Computational Fluid Dynamics

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Preface and recommended literature

These lecture notes have been prepared for a new graduate course on compuational fluid dynamics in the Guelph–Waterloo Institute for Physics (GWIP), the joint graduate school of the Universities of Guelph and Waterloo, as well as Perimeter Institute for Theoretical Physics (Fall 2020). Many textbooks helped me compose these lectures and lecture notes. Some suggested textbooks include:

Numerical Methods

- Toro (2009): *Riemann Solvers and Numerical Methods for Fluid Dynamics* (Springer, 3rd edition, 2009)
- Leveque (2002): *Finite Volume Methods for Hyperbolic Problems* (Cambridge Univ. Press, Cambridge Texts in Applied Mathematics, 2002)

Mathematically inclined literature:

- Kröner (1997): *Numerical Schemes for Conservation Laws* (Wiley, 1997)
- Bressan (2000): *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem* (Oxford Lecture Series in Mathematics and its Applications, Oxford Univ. Press, 2000)
- Evans (2010): *Partial Differential Equations* (Graduate Studies in Mathematics, American Mathematical Society, 2nd edition, 2010)

Other recommended literature

- Bodenheimer et al. (2006): *Numerical Methods in Astrophysics* (Taylor & Francis, 2007)
- Anile (1989): *Relativistic fluids and magneto-fluids* (Cambridge Univ. Press, 1990)
- Rieutord (2015): *Fluid Dynamics* (Springer, 2015)
- LeVeque (2007): *Finite Difference Methods for Ordinary and Partial Differential Equations* (SIAM, 2007)
- Morton & Mayers (2005): *Numerical Solution of Partial Differential Equations* (Cambridge Univ. Press, 2nd edition, 2005)

Chapter 1

Basic Notions of Partial Differential Equations

1.1 PDEs of 2nd order

Recommended reading: Evans (2010) Chap. 1, Chap. 6.1, 7.1.1, 7.2.1, Strauss (2007) Chap. 1.6, most introductory books on PDEs

1.2 PDEs of 1st order

Recommended reading: Evans (2010) Chap. 7.3.1, 11.1, Toro (2009) Chap. 2

1.3 Some properties of 1st order hyperbolic systems

Recommended reading: Toro (2009) Chap. 2, Leveque (2002) Chap. 2.9–2.11, 3.1–3.6, 11.2, 18.5

1.3.1 Characteristics

1.3.2 Domain of dependence and range of influence

Chapter 2

Basic Equations of Computational Fluid Dynamics

Recommended reading: Bodenheimer et al. (2006) Chap. 1, Rieutord (2015) Chap. 11, Lifshitz & Pitaevskii (1981) Chap. 1, Anile (1989) Chap. 2

2.1 Continuous media and the Boltzmann equation

More on the N-body approach: Bodenheimer et al. (2006) Chap. 3
A compilation of the largest cosmological N-body simulations accomplished to date can be found in Tab. II of Cheng et al. (2020).

2.2 From the Boltzmann equation to the Euler equations

2.3 Navier-Stokes equations

More on viscosity: Landau & Lifshitz (2004) §15, Shakura & Sunyaev (1973), Balbus & Hawley (1991)

2.4 Magnetohydrodynamics

2.5 Radiation transfer

2.6 Relativistic Hydrodynamics

Recommended reading: Gourgoulhon (2012) Chaps. 4 & 6, Gourgoulhon (2006), Baumgarte & Shapiro (2010) Chap. 5, Alcubierre (2008) Chap. 2.2 & 7

2.7 Relativistic radiation transfer

Recommended reading: Thorne (1981), Shibata et al. (2011), Straumann (2013) Sec. 3.11.

Chapter 3

Finite Difference methods for PDEs

Recommended reading: LeVeque (2007) Chap. 1, 9, 10, Bodenheimer et al. (2006) Chap. 2, Choptuik (2006) Sec. 1., Kröner (1997) Chap. 2.4, Toro (2009) Chap. 5.1

3.1 Basic notions of discretization

3.2 Finite difference approximations

3.2.1 Partial derivatives & differential operators

3.2.2 Sample discretizations

3.3 Consistency, stability, convergence

Recommended reading: Morton & Mayers (2005) Chap. 5.

