BIRLA INSTITUTE OF TECHNOLOGY SCIENCE, PILANI

A Project report on

Magnetohydrodynamic Simulations

30th April, 2021

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Academic year 2020-2021

Acknowledgment

I would like to express my gratitude towards Dr. Kinjal Banerjee, the project mentor, for his constant guidance and support during the entirety of the project, providing me with the relevant resources to learn the theory behind Magneto-hydrodynamics and his help in reviewing my understanding of the complex AREPO code.
Lastly, I would like to express my sincere thanks to BITS Pilani K.K. Birla Goa Campus, for providing me the opportunity to work on this project.
Thanks a lot.

Abstract:

Humans have looked at stars for millennia, and wondered what lies in the great expanse of the night sky. In recent decades, we have looked at the Universe with ever growing interest and the technological advances have helped us in our ever complicated journey.

But, with the unimaginably immense size of the universe and the finite speed of light, we can see only so far, both spatially and temporally.

A major question that has always driven astronomers and astrophysicists alike, was how the universe came into being, what were the initial conditions and how did it evolve from that initial state to what we see now.

With the current astronomical advancements, we have a huge amount of data on how the universe looks right now, for us. Extrapolating it to what it could have been in the past and possibly, the future is something scientists have been working on for decades. Naturally, we can only theorize so much, we need the theories to fit in with the actual data we observe.

One way to work with theories is by feeding in the physics that governs the universe at large scales, to simulate it with some initial conditions and let it evolve. We can then compare our results with what we see, update our theory and by fine tuning various parameters, rerun it and repeat.

This is what we achieve to do with Magneto-hydrodynamic simulations. We first try to write the equations of physics that might be used for any cosmologically appropriate setting (Star Clusters, Galaxy evolution, Galaxy Formation, etc) and with a framework in mind, as to how to have a background grid on which we can do simulation, we discretize the above said relations for maximum accuracy possible and code them.

After having a generic system which can run a range of simulations with various initial conditions and other parameters, we run our simulations and try to compare with real observation data.

In this project, we work towards learning the theory behind Astrophysical Fluid Dynamics, (especially the Newtonian MHD limit) and then, using the AREPO Code, try to understand the implementation of MHD with a finite volume approach using Voronoi mesh and TreePM method for gravity and then create and run some custom simulations.

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1 Introduction

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, formation, evolution and merger of galaxies, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.
The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. We look at the mathematical structure of the governing equations and the associated conservation laws and look at their physical interpretation.
Astrophysical fluid dynamics is a theory relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, and also the interstellar medium, the intergalactic medium and cosmology itself.
A fluid description is not applicable

- in solidified regions, like rocky or icy cores of giant planets (under certain conditions) and the crusts of neutron stars
- in regions where the medium is not sufficiently collisional.

A range of important areas of application include:

- Instabilities in astrophysical fluids
- Differential rotation in stars
- Magnetospheres of stars, planets and black holes
- Interacting binary stars
- Tidal disruption and stellar collisions
- Supernovae
- Planetary nebulae
- Star formation and the physics of the interstellar medium
- Astrophysical discs

- Galaxy clusters and the physics of the intergalactic medium
- Cosmology and structure formation
- Jets and winds from stars and discs

There are various types of AFD:

- Hydrodynamics (HD) or gas dynamics: basic model, involves a compressible, inviscid fluid and is Newtonian (i.e. non-relativistic) (The thermal physics of the fluid may be treated in different ways, either by assuming it to be isothermal or adiabatic, or by including radiative processes in varying levels of detail.)
- Magnetohydrodynamics (MHD): generalizes HD by including the dynamical effects of a magnetic field. Often the fluid is assumed to be perfectly electrically conducting (ideal MHD)
- Radiation (magneto)hydrodynamics (RMHD): MHD+ dynamical (rather than thermal) effects of radiation. Dissipative effects such as viscosity and resistivity can be included.

One can also include general relativistic effects (GMHD/GRMHD).
Compressibility and gravitation are often important in AFD, while magnetic fields, radiation forces and relativistic phenomena are important in some applications. Effects that are often unimportant in AFD include viscosity, surface tension and the presence of solid boundaries.

2 Theory behind Astrophysical Fluid Dynamics

For majority of the theoretical understanding, we follow the Lecture Notes on Astrophysical Fluid Dynamics by Gordon Oglivie [1].

2.1 Ideal Gas Dynamics

A fluid is characterized by a velocity field $\vec{u}(\vec{x}, t)$ and two independent thermodynamic properties. Most useful are the dynamical variables: the pressure $p(\vec{x}, t)$ and the mass density $\rho(, t)$. Other properties, e.g. temperature T, can be regarded as functions of p and ρ. The specific volume (volume per unit mass) is $v = 1/\rho$. We neglect the possible complications of variable chemical composition associated with chemical and nuclear reactions, ionization and recombination.

We now look at the Eulerian and the Lagrangian viewpoints.
In the Eulerian viewpoint we consider how fluid properties vary in time at a point that is fixed in space, i.e. attached to the (usually inertial) coordinate system. The Eulerian time-derivative is simply the partial differential operator:

$$\textit{Eulerian time-derivative} \implies \frac{\partial}{\partial t}$$

In the Lagrangian viewpoint we consider how fluid properties vary in time at a point that moves with the fluid at velocity $\vec{u}(\vec{x}, t)$. The Lagrangian time-derivative is then:

$$\textit{Lagrangian time-derivative} \implies \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}.\vec{\nabla} \tag{1}$$

We now proceed to defining material structures and their evolution. A material point is an idealized fluid element, a point that moves with the bulk velocity $\vec{u}(\vec{x}, t)$ of the fluid. The true particles of which the fluid is composed have in addition a random thermal motion. Material curves, surfaces and volumes are geometrical structures composed of fluid elements; they move with the fluid flow and are distorted by it.
An infinitesimal material line element, represented by $\delta\vec{x}$, as seen in figure 1 evolves as:

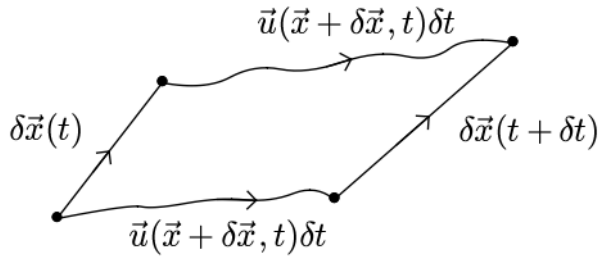

Figure 1: Evolution of an Infinitesimal Line Element

$$\begin{aligned}\frac{D(\delta\overrightarrow{x})}{Dt} &= \frac{\partial(\delta\overrightarrow{x})}{\partial t} + \overrightarrow{u}.\overrightarrow{\nabla}(\delta\overrightarrow{x}) \\ &= \frac{\partial(\delta\overrightarrow{x})}{\partial t} + u_j\frac{\partial\delta x_i}{\partial x_j} \\ &= \delta\overrightarrow{u}\end{aligned}$$

Using the figure 1, we can write $\delta\overrightarrow{u}$ as:

$$\begin{aligned}\delta\overrightarrow{u} &= \frac{\overrightarrow{u}(\overrightarrow{x}+\delta\overrightarrow{x},t)\delta t - \overrightarrow{u}(\overrightarrow{x},t)\delta t}{\delta t} \\ &= \overrightarrow{u}(\overrightarrow{x},t) + (\delta\overrightarrow{x}.\overrightarrow{\nabla}\overrightarrow{u}) - \overrightarrow{u}(\overrightarrow{x},t)\end{aligned}$$

To get:

$$\delta\overrightarrow{u} = (\delta\overrightarrow{x}.\overrightarrow{\nabla})\overrightarrow{u} \tag{2}$$

Similarly, we can define the Infinitesimal material surface and volume elements from two or three material line elements according to the vector product and the triple scalar product as:

$$\delta\overrightarrow{S} = \delta\overrightarrow{x}^{(1)} \times \delta\overrightarrow{x}^{(2)}, \quad \delta V = \delta\overrightarrow{x}^{(1)} \cdot \delta\overrightarrow{x}^{(2)} \times \delta\overrightarrow{x}^{(3)}. \tag{3}$$

As shown in figure 2.

They therefore evolve according to:

$$\frac{\mathrm{D}\delta\overrightarrow{S}}{\mathrm{D}t} = (\overrightarrow{\nabla}\cdot\overrightarrow{u})\delta\overrightarrow{S} - (\overrightarrow{\nabla}\overrightarrow{u})\cdot\delta\overrightarrow{S}, \quad \frac{\mathrm{D}\delta\mathrm{V}}{\mathrm{D}t} = (\overrightarrow{\nabla}\cdot\overrightarrow{u})\delta\mathrm{V} \tag{4}$$

Let us now talk about the equation of mass conservation. It has the typical form:

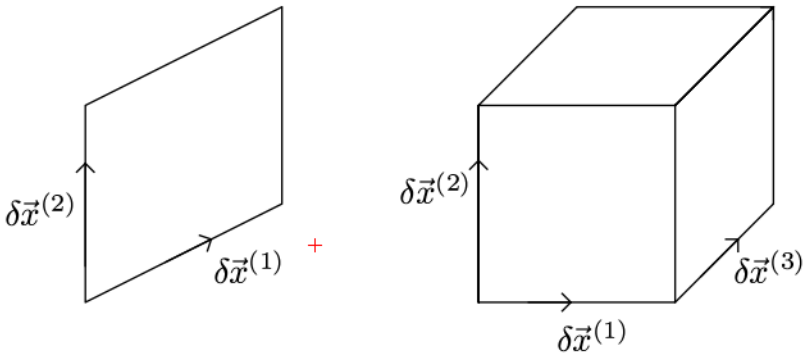

Figure 2: Material Surface and Volume elements

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overrightarrow{u}) = 0 \tag{5}$$

ρ is the mass density (mass per unit volume) and $\rho \overrightarrow{u}$ is the mass flux density (mass flux per unit area). An alternative form of the same equation is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \overrightarrow{\nabla} \cdot \overrightarrow{u} \tag{6}$$

If $\delta m = \rho \delta V$ is a material mass element, it can be seen that mass is conserved in the form

$$\frac{\mathrm{D}\delta m}{\mathrm{D}t} = 0$$

Now we will look at the equation of motion. We use Newton's second law per unit volume with gravitational and pressure forces to write

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p \tag{7}$$

$\Phi(\overrightarrow{x}, t)$ is the gravitational potential and $g = -\nabla\Phi$ is the gravitational field. The force due to pressure acting on a volume V with bounding surface S is

$$-\int_S p \, \mathrm{d}\overrightarrow{S} = \int_V (-\overrightarrow{\nabla} p) \mathrm{d}V.$$

Since we are dealing with ideal gas dynamics, Viscous forces are neglected.

