# Galaxy formation and evolution
# PAP 318, 5 op, autumn 2020

on Zoom

## Lecture 7: Non-linear evolution of dark matter haloes, 16/10/2020

# On this lecture we will discuss

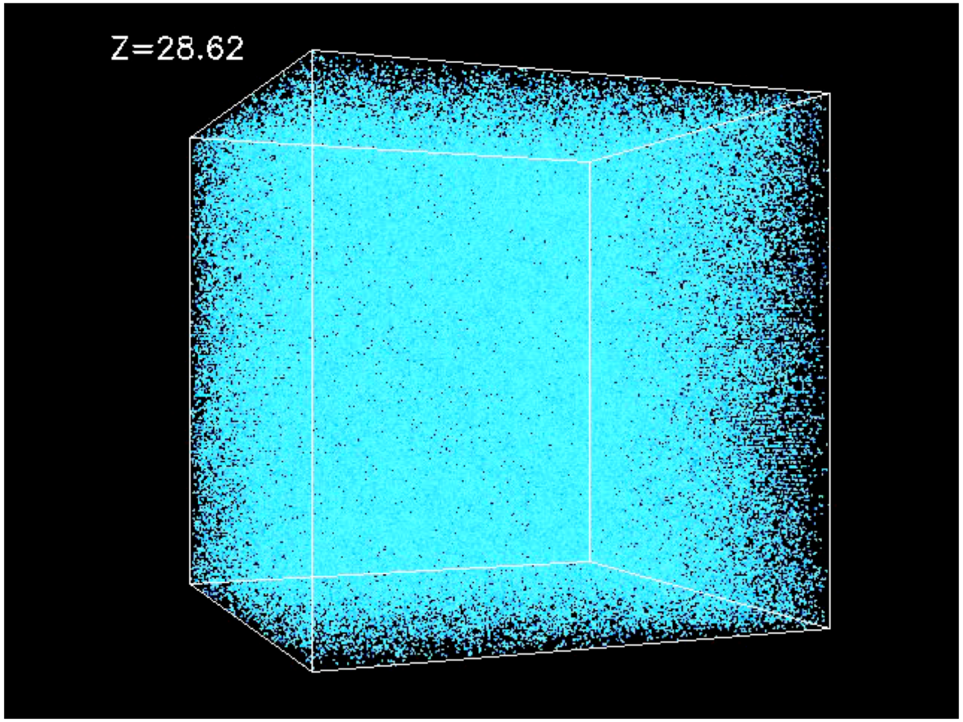
1. The post-recombination Universe. The non-linear evolution of density perturbations.
2. The isotropic top-hat collapse of a spherical perturbation.
3. Violent relaxation and the virial equilibrium.
4. The Zeldovich approximation and its application.
5. The mass function of dark matter haloes and the Press-Schechter halo formalism.
6. The density profiles of dark matter haloes, the NFW profile.
7. The lecture notes correspond to: L: p. 471-477, 482-489 (§16.1, §16.3)
   MBW: p. 177-178, 215-218, 326-335 (§4.1.8, §5.1.1, §7.2)

# 7.1 The post-recombination Universe

- Up to this point we have studied the simple evolution of linear perturbations. Now we turn to study the non-linear evolution of density perturbations which is crucial for forming galaxies.
- The post-recombination Universe, which began at z~1100 after the formation of the CMB can to first approximation be divided into two major phases.
- The first phase corresponds to the time between the epoch of recombination at z~1100 and the epoch of reionisation at z~8-15 when the intergalactic gas was reionised by the first generation of massive stars. The era between z~1100-15 is also called the Dark age of the Universe.
- The second era from z~8-15 to z=0 may be termed the observable Universe of galaxies and from this era we have direct observations.

# The non-linear collapse

- The study of non-linear density perturbations must be carried out using supercomputer simulations in the general case.
- Only certain special cases can be treated analytically, but these analytic calculations are very important as they help us in interpreting and understanding the results of the full numerical simulations.

