Find EOM & $T_{\mu\nu}$ for:-

$$S_M = \int d^4x \sqrt{|g|}\left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]$$

$$\delta_\phi S_M = S[\phi+\delta\phi] - S[\phi]$$

$$= \int d^4x \sqrt{|g|}\left[\frac{1}{2} g^{\mu\nu}\partial_\mu(\phi+\delta\phi)\cdot\partial_\nu(\phi+\delta\phi) - V(\phi+\delta\phi) - \frac{1}{2} g^{\mu\nu}\partial_\mu(\phi)\partial_\nu\phi + V(\phi)\right]$$

$$= \int d^4x \sqrt{|g|}\left[\frac{1}{2} g^{\mu\nu}\left(\partial_\mu(\phi+\delta\phi)\cdot\partial_\nu(\phi+\delta\phi) - \partial_\mu\phi\,\partial_\nu\phi\right) - \left(V(\phi+\delta\phi) - V(\phi)\right)\right]$$

$$(\partial_\mu\phi + \partial_\mu\delta\phi)(\partial_\nu\phi + \partial_\nu\delta\phi) - \partial_\mu\phi\,\partial_\nu\phi$$ (Taking 1st order)

$$= \partial_\nu\phi\,\partial_\mu\delta\phi + \partial_\mu\phi\,\partial_\nu\delta\phi + \cancel{(\partial_\mu\delta\phi)(\partial_\nu\delta\phi)}$$

$$= \underline{2\,\partial_\mu\phi\,\partial_\nu\delta\phi}$$

& $$V(\phi+\delta\phi) - V(\phi) = \cancel{V(\phi)} + \frac{\partial V}{\partial\phi}\delta\phi - \cancel{V(\phi)}$$

$$= \frac{\partial V}{\partial\phi}\cdot\delta\phi$$

$$\Rightarrow \delta_\phi S_M = \int d^4x \sqrt{|g|}\left[\frac{1}{\cancel{2}}\cancel{2}\, g^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu(\delta\phi) - \frac{\partial V}{\partial\phi}\delta\phi\right] = 0$$

$$= \int d^4x \sqrt{|g|}\left[(\partial_\mu\partial_\nu\phi)\, g^{\mu\nu}\,\delta\phi - \frac{\partial V}{\partial\phi}\delta\phi\right] = 0$$

from integrⁿ by parts ←

$$= \int d^4x \sqrt{|g|}\underbrace{\left[g^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{\partial V}{\partial\phi}\right]}_{=0}\delta\phi = 0$$

$$\Rightarrow \boxed{g^{\mu\nu}\partial_\mu\partial_\nu\phi = \frac{\partial V}{\partial\phi}}$$

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Here, $\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$

$\Rightarrow T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L} g_{\mu\nu}$

$= \frac{\cancel{2} \times 1}{\cancel{2}} \frac{\partial g^{\alpha\beta}}{\partial g^{\alpha\beta}} \partial_\alpha \phi \partial_\beta \phi$

$\frac{\cancel{2} \times 1}{\cancel{2}} \frac{\partial g^{\alpha\beta}}{\partial g^{\mu\nu}} \partial_\alpha \phi \partial_\beta \phi - \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi)\right) \times g_{\mu\nu}$

$= \frac{1}{2}\left(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu\right) \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g^{\alpha\beta} g_{\mu\nu} \partial_\alpha \phi \partial_\beta \phi$

$+ g_{\mu\nu} V(\phi)$

$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\alpha\beta} g_{\mu\nu} \partial_\alpha \phi \partial_\beta \phi + g_{\mu\nu} V(\phi)$

(b) $S_M = \int d^4x \sqrt{|g|}\, F_{\mu\nu} F_{\alpha\beta} \cdot g^{\mu\alpha} g^{\nu\beta}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\Rightarrow S_M = \int d^4x \sqrt{|g|} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) g^{\mu\alpha} g^{\nu\beta}$

