ASTR 610
Theory of Galaxy Formation

Lecture 11: Structure of Dark Matter Halos

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YALE UNIVERSITY, FALL 2020

The Structure of Dark Matter Halos

In this lecture we examine the detailed structure of dark matter haloes in numerical simulations. We will discuss their density profiles, their shapes, their angular momentum, and their substructure. We will also discuss observational constraints on these quantities.

Topics that will be covered include:

- Virial Relations
- Halo Density Profiles
- NFW and Einasto Profiles
- Halo shapes
- Tidal Torque Theory
- Angular Momentum of Halos
- Halo Substructure

Virial Relations

Before we focus on the results of numerical simulations, it is useful to derive some very general scaling relations for dark matter haloes.

According to SC model, dark matter haloes have an average overdensity well fitted by

$$\boxed{\Delta_{\rm vir} \simeq \frac{18\pi^2 + 82\,x - 39\,x^2}{x+1} \qquad \text{where} \qquad x = \Omega_{\rm m}(z) - 1}$$

(ΛCDM only)

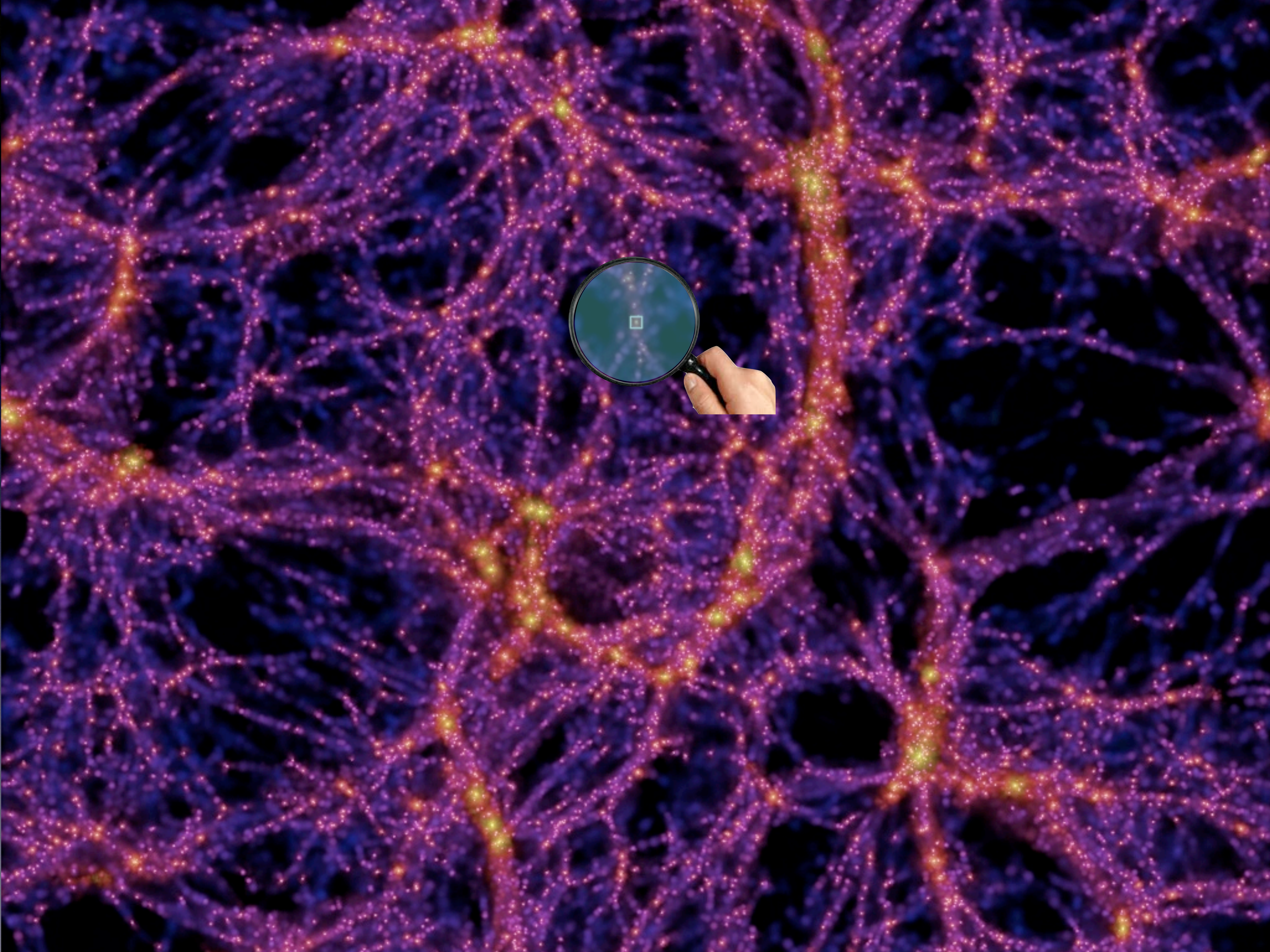
It is common practice to refer to the mass, radius and circular velocity of the halo thus defined as the virial mass, $M_{\rm vir}$, virial radius, $r_{\rm vir}$, and virial velocity, $V_{\rm vir}$.

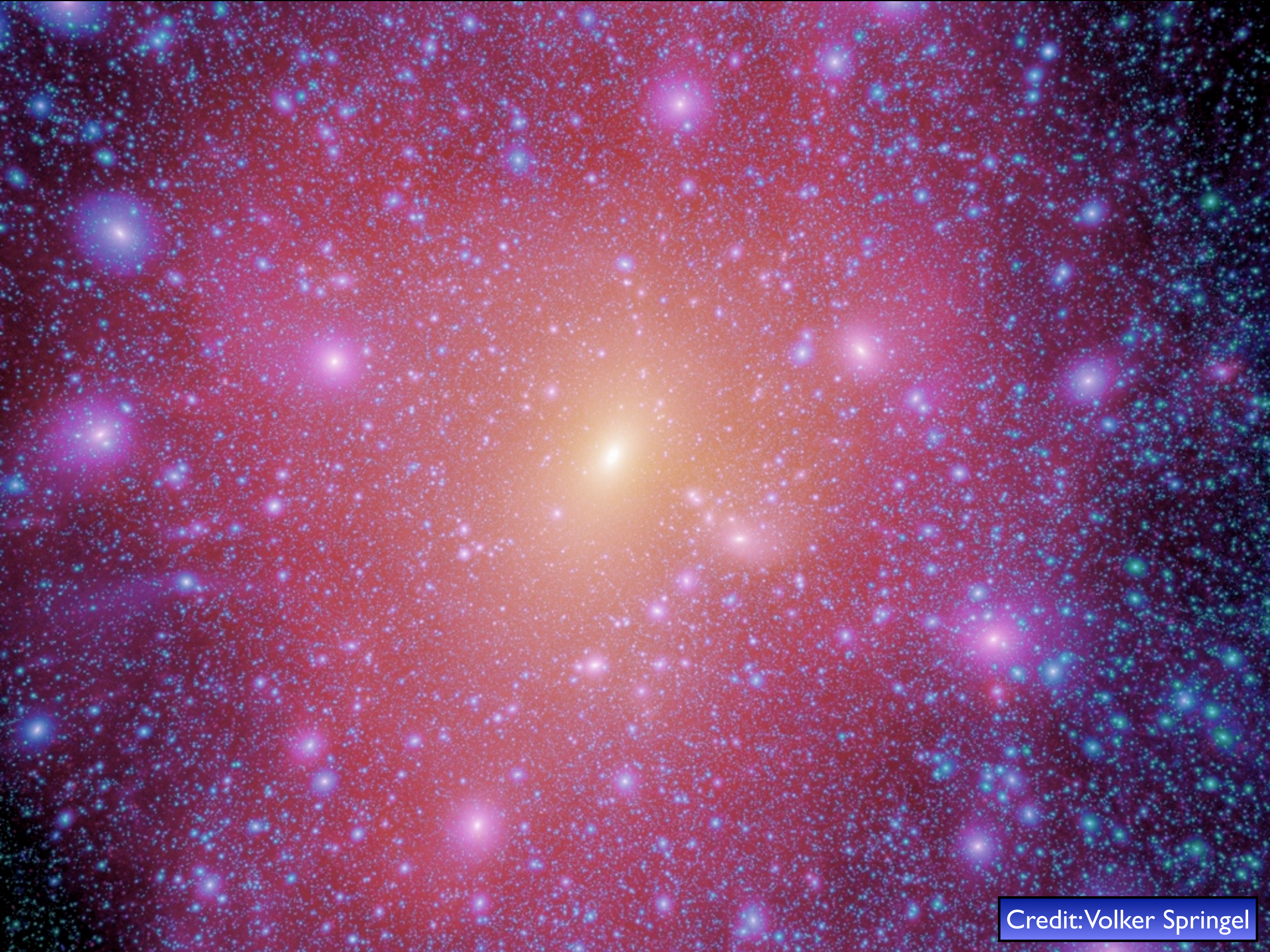
$$\boxed{\bar{\rho}_{\rm h} = \frac{3M_{\rm vir}}{4\pi r_{\rm vir}^3} = \Delta_{\rm vir}(z)\,\Omega_{\rm m}(z)\,\frac{3H^2(z)}{8\pi G}}$$

Using that $V_{\rm vir} \equiv \sqrt{G\,M_{\rm vir}/r_{\rm vir}}$ we then have that

$$r_{\rm vir} \simeq 163\,h^{-1}{\rm kpc}\left[\frac{M_{\rm vir}}{10^{12}h^{-1}M_\odot}\right]^{1/3}\left[\frac{\Delta_{\rm vir}}{200}\right]^{-1/3}\Omega_{\rm m,0}^{-1/3}\,(1+z)^{-1}$$

$$V_{\rm vir} \simeq 163\,{\rm km/s}\left[\frac{M_{\rm vir}}{10^{12}h^{-1}M_\odot}\right]^{1/3}\left[\frac{\Delta_{\rm vir}}{200}\right]^{1/6}\Omega_{\rm m,0}^{1/6}\,(1+z)^{1/2}$$




Credit: Volker Springel

Halo Density Profiles

Density Profiles

As we have seen in Lecture 6, secondary infall models `predict' that dark matter haloes have a power-law density distribution

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-\gamma} \quad \text{with} \quad \gamma = \frac{9\,\varepsilon}{1+3\varepsilon}$$

Gunn & Gott (1972)

at least if the initial condition is also a power-law, i.e., $\delta_{\rm i} \propto r_{\rm i}^{-3\varepsilon} \propto M^{-\varepsilon}$

However, single power-law density distributions are unphysical

$$M(<r) = 4\pi \int_0^r \rho(r')\, r'^2\, {\rm d}r' = \frac{4\pi\rho_0 r_0^{\gamma}}{3-\gamma}\, r^{3-\gamma}$$

For $\gamma \leq 3$ the total mass is infinite, i.e., $\lim_{r\to\infty} M(<r) = \infty$

Similarly, for $\gamma \geq 3$ the enclosed mass is infinite for <u>any</u> radius, i.e., $M(<r) = \infty$

Pure power-law density distributions cannot exist in nature...

Density Profiles

A more realistic density profile consists of a double power-law:

$$\rho(r) = \frac{\rho_0}{(r/r_0)^\gamma \left[1 + (r/r_0)^\alpha\right]^{(\beta-\gamma)/\alpha}}$$

$$\rho \propto r^{-\gamma} \qquad r \ll r_0$$

$$\rho \propto r^{-\beta} \qquad r \gg r_0$$

The parameter α controls the sharpness of the break. In order for the mass to be finite requires $\gamma < 3$ and $\beta > 3$

The following is a list of double power-law density profiles frequently used in astronomy.

(α, β, γ)	Name	Reference
(2, 5, 0)	Plummer Profile	Plummer, 1911, MNRAS, 71, 460
(2, 4, 0)	Perfect Sphere	de Zeeuw, 1985, MNRAS, 216, 273
(2, 3, 0)	Modified Hubble Profile	Binney & Tremaine, 1987
(2, 3, 0)	Modified Isothermal Sphere	Sacket & Sparke, 1990, ApJ, 361, 409
(1, 3, 1)	NFW Profile	Navarro, Frenk & White, 1997, ApJ, 490, 493
(1.5, 3, 1.5)	Moore Profile	Moore, 1999, MNRAS, 310, 1143
(1, 4, 1)	Hernquist Profile	Hernquist, 1990, ApJ, 356, 359
(1, 4, 2)	Jaffe Profile	Jaffe, 1983, MNRAS, 202, 995

The NFW Profile

In 1997, Navarro, Frenk & White wrote a seminal paper in which they showed that CDM haloes in N-body simulations have a universal density profile, well fit by a double power-law...

A UNIVERSAL DENSITY PROFILE FROM HIERARCHICAL CLUSTERING

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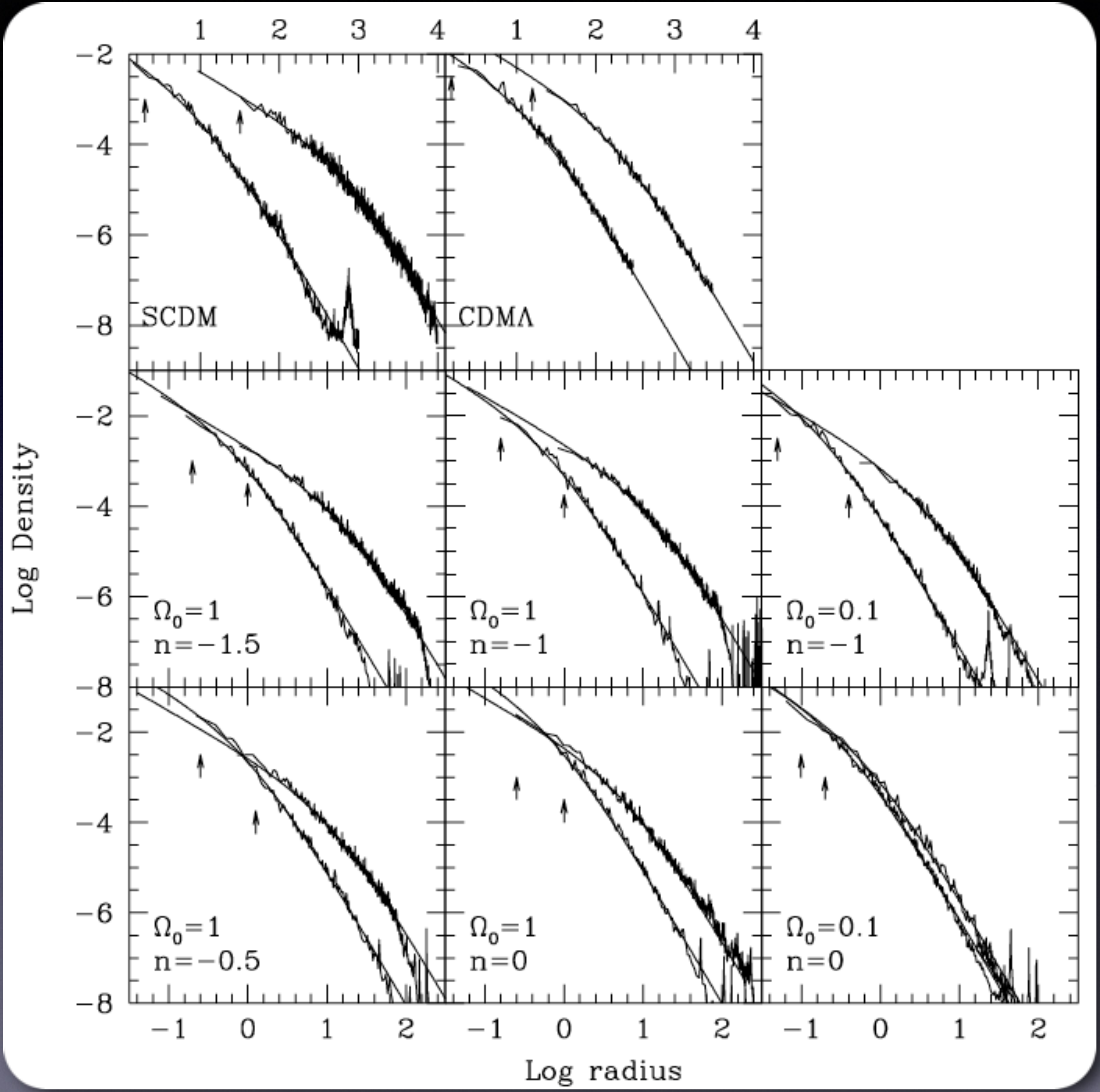
ABSTRACT

We use high-resolution *N*-body simulations to study the equilibrium density profiles of dark matter halos in hierarchically clustering universes. We find that all such profiles have the same shape, independent of the halo mass, the initial density fluctuation spectrum, and the values of the cosmological parameters. Spherically averaged equilibrium profiles are well fitted over two decades in radius by a simple formula originally proposed to describe the structure of galaxy clusters in a cold dark matter universe. In any particular cosmology, the two scale parameters of the fit, the halo mass and its characteristic density, are strongly correlated. Low-mass halos are significantly denser than more massive systems, a correlation that reflects the higher collapse redshift of small halos. The characteristic density of an equilibrium halo is proportional to the density of the universe at the time it was assembled. A suitable definition of this assembly time allows the same proportionality constant to be used for all the cosmologies that we have tested. We compare our results with previous work on halo density profiles and show that there is good agreement. We also provide a step-by-step analytic procedure, based on the Press-Schechter formalism, that allows accurate equilibrium profiles to be calculated as a function of mass in any hierarchical model.

Subject headings: cosmology: theory — dark matter — galaxies: halos — methods: numerical

The NFW Profile

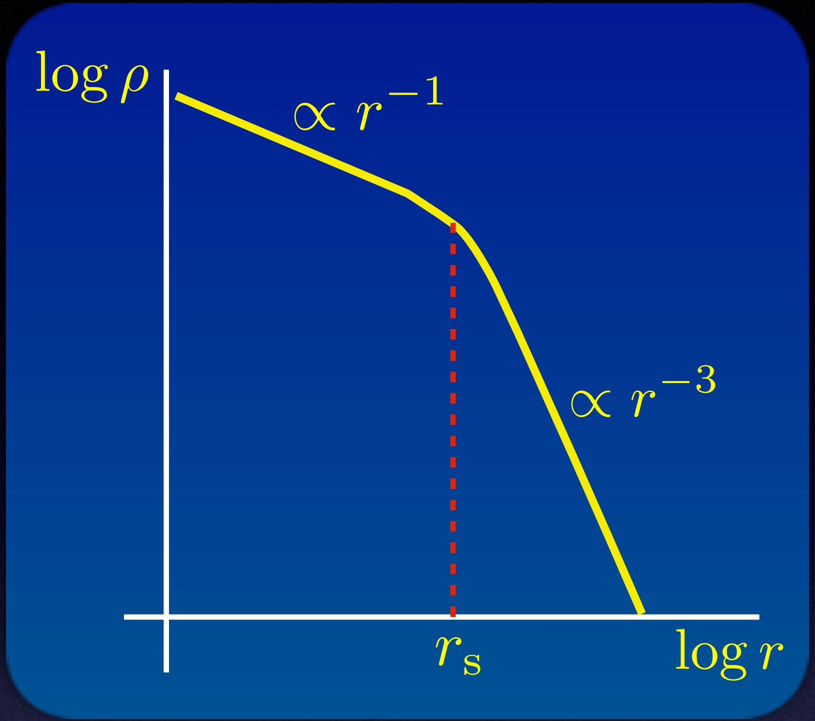


Using a suite of simulations, of different cosmologies, they showed that the density profiles of the dark matter haloes can always be fit by a universal fitting function:
the NFW profile

$$\rho(r) = \rho_{\rm crit} \frac{\delta_{\rm char}}{(r/r_{\rm s})\,(1+r/r_{\rm s})^2}$$

The NFW Profile

The NFW profile is given by $$\rho(r) = \rho_{\rm crit} \frac{\delta_{\rm char}}{(r/r_{\rm s})\,(1+r/r_{\rm s})^2}$$

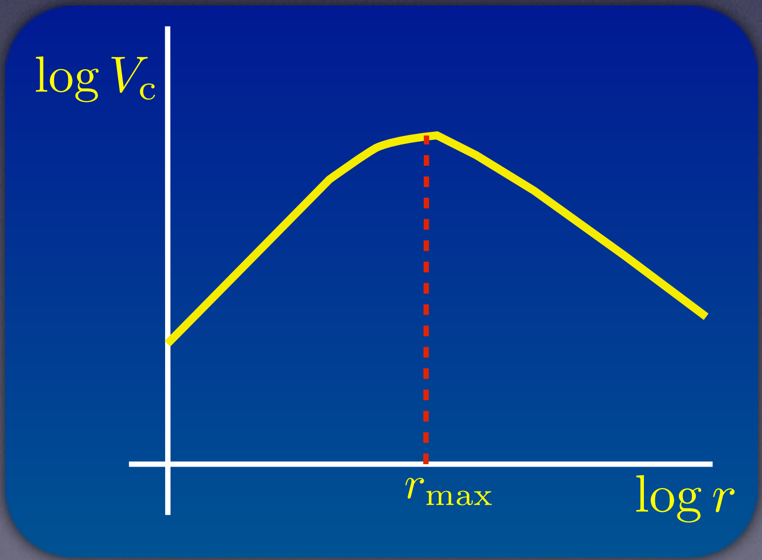


It is completely characterized by the mass $M_{\rm vir}$ and the concentration parameter $c = r_{\rm vir}/r_{\rm s}$, which is related to the characteristic overdensity according to:

$$\delta_{\rm char} = \frac{\Delta_{\rm vir}\,\Omega_{\rm m}}{3}\,\frac{c^3}{f(c)}$$

where $f(x) = \ln(1+x) + x/(1+x)$

The corresponding mass profile is $M(r) = 4\pi\rho_{\rm crit}\delta_{\rm char} r_{\rm s}^3\, f(c) = M_{\rm vir}\frac{f(cx)}{f(c)}$, where $x = r/r_{\rm vir}$



The circular velocity of an NFW profile is $$V_{\rm c}(r) = V_{\rm vir}\sqrt{\frac{f(cx)}{x\,f(c)}}$$

which has a maximum $V_{\rm max} \simeq 0.465 V_{\rm vir}\sqrt{c/f(c)}$ at $r_{\rm max} \simeq 2.163 r_{\rm s}$

For example, for $c = 10$ one has that $V_{\rm max} \sim 1.2 V_{\rm vir}$.

