Computational Fluid Dynamics
Problem Set 6

Daniel M. Siegel
Perimeter Institute for Theoretical Physics
Department of Physics, University of Guelph

Problem 1 (Project: Godunov scheme for the Euler equations)

Using your exact Riemann solver from Problem Set 5, implement (in a language of your choice) the Godunov scheme for the one-dimensional time-dependent Euler equations for ideal gases as discussed in the lectures. We shall focus here on the evolution of Riemann problems, so that the numerical results of the Godunov scheme can be compared to an 'exact' solution (which can be computed with the Riemann solver of Problem Set 5). Instructions:

- Consider a discontinuity located at $x = x_0$, which separates two constant states defined by the set of primitive values $w_\mathrm{l} = (\rho_\mathrm{l}, v_\mathrm{l}, p_\mathrm{l})$ and $w_\mathrm{r} = (\rho_\mathrm{r}, v_\mathrm{r}, p_\mathrm{r})$. Specifically, consider an ideal gas with adiabatic constant $\gamma = 1.4$ and the initial conditions already discussed on Problem Set 5:

Problem	ρ_l	v_l	p_l	ρ_r	u_r	p_r	x_0	t
1	1.0	0.75	1.0	0.125	0.0	0.1	0.3	0.2
2	1.0	-2.0	0.4	1.0	2.0	0.4	0.5	0.15

- Consider a spatial domain $x \in [0, 1]$, resolved by $M = 100$ *cells* $x_i = 1, \ldots, M$. We need to introduce *ghost cells* x_0 and x_{M+1} outside the computational domain to provide boundary conditions (i.e., flux values $f_{\frac{1}{2}}$ at $x = 0$ and $f_{M+\frac{1}{2}}$ at $x = 1$). Apply *transmissive* boundary conditions for the primitive variables, i.e., set $w_0^n = w_1^n$ and $w_M^n = w_{M+1}^n$, where w^n represents the primitives at $t = t^n$. This results in trivial Riemann problems at the boundary interfaces without any waves being generated.

- In order to compute the timestep Δt for evolution, use $\Delta t = C_\mathrm{CFL} \frac{\Delta x}{S^n_\mathrm{max}}$ with $C_\mathrm{CFL} = 0.7$ and the simplified estimate $S^n_\mathrm{max} = \max_i\{|v_i^n| + c_i^n\}$. Here, c_i^n denotes the sound speed.

Tasks:

(*a*) Plot density, pressure, velocity and specific internal energy as a fuction of x for the problems and corresponding times t provided in the table above, and compare to the exact solutions using the Riemann solver from Problem Set 5. Use dots for the numerical solution of the Godunov scheme and overplot the exact solution as a solid line. How do you interpret the behavior of the specific internal energy in Problem 2 when using the Godunov method?

(*b*) Repeat (*a*), but using the Lax-Friedrichs method discussed in the lectures instead of the Godunov scheme. The Lax-Friedrichs numerical flux is given by

$$g(u_{i-1}^n, u_i^n) = \frac{1}{2}\left[f(u_{i-1}^n) + f(u_i^n)\right] - \frac{\Delta x}{2\Delta t}(u_i^n - u_{i-1}^n). \tag{1}$$

Comment on the level of 'numerical smearing' across discontinuieties the Godunov and Lax-Friedrich schemes give rise to. Do both methods recover the correct non-linear wave speeds (why/why not)?