$\cancel{\Rightarrow \delta_A S_M = \int d^4x \sqrt{|g|} \Big[}$

$\Rightarrow \delta_A S_M = \int d^4x \sqrt{|g|} \Big[\left(\partial_\mu (A_\nu + \delta A_\nu) - \partial_\nu (A_\mu + \delta A_\mu)\right)\left(\partial_\alpha (A_\beta + \delta A_\beta) - \partial_\beta (A_\alpha + \delta A_\alpha)\right) - \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\Big] g^{\mu\alpha} g^{\nu\beta}$

$= \int d^4x \sqrt{|g|} \Big[\left(\partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha + \partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha\right) - \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\Big] g^{\mu\alpha} g^{\nu\beta}$

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$$= \int d^4x \sqrt{|g|} \Big[(F_{\mu\nu} + \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)(F_{\alpha\beta} + \partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha) - F_{\mu\nu} F_{\alpha\beta} \Big] \times g^{\mu\alpha} g^{\nu\beta}$$

$$= \int d^4x \sqrt{|g|} \Big[F_{\mu\nu} F_{\alpha\beta} + (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F_{\alpha\beta} + (\partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha) F_{\mu\nu} - F_{\mu\nu} F_{\alpha\beta} + \text{higher order terms} \Big] \times g^{\mu\alpha} g^{\nu\beta}$$

$$= \int d^4x \sqrt{|g|} \Big[2 (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F_{\alpha\beta}\, g^{\mu\alpha} g^{\nu\beta} \Big]$$

↳ Using the property that g is always symmetric & replacing $(\alpha, \beta) \leftrightarrow (\mu, \nu)$ in 2^{nd} term

$$= 2 \int d^4x \sqrt{|g|} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F_{\alpha\beta}\, g^{\mu\alpha} g^{\nu\beta}$$

$$\cancel{= 2 \int d^4x \sqrt{|g|} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)}$$

$$\Rightarrow \partial_\alpha F^{\alpha\beta} = \partial_\alpha (g^{\beta\mu} F_{\mu\nu} g^{\nu\alpha} |g|)$$

$$\& \; T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L} g_{\mu\nu}$$

$$\cancel{= 2 F_{\alpha\gamma} F_{\alpha\beta} \left(\frac{\partial g^{\mu\alpha}}{\partial g^{\mu\nu}} \right)}$$

$$= 2 F_{\gamma\delta} F_{\alpha\beta} \left(\frac{\partial g^{\gamma\alpha}}{\partial g^{\mu\nu}} g^{\delta\beta} + \frac{\partial g^{\delta\beta}}{\partial g^{\mu\nu}} g^{\gamma\alpha} \right) - F_{\gamma\delta} F_{\alpha\beta} g^{\gamma\alpha} g^{\delta\beta} g_{\mu\nu}$$

$$= F_{\gamma\delta} F_{\alpha\beta} \Big\{ \Big[(\delta^\gamma_\mu \delta^\alpha_\nu + \delta^\gamma_\nu \delta^\alpha_\mu) g^{\delta\beta} + (\delta^\delta_\mu \delta^\beta_\nu + \delta^\delta_\nu \delta^\beta_\mu) g^{\gamma\alpha} \Big] - g^{\gamma\alpha} g^{\delta\beta} g_{\mu\nu} \Big\}$$

$$= \Big[F_{\mu\delta} F_{\nu\beta}\, g^{\delta\beta} + F_{\nu\delta} F_{\mu\beta}\, g^{\delta\beta} + F_{\gamma\mu} F_{\alpha\nu}\, g^{\gamma\alpha} + F_{\gamma\nu} F_{\alpha\mu}\, g^{\gamma\alpha} - g^{\gamma\alpha} g^{\delta\beta} g_{\mu\nu} \Big]$$

$$= 4 F_{\mu\alpha} F_{\nu\beta}\, g^{\alpha\beta} - F_{\gamma\delta} F_{\alpha\beta}\, g^{\gamma\alpha} g^{\delta\beta} g_{\mu\nu}$$