For $r \ll r_{\rm max}$ the NFW profile has $V_{\rm c} \propto r^{1/2}$.

The Concentration-Mass Relation

NFW97 showed that the characteristic overdensity, $\delta_{\rm char}$, is closely related to the halo's formation time: haloes that form (assemble) earlier are more concentrated....

Since more massive haloes assemble later (on average) they are expected to be less concentrated, giving rise to an inverted concentration-mass relation. Furthermore, because of large scatter in MAHs one expects significant scatter in this relation.

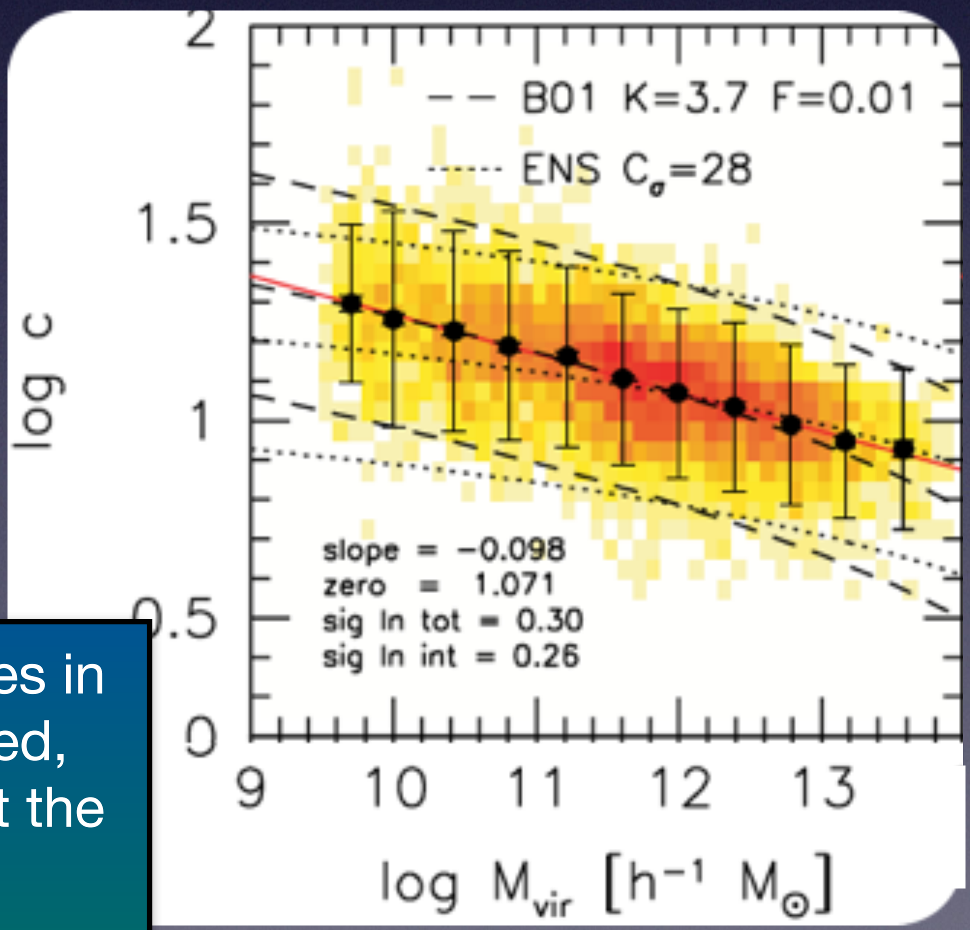
Simulations have shown that halo concentrations follow a log-normal distribution:

$$\boxed{\mathcal{P}(c|M)\,{\rm d}c = \frac{1}{\sqrt{2\pi}\,\sigma_{\rm lnc}} \exp\left[-\frac{(\ln c - \ln \bar{c})^2}{2\sigma_{\rm lnc}^2}\right] \frac{{\rm d}c}{c}}$$

with $\bar{c} = \bar{c}(M)$ and $\sigma_{\rm lnc} \simeq 0.25$.

Simulations have also shown that even at fixed mass, halo concentration is correlated with assembly time.

(e.g., Wechsler et al. 2002; Zhao et al. 2003)



Source: Maccio et al. 2007, MNRAS, 378, 55

The concentration-mass relation of dark matter haloes in a series of N-body simulations. Note that, as expected, more massive haloes are less concentrated, and that the relation has an appreciable amount of scatter...

The Concentration-Mass Relation

Several models have been developed to compute the mean concentration as function of halo mass and cosmology. All these models assume that a halo's characteristic density is related to the mean cosmic density at some characteristic epoch in the halo's history.

(e.g., Bullock et al. 2001; Eke Navarro & Steinmetz 2001; Maccio et al. 2008; Zhao et al. 2009; Diemer & Kravtsov 2015)

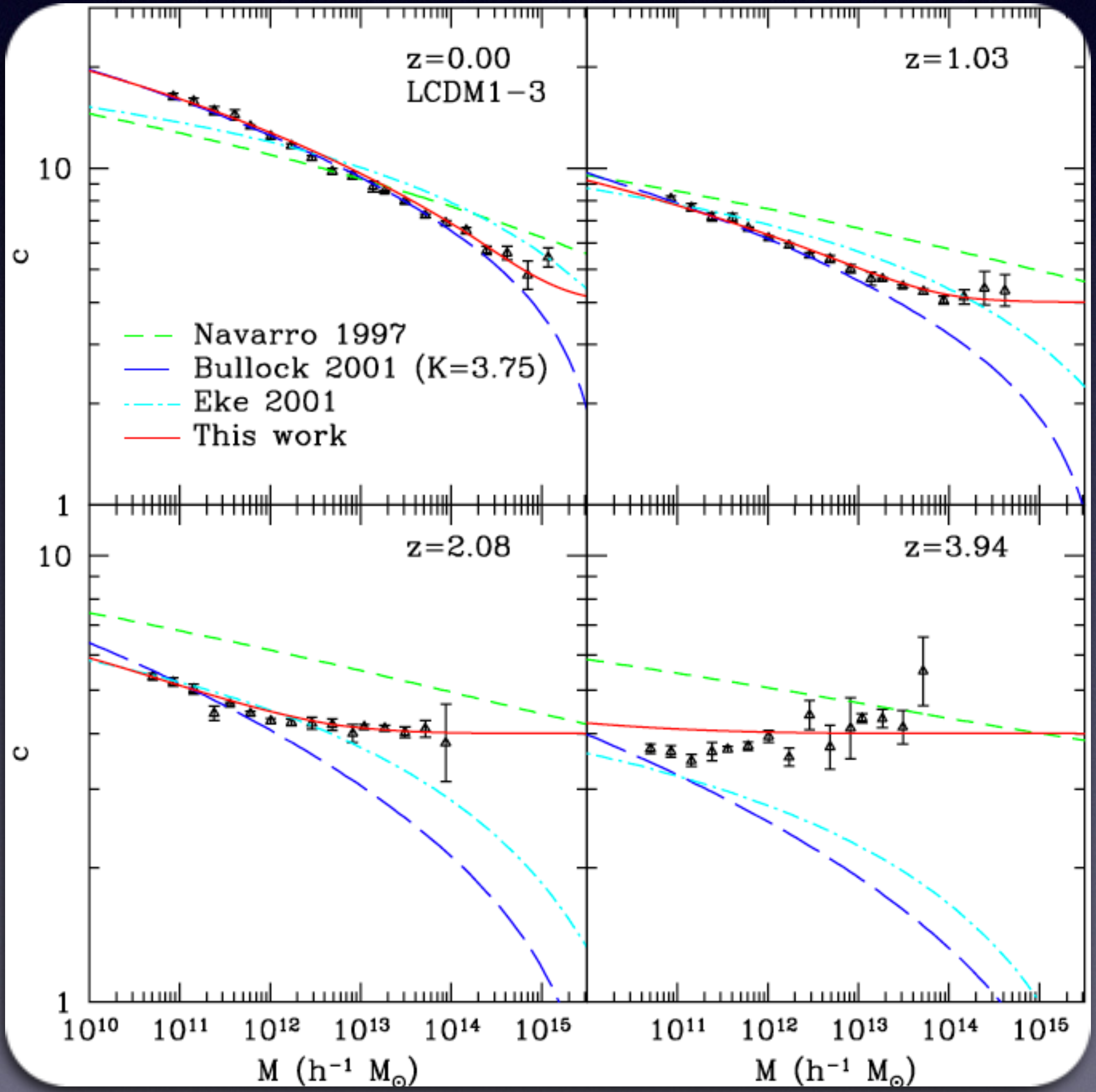
A nice example is the model of Zhao et al. (2009), according to which the average concentration is

$$\boxed{\bar{c}(M,t) = 4 \times \left\{ 1 + \left[\frac{t}{3.75\, t_{0.04}(M,t)} \right]^{8.4} \right\}^{1/8}}$$

Here $t_{0.04}(M,t)$ is the time at which the main progenitor had acquired 4% of its final mass M.

This model is based on the following empirical fact (observed in simulations):

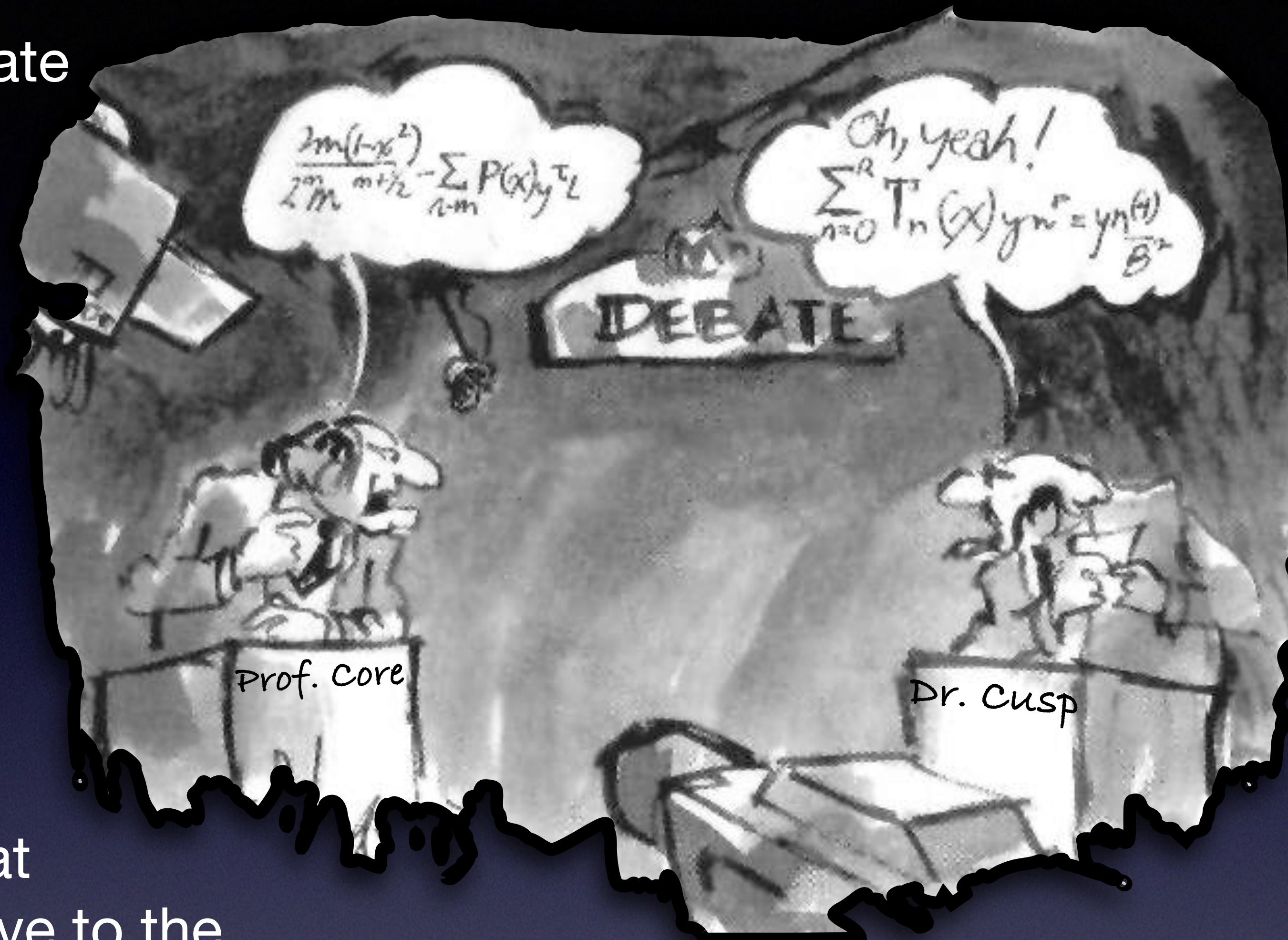
- central structure of halo is established through violent relaxation at early phase of rapid major mergers, leading to NFW profile with c~4.
- subsequent accretion increases mass & size of halo without adding much matter to center, causing concentration to increase with time...



Source: Zhao et al. 2009, ApJ, 707, 354

The Cusp-Core Controversy

Around the turn of the millenium, a lively debate broke out among simulators and observers regarding the actual inner density slopes of dark matter haloes:



According to the NFW profile, dark matter haloes have central cusps with $\rho \propto r^{-1}$

However, several studies claimed that simulated dark matter haloes have cusps that are significantly steeper. A `popular' alternative to the NFW profile was the Moore profile, which has $\gamma = 1.5$

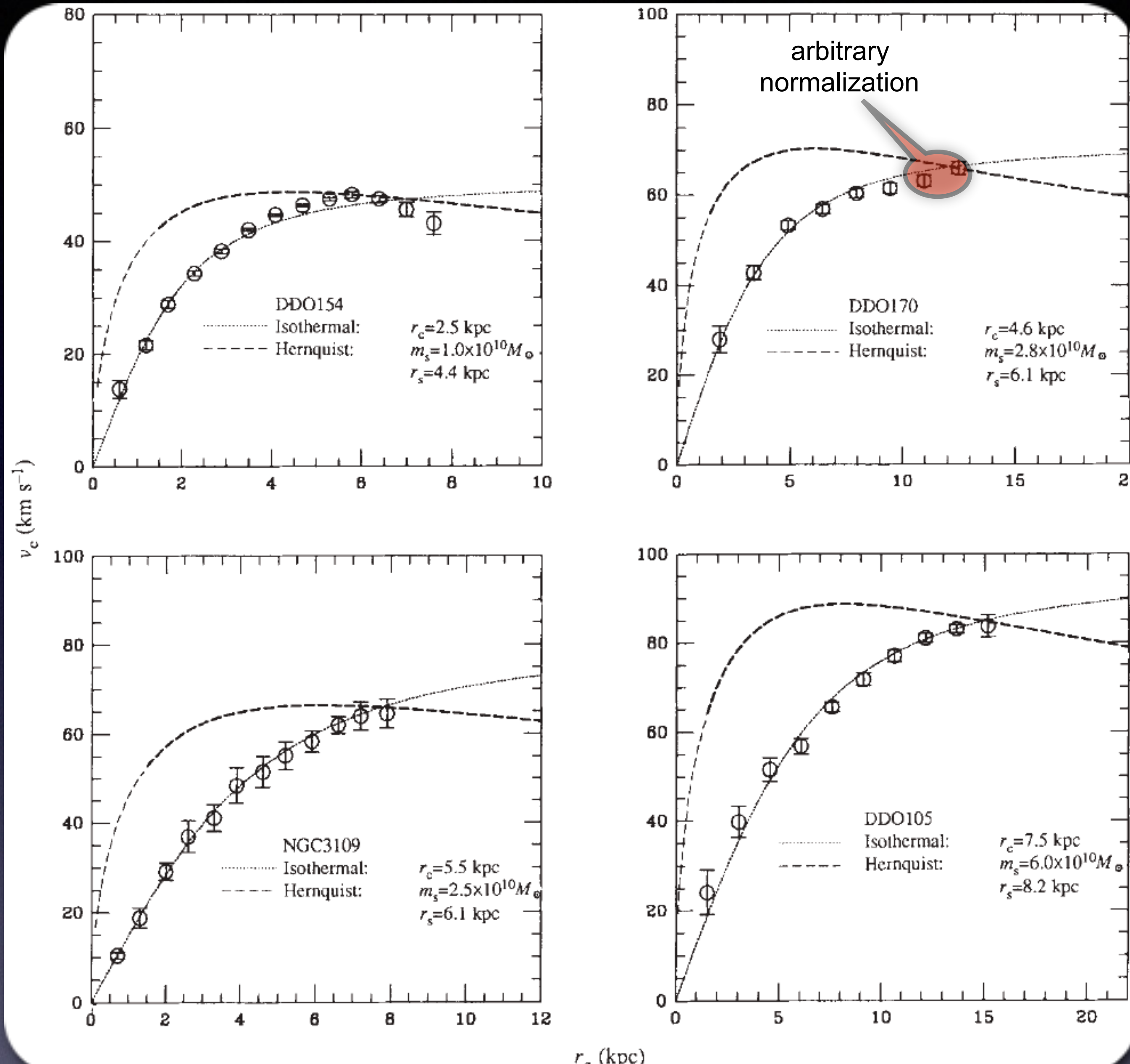
(e.g., Moore et al. 1998, ApJ, 499, L5; Fukushige & Makino, 2001, ApJ, 557, 533)

At around the same time, however, numerous studies claimed that the observed rotation curves of dwarf galaxies and low-surface brightness (LSB) disk galaxies indicate dark matter haloes with central cores; i.e., $\gamma = 0$

(e.g., Moore 1994; Flores & Primack 1994; McGaugh & de Blok 1998)

The Cusp-Core Controversy

Source: Moore, 1994, Nature, 370, 629



Direct comparison of observed rotation curves with circular velocity curves of dark matter haloes reveals inconsistency....

Evidence against dissipation-less dark matter from observations of galaxy haloes

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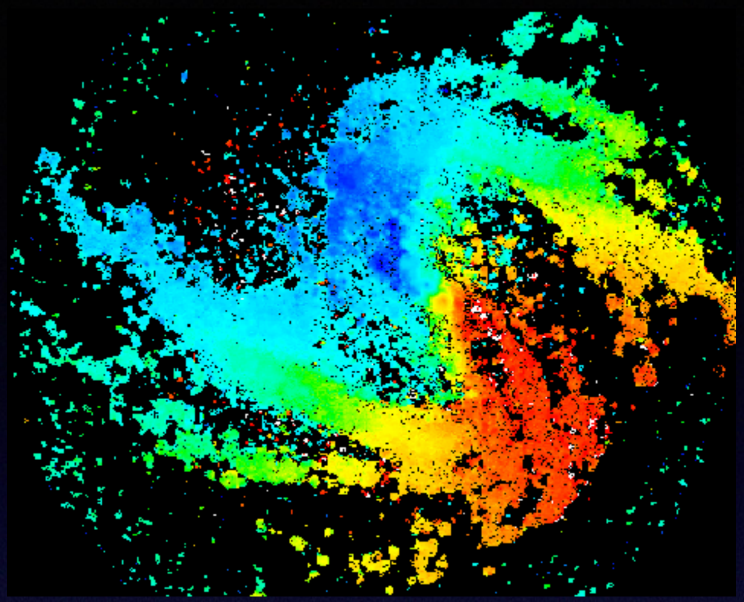
THERE are two different types of missing (dark) matter: the unseen matter needed to explain the high rotation velocities of atomic hydrogen in the outer parts of spiral galaxies[1,2], and the much larger amount of (non-baryonic) matter needed to prevent the universe from expanding forever[1] (producing either a 'flat' or a 'closed' Universe)[3]. Several models have been proposed to provide the dark matter required within galaxy haloes for a flat universe, of which cold dark matter (CDM) has proved the most successful at reproducing the observed large-scale structure of the Universe[4–6]. CDM belongs to a class of non-relativistic particles that interact primarily through gravity, and are named dissipationless because they cannot dissipate energy (baryonic particles can lose energy by emitting electromagnetic radiation). Here I show that the modelled small-scale properties of CDM[7–9] are fundamentally incompatible with recent observations[10–13] of dwarf galaxies, which are thought to be completely dominated by dark matter on scales larger than a kiloparsec. Thus, the hypothesis that dark matter is predominantly cold seems hard to sustain.

