Find $T_{\mu\nu}$ for a collec^n of N free particles (dust)

$$S_M = -\sum_{q=1}^{N} m_q \int d\tau \sqrt{g_{\mu\nu}[z_q(\tau)]\, \dot z_q^{\mu}(\tau)\, \dot z_q^{\nu}(\tau)}$$

We can write :-

$$S_M = -\sum_{q=1}^{N} m_q \int ds_q$$

Where :

$$\int ds_q = \int d\tau \sqrt{g_{\mu\nu}[z_q(\tau)]\, \dot z_q^{\mu}(\tau)\, \dot z_q^{\nu}(\tau)}$$

So, as calculated earlier in the video for a particular q :-

$$\int ds_q = \int d^4x \sqrt{|g(x)|} \int d\tau \, \frac{\delta^4(x - z_q(\tau))}{\sqrt{|g(z_q)|}} \sqrt{g_{\mu\nu}(z_q)\, \dot z_q^{\mu}\, \dot z_q^{\nu}}$$

& $\delta_g S_{M_q} = \frac{1}{2} \int d^4x \sqrt{|g|}\, (T_{\mu\nu})_q \, \delta g^{\mu\nu}$.

Where $(T_{\mu\nu})_q = 2 \frac{\partial \mathcal{L}_q}{\partial g^{\mu\nu}} - \mathcal{L}_q g_{\mu\nu}$

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Thus, from S_m!

$$\delta S_m = -\delta\left(\sum_{q=1}^{N} m_q \int ds_q\right)$$

$$= -\sum_{q=1}^{N} m_q \left(\delta \int ds_q\right)$$

$$= -\sum_{q=1}^{N} m_q \left(+\frac{1}{2}\int d^4x \sqrt{|g|}\,(T_{\mu\nu})_q\,\delta g^{\mu\nu}\right)$$

$$= \frac{1}{2}\int d^4x\sqrt{|g|}\left(-\sum_{q=1}^{N} m_q (T_{\mu\nu})_q\right)\delta g^{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} = -\sum_{q=1}^{N} m_q (T_{\mu\nu})_q$$

$$(T_{\mu\nu})_q = 2\frac{\partial \mathcal{L}_q}{\partial g^{\mu\nu}(z_q)} - \mathcal{L}_q\, g_{\mu\nu}(z_q)$$

$$\mathcal{L}_q = \int d\tau\,\frac{\delta^4[x-z_q(\tau)]}{\sqrt{|g(z_q)|}}\sqrt{g_{\mu\nu}(z_q)\dot{z}_q^\mu(\tau)\dot{z}_q^\nu(\tau)}$$

$$\frac{\partial \mathcal{L}_q}{\partial g^{\mu\nu}(z_q)} = \int d\tau\,\frac{\delta^4[x-z_q(\tau)]}{|g(z_q)|}\left(\sqrt{|g(z_q)|}\frac{\partial}{\partial g^{\mu\nu}(z_q)}\left(\sqrt{g_{\mu\nu}(z_q(\tau))\dot{z}_q^\mu(\tau)\dot{z}_q^\nu(\tau)}\right) - \sqrt{g_{\mu\nu}(z_q)\dot{z}_q^\mu\dot{z}_q^\nu}\,\frac{\partial\sqrt{|g(z_q)|}}{\partial g^{\mu\nu}(z_q)}\right)$$

$$\Rightarrow \frac{\partial}{\partial g^{\alpha\beta}(z_q)}\left(\sqrt{g_{\mu\nu}(z_q)\dot{z}_q^\mu(\tau)\dot{z}_q^\nu(\tau)}\right)$$

$$= \frac{1\;\dot{z}_q^\mu(\tau)\dot{z}_q^\nu(\tau)}{2\sqrt{g_{\mu\nu}(z_q)\dot{z}_q^\mu\dot{z}_q^\nu}}\frac{\partial}{\partial g^{\alpha\beta}}g_{\mu\nu}$$

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$$\Rightarrow \frac{\partial \mathcal{L}_q}{\partial g^{\mu\nu}(z_q)} = -\frac{1}{2}\int ds\,\frac{\dot{z}_q^\mu\dot{z}_q^\nu\,\delta(\tau-\tau_q)\,\delta^3(x-z_q(\tau))}{\sqrt{g_{\mu\nu}(z_q)\dot{z}_q^\mu\dot{z}_q^\nu}}$$

$$= -\frac{1}{2}\dot{z}_q^\mu\,\dot{z}_q^\nu\,\frac{ds}{d\tau}\,\delta^3(x-z_q(\tau))$$

$$\cancel{\Rightarrow (T_{\mu\nu})_q = m\, \dot{z}_q^{\mu}}$$

$$\Rightarrow \quad (T_{\mu\nu})_q = \dot{z}_{q\mu}\, \dot{z}_{q\nu}\, \frac{dt}{d\tau}\, \delta^3(x - z_q(\tau))$$

$$\Rightarrow \quad T_{\mu\nu} = -\sum_{q=1}^{N} m_q\, \dot{z}_{q\mu}\, \dot{z}_{q\nu}\, \frac{dt}{dt}\, \delta^3(x - z_q(\tau))$$

$$\rho(x) = T^{\mu\nu}(x)\, u_\mu(x)\, u_\nu(x)$$

$$\Rightarrow \rho(x) = \left(-\sum_{q=1}^{N} m_q\, \dot{z}_q^{\mu}\, \dot{z}_q^{\nu}\, \frac{dt}{dt}\, \delta^3(x - z_q(\tau))\right) U_\mu(x)\, U_\nu(x)$$