Linear Algebra 1 Dictionary

by Andrew Lehmann
Last updated: June 21, 2024
Typeset in $\LaTeX\,2_\varepsilon$.

List of Definitions

Chapter 1: Euclidean Vectors

Definition 1.1 (Euclidean vector or tuple).
A Euclidean vector is a list of n real numbers, also called an n-tuple. We write this list in parentheses, for example $(1, 3, -2, \dots, 0)$, and we say that this object belongs to $\mathbb{R}^n$. An arbitrary tuple can be written $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ where the components $v_i \in \mathbb{R}$ for any index i.

Definition 1.2 (Tuple addition).
Euclidean vectors are added to each other component by component. In symbols

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

Note: this means you can only add two tuples together of the same size. It makes no sense to add a 3-tuple to a 5-tuple.

Definition 1.3 (Scalar multiplication).
Let $c \in \mathbb{R}$, called a scalar quantity, and $\mathbf{v} \in \mathbb{R}^n$ with components v_i. Then the scalar multiplication $c\mathbf{v}$ gives a vector $\mathbf{w}$ with components $w_i = cv_i$ for every index i. In tuple form

$$c(v_1, v_2, \dots, v_n) = (cv_1, cv_2, \dots, cv_n).$$

Definition 1.4 (Canonical Euclidean unit vectors).
The canonical Euclidean vectors in $\mathbb{R}^n$ are the n vectors of the form

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \dots, 0) \\ \mathbf{e}_2 &= (0, 1, \dots, 0) \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \dots, 1). \end{aligned}$$

More compactly

$$\mathbf{e}_k = (\alpha_1, \alpha_2, \dots, \alpha_n) \quad \textit{where} \quad \alpha_j = \begin{cases} 1 & \textit{for } j = k, \\ 0 & \textit{for } j \neq k. \end{cases}$$

Definition 1.5 (Dot product).
For two n-tuples $\mathbf{a}$ and $\mathbf{b}$, their dot product, also called scalar product and Euclidean inner product, is the real number given by the addition of component by component multiplication

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

Definition 1.6 (Euclidean Norm).
The norm of an n-tuple $\mathbf{v}$, denoted $\|\mathbf{v}\|$, is given by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_2^2}.$$

Definition 1.7 (Orthogonal Euclidean vectors).
Two vectors in $\mathbb{R}^n$ are orthogonal if and only if their dot product equals zero.

Definition 1.8 (Displacement vector).
Given two Euclidean vectors $\mathbf{a}$ and $\mathbf{b}$, the displacement vector pointing from $\mathbf{a}$ to $\mathbf{b}$ is given by $\mathbf{r} = \mathbf{b} - \mathbf{a}$ as pictured below. Of course we can also create the displacement vector in the other direction, from $\mathbf{b}$ to $\mathbf{a}$, given by $\mathbf{a} - \mathbf{b}$.

Definition 1.9 (Vector form of a straight line).
The set of vectors in $\mathbb{R}^n$ of the form $\mathbf{v} = \mathbf{a} + t\mathbf{r}$ for a parameter $t \in \mathbb{R}$ represents a straight line through the space $\mathbb{R}^n$. That is,

$$\{(x, y) \,|\, \forall x \in \mathbb{R} \; and \, y = mx + b\} = \{\mathbf{a} + t\mathbf{r} \,|\, \forall t \in \mathbb{R}\}$$

where $\mathbf{a}$ is an arbitrary pair $(x, mx+b)$ and $\mathbf{r}$ is a displacement vector between any two distinct pairs $(x_1, mx_1 + b)$ and $(x_2, mx_2 + b)$.

Chapter 2: Matrix Algebra

Definition 2.1 (Matrix)**.**
A matrix is a collection of numbers from a field $\mathbb{F}$ (e.g. rational numbers) usually represented by a rectangular array. For example, an $m \times n$ (said m by n) matrix A with coefficients $a_{ij} \in \mathbb{F}$ would be represented by an array with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{12} & a_{13} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}.$$

Sometimes it is convenient to refer to the coefficients in the array like so: $a_{ij} = (A)_{ij}$.

Definition 2.2 (Set of all $m \times n$ matrices)**.**
We write the set of all $m \times n$ matrices with coefficients in $\mathbb{F}$ as

$$\mathcal{M}_{m,n}(\mathbb{F})$$

Definition 2.3 (Matrix columns and rows)**.**
For a matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$ we denote its j^{th} column and i^{th} row

$$A^{(j)} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix}, \qquad A_{(i)} = \begin{pmatrix} a_{i1} & a_{12} & a_{13} & \cdots & a_{ij} & \cdots & a_{in} \end{pmatrix}$$

Definition 2.4 (Transpose of a matrix)**.**
The transpose of an $m \times n$ matrix, A, is an $n \times m$ matrix, denoted A^T, with rows equal to the columns of A. That is, $\left(A^T\right)_{ij} = (A)_{ji}$ for all combinations of i and j.

