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Ans: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$\Rightarrow g^{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}$ & $|g| \approx 1+h$; $h \equiv h^{\mu}_{\mu}$

We also know that $\Gamma^{\mu}_{\nu\alpha} = \frac{1}{2}\eta^{\mu\beta}\left[\partial_\nu h_{\alpha\beta} + \partial_\alpha h_{\beta\nu} - \partial_\beta h_{\nu\alpha}\right]$

now: $R^{\mu}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\gamma}_{\nu\beta} - \Gamma^{\mu}_{\gamma\beta}\Gamma^{\gamma}_{\nu\alpha}$

Since we need to linear order of h & $\Gamma \sim O(h)$

we can ignore $\Gamma\Gamma$ terms as they will be $O(h^2)$

$\Rightarrow R^{(1)\,\mu}_{\nu\alpha\beta} \approx \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha}$ → Using this since we want $O(h)$

& Thus $R^{(1)}_{\mu\nu\alpha\beta} = \eta_{\mu\gamma} R^{(1)\gamma}_{\nu\alpha\beta}$

$= \eta_{\mu\gamma}\left[\partial_\alpha \Gamma^{\gamma}_{\nu\beta} - \partial_\beta \Gamma^{\gamma}_{\nu\alpha}\right]$

$= \eta_{\mu\gamma}\Big[\partial_\alpha\Big(\frac{1}{2}\eta^{\gamma\delta}\left(\partial_\nu h_{\delta\beta} + \partial_\beta h_{\delta\nu} - \partial_\delta h_{\nu\beta}\right)\Big)$
$\quad - \partial_\beta\Big(\frac{1}{2}\eta^{\gamma\delta}\left(\partial_\nu h_{\delta\alpha} + \partial_\alpha h_{\delta\nu} - \partial_\delta h_{\nu\alpha}\right)\Big)\Big]$

$= \frac{1}{2}\left[\partial_\nu\partial_\alpha h_{\mu\beta} + \partial_\mu\partial_\beta h_{\nu\alpha} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\beta h_{\mu\alpha}\right]$

$\Rightarrow R^{(1)}_{\mu\nu} = R^{(1)\alpha}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha}$

$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}\eta^{\alpha\beta}\left[\partial_\mu h_{\alpha\beta} + \partial_\alpha h_{\beta\mu} - \partial_\beta h_{\mu\alpha}\right]$ → $\partial_\alpha h_{\beta\mu} - \partial_\beta h_{\mu\alpha} = 0$

$\Rightarrow R^{(1)}_{\mu\nu} = \frac{1}{2}\Big[\partial_\alpha\left(\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right)\right)$
$\quad - \partial_\nu\left(\eta^{\alpha\beta}\left(\partial_\mu h_{\beta\alpha} + \partial_\alpha h_{\beta\mu} - \partial_\beta h_{\mu\alpha}\right)\right)\Big]$ → $\partial_\alpha h_{\beta\mu} - \partial_\beta h_{\mu\alpha} = 0$

$= \frac{1}{2}\left(\eta^{\alpha\beta}\partial_\alpha\left(-\partial_\beta h_{\mu\nu}\right)\right)$

$= -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}$

$= -\frac{1}{2}\Box h_{\mu\nu}$