Q) ~~...~~ Derive . $\ddot{z}^\mu + \Gamma^\mu_{\nu\sigma} \dot{z}^\nu \dot{z}^\sigma = 0$ from $0 = \delta \int d\tau . g_{\mu\nu}[z(\tau)] \dot{z}^\mu \dot{z}^\nu$
& find Christoffel symbols of $ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2 d\Omega^2$

Ans) $0 = \delta \int d\tau \, g_{\mu\nu}[z(\tau)] \dot{z}^\mu \dot{z}^\nu$

$$= \int d\tau \Big[(\delta g_{\mu\nu}[z(\tau)]) \dot{z}^\mu \dot{z}^\nu + \underbrace{g_{\mu\nu} (\delta \dot{z}^\mu) \dot{z}^\nu + g_{\mu\nu} \dot{z}^\mu \delta \dot{z}^\nu}_{} \Big]$$

$\cancel{= \int d\tau \Big[}$ rearranging dummy indices

$$= \int d\tau \big[(\delta g_{\mu\nu}[z]) \dot{z}^\mu \dot{z}^\nu + 2 g_{\mu\nu} \dot{z}^\mu \delta \dot{z}^\nu \big]$$

$$\delta(g_{\mu\nu}[z]) = g_{\mu\nu}(z+\delta z) - g_{\mu\nu}.$$

$$= \cancel{g_{\mu\nu}[z]} + \partial_\sigma g_{\mu\nu} \delta z^\sigma - \cancel{g_{\mu\nu}[z]}$$

$$= \partial_\sigma g_{\mu\nu} \delta z^\sigma$$

$2 g_{\mu\nu} \dot{z}^\mu \delta \dot{z}^\nu = \cancel{2} \cdot \cancel{2} \; 2 \ddot{z}^\nu g_{\mu\nu} \delta z^\mu + 2 \dot{z}^\nu \left(\frac{\partial}{\partial \tau} g_{\mu\nu} \right) \delta z^\mu - \frac{\partial}{\partial \tau} (2 \dot{z}^\nu g_{\mu\nu} \delta z^\mu)$

from chain rule. ↗

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$$\Rightarrow 0 = \int d\tau \Big[(\partial_\alpha g_{\mu\nu}\, \delta z^{\alpha}) \times \dot{z}^\mu \dot{z}^\nu + 2\ddot{z}^\nu g_{\alpha\nu}\, \delta z^\alpha + 2\dot{z}^\nu \left(\frac{\partial}{\partial\tau} g_{\alpha\nu}\right)\delta z^\alpha - \frac{\partial}{\partial\tau}(2\dot{z}^\nu g_{\mu\nu}\,\delta z^\mu)\Big]$$

upon integ this goes to 0

$$\Rightarrow 0 = \int d\tau \left[(\partial_\alpha g_{\mu\nu}) \dot{z}^\mu \dot{z}^\nu + 2\ddot{z}^\nu g_{\alpha\nu} + 2\dot{z}^\nu \left(\frac{\partial}{\partial\tau} g_{\alpha\nu}\right)\right]\delta z^\alpha$$

$$\Rightarrow (\partial_\alpha g_{\mu\nu}) \dot{z}^\mu \dot{z}^\nu + 2\ddot{z}^\nu g_{\alpha\nu} + 2\dot{z}^\nu \frac{\partial}{\partial z^\mu} g_{\alpha\nu} \cdot \frac{\partial z^\mu}{\partial\tau} = 0$$

$$\Rightarrow (\partial_\alpha g_{\mu\nu}) \dot{z}^\mu \dot{z}^\nu + 2\ddot{z}^\nu g_{\alpha\nu} + 2\dot{z}^\nu \partial_\mu g_{\alpha\nu} \cdot \dot{z}^\mu = 0$$

multiplying with ~~$g_{\alpha\nu}$~~ $\frac{1}{2} g^{\alpha\nu}$:-

~~$\ddot{z}^\nu$~~

~~$\Rightarrow \ddot{z}^\nu g_{\alpha\nu} + \frac{1}{2}\partial_\alpha g_{\mu\nu} + \partial_\mu$~~

$$\Rightarrow \ddot{z}^\mu g_{\alpha\mu} + \left[\frac{1}{2}(\partial_\alpha g_{\mu\nu}) + \partial_\mu g_{\alpha\nu}\right] \dot{z}^\mu \dot{z}^\nu = 0$$

