Linear Algebra

System of Linear Equations

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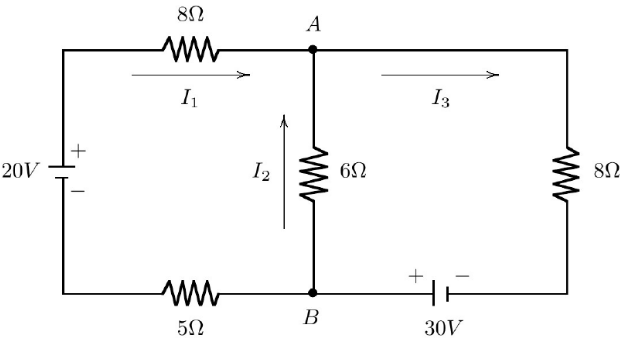
École d'ingénieurs du numérique

Example - Electronic circuits

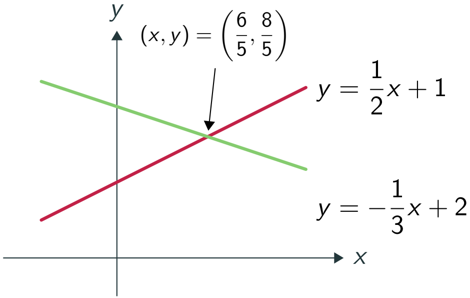


Kirchhoff's laws give a system of equations:

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 13 I_1 - I_2 &= 20 \\ 6 I_2 + 8 I_3 &= 30 \end{aligned} \quad \overset{\text{"techniques"}}{\Longrightarrow} \quad \begin{aligned} I_1 &= 2 \text{ A} \\ I_2 &= 1 \text{ A} \\ I_3 &= 3 \text{ A} \end{aligned}$$

Example - Intersection of lines

Lines form a simple system of equations if we want to find their intersection. For example:



These two equations are simultaneously true at just 1 point, where the two lines overlap: $x = 6/5$, $y = 8/5$.

Basic Concepts

Definition - System of Linear Equations

A system of m linear equations with n unknowns, denoted (S), has the general form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m \end{cases} \qquad (S)$$

where the x_j are the unknowns we want to find, a_{ij} are the *coefficients* and the y_i are the *constant terms*.

We group the coefficients into a matrix $A = (a_{ij})$, the unknowns and constants into columns X and Y so that the system can be written as a matrix equation $AX = Y$.

Definition - System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m \end{cases} \qquad (S)$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}}_{Y}$$

Goal

The goal is to find all the possible collections of x_i, called a *solution*, that satisfy the equation $AX = Y$. That is, the possible columns X. We denote the set of solutions of (S) as $\mathcal{S}$, which is a subset of all columns of size n.

Note: it is possible that $\mathcal{S}$ is an empty set, which would mean there are no solutions of $AX = Y$. For example, if two parallel lines are offset by some distance, there is no point of intersection.

Example

Definition - Associated homogeneous system

For any system of linear equations, (S), given by $AX = Y$, we associate the **homogeneous system**, denoted (H):

$$AX = 0_m$$

for the column

$$0_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{m,1}$$

We will denote the solution set of (H) by $\mathcal{H}$.

Note: the homogeneous system always admits *at least one* solution, the trivial solution $X = 0_n$.

Properties - Homogeneous solutions

If X is a solution of (H), then any scalar multiple of X is also a solution of (H).

Proof.
Let $AX = 0_m$ and $k \in \mathbb{R}$. Then

$$\begin{aligned} A(kX) &= k(AX) \\ &= k0_m \\ &= 0_m \end{aligned}$$

Therefore kX is a solution of (H). $\square$

Example

Properties - Homogeneous solutions

If X and X' are solutions of (H), then their addition $X + X'$ is also a solution of (H).

