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Q) Prove Bianchi identity:-

$$R^{\mu}{}_{\nu\alpha\beta;\gamma} + R^{\mu}{}_{\nu\gamma\alpha;\beta} + R^{\mu}{}_{\nu\beta\gamma;\alpha} = 0$$

& prove:-

$$\cancel{R} R^{\nu}{}_{\mu;\nu} = \frac{1}{2}\partial_\mu R$$

Ans We prove the identity for ~~a~~ a Local Minkowski Reference System.

Since $\Gamma^{\mu}_{\nu\alpha} = 0$; Covariant derivative reduces to normal derivative.

We first prove:-

$$R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\gamma\alpha;\beta} + R_{\mu\nu\beta\gamma;\alpha} = 0$$

& the raise the first index.

$$R_{\mu\nu\alpha\beta;\gamma} = R_{\mu\nu\alpha\beta,\gamma} \quad \text{(Since Christoffel symbol = 0)}$$

$$\Rightarrow \partial_\gamma R_{\mu\nu\alpha\beta} = \frac{1}{2}\partial_\gamma\left[\partial^2_{\nu\alpha} g_{\mu\beta} - \partial^2_{\nu\beta} g_{\mu\alpha} - \partial^2_{\mu\alpha} g_{\nu\beta} + \partial^2_{\mu\beta} g_{\nu\alpha}\right]$$

$$= \frac{1}{2}\left[\partial^3_{\gamma\nu\alpha} g_{\mu\beta} - \partial^3_{\gamma\nu\beta} g_{\mu\alpha} - \partial^3_{\gamma\mu\alpha} g_{\nu\beta} + \partial^3_{\gamma\mu\beta} g_{\nu\alpha}\right] \quad \text{— (1)}$$

$$\Rightarrow \partial_\beta R_{\mu\nu\gamma\alpha} = \frac{1}{2}\left[\partial^3_{\beta\nu\gamma} g_{\mu\alpha} - \partial^3_{\beta\nu\alpha} g_{\mu\gamma} - \partial^3_{\beta\mu\gamma} g_{\nu\alpha} + \partial^3_{\beta\mu\alpha} g_{\nu\gamma}\right] \quad \text{— (2)}$$

$$\Rightarrow \partial_\alpha R_{\mu\nu\beta\gamma} = \frac{1}{2}\left[\partial^3_{\alpha\nu\beta} g_{\mu\gamma} - \partial^3_{\alpha\nu\gamma} g_{\mu\beta} - \partial^3_{\alpha\mu\beta} g_{\nu\gamma} + \partial^3_{\alpha\mu\gamma} g_{\nu\beta}\right] \quad \text{— (3)}$$

(1) + (2) + (3) $\Rightarrow$

$$\frac{1}{2}\left[\begin{array}{l}\partial^3_{\gamma\nu\alpha} g_{\mu\beta} - \partial^3_{\gamma\nu\beta} g_{\mu\alpha} - \partial^3_{\gamma\mu\alpha} g_{\nu\beta} + \partial^3_{\gamma\mu\beta} g_{\nu\alpha} + \\ \partial^3_{\beta\nu\gamma} g_{\mu\alpha} - \partial^3_{\beta\nu\alpha} g_{\mu\gamma} - \partial^3_{\beta\mu\gamma} g_{\nu\alpha} + \partial^3_{\beta\mu\alpha} g_{\nu\gamma} + \\ \partial^3_{\alpha\nu\beta} g_{\mu\gamma} - \partial^3_{\alpha\nu\gamma} g_{\mu\beta} - \partial^3_{\alpha\mu\beta} g_{\nu\gamma} + \partial^3_{\alpha\mu\gamma} g_{\nu\beta}\end{array}\right] = 0$$

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Thus, $R_{\mu\nu\sigma\beta;\gamma} + R_{\mu\nu\gamma\sigma;\beta} + R_{\mu\nu\beta\gamma;\sigma} = 0$

Multiplying with $g^{\sigma\mu}$:-

$$R^{\sigma}{}_{\nu\sigma\beta;\gamma} + R^{\sigma}{}_{\nu\gamma\sigma;\beta} + R^{\sigma}{}_{\nu\beta\gamma;\sigma} = 0$$

Hence proved.

Now :-

$$R^{\mu}{}_{\nu\alpha\beta;\gamma} + R^{\mu}{}_{\nu\gamma\alpha;\beta} + R^{\mu}{}_{\nu\beta\gamma;\alpha} = 0$$

setting $\alpha = \mu$:-

$$R^{\mu}{}_{\nu\mu\beta;\gamma} + R^{\mu}{}_{\nu\gamma\mu;\beta} + R^{\mu}{}_{\nu\beta\gamma;\mu} = 0.$$

$$R^{\mu}{}_{\nu\mu\beta} = R_{\nu\beta} \quad \text{(Ricci tensor)}$$

$$\& \; R^{\mu}{}_{\nu\gamma\mu} = -R^{\mu}{}_{\nu\mu\gamma} = -R_{\nu\gamma}$$

$$\Rightarrow \quad R_{\nu\beta;\gamma} - R_{\nu\gamma;\beta} + R^{\mu}{}_{\nu\beta\gamma;\mu} = 0$$

Raising one index by multiplying with $g^{\alpha\nu}$:-

$$R^{\alpha}{}_{\beta;\gamma} - R^{\alpha}{}_{\gamma;\beta} + R^{\alpha\mu}{}_{\beta\gamma;\mu} = 0.$$

Setting $\alpha = \gamma$

$$\Rightarrow R^{\alpha}{}_{\beta;\alpha} - R^{\alpha}{}_{\alpha;\beta} + R^{\alpha\mu}{}_{\beta\alpha;\mu} = 0.$$

$$\Rightarrow R^{\alpha}{}_{\alpha} = R \Rightarrow \text{Ricci Scalar.}$$

Rewriting $\alpha = \nu$; $\beta = \mu$ & re-labelling dummy index:

$$R^{\nu}{}_{\mu;\nu} - R_{;\mu} + R^{\alpha\nu}{}_{\mu\alpha;\nu} = 0.$$

$$R^{\alpha\nu}{}_{\mu\alpha} = R^{\nu}{}_{\mu}$$

$$\Rightarrow R^{\nu}{}_{\mu;\nu} - R_{;\mu} + R^{\nu}{}_{\mu;\nu} = 0$$

$$\Rightarrow R^{\nu}{}_{\mu;\nu} = \frac{1}{2} R_{;\mu} = \underline{\frac{1}{2}\partial_\mu R}$$

(Since for scalars covariant = normal derivative)