Linear Algebra

Determinants

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École d'ingénieurs du numérique

Suppose we have some square matrix $A \in \mathcal{M}_{n,n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Then the *determinant of A*, denoted $\det(A)$ or $|A|$, is some number that, among other uses, tells us if A is invertible. If the determinant of a matrix is non-zero, then it is invertible. Recall that only square matrices are invertible.

Definition - Determinant of a 1 by 1 matrix

Suppose we have a general 1 by 1 matrix $A = \begin{pmatrix} a \end{pmatrix}$. It shouldn't be too difficult to see that the inverse matrix must be $A^{-1} = \begin{pmatrix} \frac{1}{a} \end{pmatrix}$ because the multiplication $\begin{pmatrix} a \end{pmatrix} \begin{pmatrix} \frac{1}{a} \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$, the identity matrix. Since $1/a$ is well defined only if $a \neq 0$, the value of a itself satisfies this idea "if the determinant is non-zero, the matrix is invertible". So we can say, the determinant of any 1 by 1 matrix is given by its only value:

$$\det \begin{pmatrix} a \end{pmatrix} = a$$

Before we define how to calculate determinants of larger square matrices, we must first introduce the concept of a submatrix.

Definition - Submatrix

From a matrix A we generate the *submatrix* A_{ij} by deleting the *ith* row and *jth* column:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & \cdots & a_{m,j-1} & a_{m,j} & a_{m,j+1} & \cdots & a_{m,n} \end{pmatrix} \to A_{ij} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & \cdots & a_{m,j-1} & a_{m,j+1} & \cdots & a_{m,n} \end{pmatrix}$$

Note: we generally have to specify in words that we are creating a submatrix. The notation A_{ij} is a little ambiguous without being explicit.

Examples - Submatrices

Definition - Determinant

The determinant of a n by n square matrix A is given by either of the following equivalent summations

$$\det(A) = \underbrace{\sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})}_{\text{for any choice of } j} = \underbrace{\sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})}_{\text{for any choice of } i}$$

We will soon learn why the left summation will be "along a column" and the right summation will be "along a row".

This definition should strike you as odd. It defines the determinant in terms of the addition of other determinants! Let's try to understand this with an example.

Definition - Determinant of a 2 by 2 matrix

Suppose we have a general 2 by 2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The determinant is therefore (choosing $j = 1$ for the first summation)

$$\begin{aligned} \det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(A_{i1}) &= (-1)^2 a_{11} \det(A_{11}) + (-1)^3 a_{21} \det(A_{21}) \\ &= a \det \begin{pmatrix} d \end{pmatrix} - c \det \begin{pmatrix} b \end{pmatrix}. \end{aligned}$$

We defined earlier the determinants of 1 by 1 matrices, so we have a formula worth remembering

$$\boxed{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc}$$

I think of this formula as "Multiplication of on-diagonal minus multiplication of off-diagonal".

Example - Determinant of a 3 by 3 matrix

Example - Determinant of a 3 by 3 matrix along a row

Properties

In the determinant expression there is this term $(-1)^{i+j}$ that appears in both summations:

$$(-1)^{i+j} = \begin{cases} +1 & \text{if } i+j \text{ is even} \\ -1 & \text{if } i+j \text{ is odd} \end{cases}$$

This determines the following $\pm$ pattern to the matrix

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Noticing this pattern lets you avoid having to explicitly write $(-1)^{i+j}$ during the calculations.

Exercices

Calculate the following determinants

$$\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 5 & 10 \\ 1 & 2 \end{vmatrix} =$$

$$\begin{vmatrix} 2 & 4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 2 & 4 & 0 \\ 1 & 2 & -3 \\ 0 & 0 & 13 \end{vmatrix} =$$

Theorem - Determinant when column is multiplied by a constant

If we multiply a column by a constant, k, then the determinant is multiplied by that constant.

Proof: Let's multiply the jth column of a matrix A by k to get the new matrix

$$A' = \begin{pmatrix} a_{11} & \cdots & ka_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & ka_{2j} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & ka_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Then, calculate the determinant of A', choosing to compute it along the jth column

$$\det(A') = \sum_{i=1}^{n} (-1)^{i+j} (ka_{ij}) \det(A'_{ij}) = k \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A'_{ij}) = k \det(A)$$

Theorems

Given two matrices $A, B \in \mathcal{M}_{n,n}$, we have

$$\det(AB) = \det(A)\det(B)$$

Let $A \in \mathcal{M}_{n,n}$ be a diagonal matrix. Then

$$\det(A) = a_{11} \times a_{22} \times \cdots \times a_{nn}$$

Let $A \in \mathcal{M}_{n,n}$ be a triangular matrix. Then

$$\det(A) = a_{11} \times a_{22} \times \cdots \times a_{nn}$$

Let $\Lambda, A \in \mathcal{M}_{n,n}$ with $\Lambda = \lambda I$ a diagonal matrix. Then

$$\det(\Lambda A) = \lambda^n \det(A)$$

Theorem Proof

$$\text{If } A \text{ is triangular then } \det(A) = a_{11} \times a_{22} \times \cdots \times a_{nn}$$