Raising index, we get :-

$$\ddot{z}^\mu + \Gamma^\mu_{\nu\alpha} \dot{z}^\nu \dot{z}^\alpha = 0$$

Now, for $ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2 d\Omega^2$

$$= e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

~~$\Rightarrow g^{\mu\nu}$~~ $g_{\mu\nu}$:-

$g_{00} = e^{\nu(r,t)}$, $g_{11} = -e^{\lambda(r,t)}$; $g_{22} = -r^2$; $g_{33} = -r^2 \sin^2\theta$

$$\Rightarrow g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu = g_{00} \dot{z}^0 \dot{z}^0 + g_{11}(\dot{z}^1)^2 + g_{22}(\dot{z}^2)^2 + g_{33}(\dot{z}^3)^2$$

$$= e^{\nu(r,t)} (\dot{t})^2 - e^{\lambda(r,t)} (\dot{r})^2 - r^2 \dot{\theta}^2 - r^2 \sin^2\theta\, \dot{\phi}^2$$

$$= e^{\nu(r,t)} - e^{\lambda(r,t)} (\dot{r})^2 - r^2 \dot{\theta}^2 - r^2 \sin^2\theta\, \dot{\phi}^2$$

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$\Rightarrow 0 = \delta \int d\tau \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]$

$\Rightarrow 0 = \int d\tau . \delta \left[e^\nu - e^\lambda (\dot{r})^2 - r^2 \dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2 \right]$

→ Variation in t i.e. $\underline{\delta t}$ i.e. $t \to t + \delta t$

$\delta(e^{\nu(r,t)}) = e^{\nu(r,t+\delta t)} - e^{\nu(r,t)}$

$= e^{\nu(r,t)} + \left(\frac{\partial e^{\nu(r,t)}}{\partial t} \right) \delta t - e^{\nu(r,t)}$

$= \underline{\dot{\nu}(r,t) e^{\nu(r,t)} \delta t}$

$\delta e^\lambda = \dot{\lambda}(r,t) e^{\lambda(r,t)} \delta t$

$\Rightarrow 0 = \int d\tau \left(\dot{\nu}(r,t) e^{\nu(r,t)} - \dot{\lambda}(r,t) e^{\lambda(r,t)} \dot{r}^2 \right) \delta t$

$\Rightarrow \frac{\dot{\lambda}(r,t)}{\dot{\nu}(r,t)} e^{\lambda-\nu} \dot{r}^2 = 0$

T^0_{11} & $T^0_{\mu\nu}$ → otherwise $= 0$

→ Variation in r: $r \to r + \delta r$

$\delta(e^\nu) = \frac{\partial \nu(r,t)}{\partial r} e^\nu \delta r = \nu' e^\nu \delta r$

$\delta(e^\lambda) = \lambda' e^\lambda \delta r$

$\delta(\dot{r})^2 = 2 \dot{r} \delta \dot{r} = 2\dot{r} \frac{\partial(\delta r)}{\partial t}$

$\Rightarrow 0 = \int d\tau \left[\delta(e^\nu) - \delta(e^\lambda) \dot{r}^2 - e^\lambda \delta(\dot{r}^2) - 2r\dot{\theta}^2 \delta(r) - 2r \sin^2\theta . \dot{\phi}^2 \delta(r) \right]$

$\Rightarrow 0 = \int d\tau \left[\nu' e^\nu \delta r - \lambda' e^\lambda (\dot{r})^2 \delta r - 2 e^\lambda \dot{r} \frac{\partial(\delta r)}{\partial t} - 2r\dot{\theta}^2 \delta(r) - 2r \sin^2\theta \dot{\phi}^2 \delta(r) \right]$

$\Rightarrow 0 = \int d\tau \left[\nu' e^\nu - \lambda' e^\lambda (\dot{r})^2 - 2 \frac{\partial}{\partial t}(e^\lambda \dot{r}) - 2r\dot{\theta}^2 - 2r\sin^2\theta \dot{\phi}^2 \right] \delta r$

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$$\Rightarrow \nu' e^{\nu} - \lambda' e^{\lambda} \dot{r}^2 - 2\dot{\lambda} e^{\lambda} \dot{r} - 2e^{\lambda} \ddot{r} - 2r\dot{\theta}^2 - 2r\sin^2\theta\, \dot{\phi}^2 = 0$$

Setting $\mu = 1$

$\Gamma^1_{\mu\nu}$ values can be found :-

$$\Gamma^1_{11} = \frac{\lambda'}{2}, \quad \Gamma^1_{22} = -re^{-\lambda}; \quad \Gamma^1_{00} = \frac{\nu'}{2} e^{\nu-\lambda}$$

$$\Gamma^1_{10} = \frac{\dot{\lambda}}{2}, \quad \Gamma^1_{33} = -r\sin^2\theta\, e^{-\lambda}$$

$\rightarrow$ Similarly for varition in θ; $\theta \rightarrow \theta + \delta\theta$
& ϕ; $\phi \rightarrow \phi + \delta\phi$

We get

$$\Gamma^2_{12} = \frac{1}{r}; \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\& \; \Gamma^3_{23} = \cot\theta; \quad \Gamma^3_{13} = \frac{1}{r}$$