Q1) Calculate the expression in 4-D Minkowski space time :-

$$\epsilon_{\mu\nu\alpha\beta}\,\epsilon^{\alpha\beta\gamma\sigma}\partial_\gamma\partial^\nu \frac{|x|}{(k.x)^2}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$; $k.x \equiv k_\mu x^\mu$ & $|x| = \sqrt{x_\nu x^\nu}$; k is a constant 4-vector

Ans) We have, $g_{\mu\nu} = \eta_{\mu\nu}$; $\|\eta_{\mu\nu}\| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Looking at the free & dummy indices :-

(Say) $A_\mu{}^\sigma = \epsilon_{\mu\nu\alpha\beta}\,\epsilon^{\alpha\beta\gamma\sigma}\partial_\gamma\partial^\nu \frac{|x|}{(k.x)^2}$.

→ Using the identity for changing order of indices, we write -

$$\epsilon_{\mu\nu\alpha\beta} = (-1)^1\epsilon_{\mu\alpha\nu\beta} = (-1)^2\epsilon_{\alpha\mu\nu\beta}$$
$$= (-1)^3\epsilon_{\alpha\mu\beta\nu} = (-1)^4\epsilon_{\alpha\beta\mu\nu}$$
$$= \epsilon_{\alpha\beta\mu\nu}.$$

→ Now, using the general identity :-

$$\epsilon_{i_1 \dots i_k i_{k+1} \dots i_n}\,\epsilon^{i_1 \dots i_k j_{k+1} \dots j_n} = k!\,\delta^{j_{k+1} \dots j_n}_{i_{k+1} \dots i_n}$$

We get :-

$$\epsilon_{\alpha\beta\mu\nu}\,\epsilon^{\alpha\beta\gamma\sigma} = 2!\,\delta^{\gamma\sigma}_{\mu\nu} = \underline{2\,\delta^{\gamma\sigma}_{\mu\nu}} = 2\begin{vmatrix} \delta^\gamma_\mu & \delta^\gamma_\nu \\ \delta^\sigma_\mu & \delta^\sigma_\nu \end{vmatrix}$$

So, we get :-

$$A_\mu{}^\sigma = 2\left(\delta^\gamma_\mu\delta^\sigma_\nu - \delta^\gamma_\nu\delta^\sigma_\mu\right)\partial_\gamma\partial^\nu \frac{|x|}{(k.x)^2}$$
$$= 2\left[\delta^\gamma_\mu\delta^\sigma_\nu\partial_\gamma\partial^\nu \frac{|x|}{(k.x)^2} - \delta^\gamma_\nu\delta^\sigma_\mu\partial_\gamma\partial^\nu \frac{|x|}{(k.x)^2}\right]$$

$$\Rightarrow A_\mu^{\sigma} = 2\left[\partial_\mu \partial^\sigma \frac{|x|}{(k.x)^2} - \left(\partial_\nu \partial^\nu \frac{|x|}{(k.x)^2}\right)\delta^\sigma_\mu\right]$$

$$\Rightarrow \text{For: } \partial^\sigma\left(\frac{|x|}{(k.x)^2}\right) = \frac{(k.x)^2\,\partial^\sigma|x| - |x|\,\partial^\sigma(k.x)^2}{(k.x)^4}$$

$$= \frac{(k.x)^{\cancel{2}}\,\partial^\sigma|x| - 2|x|\cancel{(k.x)}\,\partial^\sigma(k.x)}{(k.x)^{\cancel{4}\,3}}$$

$$= \frac{(k.x)\,\partial^\sigma|x| - 2|x|\,\partial^\sigma(k.x)}{(k.x)^3}$$

$$\partial^\sigma|x| = \partial^\sigma\sqrt{x_\mu x^\mu} = \frac{1}{2}\,\frac{1}{|x|}\,\partial^\sigma(x_\mu x^\mu)$$

$$= \frac{1}{2|x|}\left[(\partial^\sigma x_\mu)x^\mu + x_\mu(\partial^\sigma x^\mu)\right]$$

$$= \frac{1}{2|x|}\left[\left(\frac{\partial x_\mu}{\partial x_\sigma}\right)x^\mu + x_\mu \eta^{\sigma\nu}\partial_\nu x^\mu\right]$$

$$= \frac{1}{2|x|}\left[\left(\frac{\partial x_\mu}{\partial x_\sigma}\right)x^\mu + \eta^{\sigma\nu}x_\mu\left(\frac{\partial x^\mu}{\partial x^\nu}\right)\right]$$