Proof.
Let $AX = 0_m$ and $AX' = 0_m$. Then

$$\begin{aligned} A(X + X') &= AX + AX' && \text{(distributivity)} \\ &= 0_m + 0_m && \text{(using the premise)} \\ &= 0_m && \text{(definition of zero matrix)} \end{aligned}$$

Therefore $X + X'$ is a solution of (H). $\square$

Theorem - Homogeneous solutions

Homogeneous systems have either only the trivial solution or infinite solutions.

In effect, if the system (H) has a non-zero solution X, then all columns of the form kX for $k \in \mathbb{R}$ are distinct solutions of (H), and there are infinitely many of them.

Theorem - Solution set of a system

For any system (S), if it has at least one solution, call it X_p, then the set of solutions can be written

$$\mathcal{S} = \{X_p + X_h \quad | \quad \forall X_h \in \mathcal{H}\}$$

where $\mathcal{H}$ is the set of solutions of the associated homogeneous system of (S).

Theorem - Solution set of a system

Proof.
Let X_p be any solution of the system $AX = Y$ and X_h be any solution of the associated homogeneous system. Then we have

$$A(X_p + X_h) = \underbrace{AX_p}_{Y} + \underbrace{AX_h}_{0_m} = Y + 0_m = Y$$

Hence $X_p + X_h$ is also a solution of $AX = Y$. $\square$

Corollary - Solution set of a system

Any system, (S), has either

- no solutions,
- 1 unique solution,
- infinite solutions.

Theorem - Solution with inverse

Let A be a given square matrix and (S) be the system $AX = Y$. If A is invertible, then (S) has 1 unique solution given by

$$X = A^{-1}Y.$$

Theorem - Solution with inverse

Proof.

Is $A^{-1}Y$ a solution to $AX = Y$?

$$\begin{aligned} A(A^{-1}Y) &= (AA^{-1})Y \\ &= IY \\ &= Y \end{aligned}$$

Yes.

Is $A^{-1}Y$ unique?

Let X_p be some solution of $AX = Y$.
Then

$$\begin{aligned} AX_p &= Y \\ A^{-1}AX_p &= A^{-1}Y \\ IX_p &= A^{-1}Y \\ X_p &= A^{-1}Y \end{aligned}$$

Yes.

$\square$

Theorem

Let A be a square matrix. If the system of equations, (S), represented by $AX = Y$ has a solution for every possible Y, then A is invertible.

Theorem

Let A be an invertible square matrix. For any two columns X and X' satisfying $AX = AX'$, we then have $X = X'$.

Proof.

$$
\begin{aligned}
AX &= AX' \\
A^{-1}AX &= A^{-1}AX' \\
IX &= IX' \\
X &= X'
\end{aligned}
$$

□

Gaussian reduction

Definition - Equivalent systems

Two systems of linear equations are **equivalent** if they share the same set of solutions.

For example:

$$(S_1) \begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \\ 6I_2 + 8I_3 = 30 \end{cases} \quad \text{and} \quad (S_2) \begin{cases} I_1 + I_2 - I_3 = 0 \\ 6I_2 + 8I_3 = 30 \\ 13I_1 - 6I_2 = 20 \end{cases}$$

(S_2) is simply a reordering of the equations of (S_1), so it obviously has the same solution, $(I_1, I_2, I_3) = (3, 2, 1)$. Hence (S_1) and (S_2) are equivalent systems.

Definition - Elementary operations

There are **elementary operations** that we can do to systems of equations that give new systems that remain equivalent to the old.

Exchanging two equations

$$(S_1)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \end{cases} \equiv \quad (S_2)\begin{cases} 13I_1 - 6I_2 = 20 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

Multiplying one equation by a non-zero constant

$$(S_1)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \end{cases} \equiv \quad (S_2)\begin{cases} I_1 + I_2 - I_3 = 0 \\ I_1 - (6/13)I_2 = (20/13) \end{cases}$$

Adding a multiple of one equation to another equation

$$(S_1)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 13I_1 - 6I_2 = 20 \end{cases} \equiv \quad (S_2)\begin{cases} I_1 + I_2 - I_3 = 0 \\ 15I_1 - 4I_2 - 2I_3 = 20 \end{cases}$$

These elementary operations will be used in a method to solve systems of equations, called **Gaussian reduction**. Before summarising the algorithm, we'll use it on a concrete example.