We will now look at the Poisson's equation. The gravitational potential is related to the mass density by Poisson's equation,

$$\nabla^2\Phi = 4\pi G\rho, \tag{8}$$

where G is Newton's constant. The solution

$$\Phi(\overrightarrow{x}, t) = \Phi_{\rm int} + \Phi_{\rm ext} = -G\int_V \frac{\rho\,(\overrightarrow{x}', t)}{|\overrightarrow{x}' - \overrightarrow{x}|}\mathrm{d}^3\overrightarrow{x}' - G\int_{\hat{V}} \frac{\rho\,(\overrightarrow{x}', t)}{|\overrightarrow{x}' - \overrightarrow{x}|}\mathrm{d}^3\overrightarrow{x}' \tag{9}$$

generally involves contributions from both the fluid region V under consideration and the exterior region $\hat{V}$. A non-self-gravitating fluid is one of negligible mass for which $\Phi_{\rm int}$ can be neglected.

Let us now look at the Thermal Energy Equation.
In the absence of non-adiabatic heating, like viscous dissipation or nuclear reactions, and cooling, like radiation or conduction, Fluid elements undergo reversible thermodynamic changes and preserve their entropy, resulting in conservation of specific entropy (s, entropy per unit mass) in the Lagrangian frame as:

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0 \tag{10}$$

This condition is violated in shocks. The thermal variables (T, s) can be related to the dynamical variables (p, ρ) via an equation of state and standard thermodynamic identities. The most important case is that of an ideal gas together with black-body radiation,

$$p = p_{\rm g} + p_{\rm r} = \frac{k\rho T}{\mu m_{\rm H}} + \frac{4\sigma T^4}{3c} \tag{11}$$

where k is Boltzmann's constant, $m_{\rm H}$ is the mass of the hydrogen atom, σ is Stefan's constant and c is the speed of light. μ is the mean molecular weight (the average mass of the particles in units of $m_{\rm H}$), equal to 2.0 for molecular hydrogen, 1.0 for atomic hydrogen, 0.5 for fully ionized hydrogen and about 0.6 for ionized matter of typical cosmic abundances. Radiation pressure is usually negligible except in the centres of high mass stars and in the immediate environments of neutron stars and black holes. The pressure of an ideal gas is often written in the form $\mathcal{R}\rho T/\mu$, where $\mathcal{R} = k/m_{\rm H}$ is a version of the universal gas constant. We define the first adiabatic exponent

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s \tag{12}$$

which is related to the ratio of specific heat capacities

$$\gamma = \frac{c_{\rm p}}{c_{\rm v}} = \frac{T\left(\frac{\partial s}{\partial T}\right)_p}{T\left(\frac{\partial s}{\partial T}\right)_v}$$

by

$$\Gamma_1 = \chi_\rho \gamma_f \tag{13}$$

where

$$\chi_\rho = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_T \tag{14}$$

can be found from the equation of state. We can then rewrite the thermal energy equation as

$$\frac{{\rm D}p}{{\rm D}t} = \frac{\Gamma_1 p}{\rho}\frac{{\rm D}\rho}{{\rm D}t} = -\Gamma_1 p \nabla \cdot \overrightarrow{u} \tag{15}$$

For an ideal gas with negligible radiation pressure, $\chi_\rho = 1$ and so $\Gamma_1 = \gamma$. Adopting this very common assumption, we write

$$\frac{{\rm D}p}{{\rm D}t} = -\gamma p \overrightarrow{\nabla} \cdot \overrightarrow{u}$$

We now look at some simplified models.

A perfect gas may be defined as an ideal gas with constant $c_{\rm v}, c_{\rm p}, \gamma$ and μ. Equipartition of energy for a classical gas with n degrees of freedom per particle gives $\gamma = 1 + 2/n$. For a classical monatomic gas with $n = 3$ translational degrees of freedom, $\gamma = 5/3$. This is relevant for fully ionized matter. For a classical diatomic gas with two additional rotational degrees of freedom, $n = 5$ and $\gamma = 7/5$. This is relevant for molecular hydrogen. In reality Γ_1 is variable when the gas undergoes ionization or when the gas and radiation pressures are comparable. The specific internal energy (or thermal energy) of a perfect gas is

$$e = \frac{p}{(\gamma - 1)\rho} \quad \left[= \frac{n}{\mu m_{\rm H}}\frac{1}{2}kT\right].$$

(Note that each particle has an internal energy of $\frac{1}{2}kT$ per degree of freedom, and the number of particles per unit mass is $1/\mu m_{\rm H}$.)

A barotropic fluid is an idealized situation in which the relation $p(\rho)$ is known in advance. We can then dispense with the thermal energy equation. e.g. if the gas is strictly isothermal and perfect, then $p = c_{\rm s}^2 \rho$ with $c_{\rm s}$ = constant being the isothermal sound speed. Alternatively, if the gas is strictly homentropic and perfect, then $p = K\rho^\gamma$ with K = constant.

An incompressible fluid is an idealized situation in which ${\rm D}\rho/{\rm D}t = 0$, implying $\overrightarrow{\nabla} \cdot \overrightarrow{u} =$

0. This can be achieved formally by taking the limit $\gamma \to \infty$. The approximation of incompressibility eliminates acoustic phenomena from the dynamics.

The ideal gas law itself is not valid at very high densities or where quantum degeneracy is important.

We now look at the Microphysical basis which is useful to understand the way in which the fluid-dynamical equations are derived from microphysical considerations. The simplest model involves identical neutral particles of mass m of negligible size with no internal degrees of freedom.

Let us look at the Boltzmann equation. Between collisions, particles follow Hamiltonian trajectories in their six-dimensional $(\overrightarrow{x}, \overrightarrow{v})$ phase space:

$$\dot{x}_i = v_i, \quad \dot{v}_i = a_i = -\frac{\partial \Phi}{\partial x_i}$$

The distribution function $f(\overrightarrow{x}, \overrightarrow{v}, t)$ specifies the number density of particles in phase space. The velocity moments of f define the number density $n(\overrightarrow{x}, t)$ in real space, the bulk velocity $\overrightarrow{u}(\overrightarrow{x}, t)$ and the velocity dispersion $c(\overrightarrow{x}, t)$ according to

$$\int f \, \mathrm{d}^3 v = n, \quad \int v f \, \mathrm{d}^3 v = nu, \quad \int |v-u|^2 f \, \mathrm{d}^3 v = 3nc^2. \tag{16}$$

Equivalently,

$$\int v^2 f \, \mathrm{d}^3 v = n\left(u^2 + 3e^2\right) \tag{17}$$

The relation between velocity dispersion and temperature is $kT = mc^2$. In the absence of collisions, f is conserved following the Hamiltonian flow in phase space. This is because particles are conserved and the flow in phase space is incompressible (Liouville's theorem). More generally, f evolves according to Boltzmann's equation,

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + a_j \frac{\partial f}{\partial v_j} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \tag{18}$$

The collision term on the right-hand side is a complicated integral operator but has three simple properties corresponding to the conservation of mass, momentum and energy in collisions:

$$\int m \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3 v = 0, \quad \int m v \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3 v = 0, \quad \int \frac{1}{2} m v^2 \left(\frac{\partial f}{\partial t}\right)_c \mathrm{d}^3 v = 0.$$

The collision term is local in x (not even involving derivatives) although it does involve integrals over $\overrightarrow{v}$. The equation $(\partial f/\partial t)_c = 0$ has the general solution

$$f = f_{\mathrm{M}} = \left(2\pi c^2\right)^{-3/2} n \exp\left(-\frac{|\overrightarrow{v} - \overrightarrow{u}|^2}{2c^2}\right) \tag{19}$$

with parameters n, u and c that may depend on x. This is the Maxwellian distribution.

We now try to derive the fluid equations.
We start by considering a crude model of the collision operator, the *BGK* approximation:

$$\left(\frac{\partial f}{\partial t}\right)_c \approx -\frac{1}{\tau}\left(f - f_{\mathrm{M}}\right) \tag{20}$$

where f_{m} is a Maxwellian distribution with the same n, $\overrightarrow{u}$ and c as f, and τ is the relaxation time. This can be identified approximately with the mean free flight time of particles between collisions. In other words the collisions attempt to restore a Maxwellian distribution on a characteristic time-scale τ. They do this by randomizing the particle velocities in a way consistent with the conservation of momentum and energy. If the characteristic time-scale of the fluid flow is much greater than τ, then the collision term dominates the Boltzmann equation and f must be very close to f_{M}. This is the hydrodynamic limit.