Now $\dfrac{\partial x_\mu}{\partial x_\sigma} = \delta^\sigma_\mu$ & $\dfrac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$.

$$\Rightarrow \partial^\sigma|x| = \frac{1}{2|x|}\left[\delta^\sigma_\mu x^\mu + \eta^{\sigma\nu}x_\mu\delta^\mu_\nu\right]$$

$$= \frac{1}{2|x|}\left(x^\sigma + \eta^{\sigma\nu}x_\nu\right)$$

$$= \frac{1}{2|x|}\left(x^\sigma + x^\sigma\right) = \frac{x^\sigma}{|x|}$$

Similarly:-

$$\partial^\sigma(k.x) = \partial^\sigma(k^\mu x_\mu) = k^\mu\,\partial^\sigma x_\mu = k^\sigma$$

$$\cancel{= k^\sigma x^\sigma}$$

So, we get :-

$$\partial^\sigma\left(\frac{|x|}{(k.x)^2}\right) = \frac{(k.x)\,\frac{x^\sigma}{|x|} - 2|x|\,k^\sigma}{(k.x)^3}$$

$$= \cancel{(k.x)\,|x|}$$

$$= \frac{(k.x)\,x^\sigma - 2|x|^2 k^\sigma}{|x|\,(k.x)^3}$$

$$= \frac{x^\sigma}{|x|(k.x)^2} - \frac{2|x|\,k^\sigma}{(k.x)^3}$$

Now :-

$$\partial_\mu \partial^\sigma \frac{|x|}{(k.x)^2} = \partial_\mu\left(\frac{x^\sigma}{|x|(k.x)^2}\right) - 2\partial_\mu \frac{|x|\,k^\sigma}{(k.x)^3}$$

$$= \frac{|x|(k.x)^2\,\partial_\mu x^\sigma - x^\sigma\,\partial_\mu(|x|(k.x)^2)}{|x|^2(k.x)^4} - 2k^\sigma\left(\frac{(k.x)^3\,\partial_\mu(|x|) - |x|\,\partial_\mu(k.x)^3}{(k.x)^6}\right)$$

$$= \frac{|x|(k.x)^2\,\delta^\sigma_\mu - x^\sigma(\partial_\mu(|x|))(k.x)^2 - x^\sigma|x|\,\partial_\mu(k.x)^2}{|x|^2.(k.x)^4}$$

$$- 2k^\sigma\left(\frac{(k.x)^3\,x_\mu/|x| - |x|\,3(k.x)^2\,\partial_\mu(k.x)}{(k.x)^6}\right)$$

$$= \frac{|x|\,(k.x)^{\cancel{2}}\,\delta^\sigma_\mu - x^\sigma\left(x_\mu/|x|\right)(k.x)^{\cancel{2}} - 2x^\sigma|x|\,\cancel{(k.x)}\,\partial_\mu(k.x)}{|x|^2\,(k.x)^{\cancel{4}3}}$$

$$- 2k^\sigma\left(\frac{(k.x)^3 x_\mu - 3|x|^2(k.x)^2\,k_\mu}{|x|\,(k.x)^{\cancel{6}4}}\right)$$

$$= \frac{|x|^2(k.x)\,\delta^\sigma_\mu - x^\sigma x_\mu\,(k.x) - 2x^\sigma|x|^2 k_\mu}{|x|^3(k.x)^3}$$

$$- 2k^\sigma\left(\frac{(k.x)x_\mu - 3|x|^2 k_\mu}{|x|\,(k.x)^4}\right)$$

$$\Rightarrow \partial_\mu \partial^\sigma \left(\frac{|x|}{(k\cdot x)^2}\right) = \frac{\delta^\sigma_\mu}{|x|(k\cdot x)^2} - \frac{x^\sigma x_\mu}{|x|^3 (k\cdot x)^2} - \frac{2x^\sigma k_\mu}{|x|(k\cdot x)^3} - \frac{2k^\sigma x_\mu}{|x|(k\cdot x)^3} + \frac{6|x|k^\sigma k_\mu}{(k\cdot x)^4}$$