The Cusp-Core Controversy

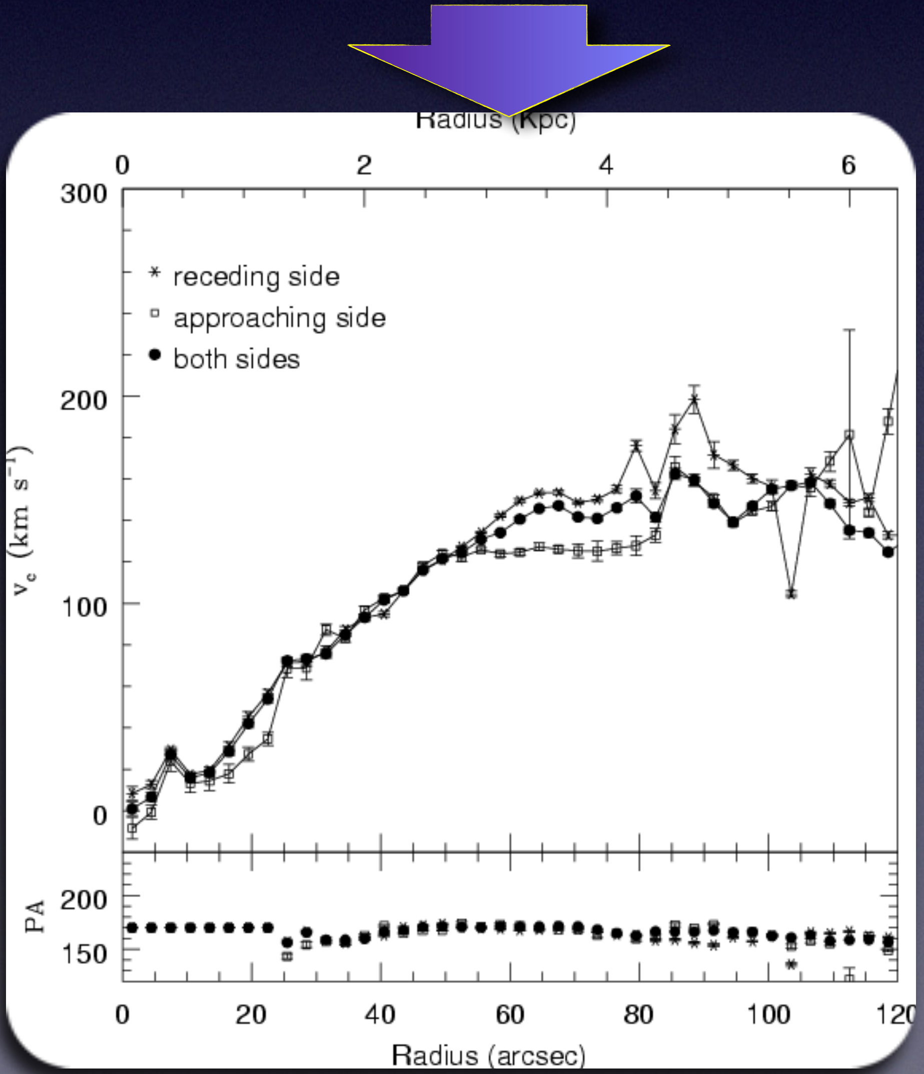
However, a direct comparison is not very meaningfull....

In general, inferring the density distribution of the dark matter halo from a rotation curve involves the following steps:



Convert observed velocity field into a rotation curve:

- find kinematic center
- correct for inclination angle
- average receding & approaching sides



$$\boxed{V_{\rm obs}^2 = V_{\rm halo}^2 + V_{\rm disk}^2}$$

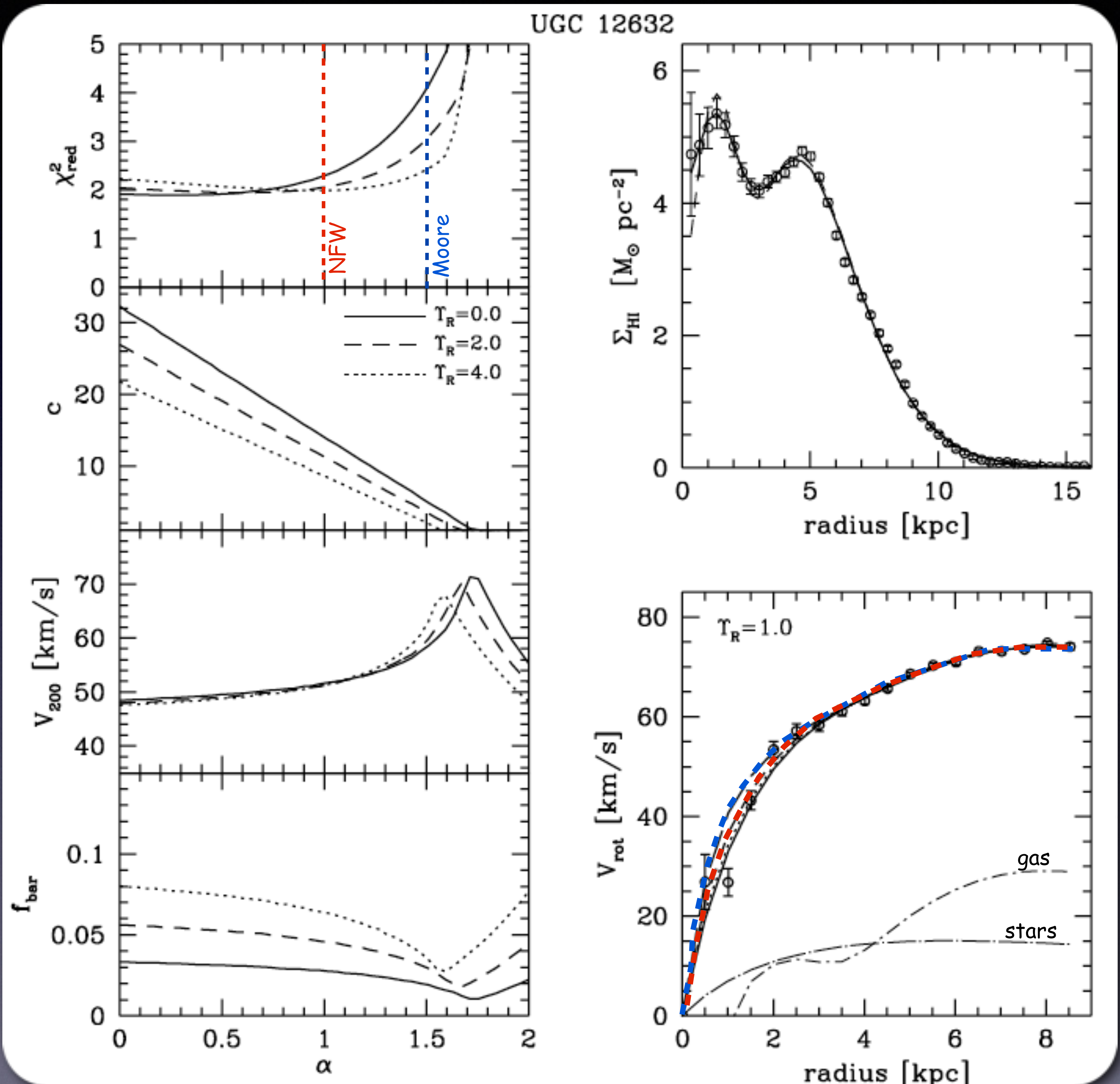
Subtract contribution due to stars & gas

- `estimate' stellar mass-to-light ratio
- `estimate' atomic & molecular gas masses

Complications

- correct for beam smearing, seeing, etc.
- correct for non-circular motions (e.g., bars)

The Cusp-Core Controversy



Source: van den Bosch & Swaters, 2001, MNRAS, 325, 1017

It soon became clear, though, that the existing data could not really discriminate between core and cusp, or between NFW and Moore profiles....

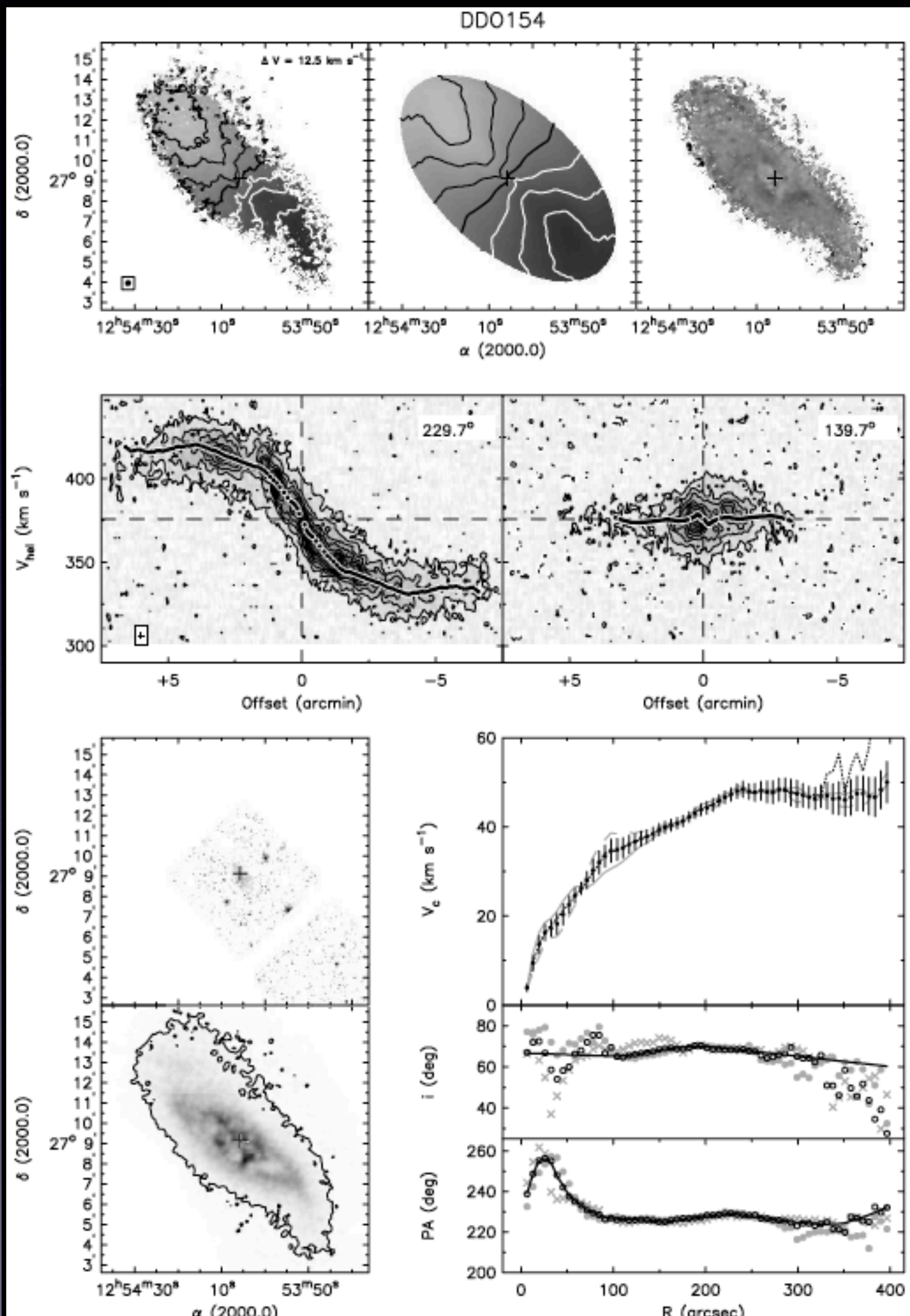
Beam smearing and uncertainties in the stellar mass-to-light ratios hamper unique mass decompositions.

Better data, of higher spatial resolution was required...

van den Bosch et al., 2000
van den Bosch & Swaters, 2001
Swaters et al., 2003
Dutton et al., 2005

The Cusp-Core Controversy

The cusp-core controversy, and the realization that existing HI data was insufficient to settle the issue, has prompted a rush to obtain high resolution Hα - rotation curves.

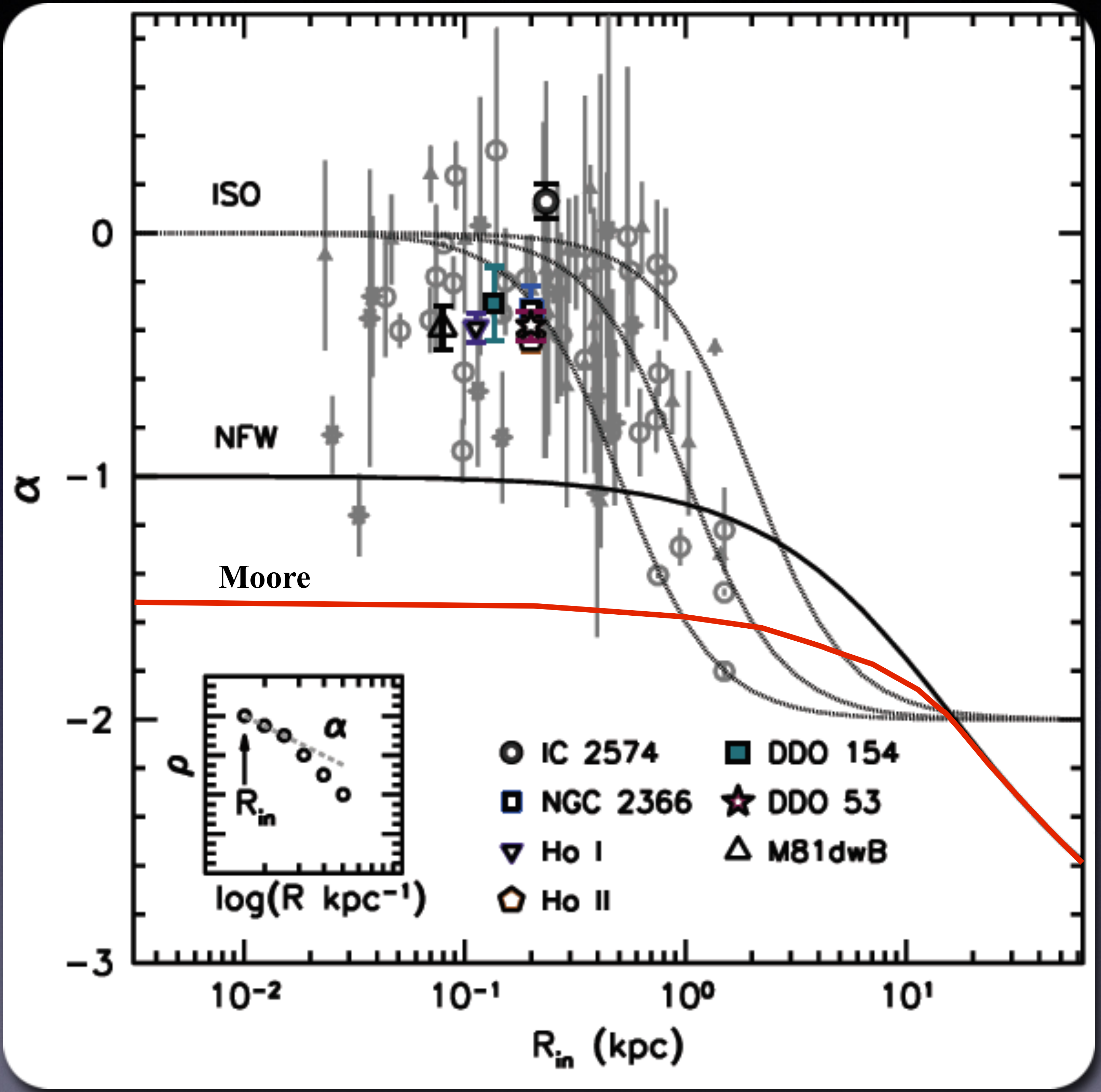


Source: de Blok et al. 2008, AJ, 136, 2648

- Data is much improved, and beam smearing is no longer an issue (see example to the left).
- Cored profiles provide, in general, a better fit to the data that cusped halo profiles.
- Only in few cases is NFW halo clearly inconsistent with the data. Often data is consistent with NFW, but cores are typically preferred...
- Moore profile is clearly inconsistent with data.
- Potential issues with non-circular motions due to bars, triaxiality, asymmetric drift, remain concern.
- But does this indicate problem for CDM???