Definition 2.5 (Diagonal matrix).
A square matrix A is said to be diagonal if all its non-diagonal elements are zero, e.g. $(A)_{ij} = 0$ *whenever* $i \neq j$.

Definition 2.6 (Symmetric matrix).
A matrix A is symmetric if it is equal to its transpose, $A = A^T$.

Definition 2.7 (Matrix addition).
Matrix addition is done coefficient by coefficient, that is, for two matrices A and B we define the i,jth coefficient of the addition as the addition of the i,jth coefficients of each matrix:

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}\,.$$

Definition 2.8 (Scalar multiplication).
Given a number $k \in \mathbb{R}$ (called a scalar) and a matrix $A \in \mathcal{M}_{m,n}$, we define matrix scalar multiplication, kA, to be a matrix $B \in \mathcal{M}_{m,n}$ with coefficients given by:

$$b_{ij} = k a_{ij},$$

that is, we multiply every coefficient by the scalar.

Definition 2.9 (Zero matrix).
The zero matrix of any shape is a matrix $M_0 \in \mathcal{M}_{m,n}$ consisting entirely of zeros as coefficients.

Definition 2.10 (Additive inverse).
Given a matrix $A \in \mathcal{M}_{ij}$, its additive inverse is the same matrix multiplied by the scalar -1. We denote the additive inverse of A as $-A$.

Definition 2.11 (Multiplication of a matrix by a column).
Consider a matrix $A \in \mathcal{M}_{m,n}$ and a column $X \in \mathcal{M}_{n,1}$. We define the product AX to result in the column $Y \in \mathcal{M}_{m,1}$ with coefficients

$$(Y)_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m = \sum_{k=1}^{m} a_{ik}x_k$$

Visually

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_m \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

$$\implies Y = x_1 A^{(1)} + x_2 A^{(2)} + \cdots + x_m A^{(m)}$$

Definition 2.12 (Rotation matrix - arbtirary angle anti-clockwise)**.**
By using a column $X \in \mathcal{M}_{2,1}$ to represent a Euclidean vector, the following matrix allows the operation of rotataion, anti-clockwise, of X by an angle θ:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where the rotated vector is represented by a column $X' \in \mathcal{M}_{2,1}$ obtained by matrix multiplication $X' = R_\theta X$.

Definition 2.13 (Multiplication of two matrices)**.**
Consider two matrices $A \in \mathcal{M}_{n,m}$ and $B \in \mathcal{M}_{m,q}$. We define the product AB to be the matrix $C \in \mathcal{M}_{n,q}$ with coefficients

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

$$\implies \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mq} \end{pmatrix}$$

$$= \left(\underbrace{b_{11} \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + b_{m1} \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}}_{\textit{first column}} \quad \cdots \quad \underbrace{b_{1q} \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + b_{mq} \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}}_{n^{th} \textit{ column}} \right)$$

Additionally, for the product

$$\overbrace{(n, m)}^{A}\overbrace{(m, q)}^{B}$$

we will call the indices for the columns of A and rows of B the inner indices (blue), whereas the indices for the rows of A and columns of B will be called the outer indices (red).

Definition 2.14 (Identity matrix).
The n-dimensional identity matrix I is a square matrix of size $n \times n$ with 1s along the diagonal and 0s elsewhere, that is,

$$(I)_{ij} = \begin{cases} 1 & \textit{whenever } i = j, \\ 0 & \textit{whenever } i \neq j. \end{cases}$$

Definition 2.15 (Invertible matrix).
A matrix A is invertible if and only if there exists a matrix B such that

$$AB = BA = I$$

This matrix B is called the inverse of A and is denoted A^{-1}. As we have commutative matrices, $AB = BA$, recall that this can only happen if A is square. So, only square matrices can have inverses.

Definition 2.16 (Determinant of a 1 by 1 matrix).
The determinant of any 1 by 1 matrix is given by its only coefficient:

$$\det\big((a)\big) = a$$

Definition 2.17 (Submatrix).
From a matrix A we generate the submatrix A_{ij} by deleting the ith row and jth column:

$$\textit{For}\, A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & \cdots & a_{m,j-1} & a_{m,j} & a_{m,j+1} & \cdots & a_{m,n} \end{pmatrix}$$

$$\textit{The submatrix}\, A_{ij} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & \cdots & a_{m,j-1} & a_{m,j+1} & \cdots & a_{m,n} \end{pmatrix}$$

Note: we generally have to specify in words that we create a submatrix. The notation A_{ij} is a little ambiguous without being explicit.

