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Optical scalars and the spherical gravitational lens

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Summary. The optical scalar equations of Sachs are used to consider the gravitational lens effect of a static, spherically symmetric mass distribution. Two approaches are used, the first of which is an exact treatment of the bending approach used by Liebes and others, but without the restrictions imposed by the limited applicability of the Einstein bending formula. The second approach emphasizes the close relationship between the optical parameters and the physical parameters associated with the mass distribution. This is useful in astrophysical applications, since we can often treat a realistic case by making small perturbations from some simple case.

1 Introduction

The gravitational lens effect of a static, spherically symmetric mass distribution has been discussed by several authors (Barnothy & Barnothy 1968, 1972; Clark 1972; Lawrence 1971; Sanitt 1971; de Silva 1970a, b, 1972, 1974), all of whom used the Einstein bending formula, or its modification for distributed-mass lenses, in a first-order ray-tracing technique. In this paper we employ the optical scalar equations (Sachs 1961) to give an exact treatment of such distributed-mass lenses. Two methods of solution are developed, the first of which is an exact treatment of the bending approach used by Liebes (1964), Refsdal (1964), and others, whose work was limited by the first-order nature of the Einstein bending formula, while ours is not limited in this way. The second method emphasizes the close connection between the optics and the physical quantities associated with the mass distribution, removing, as much as possible, the less transparent properties associated with the metric coefficients. Such an approach is very useful in astrophysical applications where we are then able to make small perturbations away from some simple case to treat, approximately, a more realistic case.

Throughout this paper, we will refer only to geometrical amplification factors. If one wishes to consider luminosity amplification these must be corrected by the standard redshift factor, due to Etherington (1933), though in many applications, this can be ignored.

2 The optical scalar equations

The geometric-optics approximation for a test electromagnetic field introduces a sequence of null hypersurfaces whose normals are null, and tangent to an irrotational congruence of null geodesics (Trautmen 1964). From the Ehlers–Sachs theorem, the cross-section of such a congruence of geodesics, i.e. a beam of light, is expanded and shared at the respective rates (Sachs 1961)

$$\theta = \tfrac{1}{2}k^a{}_{\|a} \quad \text{and} \quad |\sigma| = \{\tfrac{1}{2}k_{(a\|b)}k^{a\|b} - \theta^2\}^{1/2}, \tag{1}$$

with respect to the affine parameter, τ, along the beam. $k^a \equiv dx^a/d\tau$ is the null tangent vector to the geodesics, $_{\|a}$ denotes covariant differentiation with respect to x^a, and $_{()}$ denotes symmetrization.

The optical scalars, θ and σ, are propagated along the congruence by the equations (Sachs 1961)

$$\dot{\theta} + \theta^2 + \sigma\bar{\sigma} = \mathcal{R} \equiv \tfrac{1}{2}R_{ab}k^a k^b, \tag{2}$$

and

$$\dot{\sigma} + 2\theta\sigma = F\exp(i\beta) \equiv R_{aibj}k^a k^b \bar{t}^i \bar{t}^j, \tag{3}$$

where $^{\cdot} \equiv d/d\tau$, and $\bar{t}^i$ is the complex conjugate of t^i. R_{ab} is the Ricci tensor, R_{aibj} is the Riemann tensor, and t^i is a complex null vector parallelly propagated along the congruence such that, with a signature of -2, we have $t^a k_a = 0$ and $t^a \bar{t}_a = -1$. In fact the real and imaginary parts of t^a define orthogonal directions in a plane element, normal to the congruence, parallelly propagated along the geodesics.

From the Einstein field equations, the Ricci driving term, $\mathcal{R}$, can be written in the form

$$\mathcal{R} = -\frac{4\pi G}{c^2}T_{ab}k^a k^b, \tag{4}$$

which is always non-positive on the assumption of positive-definite local energy density. From the definition of the Weyl tensor, C_{aibj} (Einsenhart 1925), the term $F\exp(i\beta)$ can always be written as

$$F\exp(i\beta) = C_{aibj}k^a k^b \bar{t}^i \bar{t}^j. \tag{5}$$

This Weyl driving term represents the focusing of the congruence by non-local matter, whereas the Ricci driving term represents the focusing due to local matter, i.e. matter within the beam.