For review article on cusp-core problem, see http://arxiv.org/abs/0910.3538

The Cusp-Core Controversy



Source: Oh et al. 2011, AJ, 141, 193

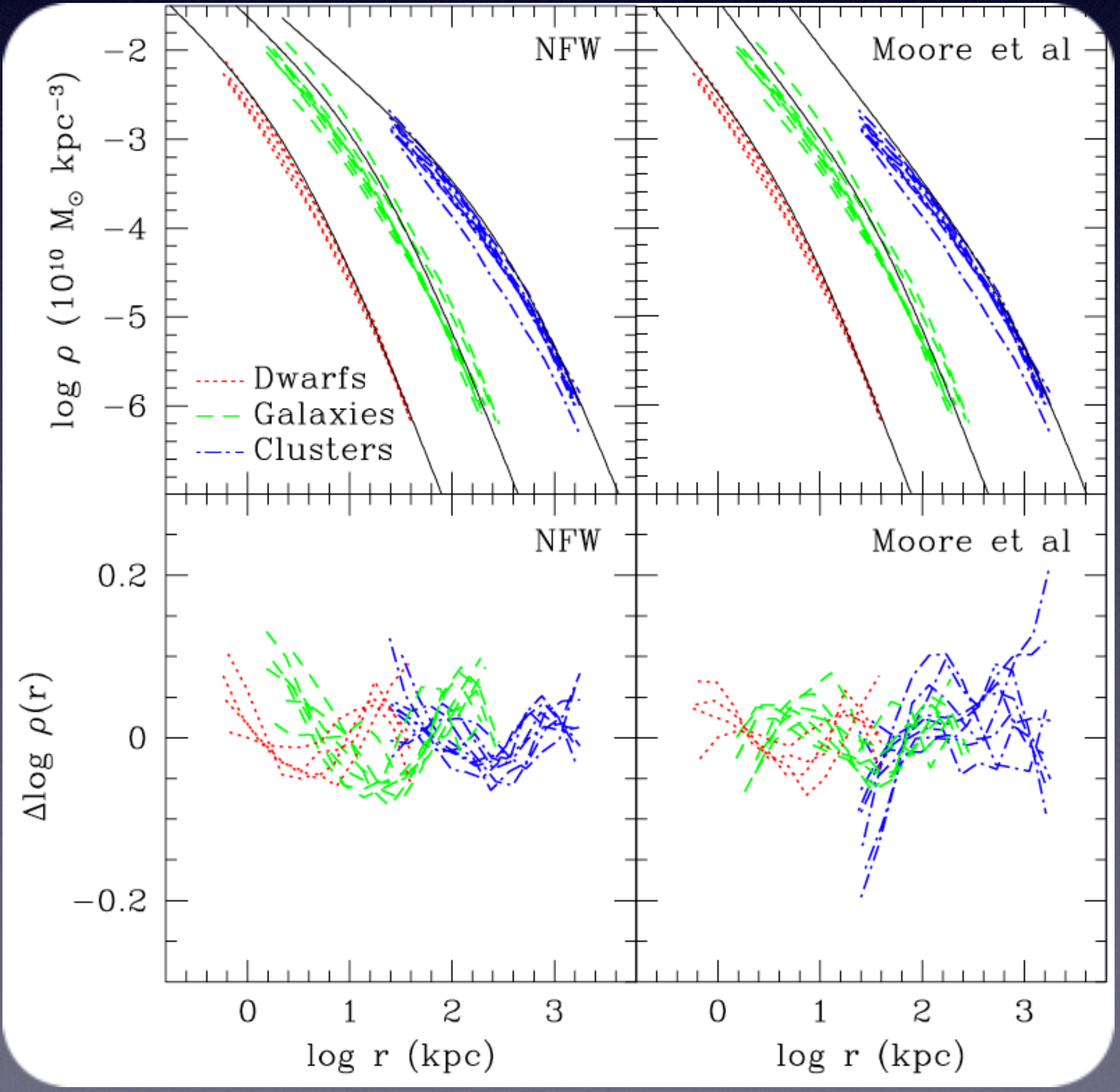
Halo Density Profiles...new insights

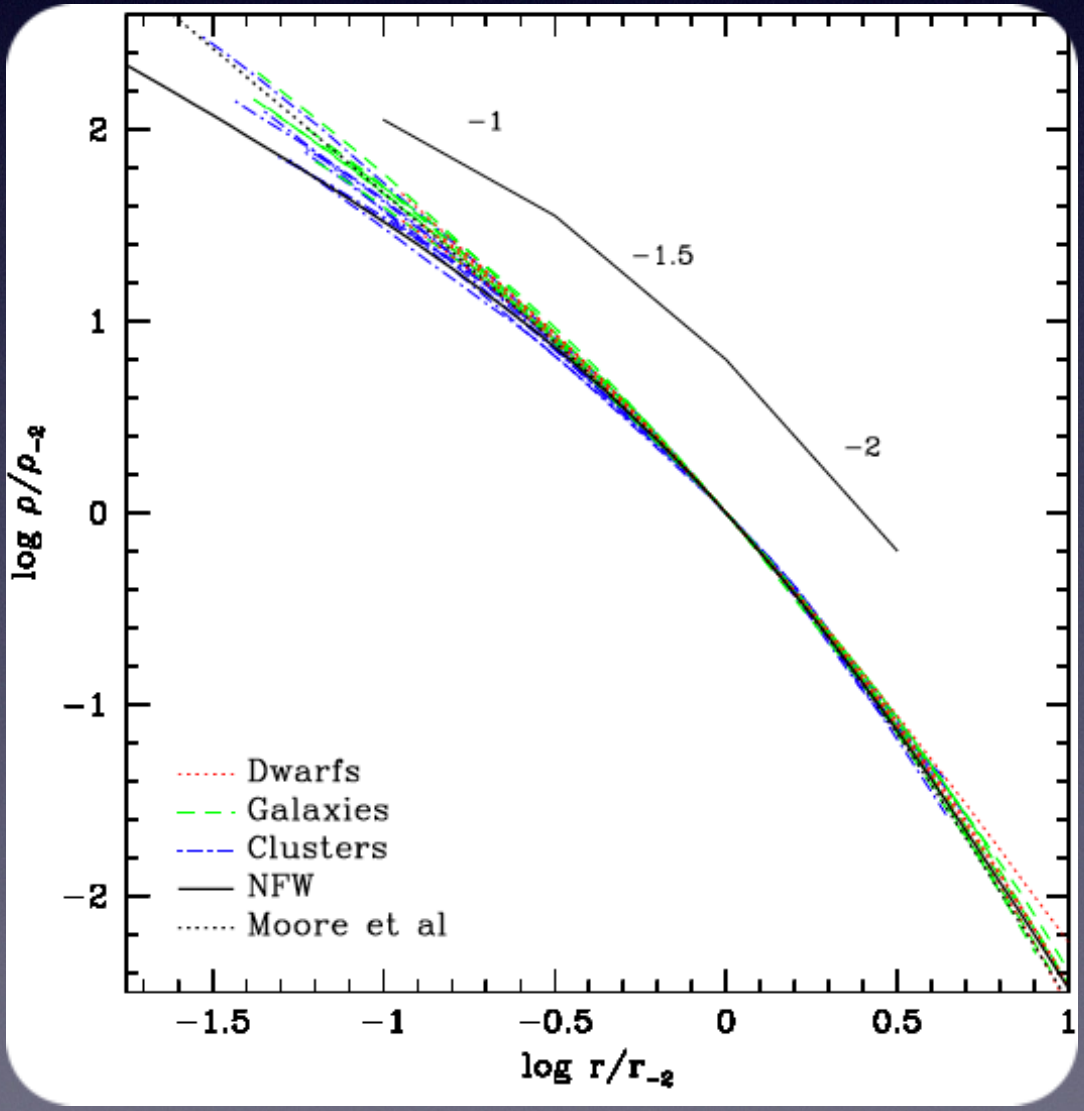
While the cusp-core controversy continues, the dispute among simulators as to the exact cusp-slope of dark matter haloes has largely been resolved...

Part of the discrepancy was related to resolution issues in the simulations.

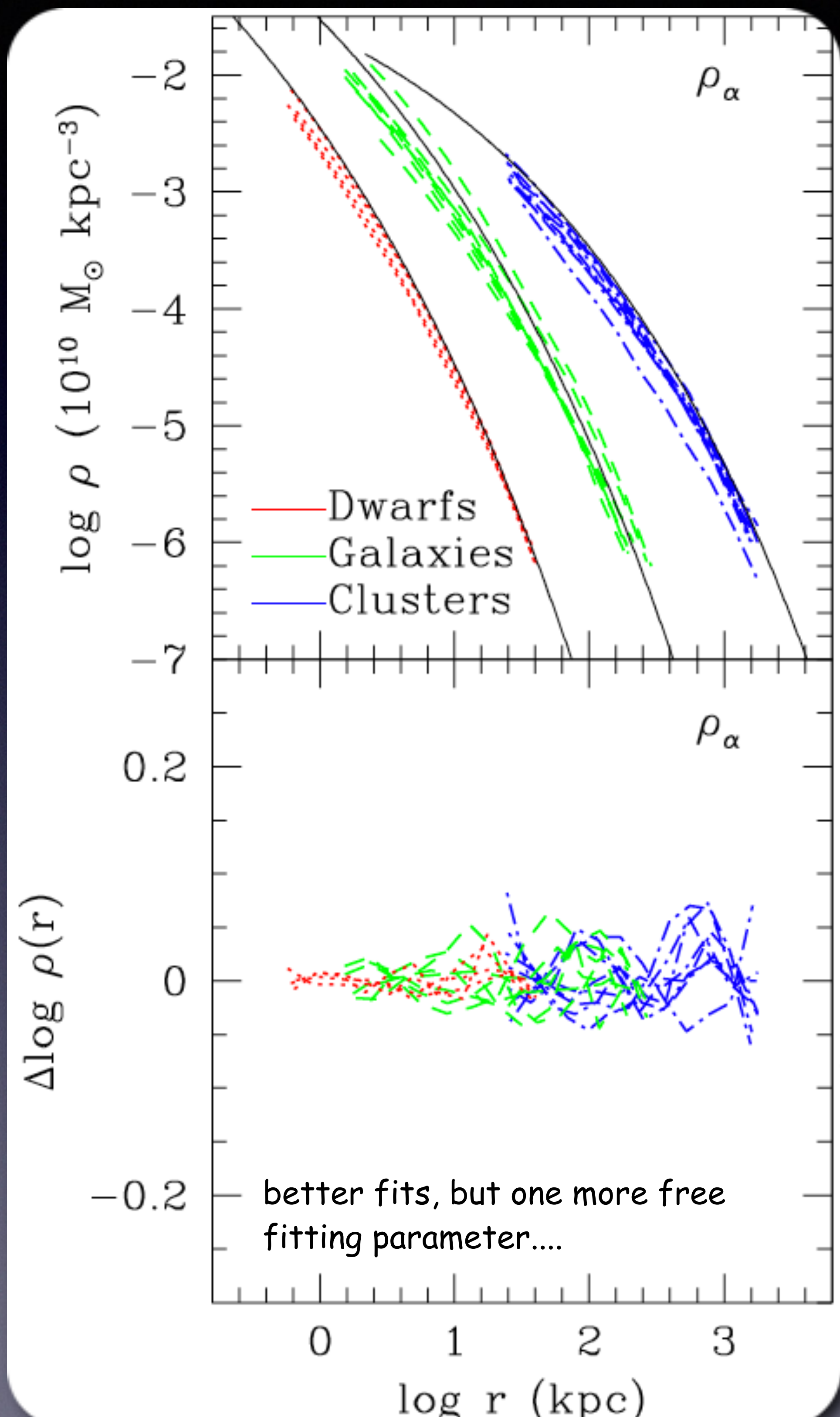
But the main solution seems to be that dark matter haloes do not have double power-law density profiles....Neither NFW- nor Moore-profile are perfect fits...

Source: Navarro et al. 2004, MNRAS, 349, 1039





The Einasto Profile



Navarro et al. (2004) showed that dark matter haloes in simulations are better fit by an Einasto profile:

$$\rho(r) = \rho_{-2} \exp\left[\frac{-2}{\alpha}\left\{\left(\frac{r}{r_{-2}}\right)^{\alpha} - 1\right\}\right]$$

The slope of the Einasto profile is a power-law function of radius:

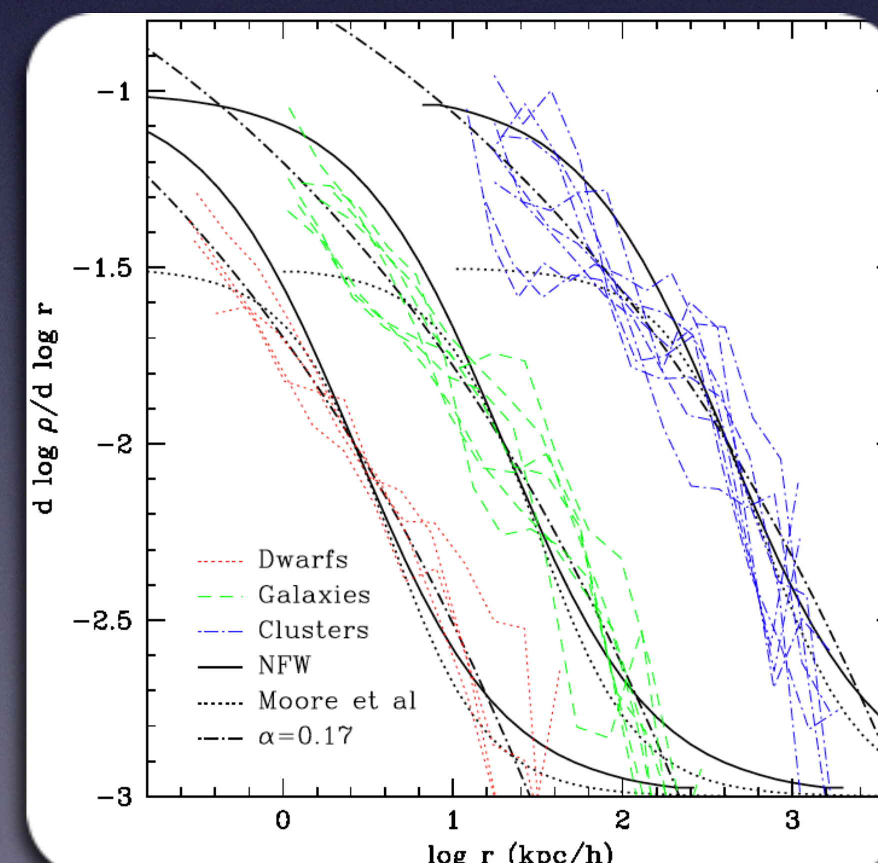
$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} = -2\left(\frac{r}{r_{-2}}\right)^{\alpha}$$

The best-fit value of α typically spans the range $0.12 < \alpha < 0.25$

(Gao et al., 2008, MNRAS, 387, 536)

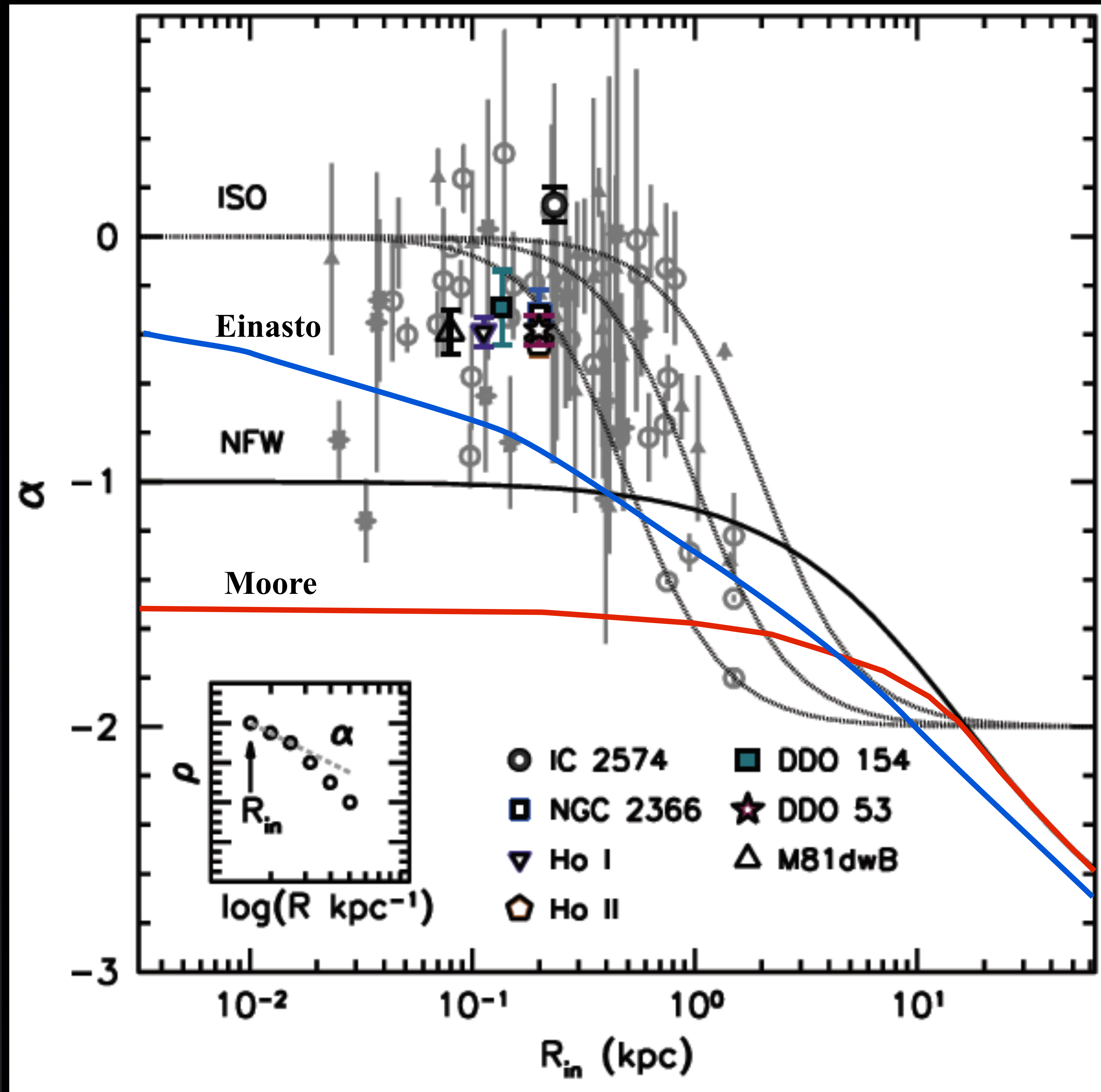
Interestingly, the Einasto profiles also seem to be in better agreement with observed rotation curves...

(Chemin et al., 2011, AJ, 142, 109)



Source: Navarro et al. 2004, MNRAS, 349, 1039

The Cusp-Core Controversy



Source: Oh et al. 2011, AJ, 141, 193

The Cusp-Core Controversy

Even *if* observed dark matter haloes have cusps, this does not necessarily rule out CDM: Baryons to the rescue!!

Baryons may have several effects:

- they can steepen the central profile via adiabatic contraction
- they can create cores via dynamical friction
- they can create cores via three-body interactions (i.e., massive binary BHs)
- they can create cores via supernova feedback

The Cusp-Core Controversy

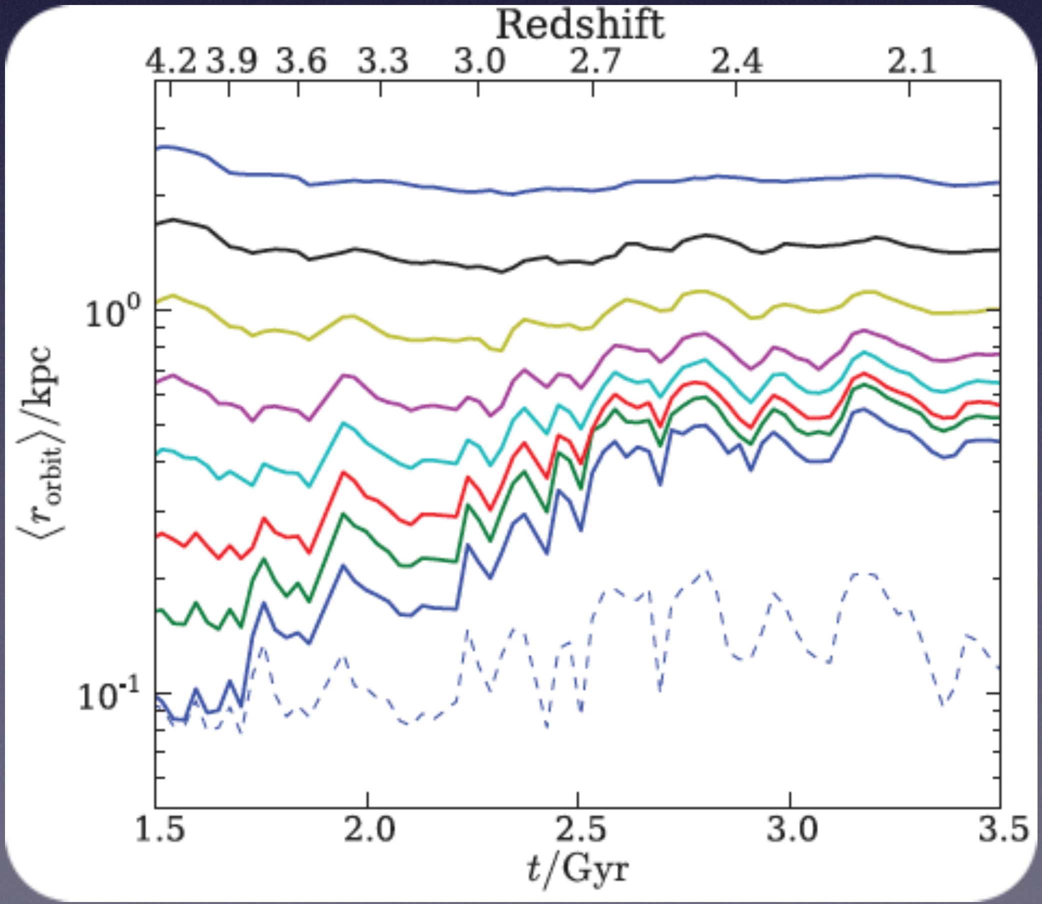
Of the various effects mentioned on the previous slide, only the supernova (SN) feedback one is likely to play a role in dwarf and LSB galaxies....

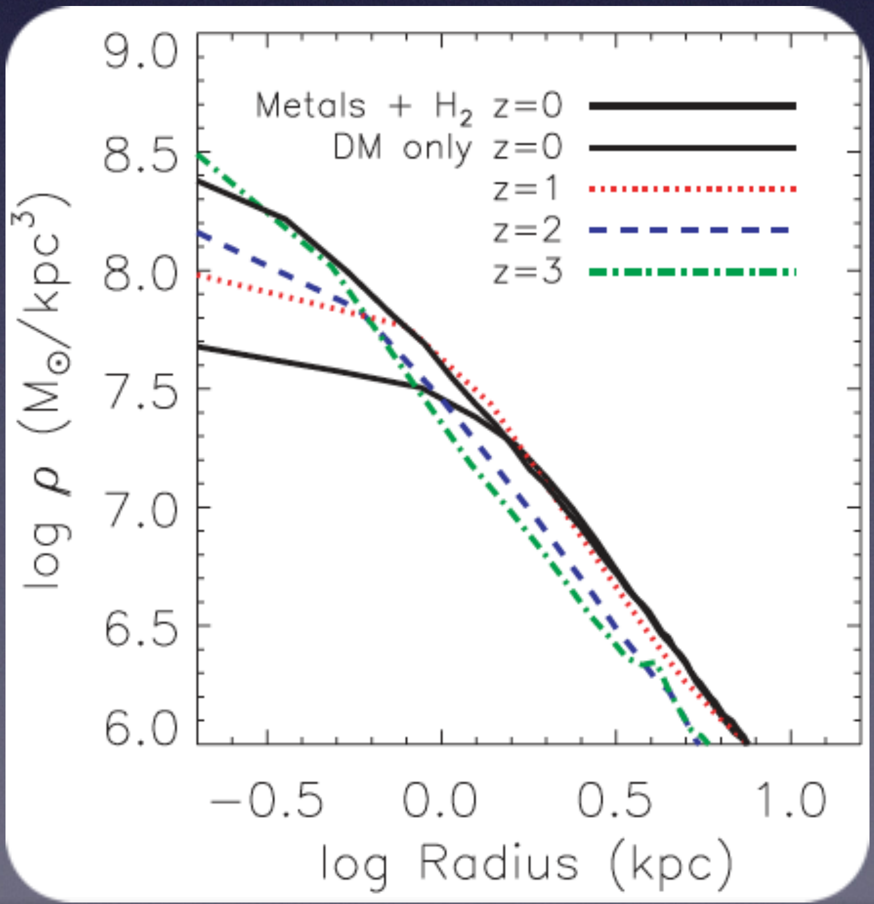
As shown by Pontzen & Governato (2012) SN feedback can result in impulsive heating of central region; since expansion speeds of winds are much faster than local circular speed, winds can cause changes in the potential that are virtually instantaneous (impulsive).