The velocity moments of f_M can be determined from standard Gaussian integrals, as:

$$\begin{aligned}
\int f_m d^3\overrightarrow{v} &= \int (2\pi c^2)^{-\frac{3}{2}} n e^{-\frac{|\overrightarrow{v} - \overrightarrow{u}|^2}{2c^2}} d^3\overrightarrow{v} \\
\textit{Doing a substitution, } \overrightarrow{v} - \overrightarrow{u} &= \overrightarrow{x} \\
\int f_m d^3\overrightarrow{v} &= n(2\pi c^2)^{-\frac{3}{2}} \int e^{-\frac{1}{2c^2}\overrightarrow{x}^2} d^3\overrightarrow{x} \\
\textit{Comparing with } \int e^{-a\overrightarrow{x}^2} d^3\overrightarrow{x} &= \left(\frac{\pi}{a}\right)^{\frac{3}{2}} \\
\int f_m d^3\overrightarrow{v} &= n
\end{aligned}$$

For the second integral, we can do an integral by parts after the same substitution to get a sum of odd integral and a non-zero part. The Integrals with higher velocity moments can be solved by comparing with:

$$\int \overrightarrow{x}^2 e^{-a\overrightarrow{x}^2} d^3\overrightarrow{x} = \frac{1}{8a^3}\left(\frac{\pi}{a}\right)^{\frac{3}{2}}$$

Finally, we get the following set of equations:

$$\begin{array}{cc} \int f_{\mathrm{M}} \mathrm{d}^3 \overrightarrow{v} = n, & \int v_i f_{\mathrm{M}} \mathrm{d}^3 \overrightarrow{v} = n u_i \\ \int v_i v_j f_{\mathrm{M}} \mathrm{d}^3 \overrightarrow{v} = n \left(u_i u_j + c^2 \delta_{ij} \right), & \int v^2 v_i f_{\mathrm{M}} \mathrm{d}^3 v = n \left(u^2 + 5c^2 \right) u_i \end{array} \tag{21}$$

We obtain equations for mass, momentum and energy by taking moments of the Boltzmann equation weighted by $\left(m, mv_i, \frac{1}{2}mv^2\right)$. In each case the collision term integrates to zero because of its conservative properties, and the $\partial/\partial v_j$ term can be integrated by parts. We replace f with f_{M} when evaluating the left-hand sides and note that $mn = \rho$:

$$\begin{array}{c} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x_i} \left(\rho u_i \right) = 0 \\ \dfrac{\partial}{\partial t} \left(\rho u_i \right) + \dfrac{\partial}{\partial x_j} \left[\rho \left(u_i u_j + c^2 \delta_{ij} \right) \right] - \rho a_i = 0, \\ \dfrac{\partial}{\partial t} \left(\dfrac{1}{2} \rho u^2 + \dfrac{3}{2} \rho c^2 \right) + \dfrac{\partial}{\partial x_i} \left[\left(\dfrac{1}{2} \rho u^2 + \dfrac{5}{2} \rho c^2 \right) u_i \right] - \rho u_i a_i = 0. \end{array} \tag{22}$$

These are equivalent to the equations of ideal gas dynamics in conservative form for a monatomic ideal gas ($\gamma = 5/3$). The specific internal energy is $e = \frac{3}{2}c^2 = \frac{3}{2}kT/m$.
This approach can be generalized to deal with molecules with internal degrees of freedom and also to plasmas or partially ionized gases where there are various species of particle with different charges and masses. The equations of MHD ean be derived by including the electromagnetic forces in Boltzmann's equation.

Let us now look at the validity of the fluid approach. The essential idea here is that deviations from the Maxwellian distribution are small when collisions are frequent compared to the characteristic time-scale of the flow. In higher-order approximations these deviations can be estimated, leading to the equations of dissipative gas dynamics including transport effects (viscosity and heat conduction). The fluid approach breaks down if the mean flight time τ is not much less than the characteristic time-scale of the flow, or if the mean free path $\lambda \approx er$ between collisions is not much less than the characteristic length-scale of the flow. λ can be very long (measured in AU or pe) in very tenuous gases such as the interstellar medium, but may still be smaller than the size of the system. Some typical order-of-magnitude estimates: Solar-type star: centre $\rho \sim 10^2 \mathrm{gcm}^{-3}, T \sim 10^7$ K; photosphere $\rho \sim 10^{-7}$ g cm^{-3} $T \sim 10^4$ K; corona $\rho \sim 10^{-15}$ g cm^{-3}, $T \sim 10^6$ K Interstellar medium: molecular clouds $n \sim 10^3$ cm^{-3}, $T \sim 10$ K; cold medium (neutral) $n \sim 10 - 100$ cm^{-3}, $T \sim 10^2 K$; warm medium (neutral/ionized) $n \sim 0.1 - 1$ cm^{-3}, $T \sim 10^4 K$; hot medium (ionized) $n \sim 10^{-3} - 10^{-2}$ cm^{-3}, $T \sim 10^6 K$. The Coulomb cross-section for 'collisions' (i.e. large-angle scatterings) between charged particles (electrons or ions) is $\sigma \approx 1 \times 10^{-4} (T/\mathrm{K})^{-2}$ cm^2. The

mean free path is $\lambda = 1/(n\sigma)$

2.2 Ideal Magnetohydrodynamics

We will start with an elementary derivation of the Magnetohydrodynamics (MHD) equations.

MHD is the dynamics of an electrically conducting fluid (a fully or partially ionized gas or a liquid metal) containing a magnetic field. It is a fusion of fluid dynamics and electromagnetism. Let us first look at Galilean electromagnetism.

The equations of Newtonian gas dynamics are invariant under the Galilean transformation to a frame of reference moving with uniform velocity $\vec{v}$,

$$\vec{x}' = \vec{x} - \vec{v}t, \quad t' = t \tag{23}$$

Under this change of frame, the fluid velocity transforms according to

$$\vec{u}' = \vec{u} - \vec{v} \tag{24}$$

while scalar variables such as p, ρ and Φ are invariant. The Lagrangian time-derivative $\mathrm{D}/\mathrm{D}t$ is also invariant, because the partial derivatives transform according to

$$\nabla' = \nabla, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + v \cdot \nabla$$

In Maxwell's electromagnetic theory the electric and magnetic fields $\vec{E}$ and $\vec{B}$ are governed by the equations

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right), \quad \vec{\nabla} \cdot \vec{E} = \frac{\rho_e}{\varepsilon_0} \tag{25}$$

where μ_0 and ε_0 are the vacuum permeability and permittivity, $\vec{J}$ is the electric current density and ρ_e is the electric charge density.

These equations are invariant under the Lorentz transformation. But they cannot be consistently coupled with those of Newtonian gas dynamics, which are invariant under the Galilean transformation. To derive a consistent Newtonian theory of MHD, valid for situations in which the fluid motions are slow compared to the speed of light, we must use Maxwell's equa-

tions without the displacement current $\varepsilon_0 \partial E/\partial t$,

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{26}$$

The fourth Maxwell equation involving $\vec{\nabla} \cdot \vec{E}$ isn't required, because the charge density is unimportant to our case. One can show that these set of equations are invariant under the Galilean transformation, provided that the fields transform according to:

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}, \quad \vec{B}' = \vec{B}, \quad \vec{J}' = \vec{J} \tag{27}$$

The proof is as follows:

$$\frac{\partial \vec{B}'}{\partial t'} = -\vec{\nabla}' \times \vec{E}'$$

$$\frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{B} = -\vec{\nabla} \times (\vec{E} + \vec{v} \times \vec{B})$$

Since $\vec{\nabla}' \cdot \vec{B}' = \vec{\nabla} \cdot \vec{B} = 0$, *we open the cross product to get*

$$\frac{\partial \vec{B}}{\partial t} + \cancel{(\vec{v} \cdot \vec{\nabla})\vec{B}} = -\vec{\nabla} \times \vec{E} + \cancel{(\vec{v} \cdot \vec{\nabla})\vec{B}}$$

Also, $\vec{\nabla}' \times \vec{B}' = \mu_0 \vec{J}' \rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ *as* $\vec{B}$ *and* $\vec{J}$ *are invariant*

These relations correspond to the limit of the Lorentz transformation for electromagnetic fields when $|\vec{v}| \ll c$ and $|\vec{E}| \ll c|\vec{B}|$. Under the pre-Maxwell theory, the equation of charge conservation takes the simplified form $\nabla \cdot J = 0$; this is analogous to the use of $\vec{\nabla} \cdot \vec{u} = 0$ as the equation of mass conservation in the incompressible (highly subsonic) limit of gas dynamics. The equation of energy conservation takes the simplified form

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\frac{\vec{E} \times \vec{B}}{\mu_0}\right) = 0 \tag{28}$$

in which the energy density, $B^2/2\mu_0$, is purely magnetic (because $|\vec{E}| \ll |\vec{B}|$), while the energy flux density has the usual form of the Poynting vector $\vec{E} \times \vec{B}/\mu_0$.

Let us now look at the induction equation. In the ideal MHD approximation we regard the fluid as a perfect electrical conductor. The electric field in the rest frame of the fluid therefore vanishes, implying that

$$\vec{E} = -\vec{u} \times \vec{B} \tag{29}$$

in a frame in which the fluid velocity is $\overrightarrow{u}(\overrightarrow{x}, t)$. This condition can be regarded as the limit of a constitutive relation such as Ohm's law, in which the effects of resistivity (i.e. finite conductivity) are neglected. From Maxwell's equations, we then obtain the ideal induction equation,

$$\frac{\partial \overrightarrow{B}}{\partial t} = \overrightarrow{\nabla} \times (\overrightarrow{u} \times \overrightarrow{B}) \tag{30}$$

This is an evolutionary equation for $\overrightarrow{B}$ alone, $\overrightarrow{E}$ and $\overrightarrow{J}$ having been eliminated. The divergence of the induction equation,

$$\frac{\partial}{\partial t}(\overrightarrow{\nabla} \cdot \overrightarrow{B}) = 0$$

ensures that the solenoidal character of $\overrightarrow{B}$ is preserved.
An important aspect of EM is the Lorentz force.
A fluid carrying a current density J in a magnetic field B experiences a bulk Lorentz force

$$\overrightarrow{F}_{\mathrm{m}} = \overrightarrow{J} \times \overrightarrow{B} = \frac{1}{\mu_0}(\overrightarrow{\nabla} \times \overrightarrow{B}) \times \overrightarrow{B} \tag{31}$$

per unit volume. This can be understood as the sum of the Lorentz forces on individual particles of charge q and velocity v,

$$\sum qv \times \overrightarrow{B} = \left(\sum q \overrightarrow{v}\right) \times \overrightarrow{B}$$

The electrostatic force can be shown to be negligible in the limit relevant to Newtonian MHD. In Cartesian coordinates:

$$\begin{aligned}\left(\mu_0 \overrightarrow{F}_{\mathrm{m}}\right)_i &= \varepsilon_{ijk}\left(\varepsilon_{jlm}\frac{\partial B_m}{\partial x_l}\right)B_k \\ &= \left(\frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i}\right)B_k \\ &= B_k\frac{\partial B_i}{\partial x_k} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{2}\right).\end{aligned}$$