Definition 2.18 (Determinant of an $n \times n$ matrix).
For any square matrix $A \in \mathcal{M}_{n,n}$, its determinant is given by

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

where the a_{ij} are coefficients of A, A_{ij} is the i, j^{th} submatrix of A and for any $1 \leq j \leq n$. We can also sum over the j index for any $1 \leq i \leq n$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

and we will show that the answer is the same.

Definition 2.19 (Cramer system).
Suppose we have the following linear system of equations (with unknowns equal to equations)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases} \qquad (S)$$

with the associated matrix form

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{Y}$$

We say that (S) is a Cramer system if $\det(A) \neq 0$.

Definition 2.20 (Cofactor matrix).
From a matrix A we generate its cofactor matrix C_A which has entries given by determinants of submatrices of A with the same plus/minus pattern as in a determinant calculation. That is, the entries of C_A are $c_{ij} = (-1)^{i+j} \det(A_{ij})$:

$$C_A = \begin{pmatrix} |A_{11}| & -|A_{12}| & |A_{13}| & \cdots \\ -|A_{21}| & |A_{22}| & -|A_{23}| & \cdots \\ |A_{31}| & -|A_{32}| & |A_{33}| & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Chapter 3: Linear Systems

Definition 3.1 (Linear system of equations).
A system of m linear equations with n unknowns, denoted (S), has the general form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m \end{cases} \qquad (S)$$

where the x_j are the unknowns we want to find, a_{ij} are the coefficients and the y_i are the constant terms.

Definition 3.2 (Homogeneous linear system).
For any system of linear equations, (S), given by $AX = Y$, we associate the ***homogeneous system****, denoted (H):*

$$AX = 0_m$$

for the column

$$0_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{m,1}$$

We will denote the solution set of (H) by $\mathcal{H}$.
Note: the homogeneous system always admits at least one solution, the trivial solution $X = 0_n$.

Definition 3.3 (Equivalent systems).
Two systems of linear equations are ***equivalent*** *if they share the same set of solutions.*

Definition 3.4 (Elementary operations).

There are ***elementary operations*** *that we can do to systems of equations that give new systems that remain equivalent to the old.*

Exchanging two equations

$$(S_1)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \end{cases} \qquad \equiv \qquad (S_2)\begin{cases} 13I_1 - 6I_2 = 20 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

Multiplying one equation by a non-zero constant

$$(S_1)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \end{cases} \quad \equiv \quad (S_2)\begin{cases} I_1 + I_2 - I_3 = 0 \\ I_1 - (6/13)I_2 = (20/13) \end{cases}$$

Adding a multiple of one equation to another equation

$$(S_1)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \end{cases} \quad \equiv \quad (S_2)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 15I_1 - 4I_2 - 2I_3 = 20 \end{cases}$$

Definition 3.5 (Overdetermined system)**.**
An overdetermined system has more equations than unknowns. We say "there are too many equations". Such a system allows solutions only if certain conditions are met.

Definition 3.6 (Underdetermined system)**.**

An ***underdetermined system*** *has less equations than unknowns. We say "there are not enough equations". Such a system has either no solutions, or infinitely many.*

Definition 3.7 (Cramer system)**.**
Suppose we have the following linear system of equations (with unknowns equal to equations)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases} \qquad (S)$$

with the associated matrix form

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{Y}$$

We say that (S) is a Cramer system if $\det(A) \neq 0$.

Chapter 4: Vector Spaces

Definition 4.1 (Vector space).
A vector space over a field $\mathbb{F}$ is a set, call it V, with elements called vectors supplied with definitions of two operations, vector addition (VA) and scalar multiplication (SM), that satisfy the following vector space axioms:

$$\forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \quad \textit{and} \quad \forall\, k, l \in \mathbb{F}$$

(VA1)	$\mathbf{u} + \mathbf{v} \in V$	(*closure under vector addition*)
(VA2)	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	(*associativity of vector addition*)
(VA3)	$\exists\, \mathbf{0} \in V$, *such that* $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	(*additive identity*)
(VA4)	$\exists\, -\mathbf{u} \in V$ *such that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	(*additive inverse*)
(VA5)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	(*commutativity of vector addition*)
(SM1)	$k\mathbf{u} \in V$	(*closure under scalar multiplication*)
(SM2)	$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$	(*distributivity over vector addition*)
(SM3)	$(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$	(*distributivity over field addition*)
(SM4)	$k(l\mathbf{u}) = (kl)\mathbf{u}$	(*compatibility of scalar and field multiplication*)
(SM5)	$1\mathbf{u} = \mathbf{u}$	(*multiplicative identity*)

Definition 4.2 (Linear Combination).
Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of vectors in a vector space V. A linear combination of these vectors is a new vector, $\mathbf{w} \in V$, of the form

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

where the α_k are real numbers.