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# 7.2 Isotropic top-hat collapse I

- The collapse of a uniform spherical density perturbation in an otherwise uniform Universe can be worked out exactly analytically. This model is usually referred to as spherical top-hat collapse.
- The dynamics are the same as those of a closed Universe with $\Omega_0>1$ and the variation of the scale factor of the perturbation $a_p$ is given by the parametric solution (see also Lecture 4):

$$a_p = A(1-\cos\theta) \quad t = B(\theta - \sin\theta)$$

$$A = \frac{\Omega_0}{2(\Omega_0 - 1)} \quad B = \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}$$

- The perturbation reached maximum radius at $\theta=\pi$ and then collapsed to infinite density at $\theta=2\pi$.

# Isotropic top-hat collapse II

- The perturbation stopped expanding, when $da_p/dt=0$ and separated out of the expanding background at $\theta=\pi$. This occurred when the scale factor of the perturbation was $a_p=a_{max}$ and time $t=t_{max}$:

$$a_{\max} = 2A = \frac{\Omega_0}{\Omega_0 - 1} \quad t_{\max} = \pi B = \frac{\pi\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}$$

- The density of the perturbation at the maximum scale factor $\rho_{max}$ can now be related to that of the background $\rho_0$, which we here again assume to be the critical model with $\Omega_0=1$.

$$\frac{\rho_{\max}}{\rho_0} = \frac{\Omega_0 \rho_c a_{\max}^{-3}}{\rho_c a^{-3}}$$

- Here $a_{max}=2A$ and $a=(3/2H_0t)^{2/3}$ should be evaluated at $t_{max}$.

# Isotropic top-hat collapse III

$$\frac{\rho_{\max}}{\rho_0} = \frac{\Omega_0 9\pi^2}{16} \frac{\Omega_0^2}{(\Omega_0 - 1)^3} \frac{(\Omega_0 - 1)^3}{\Omega_0^3} = \frac{9\pi^2}{16} \approx 5.55$$

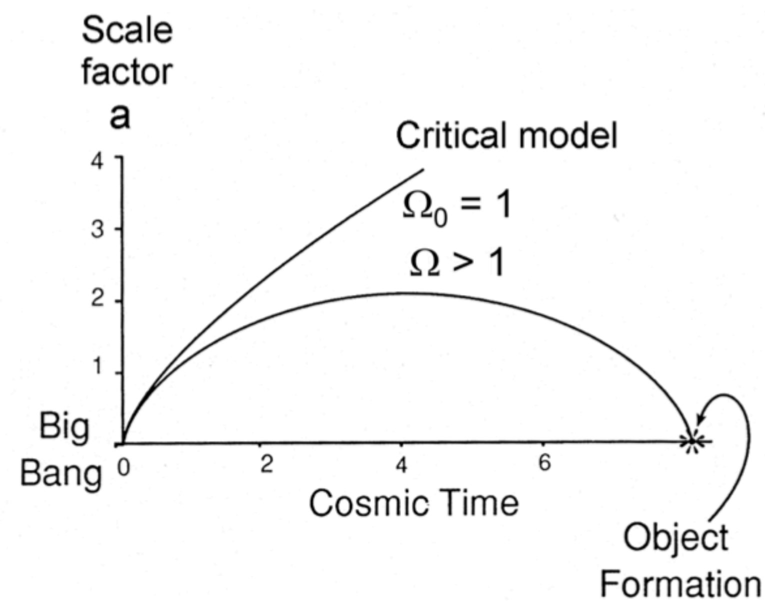
- Thus, by the time the perturbed sphere had stopped expanding, its density was already 5.55 times greater than that of the background density.
- In the spherical top-hat model, the perturbation had no internal pressure and so it collapses formally to infinite density at the time $t=2\pi B$, twice the time it took to reach the maximum expansion. Since $a \propto t^{2/3}$, it follows that the relation between the redshift of maximum expansion $z_{max}$ and the redshift of collapse $z_c$ is:

$$1 + z_c = \frac{1 + z_{\max}}{2^{2/3}}$$

E.g.: If $z_{max}=10$, $z_{coll}=5.93$, $t_{coll}=2t_{max}$.

# Interlude: Perturbing the Friedman solutions

- These results indicate why density perturbations grow only linearly with the cosmic epoch. The instability corresponds to the slow divergence between the variation of the scale factors with cosmic epoch of the model with $\Omega_0$=1 and the model with just slightly greater density.
- The instability develops only algebraically (not exponentially) and therefore galaxy formation by gravitational collapse is difficult.



