Q) Prove $0 = D_\mu T^\mu_\nu \equiv \partial_\mu T^\mu_\nu + \Gamma^\mu_{\beta\mu} T^\beta_\nu - \Gamma^\beta_{\nu\mu} T^\mu_\beta = \frac{1}{\sqrt{|g|}} \partial_\mu (T^\mu_\nu \sqrt{|g|}) - \frac{1}{2}(\partial_\nu g_{\mu\alpha}) T^{\mu\alpha}$

Ans). $\frac{1}{\sqrt{|g|}} \partial_\mu (T^\mu_\nu \sqrt{|g|}) - \frac{1}{2}(\partial_\nu g_{\mu\alpha}) T^{\mu\alpha}$

$$= \frac{1}{\cancel{\sqrt{|g|}}} (\partial_\mu T^\mu_\nu) \cancel{\sqrt{|g|}} + T^\mu_\nu (\partial_\mu \sqrt{|g|}) - \frac{1}{2}(\partial_\nu g_{\mu\alpha}) T^{\mu\alpha}$$

$$= \partial_\mu T^\mu_\nu + T^\mu_\nu \frac{1}{\cancel{2}\sqrt{|g|}} \frac{1}{2\sqrt{|g|}} \partial_\mu |g| - \frac{1}{2}(\partial_\nu g_{\mu\alpha}) T^{\mu\alpha}$$

$$= \partial_\mu T^\mu_\nu + \cancel{\tfrac{1}{4}} T^\mu_\nu \left(\frac{1}{2} \frac{1}{|g|} \partial_\mu (g_{\alpha\beta} g^{\alpha\beta}) \right) - \frac{1}{2}(\partial_\nu g_{\mu\alpha}) T^{\mu\alpha}$$

$$= \partial_\mu T^\mu_\nu + T^\mu_\nu \left(\frac{1}{2} \frac{1}{|g|} [(\partial_\mu g_{\alpha\beta}) g^{\alpha\beta} + (\partial_\mu g^{\alpha\beta}) g_{\alpha\beta}] \right) - \frac{1}{2}(\partial_\nu g_{\mu\alpha}) T^{\mu\alpha}$$

$$\Gamma^\mu_{\beta\mu} = \frac{1}{2} g^{\mu\alpha} (\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\mu\alpha} - \partial_\alpha g_{\beta\mu})$$

So, we get $\partial_\mu T^\mu_\nu + T^\mu_\nu \Gamma^\alpha_{\mu\alpha} - \Gamma^\beta_{\nu\mu} T^\mu_\beta$

→ Now, choosing ref. system at x_0 s.t. $\partial_\alpha g_{\mu\nu}(x_0) = 0$.

$\Rightarrow R^{\mu\nu}(x_0) = g^{\mu\alpha} g^{\nu\gamma} R_{\alpha\gamma}(x_0)$

$\Rightarrow R_{\alpha\gamma}(x_0) = R^\mu{}_{\alpha\mu\gamma}(x_0)$

after expanding Christoffel symbols

$$= \frac{1}{2} \Big[\partial_\rho g^{\rho\sigma} \cancel{\partial_\gamma g_{\alpha\sigma}}^{0} + \partial_\rho g^{\rho\sigma} \cancel{\partial_\alpha g_{\gamma\sigma}}^{0} - \partial_\rho g^{\rho\sigma} \cancel{\partial_\sigma g_{\alpha\gamma}}^{0}$$

$$+ g^{\rho\sigma} \partial_{\gamma\rho} g_{\alpha\sigma} + g^{\rho\sigma} \partial_{\rho\sigma} g_{\alpha\gamma} - \partial_\gamma g^{\rho\sigma} \cancel{\partial_\alpha g_{\rho\sigma}}^{0} - g^{\rho\sigma} \partial_{\alpha\gamma} g_{\rho\sigma}$$

$$- g^{\kappa\lambda} \partial_\kappa g_{\alpha\rho} g^{\rho\sigma} \cancel{\partial_\sigma g_{\gamma\lambda}}^{0} + g^{\kappa\lambda} \cancel{\partial_\kappa g_{\alpha\rho}}^{0} g^{\rho\sigma} \cancel{\partial_\gamma g_{\lambda\sigma}}^{0} \Big]$$

$$+ \frac{1}{4} \Big[g^{\kappa\lambda} \cancel{\partial_\gamma g_{\alpha\kappa}}^{0} g^{\rho\sigma} \cancel{\partial_\lambda g_{\rho\sigma}}^{0} + g^{\kappa\lambda} \cancel{\partial_\alpha g_{\gamma\kappa}}^{0} - g^{\kappa\lambda} \cancel{\partial_\kappa g_{\alpha\gamma}}^{0} g^{\rho\sigma} \partial_\lambda g_{\rho\sigma}$$

$$- g^{\kappa\lambda} \partial_\alpha g_{\kappa\rho} g^{\rho\sigma} \cancel{\partial_\gamma g_{\lambda\sigma}}^{0} \Big]$$

So, we have:-

$$R_{\alpha\gamma}(x_0) = \frac{1}{2} g^{\rho\sigma}\left[\partial_\gamma\partial_\rho g_{\alpha\sigma} - \partial_\rho\partial_\sigma g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\rho\sigma}\right]$$

$$\therefore R^{\mu\nu} = \frac{1}{2} g^{\mu\alpha} g^{\nu\gamma} g^{\rho\sigma}\left[\partial_\gamma\partial_\rho g_{\alpha\sigma} - \partial_\rho\partial_\sigma g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\rho\sigma}\right]$$

& we know:

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi K T^{\mu\nu}$$

$$\& \; R = g^{\alpha\gamma} R_{\alpha\gamma}$$
$$= \frac{1}{2} g^{\alpha\gamma} g^{\rho\sigma}\left[\partial_\gamma\partial_\rho g_{\alpha\sigma} - \partial_\rho\partial_\sigma g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\rho\sigma}\right]$$

$$\Rightarrow T^{\mu\nu}(x_0) = \frac{1}{16\pi K}\left[g^{\mu\alpha} g^{\nu\gamma} - \frac{1}{2} g^{\mu\nu} g^{\alpha\gamma}\right] g^{\rho\sigma}\left[\partial_\gamma\partial_\rho g_{\alpha\sigma} - \partial_\rho\partial_\sigma g_{\alpha\gamma} - \partial_\alpha\partial_\gamma g_{\rho\sigma}\right]$$

Since we know $\partial_\mu |g| = |g| g^{\mu\nu} \partial_\alpha g_{\mu\nu} = 0$.
We have both α & γ also as dummy indices, so we can push the g's inside:

$$T^{\mu\nu}(x_0) = \frac{1}{16\pi K}\partial_\alpha\partial_\gamma\left(g^{\mu\alpha} g^{\nu\gamma} - g^{\mu\nu} g^{\alpha\gamma}\right)$$

$$= \frac{|g|}{|g|}\frac{1}{16\pi K}\partial_\alpha\partial_\gamma\left(g^{\mu\alpha} g^{\nu\gamma} - g^{\mu\nu} g^{\alpha\gamma}\right)$$

$$= \partial_\alpha\left[\frac{1}{16\pi K}\frac{1}{|g|}\partial_\gamma\left[|g|\left(g^{\mu\alpha} g^{\nu\gamma} - g^{\mu\nu} g^{\alpha\gamma}\right)\right]\right]$$