Thus

$$\overrightarrow{F}_{\mathrm{m}} = \frac{1}{\mu_0}\overrightarrow{B} \cdot \overrightarrow{\nabla}\overrightarrow{B} - \nabla\left(\frac{B^2}{2\mu_0}\right) \tag{32}$$

The first term can be interpreted as a curvature force due to a magnetic tension $T_{\mathrm{m}} = B^2/\mu_0$ per unit area in the field lines; if the field is of constant magnitude then this term is equal to T_{m} times the curvature of the field lines, and is directed towards the centre of curvature. The

second term is the gradient of an isotropic magnetic pressure

$$p_{\mathrm{m}} = \frac{B^2}{2\mu_0}$$

which is also equal to the energy density of the magnetic field. The magnetic tension gives rise to Alfvén waves, which travel parallel to the magnetic field with characteristic speed

$$v_{\mathrm{a}} = \left(\frac{T_{\mathrm{m}}}{\rho}\right)^{1/2} = \frac{B}{\left(\mu_0\rho\right)^{1/2}} \tag{33}$$

the Alfvén speed. This is often considered as a vector Alfvén velocity,

$$v_{\mathrm{a}} = \frac{\overrightarrow{B}}{\left(\mu_0\rho\right)^{1/2}} \tag{34}$$

The magnetic pressure also affects the propagation of sound waves, which become magnetoacoustic waves (or magnetosonic waves). The combination

$$\Pi = p + \frac{B^2}{2\mu_0} \tag{35}$$

is often referred to as the total pressure, while the ratio

$$\beta = \frac{p}{B^2/2\mu_0} \tag{36}$$

is known as the plasma beta.

How consistent are the approximations we made and to what order of magnitude are they accurate?

Three effects neglected in a Newtonian theory of MHD are

- the displacement current in Maxwell's equations (compared to the electric current)
- the bulk electrostatic force on the fluid (compared to the magnetic Lorentz force)
- the electrostatic energy (compared to the magnetic energy)

We can verify the self-consistency of these approximations by using order-of-magnitude estimates or scaling relations. If the fluid flow has a characteristic length-scale L, time-scale T, velocity $U \sim L/T$ and magnetic field B, then the electric field can be estimated from equation (29) as $E \sim UB$. The electric current density and charge density can be estimated from

Maxwell's equations as $J \sim \mu_0^{-1} B/L$ and $\rho_e \sim \varepsilon_0 E/L$. Hence the ratios of the three neglected effects to the terms that are retained in Newtonian MHD can be estimated as follows:

$$\frac{\varepsilon_0 |\partial \overrightarrow{E}/\partial t|}{|\overrightarrow{J}|} \sim \frac{\varepsilon_0 UB/T}{\mu_0^{-1} B/L} \sim \frac{U^2}{c^2}$$
$$\frac{\left|\rho_e \overrightarrow{E}\right|}{|\overrightarrow{J} \times \overrightarrow{B}|} \sim \frac{\varepsilon_0 E^2/L}{\mu_0^{-1} B^2/L} \sim \frac{U^2}{c^2}$$
$$\frac{\varepsilon_0 |\overrightarrow{E}|^2/2}{|\overrightarrow{B}|^2/2\mu_0} \sim \frac{U^2}{c^2}$$

Therefore Newtonian MHD corresponds to a consistent approximation of relativistic MHD for highly subluminal flows that is correct to the leading order in the small parameter U^2/c^2.
Finally, let us summarize the MHD equations.
The full set of ideal MHD equations are:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0 \\ \rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0} (\nabla \times \overrightarrow{B}) \times \overrightarrow{B}&, \\ \frac{\mathrm{D}s}{\mathrm{D}t} &= 0, \\ \frac{\partial \overrightarrow{B}}{\partial t} &= \nabla \times (\overrightarrow{u} \times \overrightarrow{B}) \end{aligned} \tag{37}$$

$\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0$, together with the equation of state, Poisson's equation, ete., as required. Most of these equations can be written in at least one other way that may be useful in different circumstances. These equations display the essential nonlinearity of MHD. When the velocity field is prescribed, an artifice known as the kinematic approximation, the induction equation is a relatively straightforward linear evolutionary equation for the magnetic field. However, a sufficiently strong magnetic field will modify the velocity field through its dynamical effect, the Lorentz force. This nonlinear coupling leads to a rich variety of behaviour. Of course, the purely hydrodynamic nonlinearity of the $u \cdot \nabla u$ term, which is responsible for much of the complexity of fluid dynamics, is still present.
We will now try to physically interpret MHD. There are two aspects to it:

- the advection of $\overrightarrow{B}$ by $\overrightarrow{u}$ (induction equation)
- the dynamical back-reaction of $\overrightarrow{B}$ on $\overrightarrow{u}$ (Lorentz force)

Let us look at the kinematics of the magnetic field.
The ideal induction equation

$$\frac{\partial \overrightarrow{B}}{\partial t} = \overrightarrow{\nabla} \times (\overrightarrow{u} \times \overrightarrow{B})$$

has a beautiful geometrical interpretation: the magnetic field lines are 'frozen in' to the fluid and can be identified with material curves. This is sometimes known as Alfvén's theorem. One way to show this result is to use the identity

$$\nabla \times (\overrightarrow{u} \times \overrightarrow{B}) = \overrightarrow{B} \cdot \overrightarrow{\nabla}\overrightarrow{u} - \overrightarrow{B}(\overrightarrow{\nabla} \cdot \overrightarrow{u}) - \overrightarrow{u} \cdot \overrightarrow{\nabla}\overrightarrow{B} + \overrightarrow{u}(\overrightarrow{\nabla} \cdot \overrightarrow{B})$$

to write the induction equation in the form

$$\frac{\mathrm{D}B}{\mathrm{D}t} = \overrightarrow{B} \cdot \nabla\overrightarrow{u} - \overrightarrow{B}(\overrightarrow{\nabla} \cdot \overrightarrow{u})$$

and use the equation of mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\overrightarrow{\nabla} \cdot \overrightarrow{u}$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\overrightarrow{B}}{\rho}\right) = \left(\frac{\overrightarrow{B}}{\rho}\right) \cdot \nabla u. \tag{38}$$

This is exactly the same equation satisfied by a material line element δx (2). Therefore a magnetic field line (an integral curve of $\overrightarrow{B}/\rho$) is advected and distorted by the fluid in the same way as a material curve.

A complementary property is that the magnetic flux $\delta\Phi = \overrightarrow{B} \cdot \delta\overrightarrow{S}$ through a material surface element is conserved:

$$\begin{aligned}
\frac{\mathrm{D}\delta\Phi}{\mathrm{D}t} &= \frac{\mathrm{D}\overrightarrow{B}}{\mathrm{D}t} \cdot \delta\overrightarrow{S} + \overrightarrow{B} \cdot \frac{\mathrm{D}\delta\overrightarrow{S}}{\mathrm{D}t} \\
&= \left(B_j\frac{\partial u_i}{\partial x_j} - B_i\frac{\partial u_j}{\partial x_j}\right)\delta S_i + B_i\left(\frac{\partial u_j}{\partial x_j}\delta S_i - \frac{\partial u_j}{\partial x_i}\delta S_j\right) \\
&= 0.
\end{aligned}$$

By extension, we have conservation of the magnetic flux passing through any material surface.

Precisely the same equation as the ideal induction equation,

$$\frac{\partial \omega}{\partial t} = \nabla \times (\overrightarrow{u} \times \overrightarrow{\omega}), \tag{39}$$

is satisfied by the vorticity $\omega = \nabla \times u$ in homentropic or barotropic ideal fluid dynamics in the absence of a magnetic field, in which case the vortex lines are 'frozen in' to the fluid. The conserved quantity that is analogous to the magnetic flux through a material surface is the flux of vorticity through that surface, which, by Stokes's theorem, is equivalent to the circulation $\oint \overrightarrow{u} \cdot \mathrm{d}\overrightarrow{x}$ around the bounding curve. However, the fact that ω and u are directly related by the curl operation, whereas in MHD B and $\overrightarrow{u}$ are indirectly related through the equation of motion and the Lorentz force, means that the analogy between vorticity dynamics and MHD is limited in scope.

Let's look at the Lorentz force in detail.

The Lorentz force per unit volume,

$$\overrightarrow{F}_{\mathrm{m}} = \frac{1}{\mu_0} \overrightarrow{B} \cdot \overrightarrow{\nabla} \overrightarrow{B} - \overrightarrow{\nabla} \left(\frac{B^2}{2\mu_0} \right),$$

can also be written as the divergence of the Maxwell stress tensor:

$$\overrightarrow{F}_{\mathrm{m}} = \overrightarrow{\nabla} \cdot \mathbf{M}, \quad \mathbf{M} = \frac{1}{\mu_0} \left(\overrightarrow{B} \overrightarrow{B} - \frac{B^2}{2} \mathbf{I} \right), \tag{40}$$

where I is the identity tensor. (The electric part of the electromagnetic stress tensor is negligible in the limit relevant for Newtonian MHD, for the same reason that the electrostatic energy is negligible.) In Cartesian coordinates

$$\left(\overrightarrow{F}_{\mathrm{m}} \right)_i = \frac{\partial M_{ji}}{\partial x_j}, \quad M_{ij} = \frac{1}{\mu_0} \left(B_i B_j - \frac{B^2}{2} \delta_{ij} \right).$$

If the magnetic field is locally aligned with the x -axis, then

$$\mathbf{M} = \begin{bmatrix} T_{\mathrm{m}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{\mathrm{m}} & 0 & 0 \\ 0 & p_{\mathrm{mn}} & 0 \\ 0 & 0 & p_{\mathrm{m}} \end{bmatrix}$$

showing the magnetic tension and pressure. Combining the ideas of magnetic tension and a frozen-in field leads to the picture of field lines as elastic strings embedded in the fluid. Indeed there is a close analogy between MHD and the dynamics of dilute solutions of long-chain

polymer molecules. The magnetic field imparts elasticity to the fluid.
We now look at differential rotation and torsional Alfvén waves. Let us first consider the kinematic behaviour of a magnetic field in the presence of a prescribed velocity field involving differential rotation. In cylindrical polar coordinates (r, φ, z), let

$$\overrightarrow{u} = r\Omega(r,z)\overrightarrow{e}_{\varphi}$$

Consider an axisymmetric magnetic field, which we separate into poloidal (meridional: and z) and toroidal (azimuthal: φ) parts:

$$\overrightarrow{B} = \overrightarrow{B}_{\mathrm{p}}(r,z,t) + B_{\varphi}(r,z,t)\overrightarrow{e}_{\varphi}$$