# Violent relaxation

- Interpreted literally, the isotropic top-hat model would predict that the spherical perturbed region collapses to a black hole, but in practise this does not happen.
- Instead the spherical perturbed region will form some sort of bound object. The temperature of the gaseous baryonic matter will increase until internal pressure gradients become sufficient to balance the attractive force of gravity (see Lecture 8 for more on this).
- For the cold dark matter, during collapse, the gravitational potential will fluctuate wildly as a function of time, which will induce a change in the energies of the particles involved. This process called *violent relaxation* will lead to dynamical equilibrium for the collapsing region, i.e. the object will be virialised and will obey the virial theorem.

# Virial theorem

- The system will thus satisfy the Virial Theorem. At $a_{max}$, the sphere is stationary and all the energy of the system is in the form of gravitational potential energy. For a uniform sphere of radius $r_{max}$ the gravitational potential energy is $E_{grav}=-3GM^2/(5r_{max})$.
- If the system does not lose mass and collapses to half this radius its gravitational potential energy becomes $E_{grav}=-3GM^2/(5r_{max}/2)$ and by conservation of energy the acquired kinetic energy is:

$$E_k = \frac{3GM^2}{5(r_{\rm max}/2)} - \frac{3GM^2}{5r_{\rm max}} = \frac{3GM^2}{5r_{\rm max}}$$

- By collapsing a factor of two in radius from its maximum radius of expansion, the kinetic energy (=internal thermal energy) becomes half the gravitational potential energy and the condition for the virial equilibrium is satisfied: $E_{\rm grav} + 2E_k = 0$

# Final overdensity in the spherical top-hat collapse model I

- The virialisation happens when the perturbation has collapsed by a factor of 2 and the density of the perturbation has increased by a further factor of $2^3=8$.
- The scale factor of the perturbation reached value $a_{max}/2$ at time $t=(1.5+\pi^{-1})t_{max}=1.81t_{max}$. The background density was then:

$$\left(\frac{a_{\max}}{a_{\mathrm{vir}}}\right)^3 = \left(\frac{t_{\max}}{t_{\mathrm{vir}}}\right)^2 = \frac{1}{1.81^2}$$

- An alternative is to define $t_{vir}$ as the time when $a_p=0$, so that $t_{vir}=t(\theta=2\pi)=2t_{max}$. In this case the background density has dropped by $1/2^2=1/4$. The final overdensities are depending on the definition:

$$\Delta \sim 5.55 \times 8 \times 1.81^2 \approx 145$$

$$\Delta \sim 5.55 \times 8 \times 2^2 \approx 178$$

# Final overdensity in the spherical top-hat collapse model II

- Thus we can conclude that perturbations form gravitationally bound structures when they become 150-200 times as dense as the background. Often the overdensity definition of 200 is used to define the virial mass and radius of a bound structure and these results are consistent with numerical simulations, i.e. $r_{vir}=r_{200}$ and $M_{vir}=M_{200}$.
- Assuming that an overdensity of at least 100 is necessary to form structures and that the galaxies are in virial equilibrium we can estimate the redshift at which an object became virialised as follows (see Longair page 475 for details, note $v^2$ is the velocity dispersion):

$$(1+z_{\rm vir}) \leq 0.47 \left(\frac{v}{100\ {\rm kms}^{-1}}\right)^2 \left(\frac{M}{10^{12} M_\odot}\right)^{-2/3} (\Omega_0 h^2)^{-1/3}$$

- Galaxies: $v \sim 300$ kms$^{-1}$, $M \sim 10^{12}\ M_\odot$ -> $z \leq 7$.
- Galaxy clusters: $v \sim 1000$ kms$^{-1}$, $M \sim 10^{15}\ M_\odot$ -> $z \sim 1$

# 7.3 The Zeldovich approximation I

- Next we will discuss the Zeldovich approximation in which it is assumed that the density perturbations are ellipsoidal with three unequal principal axes, as expected from the superposition of Gaussian fields.
- In the Zeldovich approximation, the development of the perturbations into the non-linear regime is followed in Lagrangian coordinates. If **x** and **r** are the proper and comoving position vectors of the particles of the fluid, the Zeldovich approximation can be written as:

$$\vec{x} = a(t)\vec{r} + b(t)\vec{p}(\vec{r})$$

- The first term on the right-hand side describes the uniform expansion of the background model and the second term the perturbations of the particles' positions about the Lagrangian (comoving) coordinate **r**.

