Q1) Calc. R_{ij} for following:-

$$dl^2 = \frac{dr^2}{1-r^2} + r^2 d\Omega^2$$

$$\Rightarrow dl^2 = \frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$\Rightarrow R_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\partial^2_{\nu\alpha} g_{\mu\beta} - \partial^2_{\nu\beta} g_{\mu\alpha} - \partial^2_{\mu\alpha} g_{\nu\beta} + \partial^2_{\mu\beta} g_{\nu\alpha}\right]$$

$$\|g_{\mu\nu}\| = \mathrm{Diag}\left(\frac{1}{1-r^2}, r^2, r^2\sin^2\theta\right)$$

$$\Rightarrow \mu\beta = \mu\alpha = \nu\beta = \nu\alpha = \{11, 22, 33\}$$

$$\Rightarrow R_{\nu\beta} = g^{\mu\eta} R_{\eta\nu\mu\beta}$$

$\cancel{\text{For } \mu = \eta = 1;\ R_{1\nu1\beta} = \frac{1}{2}\left[\partial^2_{\nu1} g_{1\beta} - \partial^2_{\nu\beta} g_{11} - \partial^2_{11} g_{\nu\beta} + \partial^2_{1\beta} g_{\nu1}\right]}$

$\cancel{\Rightarrow \beta = 1, \nu = 2;\ \Rightarrow = 0}$

$\cancel{\Rightarrow \beta = 1, \nu = 3;\ \Rightarrow = 0}$

$\cancel{\Rightarrow \beta = 2, \nu = 1;\ \Rightarrow = 0}$

$\cancel{\Rightarrow \beta = 3, \nu = 3;\ \Rightarrow = 0}$

$\cancel{\Rightarrow \beta = 2, \nu = 2;}$

$$\Rightarrow R_{\nu\beta} = g^{11} R_{1\nu1\beta} + g^{22} R_{2\nu2\beta} + g^{33} R_{3\nu3\beta}$$

$$R_{1\nu1\beta} = \frac{1}{2}\left[\partial^2_{\nu1} g_{1\beta} - \partial^2_{\nu\beta} g_{11} - \partial^2_{11} g_{\nu\beta} + \partial^2_{1\beta} g_{\nu1}\right]$$

$\Rightarrow \nu = \beta = 1 \Rightarrow 0$ | $\nu = 1, \beta = 2 \Rightarrow 0$ | etc.

only non 0 $\Rightarrow \nu = \beta = 3$ | $\boxed{R_{1\nu1\beta} = -\sin^2\theta = R_{1313}}$

$\Rightarrow \nu = \beta = 2$ | $\boxed{R_{1212} = -1}$

$$\Rightarrow R_{2\nu2\beta} = \frac{1}{2}\left[\partial^2_{\nu2} g_{2\beta} - \partial^2_{\nu\beta} g_{22} - \partial^2_{22} g_{\nu\beta} + \partial^2_{2\beta} g_{\nu2}\right]$$

only non 0 $\Rightarrow \nu = \beta = 1 \Rightarrow \boxed{R_{2121} = -1}$

$\nu = \beta = 3 \Rightarrow \boxed{R_{2323} = -r^2\cos 2\theta}$

$$\Rightarrow R_{3\nu3\beta} = \frac{1}{2}\left[\partial^2_{\nu3} g_{3\beta} - \partial^2_{\nu\beta} g_{33} - \partial^2_{33} g_{\nu\beta} + \partial^2_{3\beta} g_{\nu3}\right]$$

$\Rightarrow \nu = \beta = 1 \Rightarrow$ $\boxed{R_{3131} = -\sin^2\theta}$

$\nu = \beta = 2 \Rightarrow$ $\boxed{R_{3232} = -r^2\cos 2\theta}$

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$\Rightarrow R_{11} = g^{22} R_{2121} + g^{33} R_{3131} = \frac{1}{r^2}(-1) + \frac{1}{r^2 \sin^2\theta}(-\sin^2\theta) = \frac{-2}{r^2}$

$\Rightarrow R_{22} = g^{11} R_{1212} + g^{33} R_{3232} = (1-r^2)(-1) + \frac{1}{r^2\sin^2\theta}(-r^2\cos 2\theta)$

$= r^2 \cancel{-1} - (\cot^2\theta \cancel{-1}) = r^2 - \cot^2\theta$

$\Rightarrow R_{33} = g^{11} R_{1313} + g^{22} R_{2323} = (1-r^2)(-\sin^2\theta) + \left(\frac{1}{r^2}\right)(-r^2\cos 2\theta)$

$= -r^2 \sin^2\theta - \sin^2\theta - \cos 2\theta$

$= r^2 \sin^2\theta - 1 + \sin^2\theta$

$= (r^2+1)\sin^2\theta - 1$

Similarly :- for $dl^2 = \frac{dr^2}{1+r^2} + r^2 d\Omega^2$

$R_{11} = -2/r^2$ | $R_{22} = (1+r^2)(-1) - (\cot^2\theta - 1)$

$= \cancel{-1} - r^2 - \cot^2\theta \cancel{+1} = -(r^2 + \cot^2\theta)$

$R_{33} = (1+r^2)(-\sin^2\theta) - \cos 2\theta$

$= -r^2\sin^2\theta - \sin^2\theta - \cos 2\theta$

$= (1-r^2)\sin^2\theta - 1$

For $dl^2 = d\chi^2 + \sin^2\chi \, d\Omega^2$

$= d\chi^2 + \sin^2\chi \, d\theta^2 + \sin^2\chi \sin^2\theta \, d\phi^2$

$\|g_{\mu\nu}\| = \text{Diag}(1, \sin^2\chi, \sin^2\chi \sin^2\theta)$

$\Rightarrow R_{11} = -\frac{\cos^2\chi}{\sin^2\chi} - \cancel{\frac{\cos^2\chi \sin^2\theta}{\sin^2\chi \sin^2\theta}} = 0$

$R_{22} = -\cos 2\chi - \frac{\sin^2\chi \cos 2\theta}{\sin^2\chi \sin^2\theta} = -\cos 2\chi - \cot^2\theta + 1$

$R_{33} = -\cos 2\chi \sin^2\theta - \frac{\sin^2\chi \cos 2\theta}{\sin^2\chi} = -(\cos 2\chi \sin^2\theta - \cos 2\theta)$

Similarly for $dl^2 = d\chi^2 + \sinh^2\chi \, d\Omega^2$

$R_{11} = 0$ | $R_{22} = -\cosh 2\chi - \cot^2\theta + 1$

$R_{33} = -\cosh 2\chi \sin^2\theta - \cos 2\theta$