Consider the system of linear equations with unknowns (x, y, z, t):

$$(S_1) \begin{cases} x & +3y & -z & +4t & = & 27 \\ -4x & -11y & +6z & -14t & = & -105 \\ -2x & -10y & -7z & -14t & = & -55 \\ 2x & +9y & +6z & +9t & = & 37 \end{cases}$$

The solution set, $\mathcal{S}$, is clearly a subset of $\mathbb{R}^4$, that is, a set of quadruples. We can rewrite (S_1) in matrix form

$$\underbrace{\begin{pmatrix} 1 & 3 & -1 & 4 \\ -4 & -11 & 6 & -14 \\ -2 & -10 & -7 & -14 \\ 2 & 9 & 6 & 9 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 27 \\ -105 \\ -55 \\ 37 \end{pmatrix}}_{Y}$$

We will solve this system using the method of Gauss. First keeping it in equation form, and then in shorthand with the matrix representation.

The general goal is to use the elementary operations to simplify the equations, so that there are "leading 1s", called pivots. The pivots are used to eliminate unknowns in different equations. Let's denote the lines as L_i and the notation, for example, $L_1 \to L_1 + 2L_2$ will mean that we add 2 times line 2 to line 1. Now, our system already has a pivot for the first equation

$$(S_2)\begin{cases} \boxed{1}\,x & +3y & -z & +4t & = & 27 \\ -4x & -11y & +6z & -14t & = & -105 \\ -2x & -10y & -7z & -14t & = & -55 \\ 2x & +9y & +6z & +9t & = & 37 \end{cases}$$

We use this pivot to eliminate the unknown x from the other 3 equations. For the 2nd line, add 4 of the 1st. Add 2 of the 1st to the 3rd . Add -2 of the first to the 4th.

$$(S_3)\begin{cases} \boxed{1}\,x & +3y & -z & +4t & = & 27 & \\ 0 & +1y & +2z & +2t & = & 3 & L_2 \to L_2 + 4L_1 \\ 0 & -4y & -9z & -6t & = & -1 & L_3 \to L_3 + 2L_1 \\ 0 & +3y & +8z & +t & = & -17 & L_4 \to L_4 - 2L_1 \end{cases}$$

We now use the pivot in the second equation to eliminate the unknown x from the lower 2 equations. For the 3nd line, add 4 of the 2nd. Subtract 3 of the 2nd to the 4th.

$$(S_4)\begin{cases} x & +3y & -z & +4t & = & 27 & \\ 0 & \boxed{1}\,y & +2z & +2t & = & 3 & \\ 0 & 0 & -1z & +2t & = & 11 & L_3 \to L_3 + 4L_2 \\ 0 & 0 & +2z & -5t & = & -26 & L_4 \to L_4 - 3L_2 \end{cases}$$

We now use the new pivot (-1 is as good as 1) in the third equation to eliminate the unknown x from the fourth equation. So, add 2 of the 3rd to the 4th.

$$(S_5)\begin{cases} x & +3y & -z & +4t & = & 27 & \\ 0 & y & +2z & +2t & = & 3 & \\ 0 & 0 & \boxed{-1}\,z & +2t & = & 11 & \\ 0 & 0 & 0 & -t & = & -4 & L_4 \to L_4 + 3L_3 \end{cases}$$

As we have used elementary operations in going from $(S_1) \to (S_2) \to \cdots \to (S_5)$, all of these systems are equivalent to each other. Hence the set of solutions of (S_5) is the same as for (S_1), which we originally seek. This final form of the equations lets us give a quadruple of numbers, the solution to (S_1):

$$(S_5)\begin{cases} x & +3y & -z & +4t & = & 27 \\ 0 & y & +2z & +2t & = & 3 \\ 0 & 0 & -z & +2t & = & 11 \\ 0 & 0 & 0 & -t & = & -4 \end{cases}$$

$$(S_5)\begin{cases} t & = 4 \\ z & = 2t - 11 = -3 \\ y & = 3 - 2z - 2t = 1 \\ x & = 27 - 3y + z - 4t = 5 \end{cases}$$