If we let the cross-sectional area of the beam be A, then the rate of expansion, θ, is just $\dot{A}/2A$, and the optical scalar equations can be written

$$\sqrt{A}^{\cdot\cdot}/\sqrt{A} + |\sigma|^2 = \mathcal{R}, \tag{6}$$

$$(|\sigma|A)^{\cdot} = AF\cos(\beta - \phi), \tag{7}$$

and

$$\dot{\phi} = \frac{F}{|\sigma|}\sin(\beta - \phi), \tag{8}$$

where ϕ is the phase of the shearing rate, σ. This form of the optical scalar equations has been used for a statistical consideration of a beam of light passing through a locally 'clumpy' universe (see Kantowski 1969; Dyer 1973; Dyer & Roeder 1974). In situations with much symmetry, another form, due to Kantowski (1968), is more useful. Define the real quantities $C_\pm$ and $\alpha_\pm$ in terms of the two principal curvatures of the wave front

$$\{\log C_\pm \exp(i\alpha_\pm)\}^{\cdot} = \theta \pm \sigma, \tag{9}$$

in which case the area of the beam is just C_+C_-, up to some constant factor. Requiring that the expansion rate, θ, be real, the optical scalar equations become

$$\ddot{C}_\pm/C_\pm + \dot{\alpha}^2 = \mathcal{R} \pm F\cos\beta, \tag{10}$$

and

$$\ddot{\alpha} + \dot{\alpha}\{\log C_+C_-\}^\cdot = F\sin\beta, \tag{11}$$

where we have introduced $\alpha = \alpha_+ = -\alpha_-$.

We consider a static, spherically symmetric gravitational lens, and write the line element in the form

$$ds^2 = \exp(2C)\,dt^2 - \exp(2A)\,dr^2 - \exp(2B)\,d\Omega^2, \tag{12}$$

where A, B, and C are functions of r, the radial coordinate, and $d\Omega^2$ is the line element on the unit sphere, in the standard spherical polar coordinates, (r, θ, ϕ).

From a knowledge of the Killing vectors associated with this situation, or directly from the Euler–Lagrange equations, we can write the null tangent vector, k^a, to a null geodesic in this field as

$$k^0 = \exp(-2C), \quad k^1 = \epsilon\exp[-(A+C)]\,\{1 - h^2\exp[2(C-B)]\}^{1/2},$$
$$k^2 = 0, \quad k^3 = h\exp(-2B), \tag{13}$$

where $\epsilon = \pm 1$, for incoming or outgoing rays respectively, and h is an impact parameter.

The null vector, k^a, determines, up to a phase factor, a 1-spinor, κ^A, through

$$\kappa^A\bar{\kappa}^{\dot{X}} = \pm\sigma_a^{A\dot{X}}k^a, \tag{14}$$

where $\sigma_a^{A\dot{X}}$ are the spin connections (Pirani 1964). The '+' sign will be chosen here since we are considering future-pointing null tangent vectors. The spin connections appropriate to our line element are

$$\sigma_0^{B\dot{X}} = \frac{\exp(C)}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma_1^{B\dot{X}} = \frac{\exp(A)}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\sigma_2^{B\dot{X}} = \frac{i\exp(B)}{\sqrt{2}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \sigma_3^{B\dot{X}} = \frac{\exp(B)}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{15}$$

The parallel propagation of k^a requires the parallel propagation of κ^A, and since $k^2 = 0$, it also follows that $\kappa^{①}$ and $\kappa^{②}$ must have their phase factors equal. (Circled indices are spinor indices.) This simplification is equivalent to the existence of planar orbits in the spherically symmetric field.

Upon introducing a second 1-spinor, μ^A, we can form the complex null vector

$$t^a = \sigma^a_{B\dot{X}}\kappa^B\bar{\mu}^{\dot{X}}, \tag{16}$$

which satisfies the conditions given earlier for t^a. If t^a is to undergo parallel propagation along the congruence, then so must μ^A. It can be shown that the arbitrariness in the phase factor of κ^A results in t^a being determined only up to some arbitrary phase factor, which is constant along the beam. This corresponds to the freedom in the initial reference directions in the plane element normal to the beam.

Since $\kappa^{①}$ and $\kappa^{②}$ have the same phase factor, we can always choose κ^A to be real, and we then have

$$\kappa^{①} = 2^{-1/4}S_+ \quad \text{and} \quad \kappa^{②} = 2^{-1/4}S_-, \quad \text{where} \quad S_\pm = \sqrt{\exp(-C) \pm h\exp(-B)}. \tag{17}$$