# The Zeldovich approximation II

- Zeldovich showed that, in the coordinate system of the principal axes of the ellipsoid, the motion of the particles in comoving coordinates is described by a 'deformation tensor' *D*:

$$D = \begin{bmatrix} a(t) - \alpha b(t) & 0 & 0 \\ 0 & a(t) - \beta b(t) & 0 \\ 0 & 0 & a(t) - \gamma b(t) \end{bmatrix} .$$

- Because of conservation of mass, the density ρ in the vicinity of any particle can be given as below, where ρ bar is the mean density of matter in the Universe:

$$\rho[a(t) - \alpha b(t)][a(t) - \beta b(t)][a(t) - \gamma b(t)] = \bar{\rho} a^3(t)$$

# The Zeldovich approximation III

- The clever aspect of the Zeldovich approximation is that, although the constants $\alpha$, $\beta$ and $\gamma$ vary from point to point in space depending upon the spectrum of the perturbations, the functions *a(t)* and *b(t)* are the same for all particles. In the case of the $\Omega_0$=1 model:

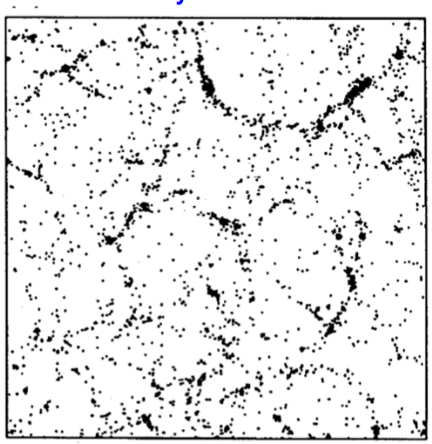
$$a(t) = \frac{1}{1+z} = \left(\frac{t}{t_0}\right)^{2/3} \qquad b(t) = \frac{2}{5}\frac{1}{(1+z)^2} = \frac{2}{5}\left(\frac{t}{t_0}\right)^{4/3}$$

- Here $t_0=2/3H_0$ and the function b(t) has exactly the same dependence upon scale factor as was derived by perturbing the Friedmann solutions (see Lecture 4, page 9).
- If we consider the case in which $\alpha>\beta>\gamma$, collapse occurs most rapidly along the **x**-axis and the density becomes infinite when $a(t)-\alpha b(t)=0$.

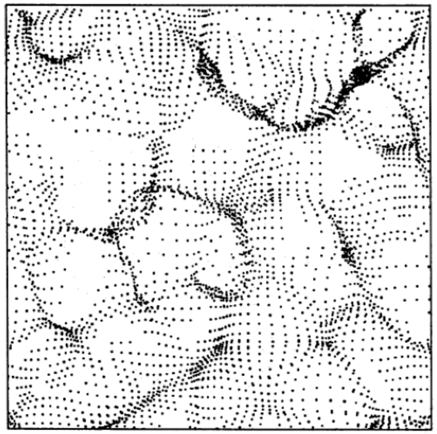
# Zeldovich approximation and comparison with simulations

- At this point, the ellipsoid has collapsed to a ‘pancake’ and the solution breaks down for later times.
- The Zeldovich approximation cannot deal with the more realistic situation in which the collapse of the gas cloud gives rise to strong shock waves, which heat the infalling matter.

