Motivation. Recall that all linear, constant-coefficient second-order hyperbolic equations can be written as

$$u_{tt} - \Delta u + \ldots = 0$$

through a change of variables, where "..." represents lower-order terms. One of the distinguishing features of the wave equation is that it has "wave-like" solutions. In particular, for the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

the general solution is given by

$$u(x,t) = f(x+ct) + g(x-ct),$$

the sum of a wave moving to the right and a wave moving to the left. The functions $f(x+ct)$ and $g(x-ct)$ are known as **travelling waves**. More generally, for the wave equation in $\mathbb{R}^n$,

$$u_{tt} - c^2 \Delta u = 0 \qquad x \in \mathbb{R}^n, \tag{8.2}$$

for any (smooth) function f and any $\xi \in \mathbb{R}^n$, $u(x,t) = f(\xi \cdot x - \sigma t)$ is a solution of (8.2) for $\sigma = \pm c|\xi|$. A solution of the form $f(\xi \cdot x - \sigma t)$ is known as a **plane wave solution**.

Motivated by the existence of plane wave solutions for the wave equation, we look for properties of the system (8.1) such that the equation will have plane wave solutions. The conditions under which plane wave solutions exist lead us to the definition of hyperbolicity given above.

First, we rewrite the wave equation as a system in the form of (8.1). First, consider the wave equation in one spatial dimension,

$$u_{tt} - u_{xx} = 0.$$

Let

$$U \equiv \begin{bmatrix} u_x \\ u_t \end{bmatrix}.$$

Then

$$U_t = \begin{bmatrix} u_{xt} \\ u_{tt} \end{bmatrix} = \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_t \end{bmatrix}_x = A_1 U_x$$

where

$$A_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In general, for $x \in \mathbb{R}^n$, consider

$$u_{tt} - \Delta u = 0.$$

Let

$$U = \begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \\ u_t \end{bmatrix}.$$

Then

$$\begin{aligned}
U_t = \begin{bmatrix} u_{x_1 t} \\ \vdots \\ u_{x_n t} \\ u_{tt} \end{bmatrix} = \begin{bmatrix} u_{tx_1} \\ \vdots \\ u_{tx_n} \\ \sum_{i=1}^n u_{x_i x_i} \end{bmatrix} &= \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \\ u_{x_1} \end{bmatrix}_{x_1} + \begin{bmatrix} 0 \\ u_t \\ \vdots \\ 0 \\ u_{x_2} \end{bmatrix}_{x_2} + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_t \\ u_{x_n} \end{bmatrix}_{x_n} \\
&= \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \\ u_t \end{bmatrix}_{x_1} + \ldots + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \\ u_t \end{bmatrix}_{x_n} \\
&= \sum_{i=1}^n A_i U_{x_i}
\end{aligned}$$

where each A_i is an $(n+1) \times (n+1)$ symmetric matrix whose entries a^i_{jk} are given by

$$a^i_{jk} = \begin{cases} 1 & j = i, k = n+1; j = n+1, k = i \\ 0 & \text{otherwise.} \end{cases}$$

Now we claim that for A_i as defined above, for each $\xi \in \mathbb{R}^n$, there are $m = n+1$ distinct plane wave solutions $U(x,t) = V(\xi \cdot x - \sigma t)$ of

$$U_t - \sum_{i=1}^n A_i U_{x_i} = 0.$$

In particular, define

$$A(\xi) = \sum_{i=1}^n A_i \xi_i.$$

Let $\lambda_i(\xi)$, $R_i(\xi)$ be the ith eigenvalue and corresponding eigenvector of $-A(\xi)$. Let $U(x,t) = f(\xi \cdot x - \lambda_i(\xi)t)R_i(\xi)$. Now

$$U_t = \lambda_i(\xi) f'(\xi \cdot x - \lambda_i(\xi)t) R_i(\xi)$$

and

$$U_{x_i} = \xi_i f'(\xi \cdot x - \lambda_i(\xi)t) R_i(\xi).$$

Therefore,

$$\begin{aligned} U_t - \sum_{i=1}^n A_i U_{x_i} &= f'(\xi \cdot x - \lambda_i(\xi)t) \left[\lambda_i(\xi) R_i(\xi) - \sum_{i=1}^n A_i \xi_i R_i(\xi) \right] \\ &= f'(\xi \cdot x - \lambda_i(\xi)t) \left[\lambda_i(\xi) R_i(\xi) - A(\xi) R_i(\xi) \right] = 0 \end{aligned}$$

because

$$-A(\xi)R_i(\xi) = \lambda_i(\xi)R_i(\xi).$$

Now notice that A_i is a symmetric matrix for $i = 1, \ldots, n$. Therefore, $A(\xi) = \sum_{i=1}^n A_i \xi_i$ is an $m \times m$ symmetric matrix for each $\xi \in \mathbb{R}^n$. Consequently, $A(\xi)$ has m real eigenvalues and m linearly independent eigenvectors $R_i(\xi)$. Therefore, for each $\xi \in \mathbb{R}^n$ and each eigenvalue/eigenvector pair $\lambda_i(\xi), R_i(\xi)$, we get a distinct plane wave solution $U(x,t) = V(\xi \cdot x - \lambda_i(\xi)t)$.

We use this fact to define hyperbolicity for systems of the form (8.1). In particular, we want to find a condition on the system (8.1) under which there will be m distinct plane wave solutions for each $\xi \in \mathbb{R}^n$. We look for a solution of (8.1) of the form $U(x,t) = V(\xi \cdot x - \sigma t)$. Plugging a function U of this form into (8.1) with $F(x,t) \equiv 0$, we see this implies

$$-\sigma V' + \sum_{i=1}^n \xi_i A_i V' = 0. \tag{8.3}$$

Now if $\sum_{i=1}^n \xi_i A_i$ is an $m \times m$ diagonalizable matrix, then (8.3) will have m solutions $V_1', \ldots, V_m'$. These solutions are the eigenvectors of $\sum_{i=1}^n \xi_i A_i$ which correspond to the m eigenvalues $\sigma_1, \ldots, \sigma_m$. As a result, if $\sum_{i=1}^n \xi_i A_i$ is diagonalizable, then we have m plane wave solutions of (8.1). This criteria gives us our definition for hyperbolicity described above.

8.2 Solving Hyperbolic Systems.

In this section, we will solve hyperbolic systems of the form

$$U_t + AU_x = F(x,t) \tag{8.4}$$

where A is a constant-coefficient matrix. Note that if (8.4) is hyperbolic, then A must be a diagonalizable matrix. Therefore, there exists an $m \times m$ invertible matrix Q and an $m \times m$ diagonal matrix Λ such that

$$Q^{-1}AQ = \Lambda.$$

In particular, Λ is the diagonal matrix of eigenvalues and Q is the matrix of eigenvectors. Therefore,

$$A = Q\Lambda Q^{-1}.$$

Substituting this into (8.4), our system becomes

$$U_t + Q\Lambda Q^{-1}U_x = F(x,t). \tag{8.5}$$