$$\Rightarrow \partial_\mu \partial^\sigma \left(\frac{|x|}{(k\cdot x)^2}\right) = \frac{\delta^\sigma_\mu}{|x|(k\cdot x)^2} - \frac{x^\sigma x_\mu}{|x|^3 (k\cdot x)^2} - \frac{2(x^\sigma k_\mu + k^\sigma x_\mu)}{|x|(k\cdot x)^3} + 6\frac{|x|k^\sigma k_\mu}{(k\cdot x)^4}.$$

We can similarly write

$$\partial_\nu \partial^\nu \left(\frac{|x|}{(k\cdot x)^2}\right) = \frac{\delta^\nu_\nu}{|x|(k\cdot x)^2} - \frac{x^\nu x_\nu}{|x|^3 (k\cdot x)^2} - \frac{2(x^\nu k_\nu + k^\nu x_\nu)}{|x|(k\cdot x)^3} + 6\frac{|x|k^\nu k_\nu}{(k\cdot x)^4}$$

$$= \cancel{\frac{1}{|x|(k\cdot x)^2}} - \cancel{\frac{1}{|x|(k\cdot x)^2}} - \frac{4}{|x|(k\cdot x)^2} + \frac{6|x||k|^2}{(k\cdot x)^4}.$$

$$\Rightarrow \partial_\nu \partial^\nu \left(\frac{|x|}{(k\cdot x)^2} \right) = \frac{6|x||k|^2}{(k\cdot x)^4} - \frac{4}{|x|(k\cdot x)^2}$$

$$= \frac{6|x|^2|k|^2 - 4(k\cdot x)^2}{|x|(k\cdot x)^4}$$

$$\Rightarrow A^\sigma{}_\mu = 2\left[\frac{\delta^\sigma_\mu}{|x|(k\cdot x)^2} - \frac{x^\sigma x_\mu}{|x|^3(k\cdot x)^2} - \frac{2(x^\sigma k_\mu + k^\sigma x_\mu)}{|x|(k\cdot x)^3} + \frac{6|x|k^\sigma k_\mu}{(k\cdot x)^4} - \frac{(6|x|^2|k|^2 - 4(k\cdot x)^2)\,\delta^\sigma_\mu}{|x|(k\cdot x)^4} \right]$$

For $\sigma = \mu$:-

$$\Rightarrow A^\mu{}_\mu = 2\left[\frac{1}{|x|(k\cdot x)^2} - \frac{\cancel{|x|^2}}{|x|^{\cancel{3}}(k\cdot x)^2} - \frac{2\cdot 2\cancel{(k\cdot x)}}{|x|(k\cdot x)^{\cancel{3}\,2}} + \frac{6|x||k|^2}{(k\cdot x)^4} - \frac{6|x|^2|k|^2}{\cancel{|x|}(k\cdot x)^4} + \frac{4\cancel{(k\cdot x)^2}}{|x|(k\cdot x)^{\cancel{4}\,2}} \right]$$

$$= 0$$

For $\sigma \neq \mu$.

$$\Rightarrow A^\sigma{}_\mu = 2\left[- \frac{x^\sigma x_\mu}{|x|^3(k\cdot x)^2} - \frac{2(x^\sigma k_\mu + k^\sigma x_\mu)}{|x|(k\cdot x)^3} + \frac{6|x|k^\sigma k_\mu}{(k\cdot x)^4} \right]$$

$$\Rightarrow A^\sigma_\mu = \begin{cases} 0 & \text{for } \sigma = \mu \\[2ex] \dfrac{12|x|k^\sigma k_\mu}{(k\cdot x)^4} - \dfrac{2x^\sigma x_\mu}{|x|^3(k\cdot x)^2} - \dfrac{4(x^\sigma k_\mu + k^\sigma x_\mu)}{|x|(k\cdot x)^3} & \text{for } \sigma \neq \mu \end{cases}$$