Applying the ideal induction equation:

$$\begin{aligned}
\frac{\partial \overrightarrow{B}}{\partial t} &= \overrightarrow{\nabla} \times (\overrightarrow{u} \times \overrightarrow{B}) \\
\overrightarrow{u} \times \overrightarrow{B} &= r\Omega(\overrightarrow{e}_{\varphi} \times \overrightarrow{B}_{p} + B_{\varphi}\cancel{\overrightarrow{e}_{\varphi} \times \overrightarrow{e}_{\varphi}}) \\
\frac{\partial \overrightarrow{B_p}}{\partial t} + \frac{\partial B_{\varphi}}{\partial t}\overrightarrow{e}_{\varphi} &= \overrightarrow{\nabla} \times (r\Omega \overrightarrow{e}_{\varphi} \times \overrightarrow{B_p}) \\
&= \varepsilon_{ijk}\varepsilon_{klm}\nabla_j(r\Omega B_{p_m})e_{\varphi_l} \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\nabla_j(r\Omega B_{p_m})e_{\varphi_l} \\
&= \nabla_j(r\Omega B_{p_j})e_{\varphi_i} - \cancel{\nabla_j(r\Omega B_{p_i})e_{\varphi_j}} \\
&= \nabla_j(r\Omega B_{p_j})\overrightarrow{e}_{\varphi}
\end{aligned}$$

The second term in second last relation is zero because the e_{φ_j} is non-zero for j=2 but the grad φ gives 0 as none of the quantities are φ dependent. Thus, we get:

$$\frac{\partial \overrightarrow{B}_{\mathrm{p}}}{\partial t} = 0, \quad \frac{\partial B_{\varphi}}{\partial t} = r\overrightarrow{B}_{\mathrm{p}} \cdot \nabla\Omega \tag{41}$$

Differential rotation winds the poloidal field to generate a toroidal field. To obtain a steady state without winding, we require the angular velocity to be constant along each magnetic field line:

$$\overrightarrow{B}_{\mathrm{p}} \cdot \nabla\Omega = 0$$

a result known as Ferraro's law of isorotation. There is an energetic cost to winding the field, as work is done against magnetic tension. In a dynamical situation a strong magnetic field tends to enforce isorotation along its length. We now generalize the analysis to allow for axisymmetric torsional oscillations:

$$\overrightarrow{u} = r\Omega(r,z,t)e_{\varphi}$$

The azimuthal component of equation of motion is

$$\rho r \frac{\partial \Omega}{\partial t} = \frac{1}{\mu_0 r} \overrightarrow{B}_{\mathrm{p}} \cdot \nabla \left(r B_\varphi \right).$$

This combines with the induction equation to give

$$\frac{\partial^2 \Omega}{\partial t^2} = \frac{1}{\mu_0 \rho r^2} B_{\mathrm{p}} \cdot \nabla \left(r^2 \overrightarrow{B}_{\mathrm{p}} \cdot \nabla \Omega \right) \tag{42}$$

This equation describes torsional Alfvén waves. For example, if $\overrightarrow{B}_{\mathrm{p}} = B_z e_z$ is vertical and uniform, then

$$\frac{\partial^2 \Omega}{\partial t^2} = v_{\mathrm{a}}^2 \frac{\partial^2 \Omega}{\partial z^2}$$

This is not strictly an exact nonlinear analysis because we have neglected the force balance (and indeed motion) in the meridional plane.

Another example is of force-free fields in regions of low density, such as the solar corona. In such cases, the magnetic field may be dynamically dominant over the effects of inertia, gravity and gas pressure. Under these circumstances we have (approximately) a force-free magnetic field such that

$$(\overrightarrow{\nabla} \times \overrightarrow{B}) \times \overrightarrow{B} = 0$$

Vector fields $\overrightarrow{B}$ satisfying this equation are known in a wider mathematical context as Beltrami fields. Since $\nabla \times \overrightarrow{B}$ must be parallel to $\overrightarrow{B}$, we may write

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \lambda \overrightarrow{B} \tag{43}$$

for some scalar field $\lambda(\overrightarrow{x})$. The divergence of this equation is

$$0 = \overrightarrow{B} \cdot \overrightarrow{\nabla} \lambda$$

so that λ is constant along each magnetic field line. In the special case $\lambda =$ constant, known as a linear force-free magnetic field, the curl of equation (43) results in the Helmholtz equation

$$-\nabla^2 \overrightarrow{B} = \lambda^2 \overrightarrow{B}$$

which admits a wide variety of non-trivial solutions. A subset of force-free magnetic fields

consists of potential or current-free magnetic fields for which

$$\nabla \times \overrightarrow{B} = 0$$

In a true vacuum, the magnetic field must be potential. However, only an extremely low density of charge carriers (i.e. electrons) is needed to make the force-free description more relevant. An example of a force-free field in cylindrical polar coordinates (r, φ, z) is

$$\begin{aligned} \overrightarrow{B} &= B_\varphi(r)e_\varphi + B_z(r)e_z \\ \overrightarrow{\nabla} \times \overrightarrow{B} &= -\frac{\mathrm{d}B_x}{\mathrm{d}r}e_\varphi + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_\varphi\right)\overrightarrow{e}_x \end{aligned}$$

Now $\overrightarrow{\nabla} \times \overrightarrow{B} = \lambda \overrightarrow{B}$ implies

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}B_z}{\mathrm{d}r}\right) = \lambda^2 B_z$$

which is the z component of the Helmholtz equation. The solution regular at $r = 0$ is

$$B_z = B_0 J_0(\lambda r), \quad B_\varphi = B_0 J_1(\lambda r),$$

where J_n is the Bessel function of order n. The helical nature of this field is typical of force-free fields with $\lambda \neq 0$.

When applied to a infinite cylinder (e.g. as a simplified model of a magnetized astrophysical jet), the solution could be extended from the axis to the first zero of J_1 and then matched to a uniform external axial field B_z. In this case the net axial current is zero. Alternatively the solution could be extended from the axis to the first zero of J_0 and matched to an external azimuthal field $B_\varphi \propto r^{-1}$ generated by the net axial current.

3 Understanding the AREPO Code

Now that we are familiar with some basic fundamentals behind MHD, we look at the AREPO Code [2] [3].

3.1 What is AREPO?

AREPO is a massively parallel gravity and magnetohydrodynamics code for astrophysics, designed for problems of large dynamic range.
One of the key features is the discretization of the MHD equations on a moving Voronoi-mesh which combines the superior shock capturing, mixing and convergence properties of grid-based schemes with the Galilean-invariance of SPH. The combination of both has been key for some types of simulations in astrophysics, in particular where the internal gas dynamics of objects moving relative to each other needs to be modeled accurately.

The basic working is as follows. AREPO implements a finite-volume magnetohydrodynamics algorithm on an unstructured, dynamic Voronoi tessellation coupled to a tree-particle-mesh algorithm for the Poisson equation either on a Newtonian or cosmologically expanding spacetime. Time-integration is performed adopting local timestep constraints for each cell individually, solving the fluxes only across active interfaces, and calculating gravitational forces only between active particles, using an operator-splitting approach. This allows simulations with high dynamic range to be performed efficiently. Arepo is a massively distributed-memory parallel code, using the Message Passing Interface (MPI) communication standard and employing a dynamical work-load and memory balancing scheme to allow optimal use of multi-node parallel computers. The employed parallelization algorithms of Arepo are deterministic and produce binary-identical results when re-run on the same machine and with the same number of MPI ranks.

3.2 Overview of the Code

A detailed analysis of the source code is done in AREPO Code Notes.
Initial setup of the simulation: First, the parameter file is read, a consistency check of parameters is done, then routines for setting units, etc are called. This function only does the setup necessary to load the IC file. After the IC file has been loaded and prepared, setup continues. The output files are opened and various modules are initialized. The next output time is de-

termined and various timers are set. Then, the main simulation loop starts.
The main simulation loop iterates over single timesteps and terminates when either
(a) the cpu-time limit is reached;
(b) a 'stop' file is found in the output directory; or
(c) when the simulation ends because we arrived at TimeMax.
If the simulation is started from Initial Conditions,
(a) A Domain decomposition is performed;
(b) The gravitational forces are computed; and
(c) Voronoi mesh is constructed.
The main loop goes as following:

- Find new timesteps
- First gravitational half-kick
- Gradients are calculated (for every cell using Green-gauss gradient estimation)
- Vertex velocities are assigned (determine the speed of the mesh-generating vertices)
- Computation of the hydro flux (compute intercell flux with Riemann solver and update the cells with the fluxes)
- (de)refinement of hydro cells
- Drifting particles to next sync point

Afterwards, the timebins are updated, so different particles might now be active than before.

- (if needed) a new domain decomposition (will also make a new chained-list of synchronized particles)
- Construction of the voronoi mesh
- Computation of the hydro flux
- Update of primitive variables (this step closes off the hydro step)
- Masses and positions are now updated, we now calculate the new forces and potentials.
- Computation of gravitational forces (closes off gravity half-step)
- Second gravitational half-kick

- Do any extra physics, Strang-split (update both primitive and conserved variables as needed)

At the final time, write a restart file, which can be used to continue the simulation beyond the final time. Also output final CPU measurements.

3.3 TreePM Algorithm

We will now look at the TreePM algorithm [4], which is used for the gravity part of the code. This method is a combination of Barnes and Hut tree code and Particle-Mesh code. It combines the automatic inclusion of periodic boundary conditions of PM simulations with the high resolution of tree codes. This is done by splitting the gravitational force into a short range and a long range component.

First, we look at the Tree part of the code.