Repeated SN-driven outflows out of the central regions of (dwarf) galaxies may therefore create cores in their dark matter haloes.

Only seems to work in an intermediate range of halo masses….. (Di Cintio+14)

Source: Pontzen & Governato, 2012, MNRAS, 412, 3464





Source: Governato et al., 2012, MNRAS, 422, 1231

Halo Shapes

Halo Shapes

As we have seen in our discussion of the Zel'dovich approximation, because of the tidal tensor $\partial^2\Phi/\partial x_i \partial x_j$ perturbations are not expected to be spherical. Since gravity accentuates non-sphericity, collapsed objects are also not expected to be spherical.

Numerous authors have fitted dark matter haloes in N-body simulations with ellipsoids, characterized by the lengths of the axes $a \geq b \geq c$

These axes can be used to specify the dimensionless shape parameters

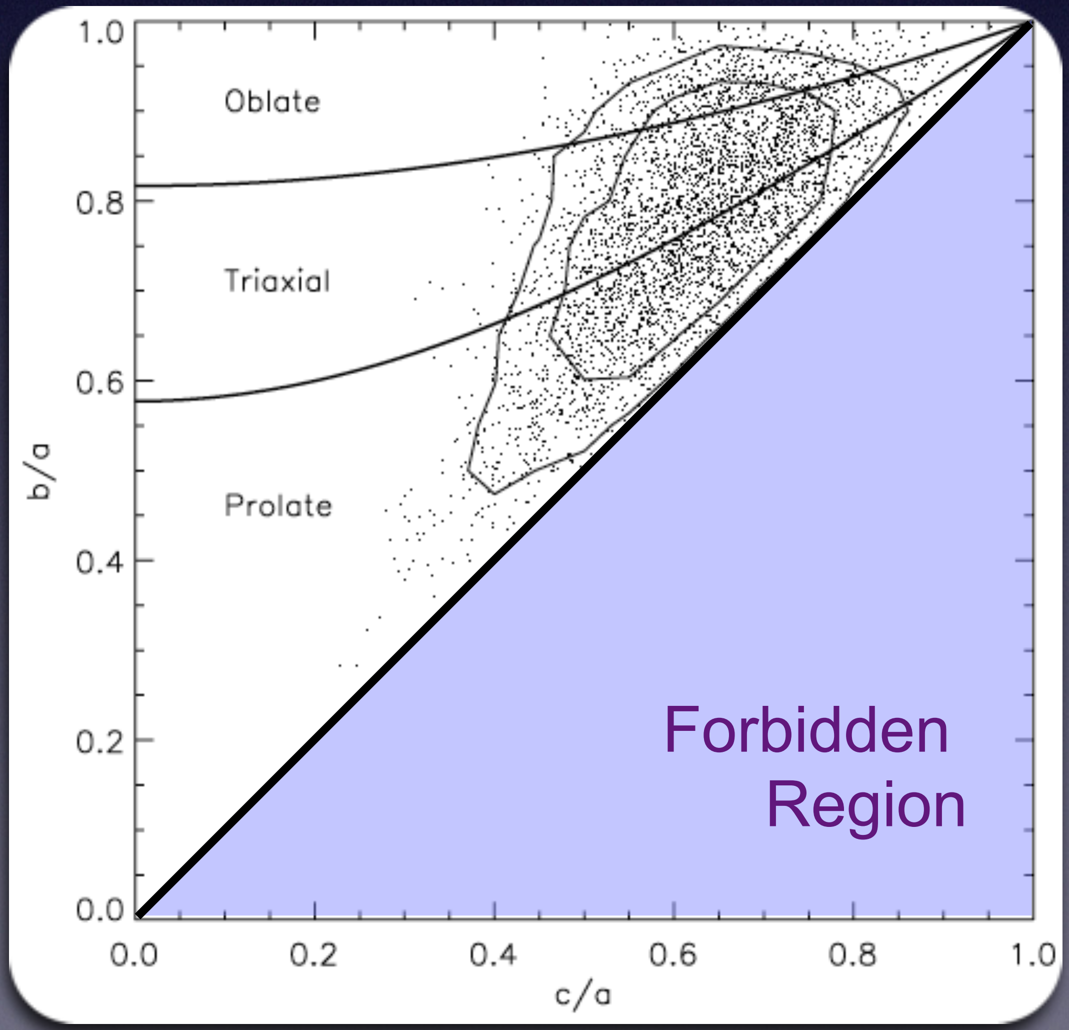
$$s = \frac{c}{a} \qquad q = \frac{b}{a} \qquad p = \frac{c}{b}$$

and/or the triaxiality parameter

$$T = \frac{a^2 - b^2}{a^2 - c^3} = \frac{1 - q^2}{1 - s^2}$$

Oblate: T = 0
Prolate: T = 1

CDM haloes in simulations typically have $0.5 < T < 0.85$



Source: Bailin & Steinmetz, 2005, ApJ, 627,647

Halo Shapes

Simulations show that more massive haloes are more aspherical (more flattened).

Allgood et al. (2006) found that the mass and redshift dependence is well characterized by

$$\langle s \rangle(M,z) = (0.54 \pm 0.03) \left[\frac{M}{M^*(z)}\right]^{-0.050 \pm 0.003}$$

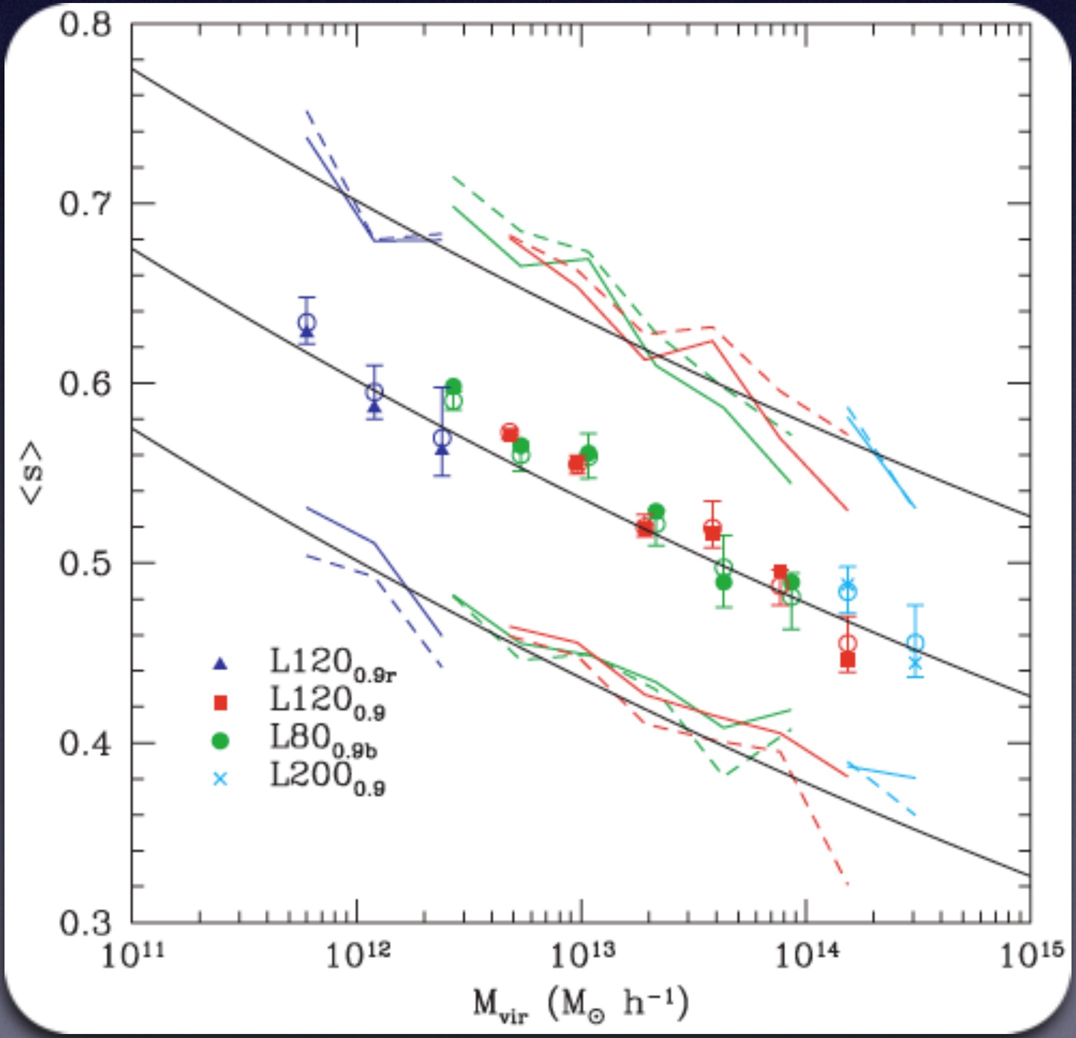
where $M^*(z)$ is the characteristic halo mass at redshift z.

Simulations suggest that the shape of a halo is tightly correlated with its merger history:

Haloes that assembled earlier are more spherical

Haloes that experienced a recent major merger are typically close to prolate, with major axis reflecting direction along which merger occurred

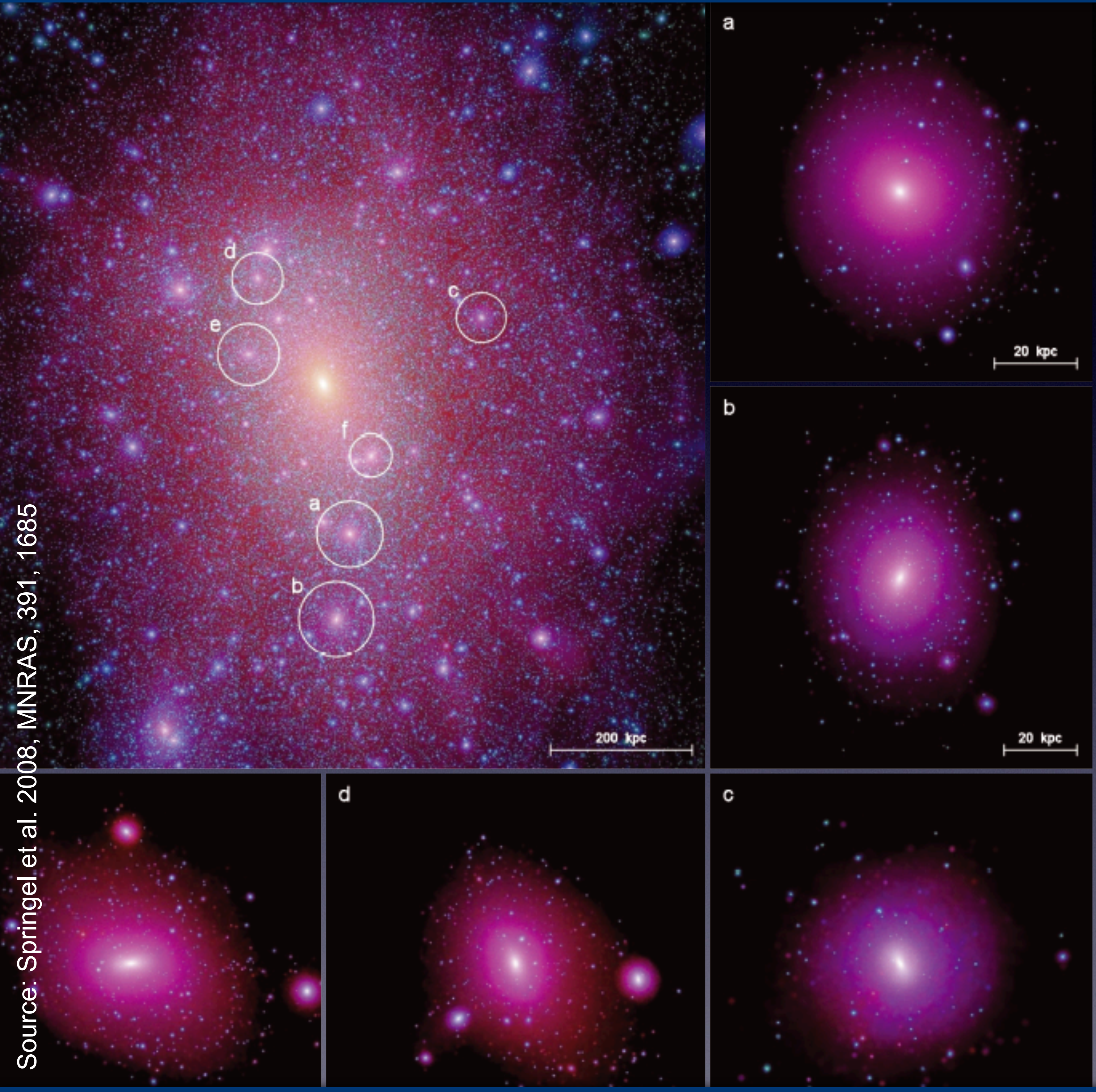
Currently there are only few observational constraints on halo shapes....



Source: Allgood et al. 2006, MNRAS, 367, 1781

Halo Substructure

Halo Substructure



Source: Springel et al. 2008, MNRAS, 391, 1685

Up until the end of the 1990s numerical simulations revealed little if any substructure in dark matter haloes.

Nowadays, faster computers allow much higher mass- and force-resolution, and simulations routinely reveal a wealth of substructure...

Dark matter subhaloes are the remnants of host haloes that survived accretion/merging into a bigger host halo.

While orbiting their hosts, they are subjected to forces that try to dissolve them: dynamical friction, impulsive encounters, and tidal forces....

The Subhalo Mass Function

The subhalo mass function, which describes the number of subhaloes of a given mass per host halo, is well fitted by a Schechter function

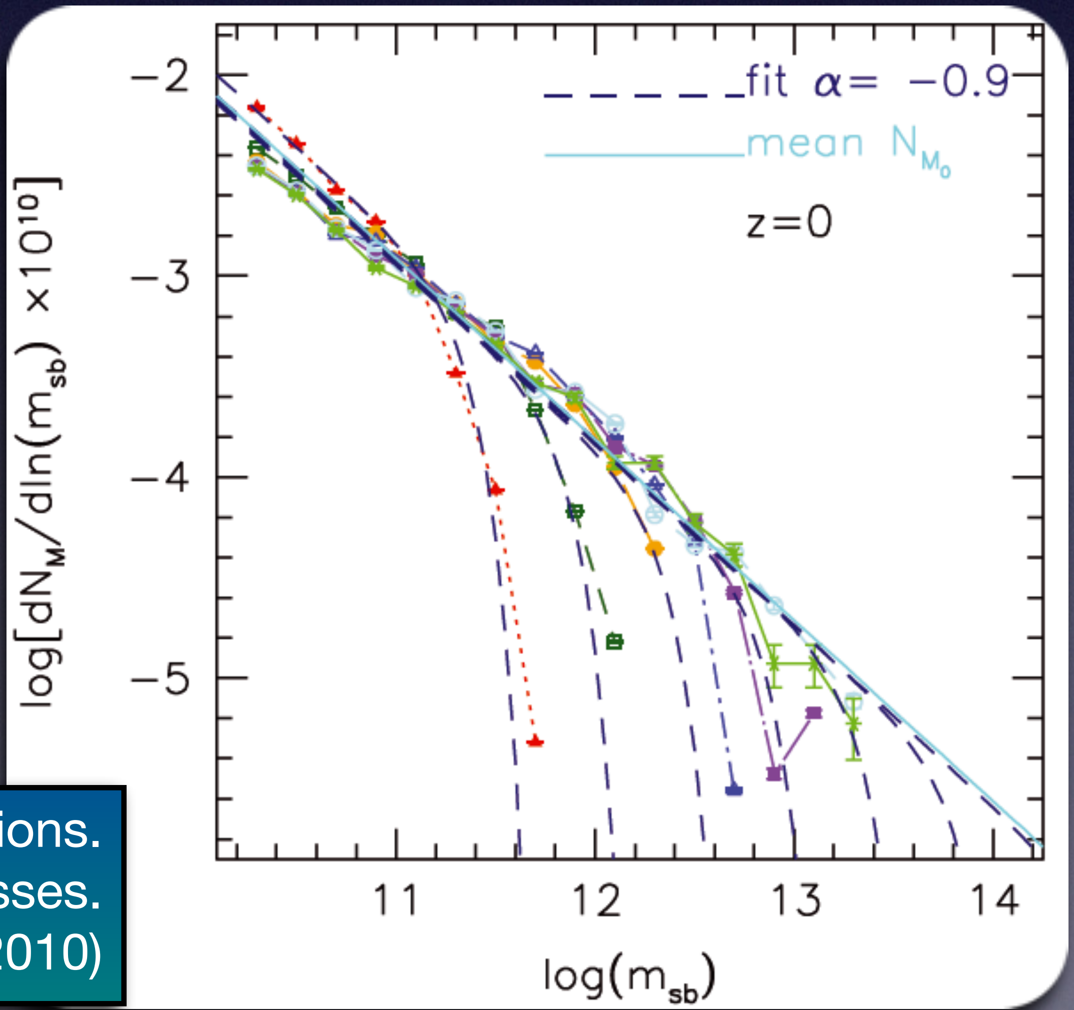
$$\frac{\mathrm{d}n}{\mathrm{d}\ln(m/M)} = \frac{f_0}{\beta\,\Gamma(1-\gamma)} \left(\frac{m}{\beta M}\right)^{-\gamma} \exp\left[-\left(\frac{m}{\beta M}\right)\right]$$

Here m and M are the masses of subhalo and host halo. Simulations indicate that $\gamma \sim 0.9 \pm 0.1$ and $0.1 < \beta < 0.5$. The large uncertainties relate to uncertainties in defining (sub)haloes in numerical simulations...

The parameter f_0 is the mean subhalo mass fraction:

$$f_0 = \frac{1}{M}\int m\frac{\mathrm{d}n}{\mathrm{d}m}\,\mathrm{d}m = \int \frac{\mathrm{d}n}{\mathrm{d}\ln(m/M)}\,\mathrm{d}\left(\frac{m}{M}\right)$$

and is difficult to measure reliably in simulations; typically one can only measure it down to the mass resolution of the simulation...



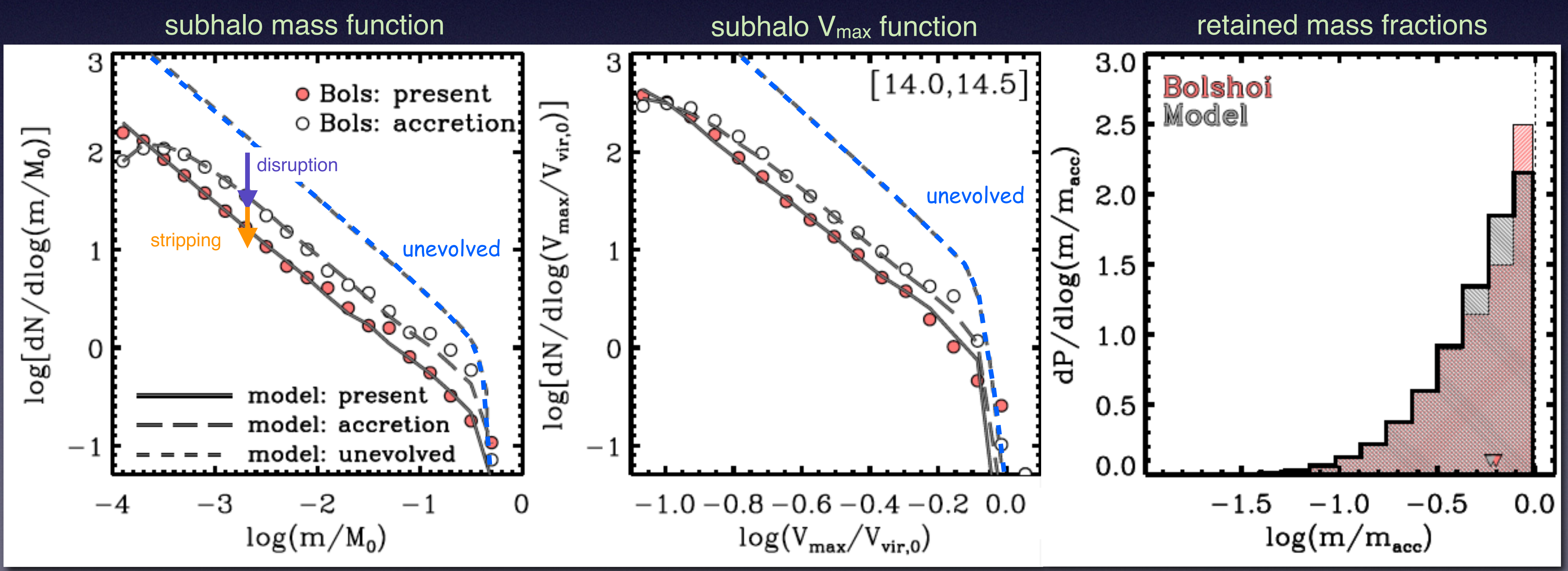
Subhalo mass functions in a series of N-body simulations. Different colors correspond to different host halo masses. From: Giocoli, Tormen, Sheth & van den Bosch (2010)

Mass Stripping

In addition to the ("evolved") subhalo mass function, which reflects the abundance of subhaloes as a function of their present-day mass, one can also define the un-evolved subhalo mass function, which measures the abundance as function of their mass at infall...