So we have a solution to the system (S_1): $(x, y, z, t) = (5, 1, -3, 4)$. And the set of solutions contains just this one, unique solution: $\mathcal{S} = \{(5, 1, -3, 4)\}$. *Note*: The set $\mathcal{S}$ is a set of 4-tuples (in this case just 1), and not a set of 4 separate numbers.

Consider the almost-same system of linear equations with different constant terms

$$(S)\begin{cases} x & +3y & -z & +4t & = & -10 \\ -4x & -11y & +6z & -14t & = & 35 \\ -2x & -10y & -7z & -14t & = & 29 \\ 2x & +9y & +6z & +9t & = & -8 \end{cases}$$

To shorthand the Gaussian reduction we convert the system into an **augmented matrix**

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & -10 \\ -4 & -11 & 6 & -14 & 35 \\ -2 & -10 & -7 & -14 & 29 \\ 2 & 9 & 6 & 9 & -8 \end{array}\right)$$

We can use this representation to make elementary row operations, without having to write down the variables in each line. So we denote rows by R_i. The matrix A is the same as the previous example, so the operations will be the same. The constant terms are the only difference between the two examples.

As previously, we use the first row's pivot to eliminate the unknown x from the other 3 rows. For the 2nd row, add 4 of the 1st. Add 2 of the 1st to the 3rd. Add -2 of the first to the 4th.

$$\begin{array}{c} \\ R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - 2R_1 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & -10 \\ 0 & 1 & 2 & 2 & -5 \\ 0 & -4 & -9 & -6 & 9 \\ 0 & 3 & 8 & 1 & 12 \end{array}\right)$$

We now use the pivot in the second row to eliminate the unknown y from the lower 2 rows. For the 3nd row, add 4 of the 2nd. Subtract 3 of the 2nd from the 4th.

$$\begin{array}{c} \\ \\ R_3 \to R_3 + 4R_2 \\ R_4 \to R_4 - 3R_2 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & -10 \\ 0 & 1 & 2 & 2 & -5 \\ 0 & 0 & -1 & 2 & -11 \\ 0 & 0 & 2 & -5 & 27 \end{array}\right)$$

We now use the new pivot in the third row to eliminate the unknown z from the fourth row. So, add 2 of the 3rd to the 4th.

$$R_4 \to R_4 + 2R_3 \qquad \left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & -10 \\ 0 & 1 & 2 & 2 & -5 \\ 0 & 0 & -1 & 2 & -11 \\ 0 & 0 & 0 & -1 & 5 \end{array}\right)$$

We now “unpack” the augmented matrix to see it again as a system of equations

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & -10 \\ 0 & 1 & 2 & 2 & -5 \\ 0 & 0 & -1 & 2 & -11 \\ 0 & 0 & 0 & -1 & 5 \end{array}\right) \longrightarrow (S') \begin{cases} x & +3y & -z & +4t & = & -10 \\ 0 & +y & +2z & +2t & = & -5 \\ 0 & +0 & -z & -2t & = & -11 \\ 0 & +0 & +0 & -t & = & 5 \end{cases}$$

And as before, a little more work to find the solution

$$(S') \begin{cases} t & = -5 \\ z & = 2t + 11 = 1 \\ y & = -5 - 2z - 2t = 3 \\ x & = -10 - 3y + z - 4t = 2 \end{cases}$$

So we have a solution to the system (S): $(x, y, z, t) = (2, 3, 1, -5)$. And the set of solutions contains just this one, unique solution: $\mathcal{S} = \{(2, 3, 1, -5)\}$.