Definition 4.3 (Vector subspace).
Suppose that V is a vector space and W is a subset of V. We call W a vector subspace if it satisfies the vector space axioms for the same definition of vector addition and scalar multiplication defined for V.

Definition 4.4 (Span).
Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of vectors from a vector space V. The span of these vectors is the set of all linear combinations of those vectors:

$$\mathit{SPAN}(\mathcal{B}) = \mathit{SPAN}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \,|\, \forall \alpha_1, \ldots, \alpha_n \in \mathbb{R}^n\}\,.$$

This set forms a vector subspace of V. It is obviously non-empty because it at least contains the vectors of $\mathcal{B}$. It is also automatically closed under vector addition and scalar multiplication because those are exactly the operations we used to create all the vectors in the span! Therefore $\mathit{SPAN}(\mathcal{B})$ is a vector subspace of V.

Definition 4.5 (Cartesian form of Euclidean vector sub spaces)**.**
Euclidean vector sub spaces can always be written as a set with some defining equations:

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \textit{equations relating the } x_k\}.$$

For example, the general form of planar vector subspaces of $\mathbb{R}^3$ *is*

$$V_P = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, ax + by + cz = 0\right\}$$

where a*,* b *and* c *are some given constants. This set is read aloud as "all the triples* (x, y, z) *such that* $ax + by + cz = 0$*".*

Definition 4.6 (Sum of subspaces (sum space))**.**
Suppose we have a vector space V *with vector subspaces* F *and* G*. We define the* ***sum of subspaces*** *(or sum space) as a new set denoted*

$$F + G = \{\boldsymbol{f} + \boldsymbol{g} \,|\, \boldsymbol{f} \in F,\ \boldsymbol{g} \in G\}$$

Note: The sum space is a subset of the parent vector space: $F + G \subset V$*.*

Definition 4.7 (Direct sum)**.**
Let F *and* G *be two vector subspaces of a vector space* V *and let* $E = F + G$ *be the sum space. We say* E *is a* ***direct sum*** *of* F *and* G *if each element of* E *has a* ***unique*** *decomposition as a sum of vectors in* F *and vectors in* G*. That is, for every* $\boldsymbol{v} \in E$*, there exists unique vectors* $\boldsymbol{f} \in F$ *and* $\boldsymbol{g} \in G$ *such that* $\boldsymbol{v} = \boldsymbol{f} + \boldsymbol{g}$*. We denote this direct sum with a new symbol*

$$E = F \oplus G$$

Definition 4.8 (Complementary vector subspaces)**.**
Let F *and* G *be two vector subspaces of* V*.* F *and* G *are called* ***complementary*** *if* V *is a direct sum of* F *and* G*. That is, if and only if*

- $V = F + G$*, and*
- $F \cap G = \{\boldsymbol{0}_V\}$

Definition 4.9 (Linear dependence)**.**
A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *from a vector space* V *is said to be linearly dependent if there exists a set of constants* $\{\alpha_1, \ldots, \alpha_n\}$ *not all zero such that*

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}_V.$$

Note: the right hand side of the equation is the zero vector, not the real number 0*.*

Definition 4.10 (Linear independence)**.**
A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *from* V *is said to be linearly independent if they are not linearly dependent. That is, the equation*

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}_V.$$

implies that the constants $\alpha_1, \ldots, \alpha_n$ *are all zero.*

Definition 4.11 (Basis)**.**
A basis of a vector space V *is a minimal set of vectors which spans the vector space. Formally, the set of vectors* $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *in a vector space* V *is a basis of* V *if it is a set of linearly independent vectors and* SPAN$(\mathbf{v}_1, \ldots, \mathbf{v}_n) = V$*. Note: bases are not unique, but they always contain the same number of vectors.*

Definition 4.12 (Dimension)**.**
The dimension of a vector space is the number of elements in a basis for that vector space.

