Computational Fluid Dynamics
Problem Set 1

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Problem 1 (Hyperbolic systems)

(*a*) This problem illustrates how the term "hyperbolic" can be motivated for 1st order systems of PDEs. Show that any linear, constant-coefficient second-order hyperbolic PDE in $\mathbb{R}^{n+1}$ of the form

$$\sum_{i,j=0}^{n} a_{ij} u_{x_i x_j} + b(\{u_{x_i}\}) = 0 \tag{1}$$

can be recast as a system of 1st order hyperbolic PDEs.
Hint: Note that it suffices to consider an equation of the form $u_{tt} - \Delta u + b(\{u_{x_i}\}) = 0$, since the principal part can always be brought into this form by a suitable coordinate transformation (as shown in the lectures). Here, $t = x_0$ and Δ only acts on $(x_1, \ldots, x_n) \in \mathbb{R}^n$. You may want to consider a vector $\mathbf{u}$ composed of derivatives u_{x_i}.

(*b*) Another way to motivate hyperbolicity for systems of m 1st order PDEs,

$$\mathbf{u}_t + \sum_{i=1}^{n} A_i(x,t)\mathbf{u}_{x_i} = 0, \tag{2}$$

is to require the existence of m independent plane wave solutions $\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\xi \cdot \mathbf{x} - \sigma t)$ for any direction $\xi \in \mathbb{R}^n$. Using this ansatz, show that hyperbolicity implies the existence of m independent plane waves.

Problem 2 (Hyperbolicity of Euler's equations)

Consider the 1D Euler equations in conservation form,

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \tag{3}$$

where $\mathbf{u} = (\rho, \rho v, E)$ and $\mathbf{f} = (\rho v, \rho v^2 + p, v(E + p))$. Assume a general equation of state $p = p(\rho, \epsilon)$. Here, $E = \frac{1}{2}\rho v^2 + e$ is the total energy density, and $\epsilon = e/\rho$ denotes the specific internal energy.

(a) (*Invariance of hyperbolicity under coordinate transformation*)
Consider $\mathbf{u}$ as a solution to a strictly hyperbolic system of the type

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = 0, \tag{4}$$

and let $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ denote a smooth coordinate transformation with inverse Φ^{-1}. Show that $\tilde{\mathbf{u}} \equiv \Phi(\mathbf{u})$ satisfies the strictly hyperbolic system

$$\tilde{\mathbf{u}}_t + \tilde{A}(\tilde{\mathbf{u}})\tilde{\mathbf{u}}_x = 0, \tag{5}$$

where $\tilde{A}(\mathbf{w}) \equiv D\Phi(\Phi^{-1}(\mathbf{w}))A(\Phi^{-1}(\mathbf{w}))D(\Phi^{-1})(\mathbf{w})$ for $\mathbf{w} \in \mathbb{R}^m$.

(b) (*Conservatives to primitives recovery*)
Write down the coordinate transformation $\Phi : \mathbf{u} \mapsto \mathbf{w}$, where $\mathbf{w} \equiv (\rho, v, \epsilon)$ are the primitive variables, and derive the Euler equations in primitive (non-conservation) form

$$\mathbf{w}_t + A(\mathbf{w})\mathbf{w}_x = 0. \tag{6}$$

(c) (*Strict hyperbolicity*)
Starting from (6) and using (a), show that the Euler equations (3) are strictly hyperbolic if the local sound speed is positive,

$$c_{\mathrm{s}} \equiv \left(\frac{p}{\rho^2}\frac{\partial p}{\partial \epsilon} + \frac{\partial p}{\partial \rho} \right)^{1/2} > 0, \tag{7}$$

i.e., provided that $p > 0$ and $\partial p/\partial\epsilon > 0$, $\partial p/\partial\rho > 0$. This shows that numerical schemes that exploit strong hyperbolicity cannot handle vacuum, but instead require at least a tenous "atmosphere" on the computational domain.

Problem 3 (Euler equations: energy evolution)

(a) From the second moment of the Boltzmann equation, and following the methods introduced in the lectures, derive the equation for the total energy density,

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}\left[(E + p)v_j\right] = -\frac{\partial h_j}{\partial x_j} + \rho v_j F_j, \tag{8}$$

where $h_j \equiv \int \frac{m}{2}\tilde{u}_j \tilde{u}^2 f d^3u$ is the heat flux.

(b) Only using the continuity and momentum equations, derive a separate evolution equation for the bulk (kinetic) energy of the fluid,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) = -v_j\frac{\partial p}{\partial x_j} + \rho v_j F_j. \tag{9}$$

Using Eq. (8) and assuming $h_j \equiv 0$ then derive the evolution equation for the internal energy density,

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x_j}(e v_j) = -p\frac{\partial v_j}{\partial x_j}. \tag{10}$$