

Computational Fluid Dynamics

Problem Set 4

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Problem 1 (Jump conditions for the Euler equations)

Consider a 3-shock with shock velocity S_3 for the 1D Euler equations in conservative form, assuming an ideal gas with adiabatic constant γ . Let $(\rho_\star, v_\star, p_\star)$ and (ρ_r, v_r, p_r) denote the sets of primitive variables on the left and right side of the shock front, respectively. Derive the Rankine-Hugoniot jump conditions

$$\frac{\rho_\star}{\rho_r} = \frac{\frac{p_\star}{p_r} + \frac{\gamma-1}{\gamma+1}}{\frac{\gamma-1}{\gamma+1} \frac{p_\star}{p_r} + 1}, \quad s_3 = v_r + c_r \sqrt{\frac{\gamma+1}{2\gamma} \frac{p_\star}{p_r} + \frac{\gamma-1}{2\gamma}}, \quad (1)$$

where $c = \sqrt{\gamma p / \rho}$ denotes the sound speed.

Problem 2 (Non-uniqueness of weak solutions)

A given initial value problem for conservation laws admits, in general, infinitely many weak solutions. Consider the Riemann problem for the inviscid Burgers' equation:

$$\begin{cases} \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0 \\ u(0, x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \end{cases} \quad (2)$$

Show that for any $\alpha \in (0, 1)$ the 'two-shock solution'

$$u_\alpha(t, x) \equiv \begin{cases} 0 & \text{if } x < \frac{\alpha}{2}t \\ \alpha & \text{if } \frac{\alpha}{2}t < x < \frac{1+\alpha}{2}t \\ 1 & \text{if } x > \frac{1+\alpha}{2}t \end{cases} \quad (3)$$

is a weak solution to the Cauchy problem (2).

Problem 3 (Entropy pair & Riemann invariants)

Consider the 1D Euler equations for an ideal gas with specific entropy $S = S_0 + c_v \ln \frac{p}{\rho^\gamma}$, where S_0 and c_v are constants and γ is the adiabatic index.

1. (*Entropy pair*) Show that the specific entropy satisfies

$$S_t + vS_x = 0 \quad (4)$$

where the solution is smooth. What is the entropy flux? Show that modulo a minus sign the physical entropy provides an entropy–entropy-flux pair for the Euler equations.

2. (*Riemann invariants*) Consider the Euler equations in primitive form using the primitives (ρ, v, S) . Show that S is a Riemann invariant. Show that there exist the following k -Riemann invariants:

- $k = 1$: $S, u + \frac{2}{\gamma-1}c$
- $k = 2$: u, p
- $k = 3$: $S, u - \frac{2}{\gamma-1}c$

3. (*Characteristic fields*) Show that the $k = 2$ characteristic field is linearly degenerate, while the $k = 1, 3$ characteristic fields are genuinely non-linear. *Hint*: It is easiest to work with the conservative formulation here.

Problem 4 (Solution to Burgers' equation)

Determine and sketch the exact entropy solution to Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$ for all $t > 0$ for initial data given by

$$u^0(x) = \begin{cases} -1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x > 1 \end{cases} \quad (5)$$

Problem 5 (Bonus question, not required)

Consider a piecewise smooth weak solution u of the system $u_t + f(u)_x = 0$, and assume that an entropy pair to the equation is given by (Φ, Ψ) . Show that the entropy condition along the discontinuity with velocity s leads to the inequality

$$s[\Phi(u_l) - \Phi(u_r)] \leq \Psi(u_l) - \Psi(u_r), \quad (6)$$

where u_l and u_r denote the values of u immediately left and right of the discontinuity. *Hint*: Consider the Rankine-Hugoniot theorem. *Remark*: This is the first step in showing the equivalence of the Lax entropy condition and the entropy inequalities.