N-body Simulation



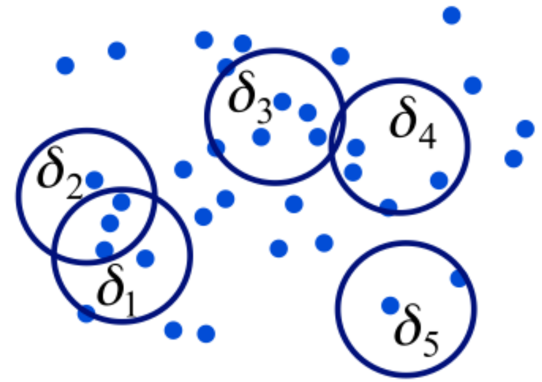
Zeldovich Approximation



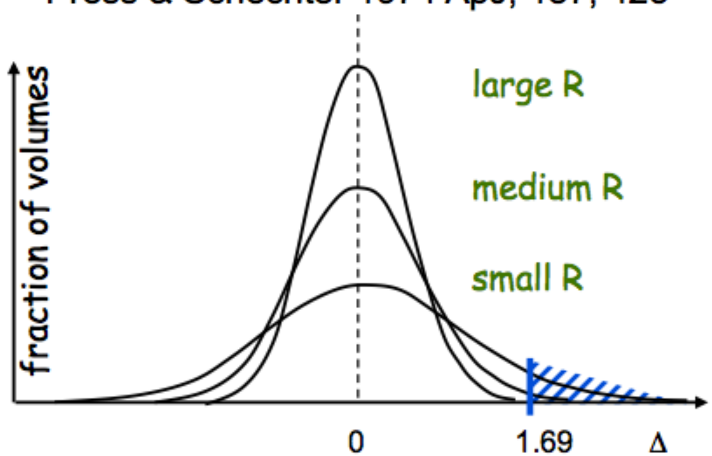
- The results of numerical N-body simu-lations have shown than the Zeldovich approximation is remarkable accurate in describing the non-linear stages of the collapse of large-scale structure up to the point that caustics (i.e. shell-crossing) are formed.

# 7.4 Mass function of collapsed haloes

- A smoothly fluctuating density field can be described by randomly scattered spheres with each having some overdensity $\delta$. Some of the spheres have a large enough linear overdensity ($\delta_c$>1.69) that they will eventually collapse and form gravitationally bound objects.
- Now the question is: "*What is the mass function of these objects at any given cosmic epoch?*"




Press & Schechter 1974 ApJ, 187, 425



# Press-Schechter mass function I

- Let us make the assumption that the primordial density perturbations were Gaussian fluctuations. Thus, the phases of the waves which made up the density distribution were random and the probability distribution of the amplitudes of the perturbations were given by:

$$p(\Delta) = \frac{1}{\sqrt{2\pi}\sigma(M)} e^{-\frac{\Delta^2}{2\sigma^2(M)}}$$

- Here $\Delta=\delta\rho/\rho$ is the density contrast associated with perturbations of mass M. Being a Gaussian distribution the mean value is zero with a finite variance $\sigma^2(M)$ and this is exactly the statistical description of the perturbations implicit in the analysis of Lecture 6:

$$\langle\Delta^2\rangle = \left\langle \left(\frac{\delta\rho}{\rho}\right)^2 \right\rangle = \sigma^2(M)$$

# Press-Schechter mass function II

- The assumption is that when the perturbations have developed an amplitude greater than the critical value $\Delta_c$ they evolve rapidly into bound objects through non-linear collapse.
- For fluctuations of a given mass M, the fraction F(M) of those which become bound at a particular epoch is given by:

$$F(M) = \frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\Delta_c}^{\infty} e^{-\frac{\Delta^2}{2\sigma^2(M)}} d\Delta = \frac{1}{2}[1 - \Phi(t_c)]$$

$$\text{where} \quad t_c = \Delta_c / \sqrt{2}\sigma \quad \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

- From Lecture 6 (slide 10) we can use the relation between the mean square density perturbation and the power spectrum of the perturbations:

$$\sigma^2(M) = \left\langle \left( \frac{\delta\rho}{\rho} \right)^2 \right\rangle = \langle \Delta^2 \rangle = AM^{-(3+n)/3}$$

# Press-Schechter mass function III

- We can now also express $t_c$ in terms of the mass distribution:

$$t_c = \frac{\Delta_c}{\sqrt{2}\sigma(M)} = \frac{\Delta_c}{\sqrt{2}A^{1/2}} M^{(3+n)/6} = \left(\frac{M}{M^*}\right)^{(3+n)/6}$$

- Assuming again $\Omega_m$=1, P(k)=$k^n$ it follows that $\Delta(M) \propto a \propto t^{2/3}$ and $\sigma^2(M) \propto \Delta^2(M) \propto t^{4/3}$, thus $A \propto t^{4/3}$ and $M^* \propto A^{3/(3+n)} \propto t^{4/(3+n)}$. This can also be written as:

$$M^* = M_0^* \left(\frac{t}{t_0}\right)^{4/(3+n)}$$

- The fraction of the perturbations with masses in the range M to M+dM:

$$dF = (\partial F/\partial M)dM$$

- In the linear regime M=mean(ρ)V and the space density N(M)dM of these masses is 1/V.