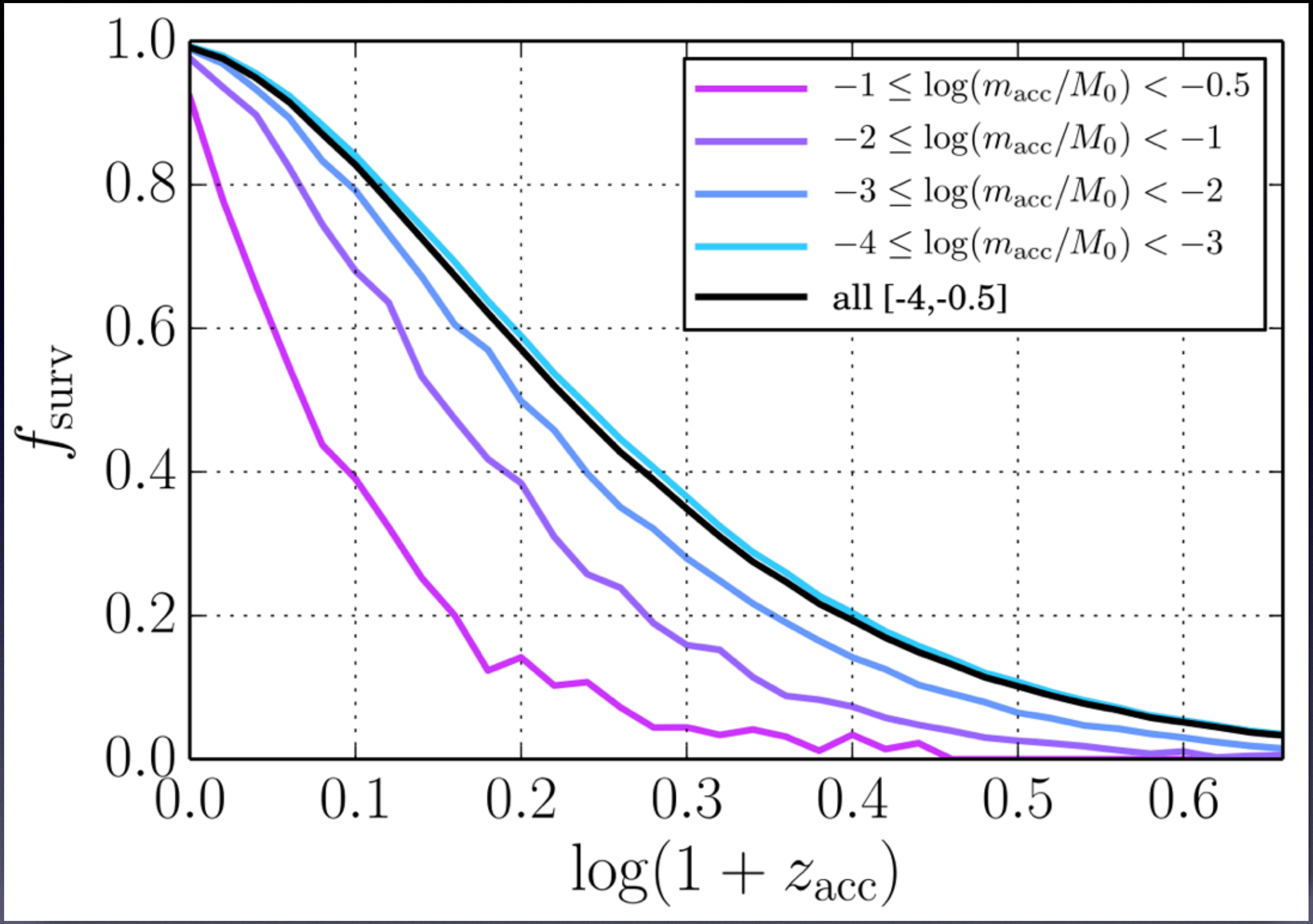
This unevolved SHMF is universal; a consequence of universal MAH of dark matter halos

Difference between evolved & un-evolved SHMFs reflects impact of tidal evolution: tidal stripping & heating causes sub halos to lose mass and (potentially) to completely disrupt…



Source: Jiang & vdBosch, 2016, MNRAS, 458, 2870

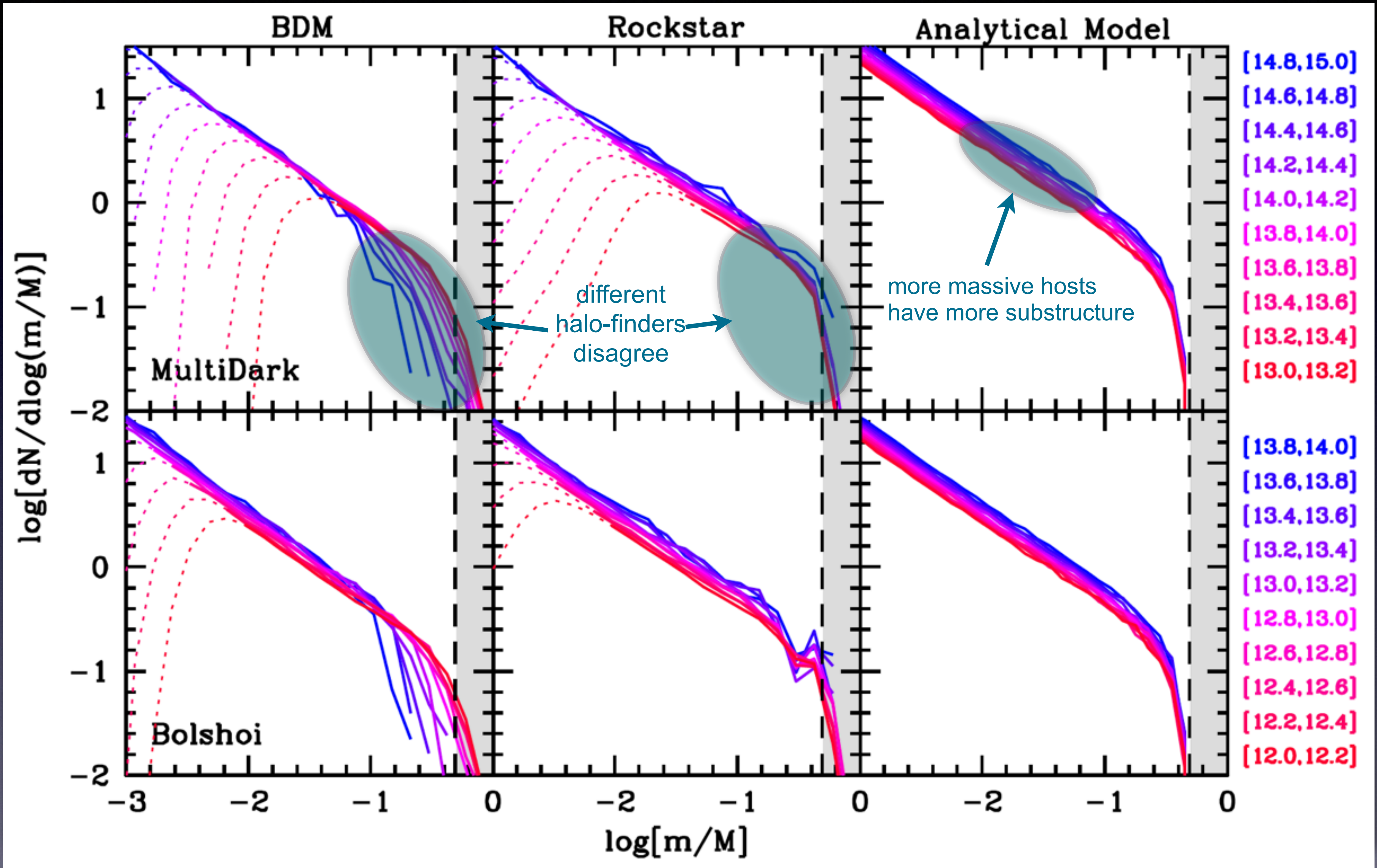
Subhalo Disruption



About 65% of subhalos accreted at z=1 have been disrupted by z=0 (Jiang & vdB 2017)

Majority of this disruption is numerical vdB & Ogiya 2018

Comparison with Simulations

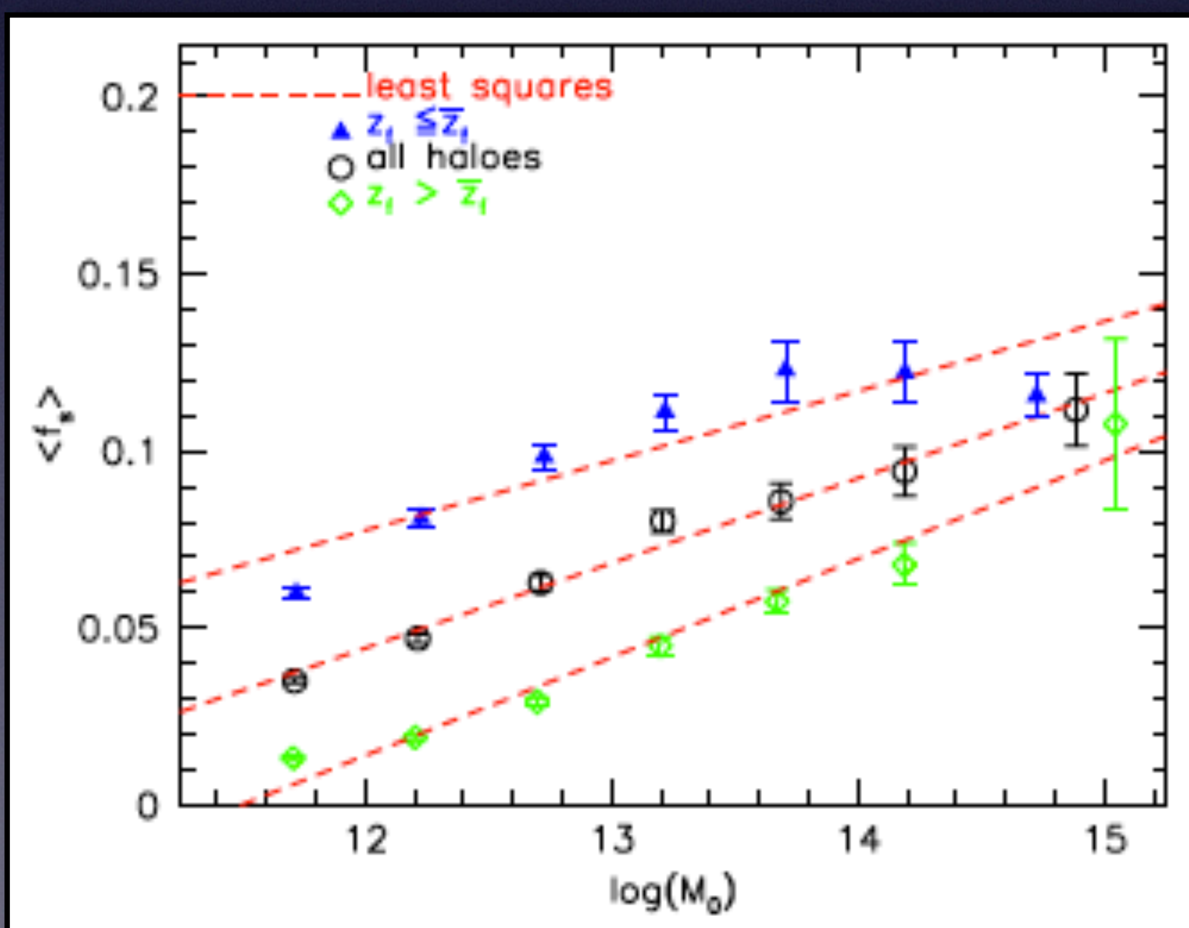


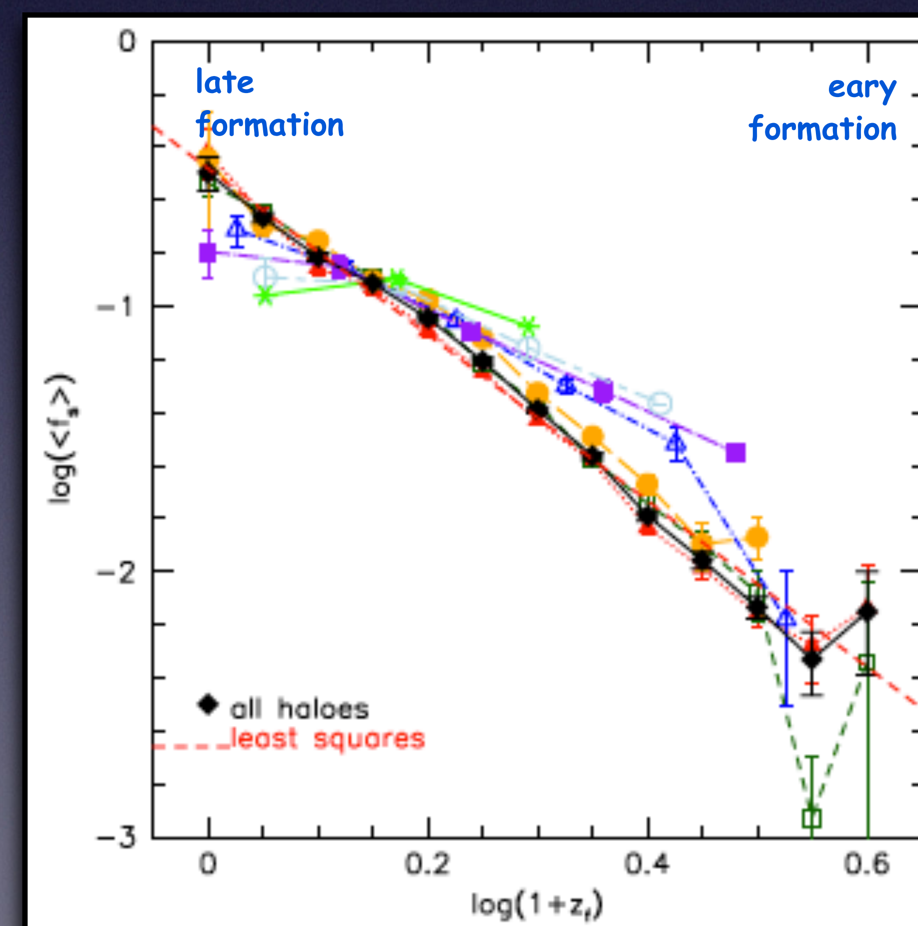
vdB & Jiang (2016)

Subhalo Mass Fractions

Simulations show that halos that assemble earlier have, at present day, less substructure. Since more massive haloes assemble later, they, on average, have more substructure.

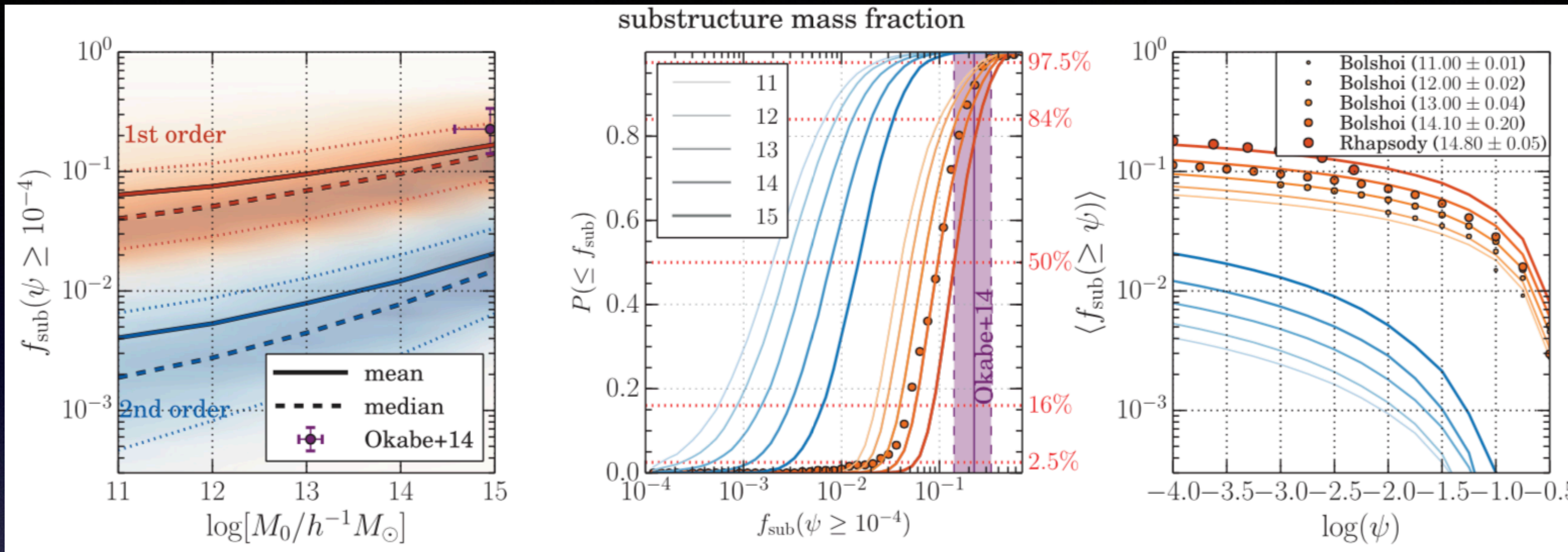
As shown in van den Bosch (2005), this is a consequence of the fact that the unevolved subhalo mass function is virtually independent of halo mass: all haloes accrete the same subhalo population (in units of m/M). Those that accrete them earlier (=assemble earlier), stripped more mass from them, resulting in lower subhalo mass fraction...