3.4 Stability analysis and the CFL condition

3.5 Diffusion and dispersion

3.6 Error analysis and convergence

Chapter 4

Properties of Conservation Laws

Recommended reading: Evans (2010) Chap. 11.1, 11.2, 11.4, Kröner (1997) Chap. 4.1., Bressan (2000) Chap. 4 & 5, Leveque (2002) Chap. 13, Toro (2009) Chap. 2.4

Recommended original literature: Lax (1957)

4.1 Local existence of classical solutions

4.2 Weak solutions

4.2.1 Definition

4.2.2 Behavior near discontinuities (Jump conditions)

4.3 Entropy condition

4.4 Riemann's Problem

4.4.1 First examples

4.4.2 Riemann invariants & characteristic fields

4.4.3 Special solutions: rarefaction waves

4.4.4 Special solutions: shocks and the Lax entropy condition

4.4.5 Special solutions: contact discontinuities

4.4.6 General solution to Riemann's problem

Chapter 5

Riemann Problem for the Euler Equations

Recommended reading: Toro (2009) Chap. 4, Leveque (2002) Chap. 14, Kröner (1997) Chap. 4.1

5.1 Overview

5.2 Solution strategy

Chapter 6

Numerical Schemes for Conservation Laws

Recommended reading: Kröner (1997) Chap. 4.4, Toro (2009) Chaps. 5.3 & 6, Leveque (2002) Chaps. 12 & 15

6.1 Conservative Schemes

6.2 Convergence: Lax-Wendroff theorem & entropy condition

6.3 Godunov's method for nonlinear systems of conservation laws

Chapter 7

Approximate Riemann Solvers

Recommended reading: Toro (2009) Chaps. 10 & 11, Kröner (1997) Chap. 4.4, Leveque (2002) Chap. 15.3

Recommended early review article on approximate Riemann Solvers: Harten et al. (1983)

7.1 The Riemann Solver of Roe

Recommended original literature: Roe (1981), Roe & Pike (1985), Einfeldt et al. (1991)

7.2 The HLL family of Riemann solvers

Recommended original literature: Harten et al. (1983), Harten (1983), Davis (1988), Einfeldt (1988), Einfeldt et al. (1991)

7.2.1 The HLL solver

7.2.2 The HLLC solver

Specific reading: Toro et al. (1994)

7.2.3 Wave speed estimates

Chapter 8

Extensions to Balance Laws and multi-D, higher-order methods

Recommended reading: Toro (2009) Chap. 15–17, Leveque (2002) Chap. 17–19

8.1 Source terms

Recommended original literature: Strang (1968)

More on time integration techniques, specifically Runge-Kutta methods: LeVeque (2007) Chap. 5 (Sec. 5.7, in particular).

8.2 Multidimensional systems of conservation laws

8.2.1 Dimensional splitting

Recommended original literature: Strang (1968)

8.2.2 Unsplit finite volume schemes

8.2.3 Higher-order multi-D schemes: MUSCL-Hancock methods

Recommended original literature: van Leer (1977a), van Leer (1977b), van Leer (1979)
PPM: Colella & Woodward (1984), Woodward & Colella (1984)
ENO: Harten & Osher (1987)
WENO: Liu et al. (1994), Jiang & Shu (1996)

Bibliography

Alcubierre, M. 2008, Introduction to 3+1 Numerical Relativity (Oxford, UK: Oxford University Press)

Anile, A. M. 1989, Relativistic fluids and magneto-fluids: with applications in astrophysics and plasma physics, Cambridge Monographs on Mathematical Physics (New York: Cambridge University Press)

Balbus, S. A. & Hawley, J. F. 1991, ApJ, 376, 214

Baumgarte, T. W. & Shapiro, S. L. 2010, Numerical Relativity: Solving Einstein's Equations on the Computer (Cambridge University Press, Cambridge UK)

Bodenheimer, P., Laughlin, G. P., Rozyczka, M., & Yorke, H. W. 2006, Numerical Methods in Astrophysics : An Introduction, Series in Astronomy and Astrophysics (CRC Press Taylor & Francis)

Bressan, A. 2000, Hyperbolic Systems of Conservation Laws: The One-Dimensional Cuachy Problem, Oxford Lecture Series in Mathematics and Its Applications (Oxford, New York: Oxford University Press)

Cheng, S., Yu, H.-R., Inman, D., et al. 2020, arXiv e-prints, 2003, arXiv:2003.03931