# Press-Schechter mass function IV

- The number density of bound objects is given by:

$$N(M)dM = \frac{1}{V} = -\frac{\bar{\rho}}{M}\frac{\partial F}{\partial M}dM$$

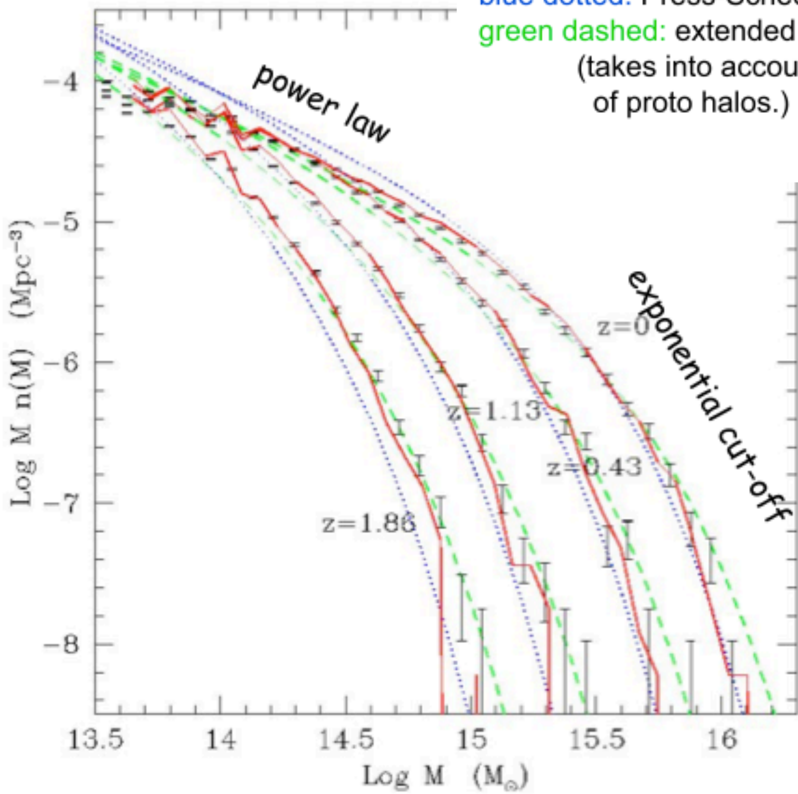
- Combining with the expression of F(M) and given that: $\frac{d\Phi}{dx} = \frac{2}{\sqrt{\pi}}e^{-x^2}$
- We finally get the expression for the mass function:

$$N(M) = \frac{1}{2\sqrt{\pi}}\left(1+\frac{n}{3}\right)\frac{\bar{\rho}}{M^2}\left(\frac{M}{M^*}\right)^{(3+n)/6}\exp\left[-\left(\frac{M}{M^*}\right)^{(3+n)/3}\right]$$

- This expression consists of a power law times an exponential. The time dependence of N(M) has been absorbed into the variation of $M^*$ with cosmic epoch.

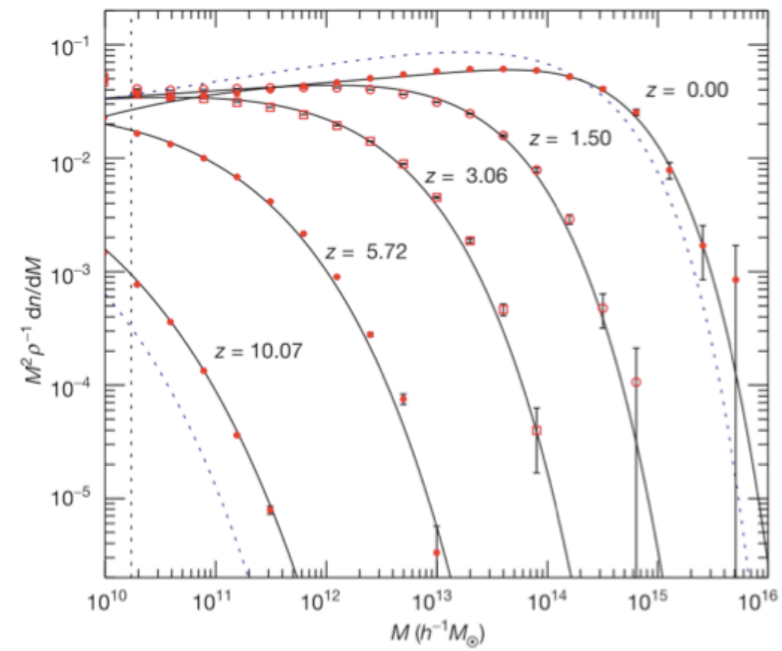
# Numerical tests of the Press-Schechter mass function