In this, the simulation volume is taken to be a cube. The tree structure is built out of cells and particles. Cells may contain smaller cells (subcells) within them. Subcells can have even smaller cells within them, or they can contain a particle. We start with the simulation volume and add particles to it. If two particles end up in the same subcell, the subcell is geometrically divided into smaller subcells until each subcell contains either subcells or at most one particle. The cubic simulation volume is the root cell. In three dimensions, each cubic cell is divided into eight cubic subcells . Cells, as structures, have attributes like total mass, location of centre of mass and pointers to subcells. Particles, on the other hand have the traditional attributes like position, velocity and mass. We maintain these as binary trees (1D), Quadtrees (2D) or Octrees (3D).

Force on a particle is computed by adding contribution of other particles or of cells. A cell that is sufficiently far away can be considered as a single entity and we can just add the force due to the total mass contained in the cell from its centre of mass. If the cell is not sufficiently far away then we must consider its constituents, subcells and particles. Whether a cell can be accepted as a single entity for force calculation is decided by the cell acceptance criterion (CAC). We compute the ratio of the size of the cell d and the distance r from the particle in question to its centre of mass and compare it with a threshold value:

$$\theta = \frac{d}{r} \leq \theta_c$$

The error in force increases with θ_c.

As for the Particle Mesh part of the code, it is obvious choice for computing long range interactions. PM codes solve for the gravitational potential in the Fourier space. These use Fast Fourier Transforms (FFT) to compute Fourier transforms, and as FFT requires data to be defined on a regular grid the concept of mesh is introduced. The density field represented by particles is interpolated onto the mesh. Poisson equation is solved in Fourier space and an inverse transform gives the potential (or force) on the grid. This is then differentiated and interpolated to the position of each particle in order to calculate the displacements. Use of a grid implies that forces are not accurate at the scale smaller than the grid cells.

Combining these two, we get the TreePM algorithm.
We wish to split the inverse square force into a long range force and a short range force. The gravitational potential can be split into two parts in Fourier space:

$$\begin{aligned}
\phi_k &= -\frac{4\pi G\varrho_k}{k^2} \\
&= -\frac{4\pi G\varrho_k}{k^2}\exp\left(-k^2r_s^2\right) - \frac{4\pi G\varrho_k}{k^2}\left(1-\exp\left(-k^2r_s^2\right)\right) \\
&= \phi_k^l + \phi_k^s \\
\phi_k^l &= -\frac{4\pi G\varrho_k}{k^2}\exp\left(-k^2r_s^2\right) \\
\phi_k^s &= -\frac{4\pi G\varrho_k}{k^2}\left(1-\exp\left(-k^2r_s^2\right)\right)
\end{aligned} \tag{44}$$

where ϕ^l and ϕ^s are the long range and the short range potentials, respectively. The splitting is done at the scale rs. G is the gravitational coupling constant and ϱ is density. The expression for the short range force in real space is:

$$\mathbf{f}^s(\mathbf{r}) = -\frac{Gm\mathrm{r}}{r^3}\left(\mathrm{erfc}\left(\frac{r}{2r_s}\right) + \frac{r}{r_s\sqrt{\pi}}\exp\left(-\frac{r^2}{4r_s^2}\right)\right) \tag{45}$$

Here, erfc is the complementary error function. These equations describe the mathematical model for force in the TreePM code. The long range potential is computed in the Fourier space, just as in a PM code, but using ϕ_k^l. This potential is then used to compute the long range force. The short range force is computed directly in real space using ϕ_k^s. In the TreePM method this is computed using the tree approximation. The short range force falls rapidly at scales r $\gg$ rs, and hence we need to take this into account only in a small region around each particle.

4 Using the AREPO Code

Now that we are familiar with a rough idea as to how the AREPO Code works in the background, we move on to doing custom simulations. The code generates a datsset for each snapshot timestep that we set in the parameter file. I wrote a stacker code which takes these snapshot images and stacks them to produce the resultant GIFs. These can be found in the GitHub Repository [5] under *test_run_plots*.

4.1 Alfven Waves

We set up the AREPO code to simulate Alfven waves in 1D with the following initial state:

$$\begin{aligned} density_0 &= 1 \\ velocity_0 &= 0 \\ pressure_0 &= 1 \\ gamma &= 5/3 \\ delta &= 10^{-6} \\ Bfield_0 &= 1 \\ k_z &= 2\pi \end{aligned}$$

where delta is the relative velocity perturbation. We let it run for 2 time units with time between snapshots as 0.25.
The pressure and Density don't change with time, whereas the magnetic fields and velocities in z and y direction are sinusoidal.

