(0) = 2.0$ and $h = 0.1$.

b) Estimate $y(0.1)$ using Euler's method, with $y(0) = 2.0$ and $h = 0.05$.

c) Estimate $y(0.1)$ using the implicit Trapezium method, with $y(0) = 2.0$ and $h = 0.1$. Use Newton's method to find the root to 4 decimal places.

d) Estimate $y(0.1)$ using the implicit Trapezium method, with $y(0) = 2.0$ and $h = 0.05$. Use Newton's method to find the root to 4 decimal places.

e) From the Trapezium method, propose an explicit method for answering the previous 2 quesitons.

f) Estimate $y(0.1)$ using the second order Runge-Kutta method, with $y(0) = 2.0$ and $h = 0.1$.

g) Estimate $y(0.1)$ using the second order Runge-Kutta method, with $y(0) = 2.0$ and $h = 0.05$.

h) Estimate $y(0.1)$ using the fourth order Runge-Kutta method, with $y(0) = 2.0$ and $h = 0.1$.

i) Estimate $y(0.1)$ using the fourth order Runge-Kutta method, with $y(0) = 2.0$ and $h = 0.05$.

j) Compare your answers to the analytic solution.

Exercise 29: Consider $y'(x) = y - 1$

Keep 4 decimal places throughout your calculations in the following questions.

a) Estimate $y(0.1)$ using Euler's method, with $y(0) = 1.1$ and $h = 0.05$.

b) Estimate $y(0.1)$ using Euler's method, with $y(0) = 0.9$ and $h = 0.05$.

c) Estimate $y(0.1)$ using the implicit Trapezium method, with $y(0) = 1.1$ and $h = 0.05$.

d) Estimate $y(0.1)$ using the implicit Trapezium method, with $y(0) = 0.9$ and $h = 0.05$.

e) From the Trapezium method, propose an explicit method for answering the previous 2 quesitons.

f) Estimate $y(0.1)$ using the second order Runge-Kutta method, with $y(0) = 1.1$ and $h = 0.05$.

g) Estimate $y(0.1)$ using the fourth order Runge-Kutta method, with $y(0) = 1.1$ and $h = 0.05$.

h) Estimate $y(0.1)$ using the second order Runge-Kutta method, with $y(0) = 0.9$ and $h = 0.05$.

i) Estimate $y(0.1)$ using the fourth order Runge-Kutta method, with $y(0) = 0.9$ and $h = 0.05$.

j) Compare your answers to the analytic solution.

Chapter 8

Matrix methods

Exercise 30: 2×2 matrices

a) For the following matrices, find L and U such that $LA_i = U$

$$A_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$$

b) For the previous matrices A_i solve for x_1 and x_2, given

$$A_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

c) For the previous matrices A_i solve for x_1 and x_2, given

$$A_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Exercise 31: 3×3 matrices

a) For the following matrices, find L and U such that $LA_i = U$

$$A_1 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & 3 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 2 & -4 \\ -2 & 2 & 1 \\ -5 & -3 & 3 \end{pmatrix}$$

b) For the previous matrices A_i solve for x_1 and x_2, given

$$A_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

c) For the previous matrices A_i solve for x_1 and x_2, given

$$A_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Exercise 32: Convergence

a) For the following system, give a range of values that k can take to guarantee the Jacobi iteration scheme converges on the true solution.

$$\begin{pmatrix} k & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

b) For the following system, give ranges of values that α, β, and γ can take to guarantee the Jacobi iteration scheme converges on the true solution?

$$\begin{pmatrix} 3 & -1 & \alpha \\ \beta & -3 & 2 \\ 1 & 1 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

Exercise 33: Schemes

For the following matrices, write the Jacobi iterative system of equations.

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 & -1 & 1 \\ 0 & -3 & 2 \\ 1 & 1 & -4 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 5 & -2 \\ 1 & -1 & -4 \end{pmatrix}$$

Exercise 34: Jacobi iteration

For the following systems, use Jacobi iteration 3 times to find $X^{(3)}$.

a)

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \quad \text{with } X^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

b)

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{with } X^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

c)

$$\begin{pmatrix} 3 & -1 & 1 \\ 0 & -3 & 2 \\ 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad \text{with } X^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

d)

$$\begin{pmatrix} 3 & -1 & 1 \\ 0 & -3 & 2 \\ 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \quad \text{with } X^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$