solid red lines: simulations
blue dotted: Press-Schechter
green dashed: extended Press-Schechter
(takes into account non-sphericity
of proto halos.)




small R medium R large R

Large number density of low mass objects, $0 < z < 10$

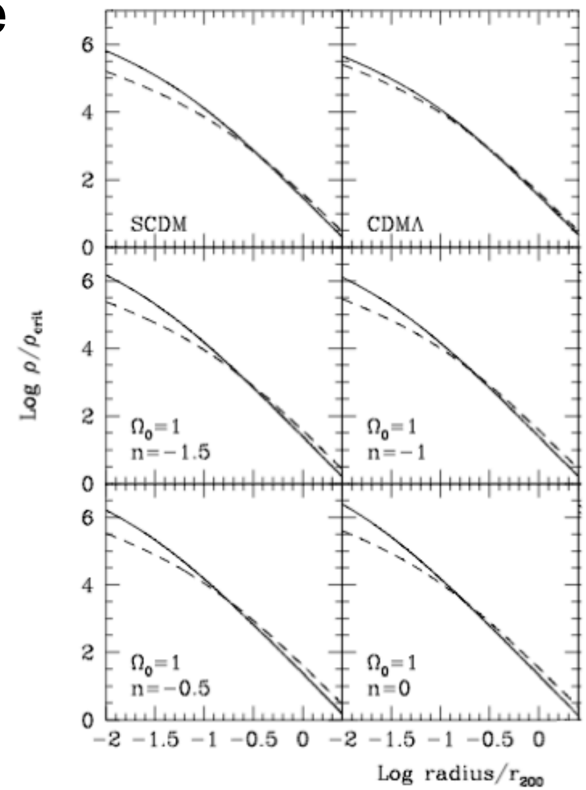



Useful tool for studying development of galaxies and clusters of galaxies in hierarchical scenario.

The Press-Schechter formalism works remarkable well and can be used to analytically predict the DM halo mass-function as a function of redshift.

# 7.5 Density profiles of dark matter haloes

- Numerical simulations have found that the halo profiles for different masses and cosmologies have the same universal functional form $\rho \sim r^{-1}$ at small and $\rho \sim r^{-3}$ at large radii.
- Lower mass haloes have higher densities at the centre because they formed earlier, and the density perturbations at earlier times of the Universe were more concentrated.
- Typically the density profiles are fitted by the **empirical** Navarro-Frenk-White (NFW) profile:




concentration parameter, $c \sim r_{virial}/r_{scale}$

# The NFW profile

- The NFW profile is defined by the equations:

$$\frac{\rho(r)}{\rho_{\mathrm{crit}}} = \frac{\delta_c}{(r/r_s)(1+r/r_s)^2}$$

$$\delta_c = \frac{200}{3}\frac{c^3}{[\ln(1+c) - c/(1+c)]}$$

- $r_s = r_{virial}/c$ is the scale radius and c is the concentration parameter, with typical values of c~5-15.

The NFW profile (Julio Navarro, Carlos Frenk and Simon White).

# What have we learned?

1. The non-linear collapse of spherical perturbation can be worked out analytically and the result is that the overdensity is 5.55 at the time of turn-around, with the final overdensity at virialization being ~150-200.
2. In the Zeldovich approximation the perturbation is assumed to be ellipsoidal. The collapse will be most rapid along the shortest axis of the ellipsoid and as a result 'pancake-structures' form. The agreement between simulations and this approximation are very good, until shell-crossing and the formation of caustics occur.
3. Assuming that the primordial density perturbations were Gaussian fluctuations, we can derive the very useful Press-Schechter formalism for the number density of haloes as a function of redshift and mass. Again, good agreement with simulation results.