One more time, let's consider the same matrix A, but with general constant terms

$$(S)\begin{cases} x & +3y & -z & +4t & = & a \\ -4x & -11y & +6z & -14t & = & b \\ -2x & -10y & -7z & -14t & = & c \\ 2x & +9y & +6z & +9t & = & d \end{cases} \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & a \\ -4 & -11 & 6 & -14 & b \\ -2 & -10 & -7 & -14 & c \\ 2 & 9 & 6 & 9 & d \end{array}\right)$$

And we proceed with the exact same row operations as before. Only the constant terms are different.

$$\begin{array}{l} \\ R_2 + 4R_1 \\ R_3 + 2R_1 \\ R_4 - 2R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & a \\ 0 & 1 & 2 & 2 & b+4a \\ 0 & -4 & -9 & -6 & c+2a \\ 0 & 3 & 8 & 1 & d-2a \end{array}\right)$$

$$
\begin{array}{c}
\\
\\
R_3+4R_2\\
R_4-3R_2
\end{array}
\left(\begin{array}{cccc|c}
1 & 3 & -1 & 4 & a\\
0 & 1 & 2 & 2 & b+4a\\
0 & 0 & -1 & 2 & c+18a+4b\\
0 & 0 & 2 & -5 & d-14a-3b
\end{array}\right)
\longrightarrow
\begin{array}{c}
\\
\\
\\
R_4+2R_3
\end{array}
\left(\begin{array}{cccc|c}
1 & 3 & -1 & 4 & a\\
0 & 1 & 2 & 2 & b+4a\\
0 & 0 & -1 & 2 & c+18a+4b\\
0 & 0 & 0 & -1 & d+22a+5b+2c
\end{array}\right)
$$

Unpacking back into equation form

$$
(S')\begin{cases}
x & +3y & -z & +4t & = & a\\
0 & +y & +2z & +2t & = & b+4a\\
0 & +0 & -z & -2t & = & c+18a+4b\\
0 & +0 & +0 & -t & = & d+22a+5b+2c
\end{cases}
$$

Solving for the unknowns

$$
\begin{aligned}
t &= -22a-5b-2c-d\\
z &= 2t-18a-4b-c = -62a-14b-5c-2d\\
y &= -2z-2t+4a+b = 172a+39b+14c+6d\\
x &= -3y+z-4t+a = -489a-111b-39c-16d
\end{aligned}
$$

In order, then, we have the solution

$$
\begin{array}{rcrrrr}
x & = & -489a & -111b & -39c & -16d \\
y & = & 172a & +39b & +14c & +6d \\
z & = & -62a & -14b & -5c & -2d \\
t & = & -22a & -5b & -2c & -d
\end{array}
$$

$$
\underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} -489 & -111 & -39 & -16 \\ 172 & 39 & 14 & 6 \\ -62 & -14 & -5 & -2 \\ -22 & -5 & -2 & -1 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}_{Y}
$$

Writing this out like this allows us to solve $AX = Y$ for any arbitrary 4-tuple Y. Recall the theorem that states that this property means A is invertible.

B **is exactly the inverse of** A.

Example - Column exchange

Consider the following system

$$(S)\begin{cases} -11x & +6y & -4z & -14t & = & -105 \\ 9x & +6y & +2z & +9t & = & 37 \\ 3x & -y & +z & +4t & = & 27 \\ -10x & -7y & -2z & -14t & = & -55 \end{cases}$$

If we wanted to form a pivot in the first line we would have divide this line by -11. This is perfectly legitimate, but introduces fractions that will make computations very messy and prone to error later. Instead we could switch lines 1 and 3, and rewrite the equations with z being the leading term:

$$(S')\begin{cases} +z & 3x & -y & +4t & = & 27 \\ +2z & 9x & +6y & +9t & = & 37 \\ -4z & -11x & +6y & -14t & = & -105 \\ -2z & -10x & -7y & -14t & = & -55 \end{cases}$$