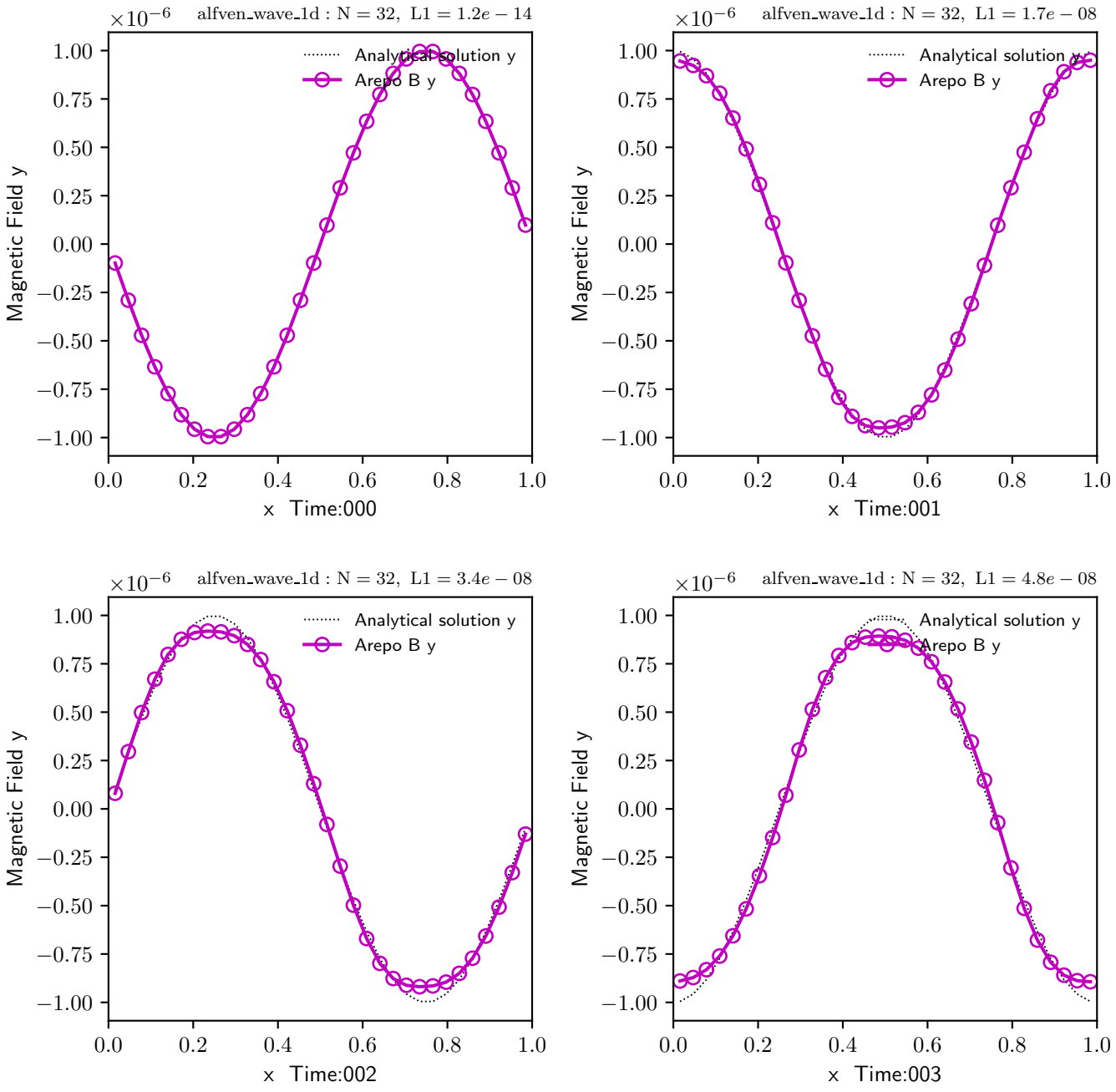

Figure 3: B field in y direction

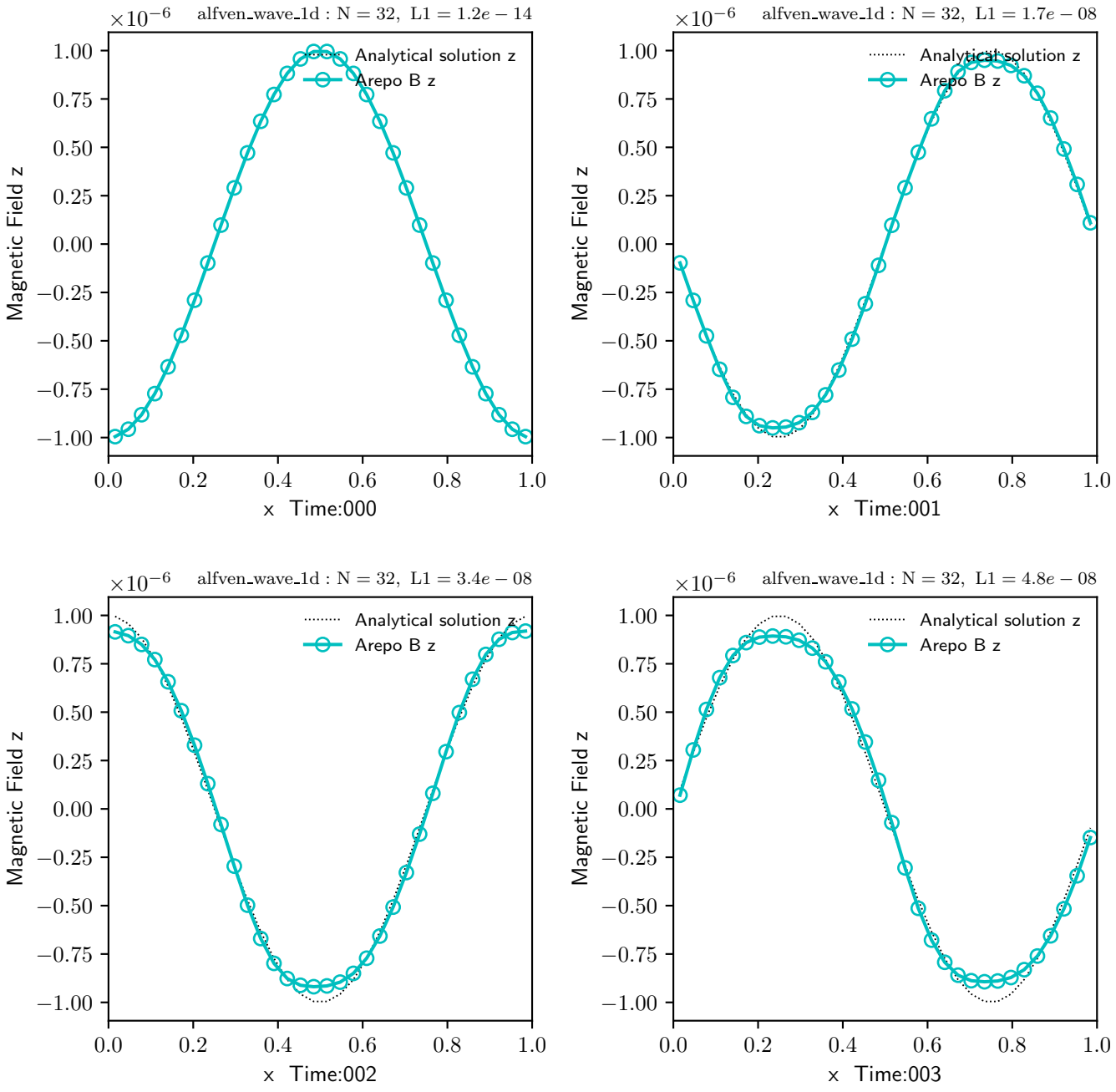

Figure 4: B field in z direction

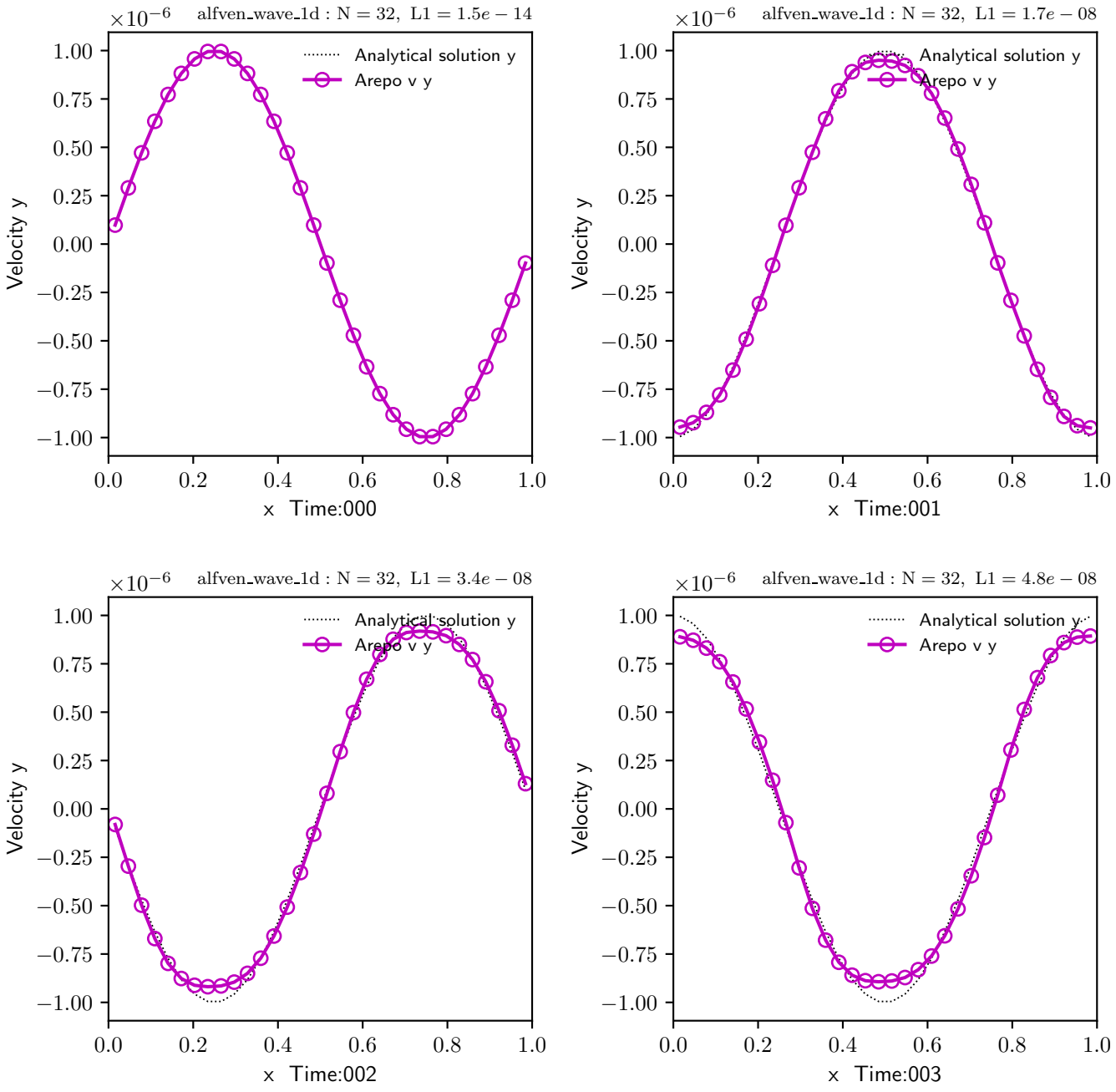

Figure 5: Velocity in y direction

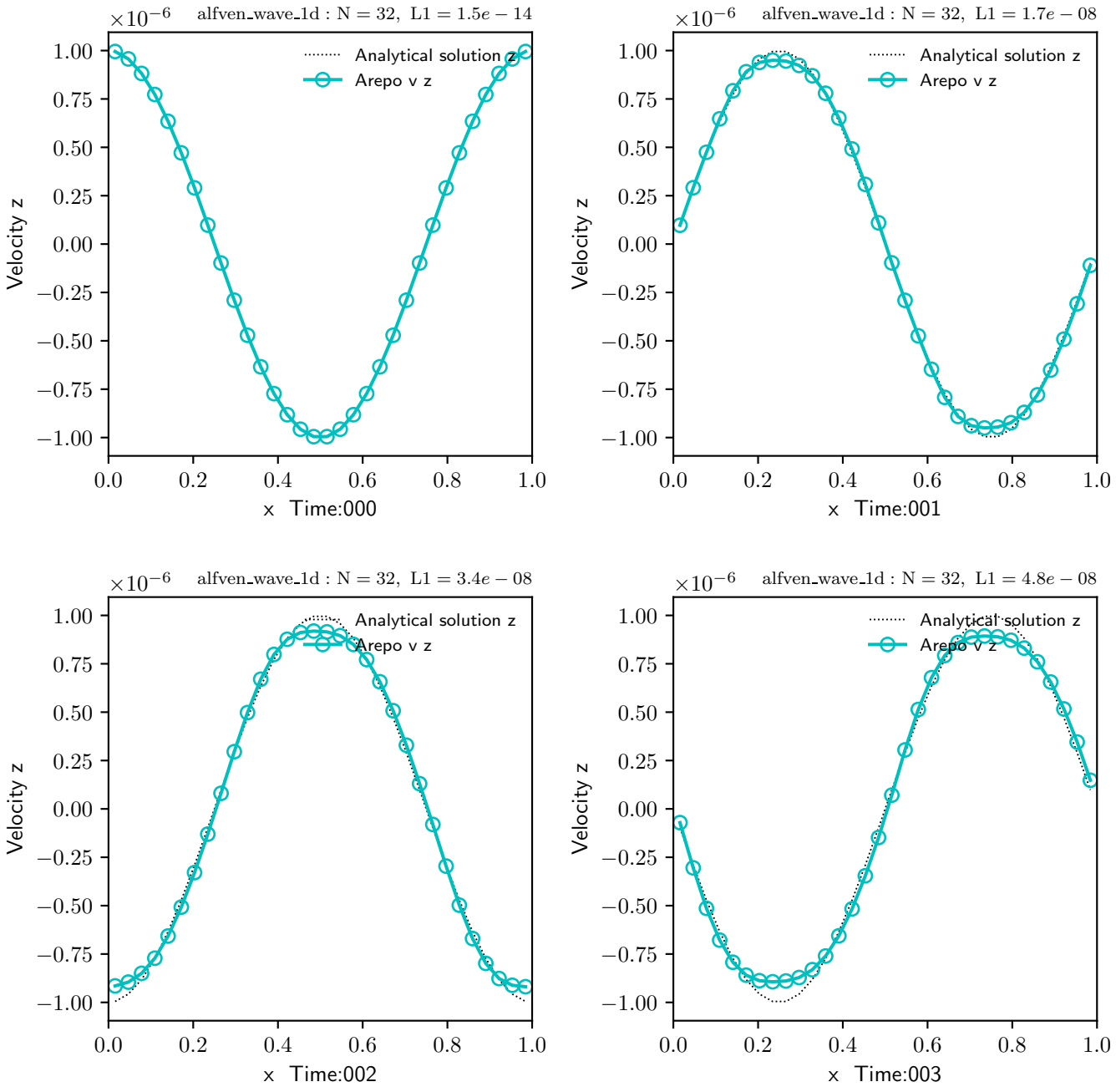

Figure 6: Velocity in z direction

4.2 Shocktube

We set up the AREPO code to simulate a Shocktube in 1D with the following initial state:

$$
\begin{aligned}
position_0 &= 10 \\
density_0 &= 1 \\
density_1 &= 0.125 \\
velocity_0 &= 0 \\
velocity_1 &= 0 \\
pressure_0 &= 1 \\
pressure_1 &= 0.1 \\
gamma &= 1.4
\end{aligned}
$$

$position_0$ is the initial position of discontinuity. We let it run 5 unit of time with 1 unit time between snapshots.
The shock given initially travels through the tube.