Partial Proof: Consider an upper triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

If we take the determinant along the first column at each step we get

$$\det(A) = a_{11} \det \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11} a_{22} \det \begin{pmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} = \cdots = a_{11} a_{22} \cdots a_{nn}$$

Theorem - Determinant of Inverse

Let $A \in \mathcal{M}_{n,n}$ be an invertible matrix. Then

$$\det(A) \neq 0 \quad \text{and} \quad \det(A^{-1}) = \frac{1}{\det(A)}$$

Proof of the second: Since A is invertible, its inverse exists and is defined by $AA^{-1} = I$. The determinant of this relation is

$$\begin{aligned} \det(AA^{-1}) &= \det(I) \\ \det(A)\det(A^{-1}) &= 1 \\ \det(A^{-1}) &= \frac{1}{\det(A)} \end{aligned}$$

Theorem

Let $P \in \mathcal{M}_{n,n}$ be an invertible matrix. Then for any matrix $A \in \mathcal{M}_{n,n}$ we have

$$\det(PAP^{-1}) = \det(A).$$

Can you prove it?

Theorems - determinants and column properties

Let $A \in \mathcal{M}_{n,n}$ be a square matrix. If any two columns of A are equal to each other then $\det(A) = 0$. For example

$$\det\begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 6 & 0 & 6 \end{pmatrix} = 0.$$

If any column is a multiple of another, then $\det(A) = 0$. For example

$$\det\begin{pmatrix} 1 & 3 & 0 \\ 2 & 6 & 3 \\ 2 & 6 & 3 \end{pmatrix} = 0.$$

Theorems - determinants and column operations

Let $A \in \mathcal{M}_{n,n}$ be a square matrix. If we exchange any two columns, then the determinant of the new matrix is -1 times the old. For example

$$\det \begin{pmatrix} 2 & 1 & 7 \\ 4 & 2 & 4 \\ 6 & 0 & 3 \end{pmatrix} = -1 \det \begin{pmatrix} 2 & 7 & 1 \\ 4 & 4 & 2 \\ 6 & 3 & 0 \end{pmatrix}$$

The determinant is unchanged if we add to a column a linear combination of other columns. For example

$$\det \begin{pmatrix} 2 & 1 & 7 \\ 4 & 2 & 4 \\ 6 & 0 & 3 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 9 \\ 4 & 2 & 8 \\ 6 & 0 & 3 \end{pmatrix}$$

(the new third column is the old third column $+$ 2 times the second column)

Theorems - determinants and row properties and operations

For every square matrix A, the previous results also hold for rows! For example:

- If two rows are equal to each other, then $\det(A) = 0$.
- If any row is a multiple of another, then $\det(A) = 0$.
- If we exchange any two rows, the new determinant is -1 times the old.
- The determinant is unchanged if we add to a row a linear combination of other rows.

Theorem - Existence of a triangular with the same determinant

For every square matrix, A, there exists a triangular matrix, T, such that $\det(A) = \det(T)$.

Proof:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Consider the first column. There are 2 possibilities:

1. All the $a_{i1} = 0$. Then $\det(A) = 0$ and the zero matrix is a triangular matrix with the same determinant.
2. There is a non-zero element in the first column, say $a_{k1} \neq 0$. We can add row k to the first row, giving a new matrix A' without changing the determinant. In this way we can always generate a matrix with a non-zero upper left element, a'_{11}, and the same determinant.

Proof continued

Now we can use this a'_{11} to guarantee it is the *only* non-zero element in the first column. To do this we subtract from every row other than the first this particular multiple of the first row:

$$R_i \to R_i - \frac{a_{i1}}{a'_{11}} R_1$$

This operation does not change the determinant, so we have:

$$\det(A) = \det \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a^{(1)}_{22} & \cdots & a^{(1)}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a^{(1)}_{n2} & \cdots & a^{(1)}_{nn} \end{pmatrix}$$

Proof continued

We can repeat this procedure for the second column, ignoring the first row, to successively generate an upper triangular matrix

$$\det(A) = \det \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & a^{(1)}_{22} & a^{(1)}_{23} & \cdots & a^{(1)}_{2n} \\ 0 & 0 & a^{(2)}_{33} & \cdots & a^{(2)}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a^{(2)}_{3n} & \cdots & a^{(2)}_{nn} \end{pmatrix} = \cdots = \det \begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & a^{(1)}_{22} & a^{(1)}_{23} & \cdots & a^{(1)}_{2n} \\ 0 & 0 & a^{(2)}_{33} & \cdots & a^{(2)}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a^{(n-1)}_{nn} \end{pmatrix}$$

Now it's much easier to calculate the determinant of this triangular matrix (multiply the diagonal). In this way, at the end of the Gaussian reduction we can know immediately if the matrix is invertible or not (whether there is a zero on the diagonal).