Example - Column exchange

$$
(S') \begin{cases} +z & 3x & -y & +4t & = & 27 \\ +2z & 9x & +6y & +9t & = & 37 \\ -4z & -11x & +6y & -14t & = & -105 \\ -2z & -10x & -7y & -14t & = & -55 \end{cases}
$$

But be careful, when solving this system using augmented matrix form you must remember that the first column now refers to the z variable. So when you find the solution, you should put it into the original order of unknowns (x, y, z, t):

$$\mathcal{S} = \{(1, -3, 5, 4)\}$$

Example - A system with no solutions

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & 27 \\ -4 & -11 & 6 & -14 & -105 \\ -2 & -10 & -7 & -14 & -55 \\ 1 & 3 & -1 & 4 & 2015 \end{array}\right)$$

$$\begin{array}{c} \\ R_2 + 4R_1 \\ R_3 + 2R_1 \\ R_4 - 1R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & 27 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & -4 & -9 & -6 & -1 \\ 0 & 0 & 0 & 0 & 1988 \end{array}\right)$$

The last row gives the absurd equation $0x + 0y + 0z + 0t = 1998$

And so this system of equations is insoluble.

The lesson: if any 2 rows of A are the same (assuming the constant terms are different), the system has no solutions.

Example - A system with infinite solutions

$$
(S) \begin{cases} x & +3y & -z & +4t & = & 27 \\ -4x & -11y & +6z & -14t & = & -105 \\ -2x & -10y & -7z & -14t & = & -55 \\ x & +3y & -z & +4t & = & 27 \end{cases}
$$

In this system, the 4th equation is identical to the 3rd. So it really reduces down to 3 independent equations.

$$
(S_1) \begin{cases} x & +3y & -z & +4t & = & 27 \\ -4x & -11y & +6z & -14t & = & -105 \\ -2x & -10y & -7z & -14t & = & -55 \end{cases}
$$

Example - A system with infinite solutions

$$(S_2)\begin{cases} x & +3y & -z & +4t & = & 27 & \\ 0 & y & +2z & +2t & = & 3 & L_2+4L_1 \\ 0 & -4y & -9z & -6t & = & -1 & L_3+2L_1 \end{cases}$$

$$(S_2)\begin{cases} x & +3y & -z & +4t & = & 27 & \\ 0 & y & +2z & +2t & = & 3 & \\ 0 & 0 & -z & +2t & = & 11 & L_3+4L_2 \end{cases}$$

This is as far as we can go. t is a **free variable**, i.e. arbitrary, and the other unknowns can be rewritten entirely in terms of it. Working from the last equation first:

$$\begin{aligned} z &= 2t-11 \\ y &= 3-2z-2t = -6t+25 \\ x &= 27-3y+z-4t = 16t-59 \end{aligned}$$

Example - A system with infinite solutions

There is a different set of 4 numbers for each t, i.e. there are infinite 4-tuples that solve (S). They can be be written in column form

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 16t - 59 \\ -6t + 25 \\ 2t - 11 \\ t \end{pmatrix} = \begin{pmatrix} 16 \\ -6 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -59 \\ 25 \\ -11 \\ 0 \end{pmatrix}$$

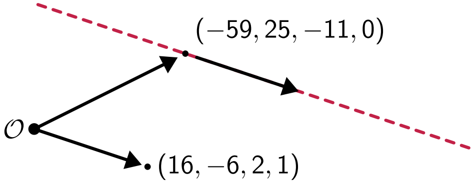
Or as the set of solutions

$$\mathcal{S} = \{(16, -6, 2, 1)t + (-59, 25, -11, 0) \,|\, \forall t \in \mathbb{R}\}$$

Example - A system with infinite solutions

The column form of the solution allows for a nice geometric interpretation of the solution space

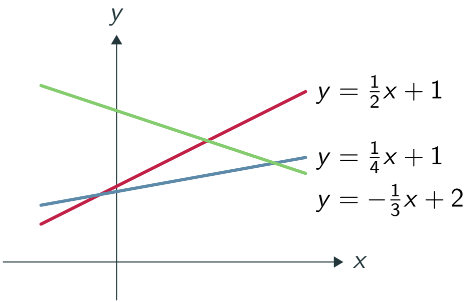
$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 16t - 59 \\ -6t + 25 \\ 2t - 11 \\ t \end{pmatrix} = \begin{pmatrix} 16 \\ -6 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -59 \\ 25 \\ -11 \\ 0 \end{pmatrix}$$



All the points on the red dashed line are solutions of $AX = Y$.