Definition 4.13 (Canonical basis of $\mathbb{R}^n$)**.**
The canonical basis of the vector space of real n-tuples, $\mathbb{R}^n$*, is the ordered set of* n n*-tuples with* k^{th} *element,* $\mathbf{c}_k = (\alpha_1, \ldots, \alpha_n)$ *such that*

$$\alpha_j = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases}$$

That is, as a set the canonical basis is

$$\mathcal{C}_n = \{(1,0,\ldots,0),\ (0,1,\ldots,0),\ \ldots,\ \underbrace{(0,0,\ldots,0,\overbrace{\ 1\ }^{k^{th}\ place},0,\ldots,0)}_{k^{th}\ tuple},\ \ldots,\ (0,0,\ldots,1)\}.$$

Definition 4.14 (Canonical basis of $\mathcal{P}_n$)**.**
The canonical basis of the vector space of polynomials with degree up to n*,* $\mathcal{P}_n$*, is the ordered set of* n *polynomials with* k^{th} *element,* $\mathbf{c}_k = x^k$*. That is, as a set the canonical basis is*

$$\mathcal{C}_n = \{1,\ x,\ x^2,\ \ldots,\ x^n\}.$$

Definition 4.15 (Coordinates of a vector)**.**
Let $\mathbf{v}$ *be a vector in a vector space* V*. The coordinates of* $\mathbf{v}$ *with respect to a given basis* $\mathcal{B}$*, denoted* $[\mathbf{v}]_{\mathcal{B}}$*, is a column of the unique set of coefficients in the linear combination of* $\mathbf{v}$ *in terms of the basis vectors.*

Chapter 5: Linear Maps

Definition 5.1 (Linear map)**.**
A mapping, f, from a vector space V to a vector space W, denoted $f : V \to W$, is called a linear map if it satisfies the following property:

$$\forall \mathbf{u},\, \mathbf{v} \in V,\, \forall \alpha, \beta \in \mathbb{R}$$
$$f(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}).$$

We say that a linear map preserves linear combinations.

Definition 5.2 (Image)**.**
The image of a linear map $f : V \to W$, denoted im(f), is the set of all possible "output" vectors of the map:

$$im(f) = \{\mathbf{w} \in W \mid \exists \mathbf{v} \in V\; f(\mathbf{v}) = \mathbf{w}\} \subseteq W.$$

Definition 5.3 (Rank)**.**
The rank of a linear map is the dimension of its image: $rank(f) = \dim(im(f))$.

Definition 5.4 (Kernel)**.**
The kernel of a linear map $f : V \to W$, denoted $\ker(f)$, *is the set of vectors that f maps to the zero vector, $\mathbf{0}_W$, of W. That is,*

$$\ker(f) = \{\mathbf{v} \in V \mid f(\mathbf{v}) = \mathbf{0}_W\}.$$

Definition 5.5 (Nullity)**.**
The nullity of a linear map is the dimension of its kernel: $nullity(f) = \dim(\ker(f))$.

Definition 5.6 (Injectivity)**.**
Let $f : V \to W$ be a linear map. We say f is injective if no two vectors of V are mapped to the same vector of W. In symbols we have two equivalent expressions

$$\forall\, \mathbf{x}, \mathbf{y} \in V, \quad (f(\mathbf{x}) = f(\mathbf{y}) \implies \mathbf{x} = \mathbf{y})$$
$$or$$
$$\forall\, \mathbf{x}, \mathbf{y} \in V, \quad (\mathbf{x} \neq \mathbf{y} \implies f(\mathbf{x}) \neq f(\mathbf{y}))$$

Definition 5.7 (Surjectivity)**.**
Let $f : V \to W$ be a linear map. We say that f is surjective if every vector in the output space has a corresponding input vector. In symbols

$$\forall\, \mathbf{w} \in W \quad \exists \mathbf{v} \in V \text{ such that } f(\mathbf{v}) = \mathbf{w}.$$

Definition 5.8 (Categories of linear maps)**.**
Let $f : V \to W$ be a linear map.

- *If $W = V$ we call f an endomorphism.*
- *If f is both injective and surjective then we say it is bijective and we call it an isomorphism.*
- *If f is both an isomorphism and an endomorphism we call it an automorphism.*

Definition 5.9 (Composition of linear maps)**.**
Composition of linear maps works exactly as you would expect if you remember the composition of regular functions. We must have a coherence between the output of one linear map and the input of another. So, two linear maps $f : A \to B$ and $g : U \to V$ can be composed as a well defined linear map $g \circ f$ ("g of f") if and only if the output space of f is the input space of g: $U = B$. For any $\mathbf{u} \in A$ the composition is written

$$g \circ f : A \to V \quad \text{and} \quad (g \circ f)(\mathbf{u}) = g(f(\mathbf{u})).$$