The equations for the parallel propagation of the 1-spinor μ^A are

$$k^a\{\partial_a\mu^A+\Gamma^A_{aB}\mu^B\}=0, \tag{18}$$

where Γ^A_{aB} are the spinor Christoffel symbols (Pirani 1964) and are, in this case

$$\Gamma^A_{0B}=\frac{C'}{2}\exp(C-A)\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \Gamma^A_{1B}=0,\quad \Gamma^A_{2B}=0,\quad\text{and}\quad \Gamma^A_{3B}=\frac{B'}{2}\exp(B-A)\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{19}$$

where $'\equiv d/dr$. Noting that $k^1=\exp(-A)S_+S_-$, the parallel propagation equations reduce to the pair

$$S_-\mu^{①'}-S'_+\mu^{②}=0,\quad\text{and}\quad S_+\mu^{②'}-S'_-\mu^{①}=0, \tag{20}$$

which suggests that $\mu^{②}$ is obtained from $\mu^{①}$ by interchanging S_+ and S_- throughout. After suitable normalization, the required solution is

$$\mu^{①}=2^{-3/4}\left\{\frac{\exp(B)}{h}S_-+HS_+\right\}\quad\text{and}\quad \mu^{②}=2^{-3/4}\left\{\frac{\exp(B)}{h}S_++HS_-\right\}, \tag{21}$$

where H is some function of r, which we do not need. The vector t^a is then

$$\begin{aligned} t^0&=\frac{\exp(-C)}{2\sqrt{2}}\left\{\frac{2}{h}S_+S_-\exp(B)+H(S_+^2+S_-^2)\right\}, & t^2&=\frac{i\exp(-B)}{\sqrt{2}}\\ t^1&=\frac{\exp(-A)}{2\sqrt{2}}\left\{\frac{\exp(B)}{h}(S_+^2+S_-^2)+2HS_+S_-\right\}, & t^3&=\frac{Hh\exp(-2B)}{\sqrt{2}}. \end{aligned} \tag{22}$$

Finally, we can form the modulus of the Weyl driving term:

$$F=\frac{h^2}{2}\exp(-4B)\{1-(C''+C'^2-C'A'-B'C'-B''+A'B')\exp[2(B-A)]\}, \tag{23}$$

and from the conditions on the phase factor of t^a, it follows that the phase factor of the Weyl driving term, $\exp(i\beta)$, is constant along the beam. Physically, this corresponds to the fact that in a spherically symmetric field, the principal shearing directions in the plane element normal to the beam do not change along the beam.

At this point it becomes clear that the $C_\pm$ formalism is ideally suited for spherically symmetric gravitational lenses, for since β is an arbitrary constant, we can set it equal to zero. Furthermore, we are free to define σ to be real initially, so that $\dot\alpha=0$ initially. It then follows from equation (11) that $\dot\alpha$ vanishes at every point, leaving us with the decoupled set of equations

$$\ddot C_\pm=(\mathcal{R}\pm F)\,C_\pm, \tag{24}$$

where now C_+ is proportional to the diameter of the beam in the radial direction, while C_- is proportional to the diameter in the tangential direction. An equivalent set of equations arises in the (ξ,η) formalism of Penrose (1966) (see Pineault 1975).

Using standard expressions for R_{ab} in terms of the functions A, B, and C, the Ricci driving term is found, so that we have

$$\mathcal{R}+F=\exp[-2(A+C)](B''+B'^2-B'C'-B'A')-h^2\exp[-2(A+B)]\times(C''+C'^2-C'B'-C'A'). \tag{25}$$

The quantity $\mathcal{R}-F$ could also be obtained, but it is not of direct interest in this form.

We have $\mathcal{R}$ and F directly in terms of the metric functions, A, B, and C, but it is often of more relevance to have $\mathcal{R}$ and F in terms of the actual mass distribution. For the spherically symmetric, static distribution of perfect fluid, the energy momentum tensor has the non-zero components

$$T_0^0=\rho, \quad T_1^1=T_2^2=T_3^3=-p/c^2, \tag{26}$$

so that the Ricci driving term is

$$\mathcal{R}=-\frac{4\pi G}{c^2}\exp(-2C)(\rho+p/c^2). \tag{27}$$

The factor $\exp(-2C)$ in $\mathcal{R}$ is equivalent to a redshift factor along the beam.

The Einstein field equations reduce to the three equations

$$\kappa\rho+\Lambda=\exp(-2B)-\exp(-2A)(2B''+3B'^2-2A'B'), \tag{28}$$

$$\kappa p/c^2-\Lambda=\exp(-2A)(B'^2+2B'C')-\exp(-2B), \tag{29}$$

and

$$\kappa p/c^2-\Lambda=\exp(-2A)(B''+B'^2+C''+C'^2+B'C'-A'B'-A'C'), \tag{30}$$

where $\kappa\equiv 8\pi G/c^2$, and Λ is the cosmological constant. Using these relations, we can write

$$F=\frac{h^2}{2}\exp(-2B)\{3\exp(-2B)\{1-B'^2\exp[2(B-A)]\}-\kappa\rho-\Lambda\}. \tag{31}$$