Definition - Overdetermined system

An overdetermined system has more equations than unknowns. We say "there are too many equations". Such a system allows solutions only if certain conditions are met. For example:



There is nowhere that all three of these lines overlap, that is, nowhere that the three equations are simultaneously true.

Example - Overdetermined system with one solution

The following overdetermined system, despite have 5 equations and 4 unknowns, has exactly one solution

$$(S)\left\{\begin{array}{ccccccc} x & +3y & -z & +4t & = & 27 \\ -4x & -11y & +6z & -14t & = & -105 \\ -2x & -10y & -7z & -14t & = & -55 \\ 2x & +9y & +6z & +9t & = & 37 \\ 2x & +18y & +26z & +23t & = & 42 \end{array}\right.$$

After many lines of the usual Gaussian method, we arrive at

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & 27 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & -1 & 2 & 11 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right)$$

Which means we've shown an equivalence between the last two equations. The 4th row can be subtracted from the 5th, and we solve the system as though only the 4 equations exist. The unique solution is then $(x, y, z, t) = (5, 1, -3, 4)$.

Example - Overdetermined system with no solutions

The following system, same as previous with just 1 change, has no solutions

$$(S)\begin{cases} x & +3y & -z & +4t & = & 27 \\ -4x & -11y & +6z & -14t & = & -105 \\ -2x & -10y & -7z & -14t & = & -55 \\ 2x & +9y & +6z & +9t & = & 37 \\ 2x & +18y & +26z & +23t & = & \boxed{45} \end{cases}$$

After many lines of the usual Gaussian method, we arrive at

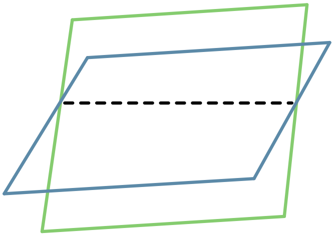
$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & 27 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & -1 & 2 & 11 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -1 & 9 \end{array}\right)$$

The last two rows represent equations $t = 6$ and $t = -9$. Of course these two equations can't be true simultaneously, so the system has no solutions.

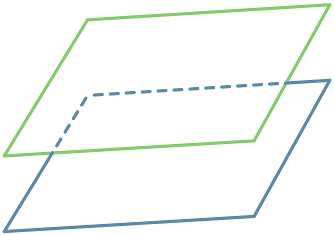
Definition - Underdetermined system

An **underdetermined system** has less equations than unknowns. We say "there are not enough equations". Such a system has either no solutions, or infinitely many.

Example: equation of a plane has 3 unknowns: $ax + by + cz = d$. So a system of 2 planes is underdetermined.



These 2 planes intersect at a line, the infinite points of which are the solutions to the two equations.



These 2 planes are simply offset and never intersect. Hence there are no solutions.

Example - Undetermined system with infinite solutions

Consider the following **underdetermined system**

$$(S)\begin{cases} x & +3y & -z & +4t & = & 27 \\ -4x & -11y & +6z & -14t & = & -105 \\ -2x & -10y & -7z & -14t & = & -55 \end{cases}$$

After some lines of the usual Gaussian method, we arrive at $\left(\begin{array}{cccc|c} 1 & 3 & -1 & 4 & 27 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & -1 & 2 & 11 \end{array}\right)$

We already solved this system.
The infinite solutions are:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 16t-59 \\ -6t+25 \\ 2t-11 \\ t \end{pmatrix} = \begin{pmatrix} 16 \\ -6 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} -59 \\ 25 \\ -11 \\ 0 \end{pmatrix}$$