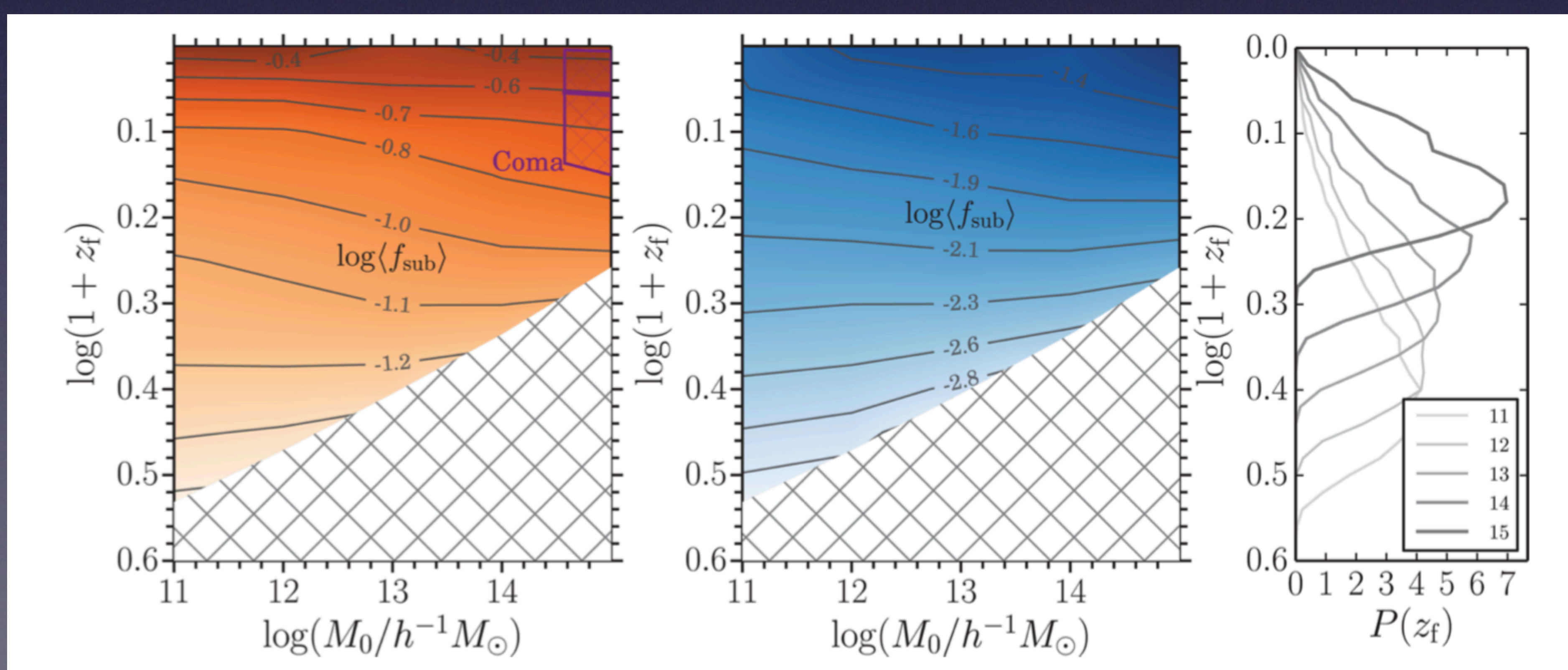




Source: Giocoli et al, 2010, MNRAS, 404, 502

Subhalo Mass Fracions





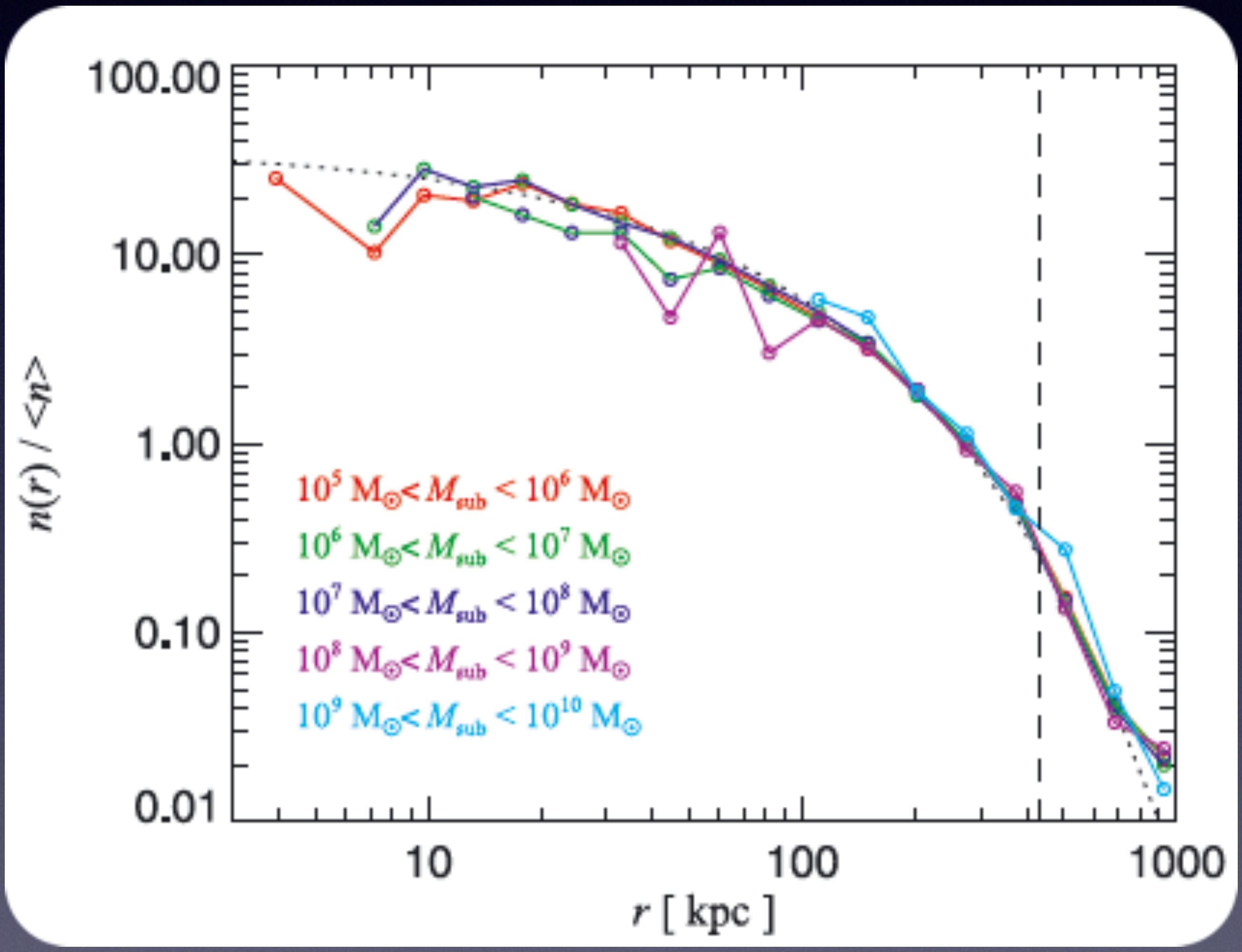
Source: Jiang & vdBosch, 2016, MNRAS, 458, 2870

Subhalo mass fraction is tightly related to the assembly time of the halo: halos that assembled later have more substructure.

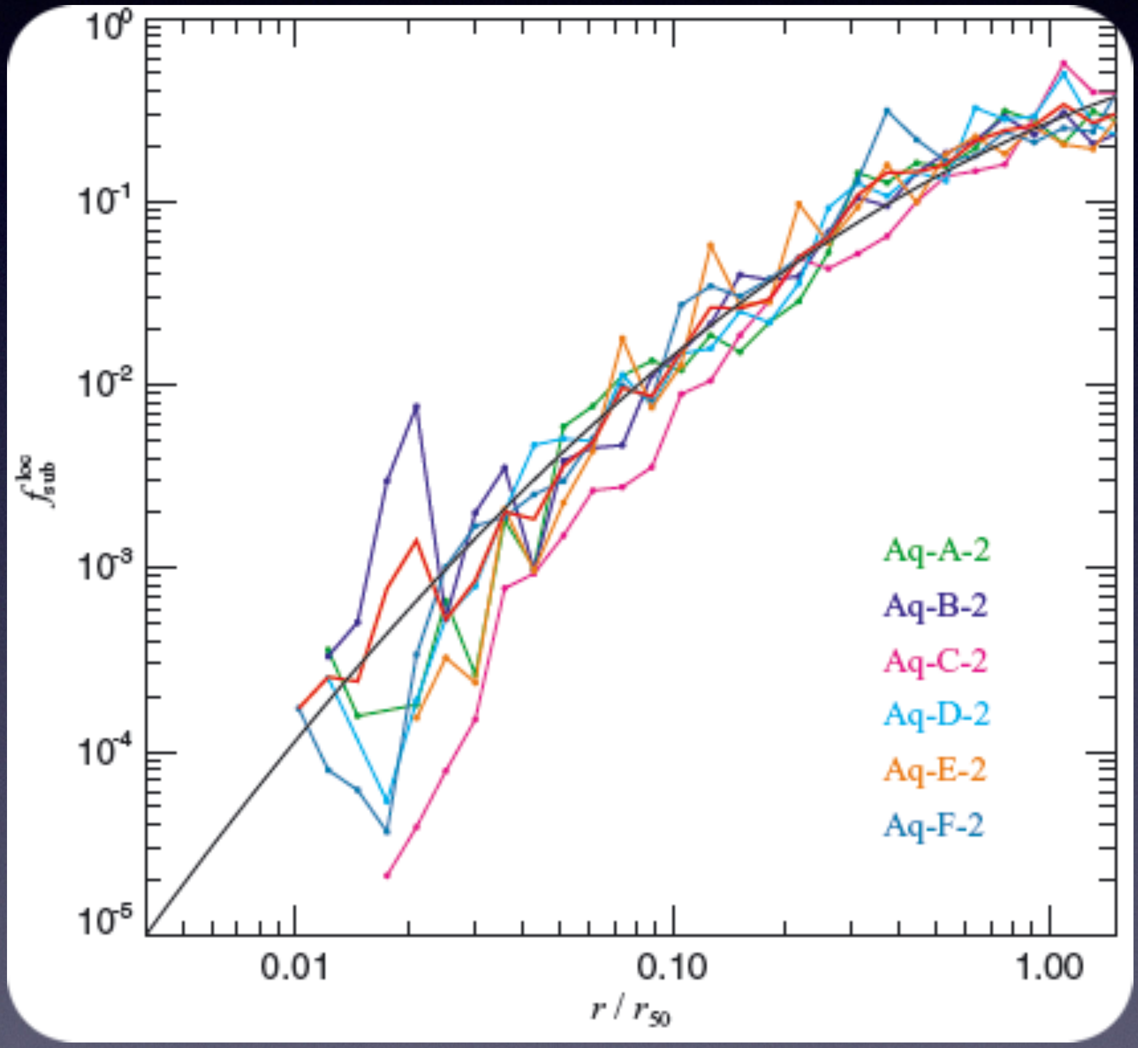
The Spatial Distribution of Subhalos

Simulations show that dark matter subhaloes are less centrally concentrated than the dark matter, and that the radial distribution is independent of subhalo mass (i.e., there is no indication of mass segregation)

Source: Springel et al. 2008, MNRAS, 391, 1685



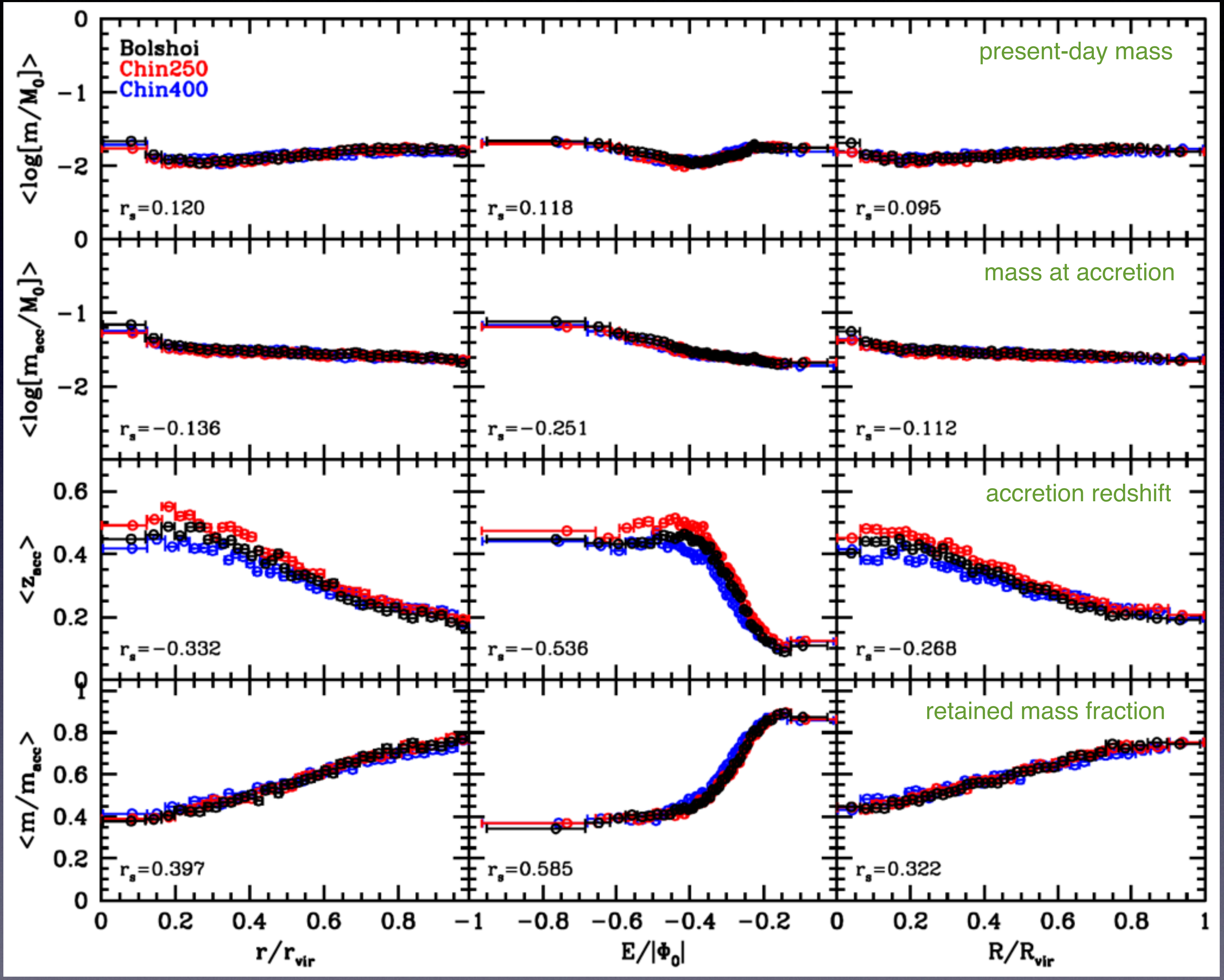
normalized radial number density profiles of dark matter subhaloes for five different mass bins. Note that there appears to be no dependence on subhalo mass.



Source: Springel et al. 2008, MNRAS, 391, 1685

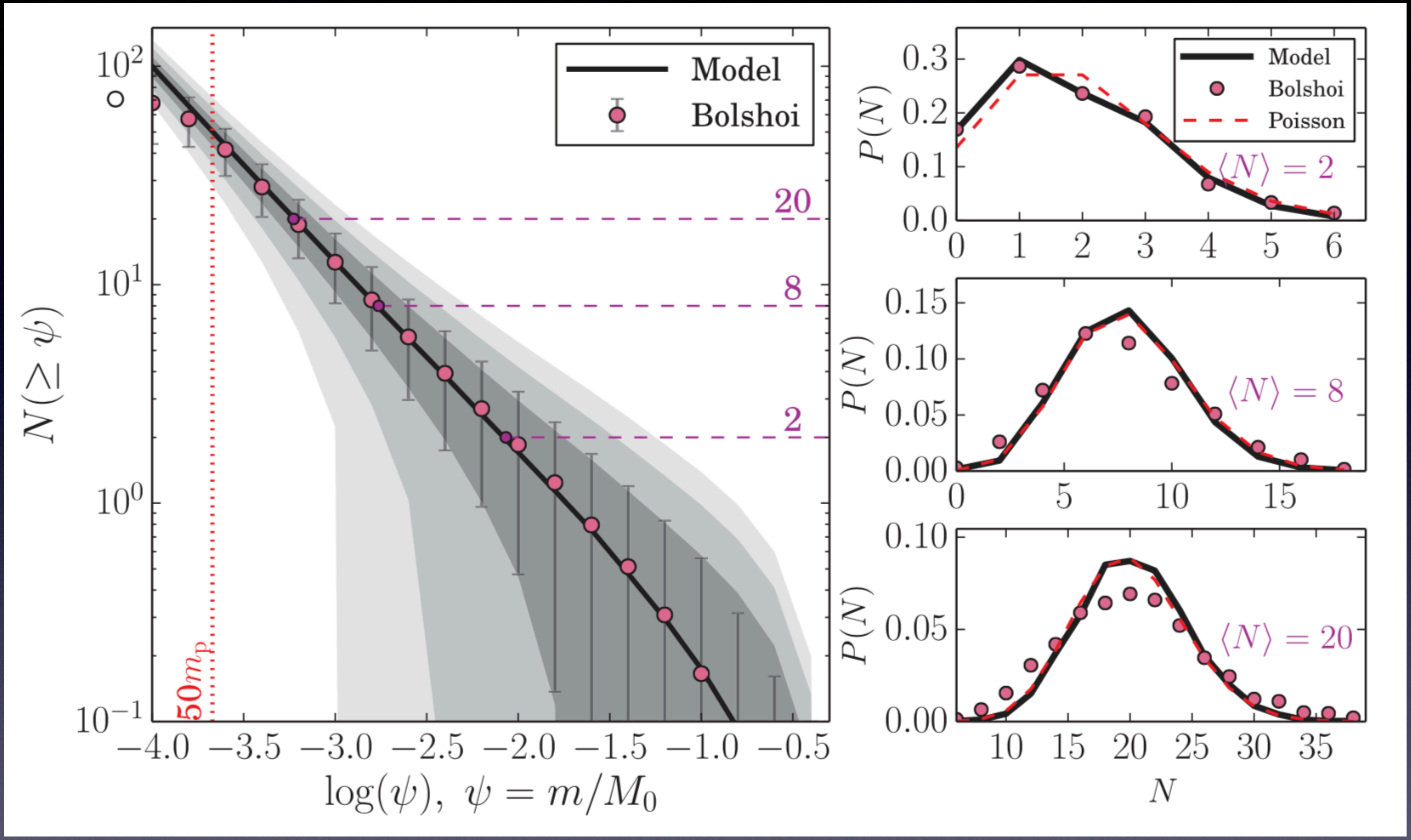
local mass fractions in subhaloes as a function of halo-centric radius. Results are shown for 6 MW-sized haloes from the Aquarius project...

Subhalo Segregation



vdB, Jiang, Campbell & Behroozi 2016

Subhalo Occupation Statistics



Source: Jiang & vdBosch, 2017, MNRAS, 472, 657

Subhalos roughly, but not exactly, follow Poisson statistics

Angular Momentum

Linear Tidal Torque Theory

See MBW §7.5.4 for more details

Dark matter haloes acquire angular momentum in the linear regime due to tidal torques from neighboring overdensities...

Consider the material that ends up as part of a virialized halo. Let V_L be the Lagrangian region that it occupies in the early Universe. The angular momentum of this material can be written as

$$\vec{J} = \int_{V_\mathrm{L}} \mathrm{d}^3\vec{x}_\mathrm{i}\, \bar{\rho}_\mathrm{m} a^3 \left(a\vec{x} - a\vec{x}_\mathrm{com}\right) \times \vec{v}$$

where $\vec{x}_\mathrm{com}$ is the center of mass (the barycenter) of the volume.

Using the Zel'dovich approximation for the velocities $\vec{v}$ inside the volume, and second-order Taylor series expansion of the potential, one finds that

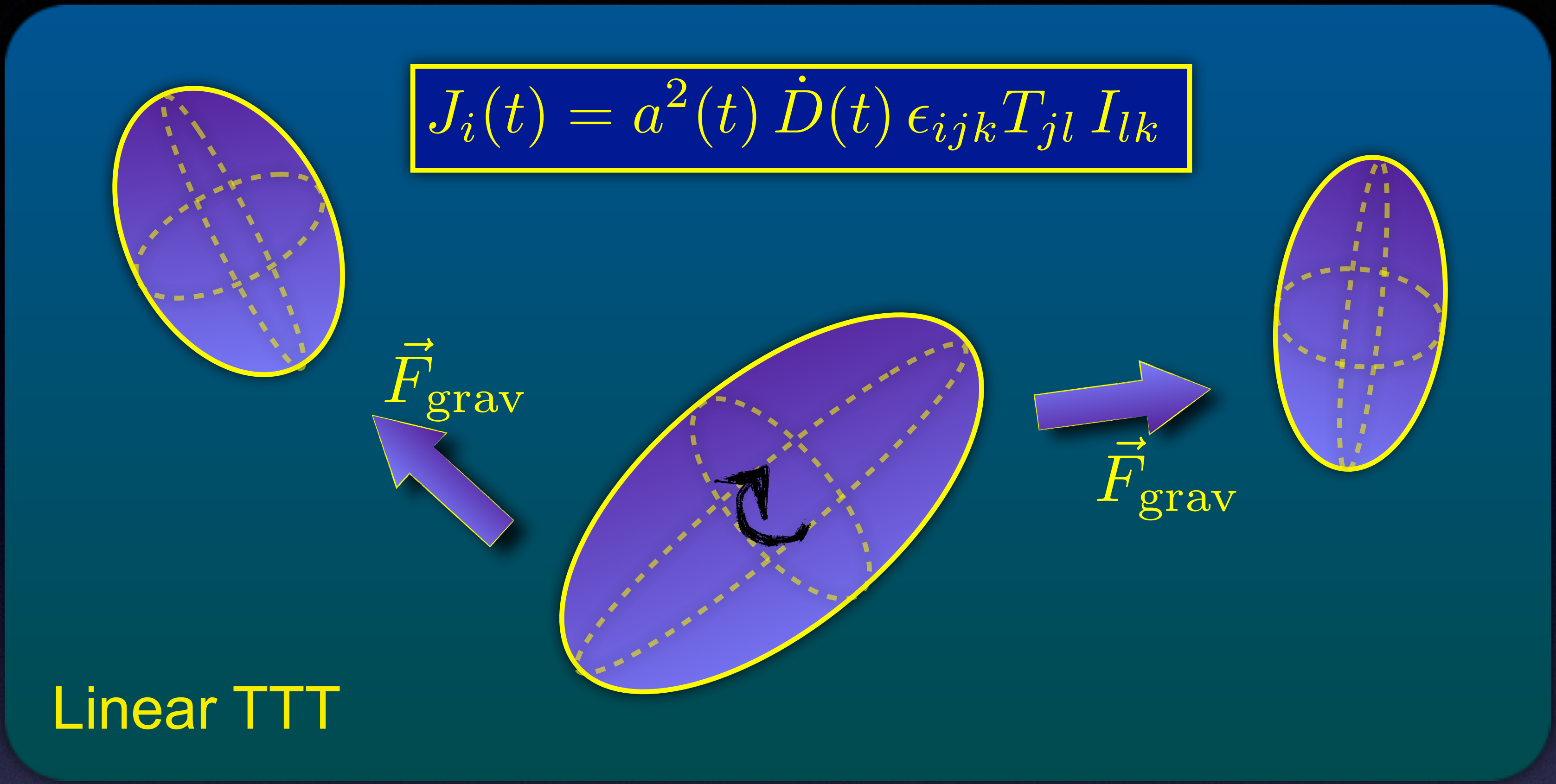
$$J_i(t) = a^2(t)\, \dot{D}(t)\, \epsilon_{ijk} T_{jl}\, I_{lk}$$

Einstein summation convention

Here $\dot{D}(t)$ is the time-derivative of the linear-growth rate, T_{ij} is the tidal tensor at the barycenter at the initial time, I_{ij} is the inertial tensor at the initial time, and ϵ_{ijk} is the 3D Levi-Civita tensor (also called the completely antisymmetric tensor).