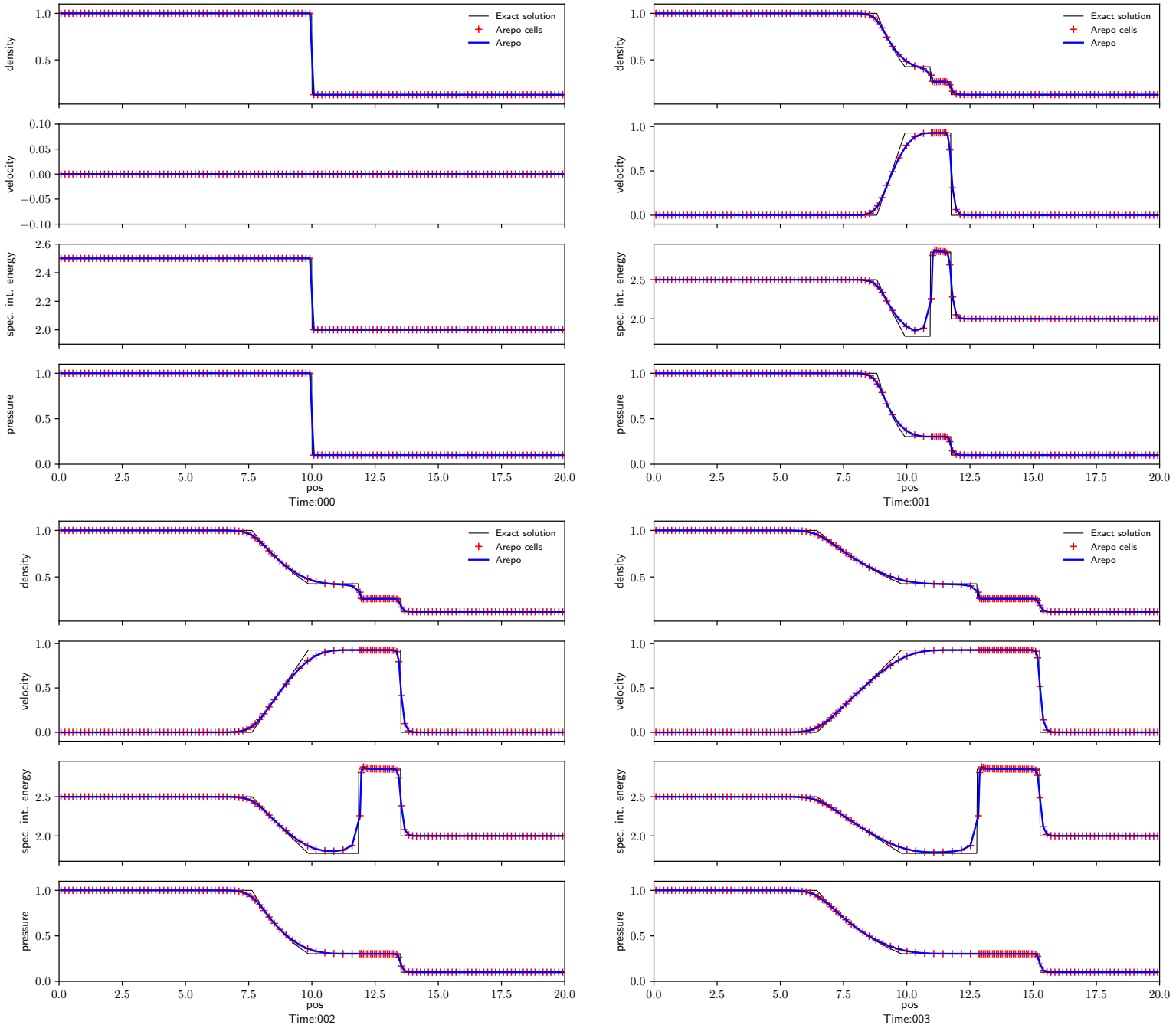

Figure 7: Shocktube Simulation

4.3 Gresho Vortex

We set up the AREPO code to simulate a Gresho vortex in 2D. It is a rotating vortex problem independent of time for the case of inviscid flow. Angular velocity u_φ depends only on radius and the centrifugal force is balanced by the gradient of the pressure p. The radial velocity is zero and the density is one everywhere.

$$\Big(u_\varphi(r), p(r)\Big) = \begin{cases} \left(5r, 5 + \frac{25}{2}r^2\right), & 0 \leq r < 0.2 \\ \left(2 - 5r, 9 - 4\ln 0.2 + \frac{25}{2}r^2 - 20r + 4\ln r\right), & 0.2 \leq r < 0.4 \\ (0, 3 + 4\ln 2), & 0.4 \leq r \end{cases} \tag{46}$$

We use Gresho to check the validity of our code as in the original formulation, the Gresho vortex is a test for the following properties:

- Accuracy of numerical scheme
- Preservation of symmetry: since the analytical solution is pure rotation, the vortex should not change its shape.
- Conservation of angular momentum: the smoothing of the azimuthal velocity profile is a measure how well the code preserves angular momentum

We have the following initial state:

$$\begin{aligned} density_0 &= 1.0 \\ velocity_0 &= 3.0 \\ gamma &= 5/3 \end{aligned}$$

$velocity_0$ is the bulk velocity. We let it run 20 unit of time with 1 unit time between snapshots.

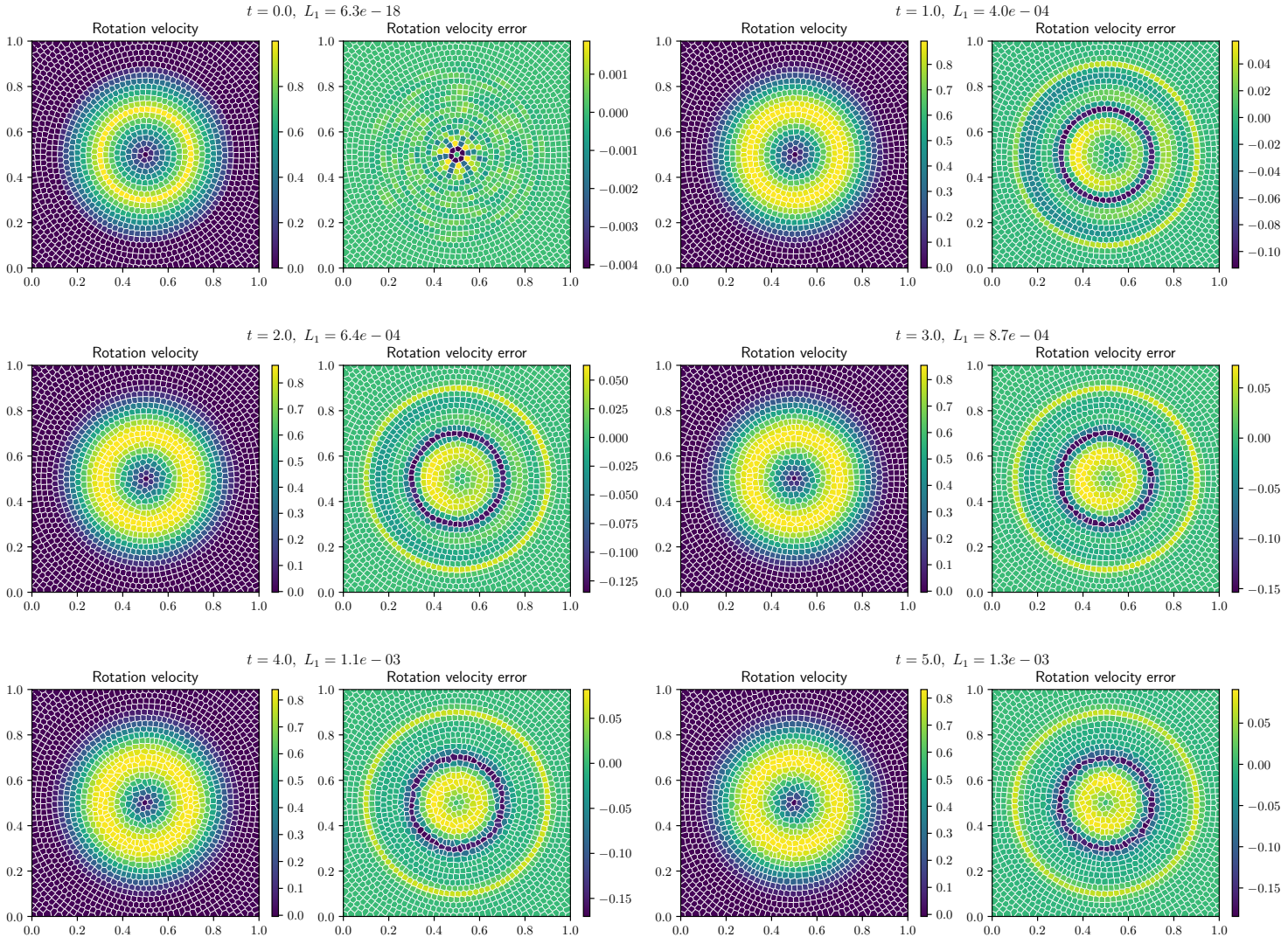

Figure 8: Gresho Vortex

5 Conclusion

The field of Magnetohydrodynamics simulation is a rich one, with years of work put in by hundreds of scientists over decades from setting up the theory to discretizing it just right for our computers to actually writing a modular and working piece of code for various purposes. Much has been achieved till date, with massive simulation runs such as Illustris-TNG, which are a series of large, cosmological magnetohydrodynamical simulations of galaxy formation, aiming to illuminate the physical processes that drive galaxy formation: to understand when and how galaxies evolve into the structures that are observed in the night sky, and to make predictions for current and future observational programs. But much is yet to be achieved as these simulations take thousands of hours even with thousands of cores parallely working together. Optimization of Code, adding more physics, making the current physics more accurate, trying out these simulations for more complex cases- a lot is yet to be explored in the field.

Moving on from these simple 1D and 2D simulations using AREPO, one can go for custom simulations of the Large Scale Structure of the Universe and compare them to real life data and draw conclusions as to how the universe would have evolved over billions of years. One can also use these to simulate galaxies individually, and also look at galaxy mergers. One can also use these simulations to check validity of various theories as well.

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