Choptuik, M. W. 2006, Lectures for VII Mexican School on Gravitation and Mathematical Physics: Relativistic Astrophysics and Numerical Relativity Numerical Analysis for Numerical Relativists, http://laplace.physics.ubc.ca/People/matt/Teaching/06Mexico/mexico06.pdf

Colella, P. & Woodward, P. R. 1984, Journal of Computational Physics, 54, 174

Davis, S. F. 1988, SIAM Journal on Scientific and Statistical Computing, 9, 445

Einfeldt, B. 1988, SIAM Journal on Numerical Analysis, 25, 294

Einfeldt, B., Roe, P. L., Munz, C. D., & Sjogreen, B. 1991, J. Comp. Phys., 92, 273

Evans, L. C. 2010, Graduate Studies in Mathematics, Vol. 19, Partial Differential Equations, 2nd edn. (American Institute of Physics)

Gourgoulhon, E. 2006, 21, 43, eAS Publications Series, eprint: arXiv:gr-qc/0603009

Gourgoulhon, E. 2012, Lecture Notes in Physics, Berlin Springer Verlag, Vol. 846, 3+1 Formalism in General Relativity

Harten, A. 1983, J. Comp. Phys., 49, 357

Harten, A., Lax, P. D., & van Leer, B. 1983, SIAM Review, 25, 35

Harten, A. & Osher, S. 1987, SIAM Journal on Numerical Analysis, 24, 279

Jiang, G.-S. & Shu, C.-W. 1996, Journal of Computational Physics, 126, 202

Kröner, D. 1997, Numerical Schemes for Conservation Laws (Wiley-Teubner)

Landau, L. D. & Lifshitz, E. M. 2004, Fluid Mechanics, Course of Theoretical Physics, Volume 6 (Oxford: Elsevier Butterworth-Heinemann)

Lax, P. D. 1957, Comm. Pure and Applied Math., 10, 537, _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160100406

Leveque, R. J. 2002, Finite Volume Methods for Hyperbolic Problems (New York: Cambridge University Press)

LeVeque, R. J. 2007, Finite Difference Methods for Ordinary and Partial Differential Equations: Steady State and Time Dependent Problems (Society for Industrial and Applied Mathematics (SIAM))

Lifshitz, E. M. & Pitaevskii, L. P. 1981, Course of Theoretical Physics, Vol. 10, Physical Kinetics (Amsterdam: Butterworth-Heinemann)

Liu, X.-D., Osher, S., & Chan, T. 1994, Journal of Computational Physics, 115, 200

Morton, K. W. & Mayers, D. F. 2005, Numerical Solution of Partial Differential Equations: An Introduction, 2nd edn. (Cambridge: Cambridge University Press)

Rieutord, M. 2015, Fluid Dynamics: An Introduction, Graduate Texts in Physics (Springer International Publishing)

Roe, P. L. 1981, J. Comput. Phys., 43, 357

Roe, P. L. & Pike, J. 1985, in Proc. of the sixth int'l. symposium on Computing methods in applied sciences and engineering, VI (Amsterdam, The Netherlands, The Netherlands: North-Holland Publishing Co.), 499–518, event-place: Versailles, France

Shakura, N. I. & Sunyaev, R. A. 1973, A&A, 24, 337

Shibata, M., Kiuchi, K., Sekiguchi, Y., & Suwa, Y. 2011, Progress of Theoretical Physics, 125, 1255

Strang, G. 1968, SIAM Journal on Numerical Analysis, 5, 12, num Pages: 12 Place: Philadelphia, United States Publisher: Society for Industrial and Applied Mathematics

Straumann, N. 2013, General Relativity (Dordrecht, Netherlands: Springer)

Strauss, W. A. 2007, Partial Differential Equations: An Introduction (John Wiley & Sons), google-Books-ID: m2hvDwAAQBAJ

Thorne, K. S. 1981, MNRAS, 194, 439

Toro, E. F. 2009, Riemann Solvers and Numerical Methods for Fluid Dynamics, 3rd edn. (Springer-Verlag)

Toro, E. F., Spruce, M., & Speares, W. 1994, Shock Waves, 4, 25

van Leer, B. 1977a, Journal of Computational Physics, 23, 263

van Leer, B. 1977b, Journal of Computational Physics, 23, 276

van Leer, B. 1979, Journal of Computational Physics, 32, 101

Woodward, P. & Colella, P. 1984, Journal of Computational Physics, 54, 115