This derivation for the growth of the angular momentum of `proto-haloes' , due to White (1984), is known as linear tidal torque theory (TTT)

Linear Tidal Torque Theory



Since principal axes of the tidal and inertia tensors are, in general, not aligned for a non-spherical volume, this linear angular momentum should be non-zero.

According to linear TTT, $J \propto a^2\,\dot{D}$, which for an EdS cosmology implies that $J \propto t$

According to linear TTT, the acquisition of angular momentum stops once a proto-halo turns around and starts to collapse: after turn-around, the moment of inertia starts to decline rapidly...Hence, according to linear TTT the final angular momentum of a virialized dark matter halo should (roughly) be equal to

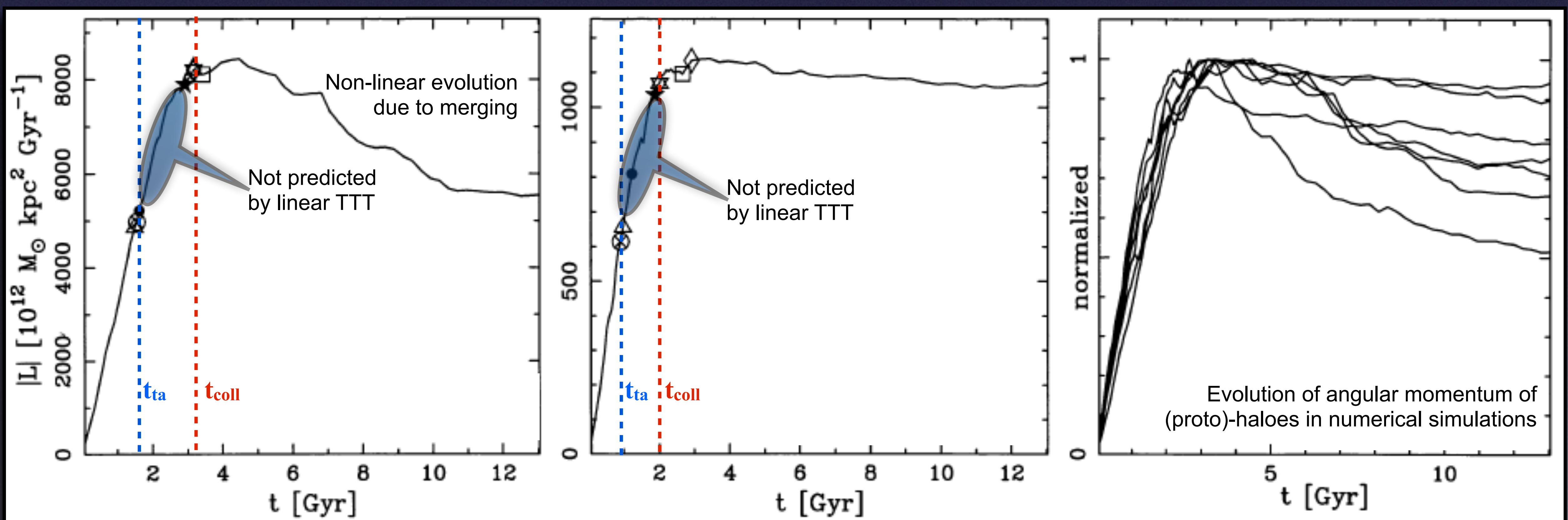
$$J_{\rm vir} = \int_0^{t_{\rm ta}} J(t)\,{\rm d}t = \epsilon_{ijk}\,T_{jl}\,I_{lk} \int_0^{t_{\rm ta}} a^2(t)\,\dot{D}(t)\,{\rm d}t$$

Testing Linear Tidal Torque Theory

Linear TTT can be tested using numerical simulations. These show that although the overall behavior of angular momentum growth of proto-haloes is consistent with TTT, it is unable to make reliable predictions for individual halos...

Two effects contribute to this `failure' :

- there is substantial angular momentum growth between turn-around and collapse, not anticipated by linear TTT
- angular momenta of haloes continue to evolve due to accretion of/ merging with other haloes (Maller et al. 2002; Vitvitska et al. 2002)



Source: Sugerman, Summers & Kamionkowski, 2000, MNRAS, 311, 762

The Halo Spin Parameter

The angular momentum of a dark matter halo is traditionally parameterized through the dimensionless spin parameter:

$$\lambda = \frac{J\,|E|^{1/2}}{G\,M^{5/2}}$$

where J, E and M are the angular momentum, energy and mass of the halo.

An alternative definition for the spin parameter, which avoids having to calculate the halo energy is:

$$\lambda' = \frac{J}{\sqrt{2}M\,V\,R}$$

where V and R are the virial velocity and viral radius, respectively. Definitions are equal if halo is singular isothermal sphere; otherwise they differ by factor of order unity....

Simulations show that PDF for spin parameter of haloes is a log-normal

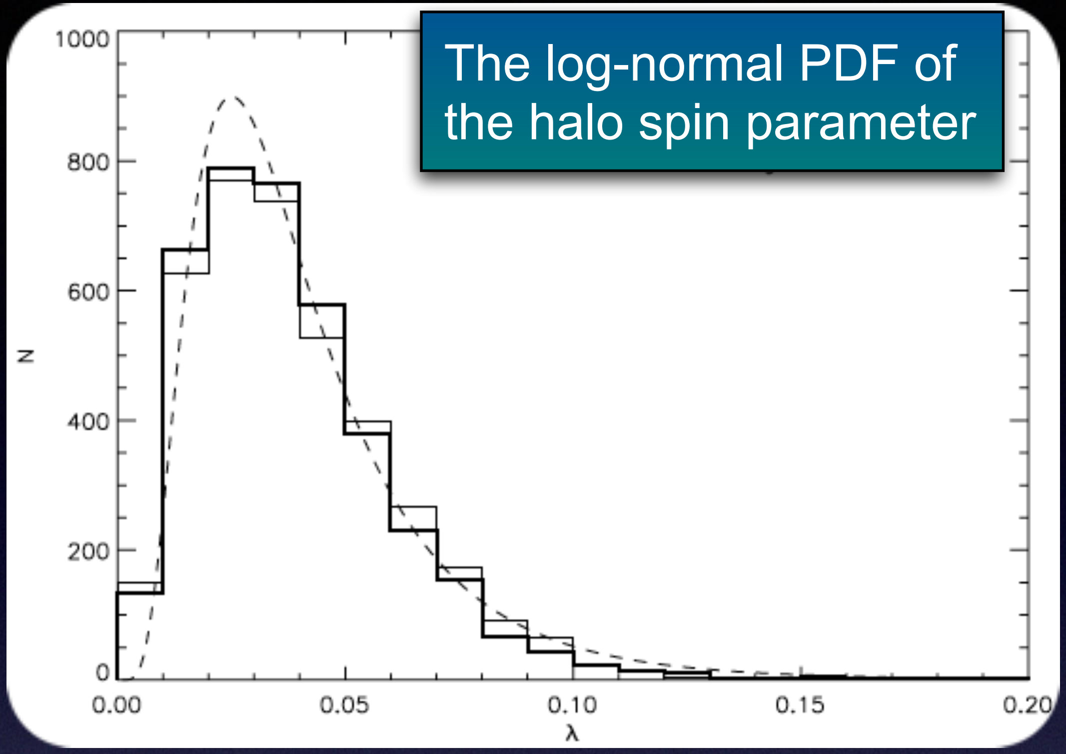
$$\mathcal{P}(\lambda)\,\mathrm{d}\lambda = \frac{1}{\sqrt{2\pi}\,\sigma_{\ln\lambda}} \exp\left(-\frac{\ln^2(\lambda/\bar{\lambda})}{2\sigma^2_{\ln\lambda}}\right)\frac{\mathrm{d}\lambda}{\lambda}$$

with $\bar{\lambda} \simeq 0.03$ and $\sigma_{\ln\lambda} \simeq 0.5$, with virtually no dependence on halo mass or cosmology...

The Halo Spin Parameter

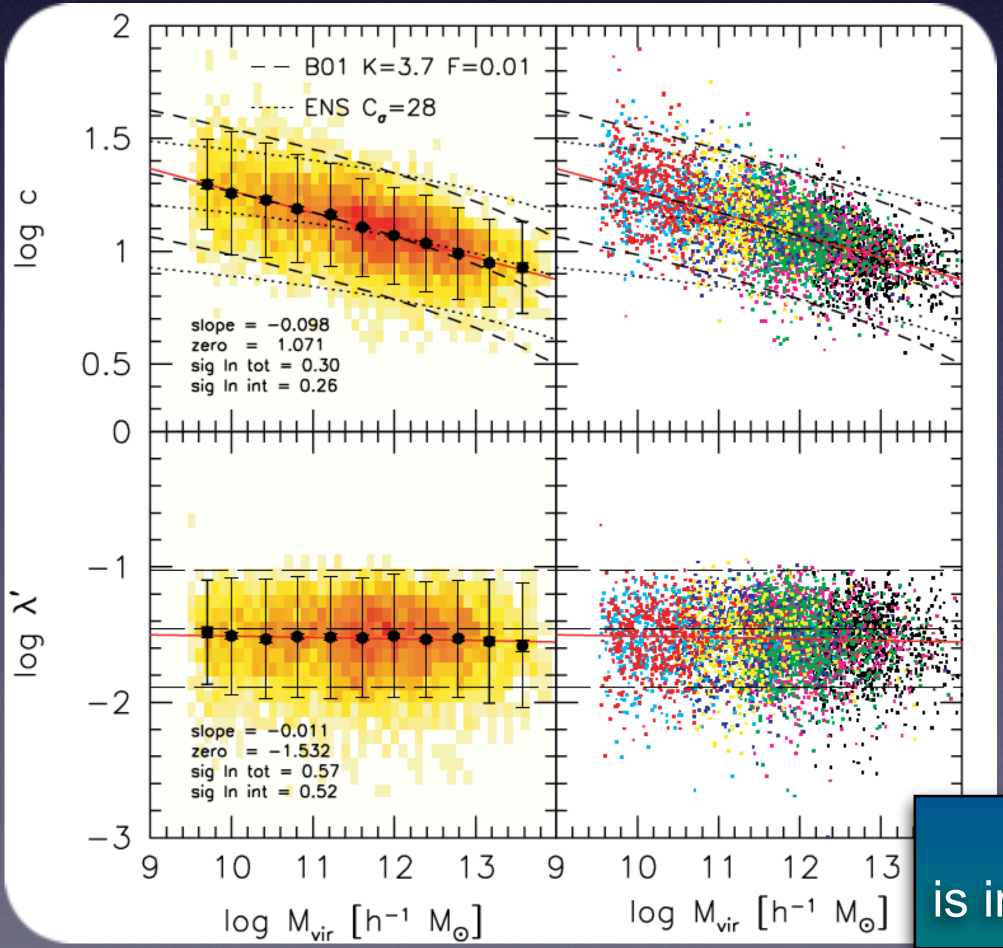
NOTE: the fact that the (median) spin parameter is so small indicates that dark matter haloes are not supported by rotation; flattening is due to velocity anisotropy, not rotation...

for comparison, the spin parameter of a typical disk galaxy is ~0.4, roughly an order of magnitude larger than that of a dark matter halo....



The log-normal PDF of the halo spin parameter

Source: Bailin & Steinmetz, 2005, ApJ, 627, 647



Source: Maccio et al. 2007, MNRAS, 378, 55

The halo spin parameter is independent of halo mass

Haloes that experienced a recent major merger have higher spin parameters than average . This reflects the large orbital angular momentum supplied by the merger

(e.g., Vitvitska et al 2001; Hetznecker & Burkert 2006)

However, this spin-merger correlation only persists for short time; virialization & accretion of new matter quickly brings spin parameter of halo back to average, non-conspicuous value

(e.g., D'Onghia & Navarro 2007)

The Angular Momentum Distribution

Using N-body simulations, Bullock et al. (2001) showed that dark matter haloes have a universal angular momentum profile with characteristic value j_0 and shape parameter μ :

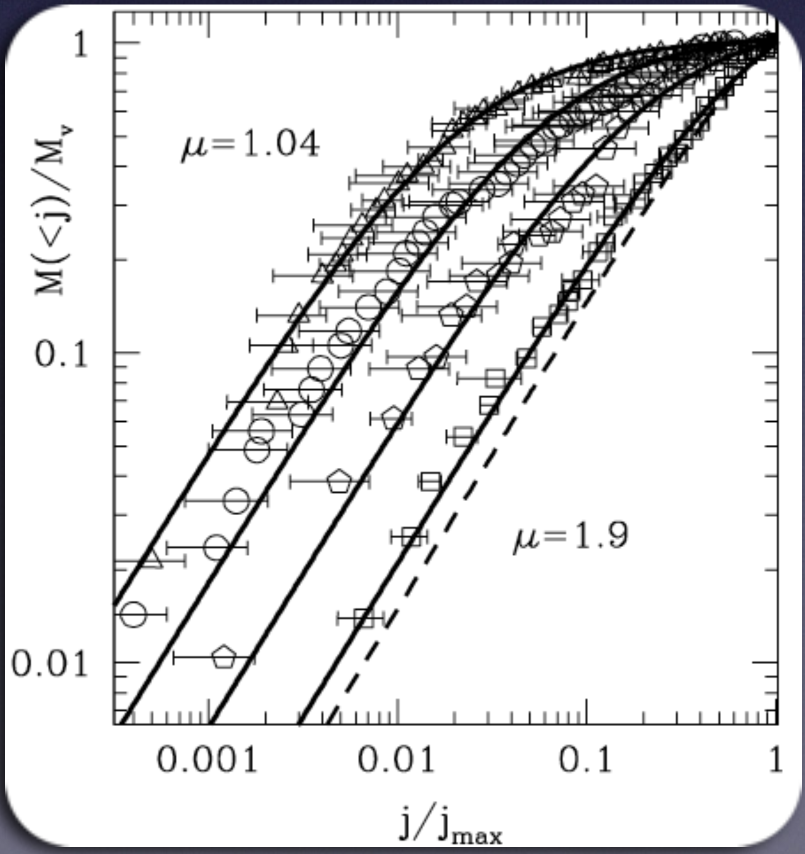
$$\mathcal{P}(j) = \frac{\mu j_0}{(j + j_0)^2} \quad \Rightarrow \quad M(<j) = M_{\rm vir} \frac{\mu j}{(j + j_0)}$$

This distribution has a maximum specific angular momentum, $j_{\rm max} = j_0/(\mu - 1)$, which is related to the halo's total specific angular momentum according to

$$j_{\rm tot} = \sqrt{2}\,\lambda'\,r_{\rm vir}\,V_{\rm vir} = j_{\rm max}\left[1 - \mu\left\{1 - (\mu - 1)\ln\left(\frac{\mu}{\mu - 1}\right)\right\}\right]$$

- The pair (λ , μ) completely specifies the angular momentum content of a dark matter halo.
- The shape parameter is characterized by a log-normal distribution with $\bar{\mu} \simeq 1.25$ and $\sigma_{\ln \mu} \simeq 0.4$.
- An alternative characterization of the angular momentum distribution within dark matter haloes is:

$$j(r) \propto r^{\alpha} \quad \text{with} \quad \alpha \simeq 1.1 \pm 0.3$$



Source: Bullock et al. 2001, ApJ, 555, 240

Lecture 11

SUMMARY

Summary: key words & important facts

Key words	
NFW/Einasto profile	Halo Concentration Parameter
Halo virial relations	Halo Spin Parameter
Cusp-Core controversy	Linear Tidal Torque Theory

- More massive haloes are less concentrated, are more aspherical, and have more substructure
  All these trends are mainly because more massive haloes assemble later
- Both concentration and spin parameter follow log-normal distributions
- The (median) spin parameter is independent of halo mass or redshift
- Dark matter halos have a universal density profile, a universal angular momentum profile, and a universal assembly history
- Subhalos reveal very little segregation by present-day mass, a weak segregation by accretion mass, and strong segregation by accretion redshift and retained mass fraction
- Dark matter haloes acquire angular momentum in the linear regime due to tidal torques from neighboring overdensities...

Summary: key equations & expressions

Halo Virial Relations

$$r_{\rm vir} \simeq 163\, h^{-1}{\rm kpc} \left[\frac{M_{\rm vir}}{10^{12} h^{-1} M_\odot}\right]^{1/3} \left[\frac{\Delta_{\rm vir}}{200}\right]^{-1/3} \Omega_{\rm m,0}^{-1/}$$

$$V_{\rm vir} \simeq 163\, {\rm km/s} \left[\frac{M_{\rm vir}}{10^{12} h^{-1} M_\odot}\right]^{1/3} \left[\frac{\Delta_{\rm vir}}{200}\right]^{1/6} \Omega_{\rm m,0}^{1/6}\,(1+z)^{1/2}$$

Subhalo Mass Function

$$\frac{{\rm d}n}{{\rm d}\ln(m/M)} \propto \left(\frac{m}{M}\right)^{-\gamma} \exp\left[-(m/\beta M)\right]$$

$$\gamma \simeq 0.9 \pm 0.1 \qquad \beta \simeq 0.3$$

Halo Density Profiles

NFW

$$\rho(r) = \frac{\rho_{\rm s}}{\left(\frac{r}{r_{\rm s}}\right)\left(1+\frac{r}{r_{\rm s}}\right)^2}$$

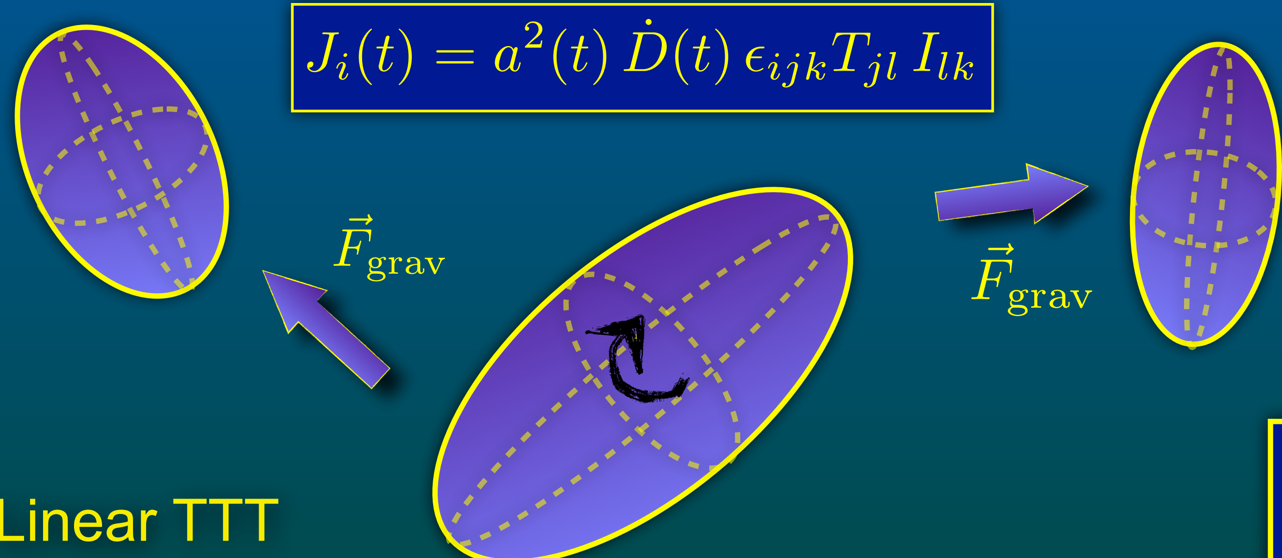
concentration parameter

$$c = r_{\rm vir}/r_{\rm s}$$

Einasto

$$\rho(r) = \rho_{-2}\, \exp\left[\frac{-2}{\alpha}\left\{\left(\frac{r}{r_{-2}}\right)^{\alpha} - 1\right\}\right] \;\Rightarrow\; \frac{{\rm d}\ln\rho}{{\rm d}\ln r} = -2\left(\frac{r}{r_{-2}}\right)^{\alpha}$$

$$J_i(t) = a^2(t)\,\dot{D}(t)\,\epsilon_{ijk} T_{jl}\, I_{lk}$$



Linear TTT

Halo Spin Parameter

$$\lambda = \frac{J\,|E|^{1/2}}{G\,M^{5/2}} \qquad \lambda' = \frac{J}{\sqrt{2} M\, V\, R}$$

$$J_{\rm vir} = \int_0^{t_{\rm ta}} J(t)\,{\rm d}t = \epsilon_{ijk}\, T_{jl}\, I_{lk} \int_0^{t_{\rm ta}} a^2(t)\,\dot{D}(t)\,{\rm d}t$$