To proceed further, we must obtain solutions to the above field equations, and this is most conveniently done by choosing the radial coordinate, r, to be a curvature coordinate, as it will be in the simple Schwarzschild exterior solution, i.e. $r=\exp(B)$. The solutions are then

$$\exp(-2A)=1-\frac{1}{3}\Lambda r^2-\frac{\kappa}{r}\int\rho r^2\,dr, \tag{32}$$

and

$$2C=\int\left\{(\kappa p/c^2-\Lambda)\,r\exp(2A)+\frac{\exp(2A)-1}{r}\right\}dr. \tag{33}$$

Of course, outside the spherical mass, we have

$$\exp(2C)=\exp(-2A)=1-\frac{2m}{r}-\frac{1}{3}\Lambda r^2, \tag{34}$$

so that F is simply

$$F=3mh^2/r^5, \tag{35}$$

where $m=GM/c^2$ is the geometrized mass of the spherical mass.

Using equation (31), we obtain for F inside the spherical mass

$$F=\frac{4\pi G}{c^2}\frac{h^2}{r^2}\left\{\frac{3}{r^3}\int_0^r\rho r^2\,dr-\rho(r)\right\}=-\frac{4\pi G}{c^2}\frac{h^2}{r^5}\int_0^r r^3\,d\rho. \tag{36}$$

A useful way to write this expression for F is

$$F = \frac{3h^2}{r^5}\{m(r) - \tilde{m}(r)\}, \tag{37}$$

where $m(r)$ is the geometrized mass out to radius r, and $\tilde{m}(r)$ is the mass of a sphere of radius r, and of uniform density equal to $\rho(r)$, the density at r. Hence the Weyl driving term arises only from the deviation of the lens from uniform density, so that a uniform sphere or homogeneous universe produces no Weyl focusing, as expected.

As one can easily verify, all components of the Weyl tensor, C_{aibj}, are either zero, or proportional to $\{m(r) - \tilde{m}(r)\}/r^3$, being effectively tidal force terms. Hence C_{aibj} can only vanish interior to r if the sphere is of uniform density interior to r. If a static distribution of perfect fluid must be spherically symmetric (see Marks 1977, who includes thermal conductivity and viscosity), it follows that the only static distribution generating a conformally flat interior is one of uniform density. Making the same assumption, Buchdahl (1971) has shown the same result for $\Lambda = 0$.

3 Exact bending solution

Consider the case of an idealized lens where we know the metric functions explicitly. From the form of the null vector k^a, we know that the ray lies in a plane defined by the observer, the source, and the lens, and the ray path in this plane is only a function of h, the impact parameter. Suppose that a ray with impact parameter h intersects a screen at position x_0^a while a radially adjacent ray, with impact parameter $h + dh$, intersects it at $x_0^a + dx_0^a$. We can then evaluate the change, dx_0^a, by considering

$$\frac{\partial x_0^a}{\partial h} = \frac{\partial}{\partial h}\int_{\text{source}}^{\text{observer}} k^a\, d\tau, \tag{38}$$

and find that the distance between the two points of intersection is proportional to

$$\Gamma = \exp(B)\,\{1 - h^2 \exp[2(C-B)]\}^{1/2}\int_{\text{source}}^{\text{observer}} \frac{\exp(A + C - 2B)\, dr}{\{1 - h^2\exp[2(C-B)]\}^{3/2}}. \tag{39}$$

This is the generalization of an approach used by Dwivedi & Kantowski (1970) to consider the luminosity of a collapsing star, restricted to the Schwarzschild vacuum field.

One can easily verify that Γ is, in fact, a solution of equation (24) for C_+ where $\mathcal{R} + F$ is given by equation (28), so that we can determine C_+ explicitly by an integration along the ray. Using the expression for $dr/d\phi$, we can write C_+ in the sometimes more useful form

$$C_+ \propto \{\exp(2B) - h^2\exp(2C)\}^{1/2}\int_{\text{source}}^{\text{observer}} \frac{\epsilon(\phi)\, d\phi}{1 - h^2\exp[2(C-B)]}, \tag{40}$$

where $\epsilon(\phi) = \pm 1$ as $dr/d\tau$ is positive or negative.

The solution for C_- is simply obtained by using the fact that all the rays from a given point source, S, and deflected by a lens, L, lie in planes, all of which contain the common line, SL (see Fig. 1). Then the separation between any two tangentially adjacent rays, and hence C_-, is simply given by

$$C_- \propto r\sin(\phi - \phi_s), \tag{41}$$

at any point (r, ϕ) along the ray. Hence C_- is also obtained directly from an integration along the central ray of the beam, i.e. the integration of $dr/d\phi$.

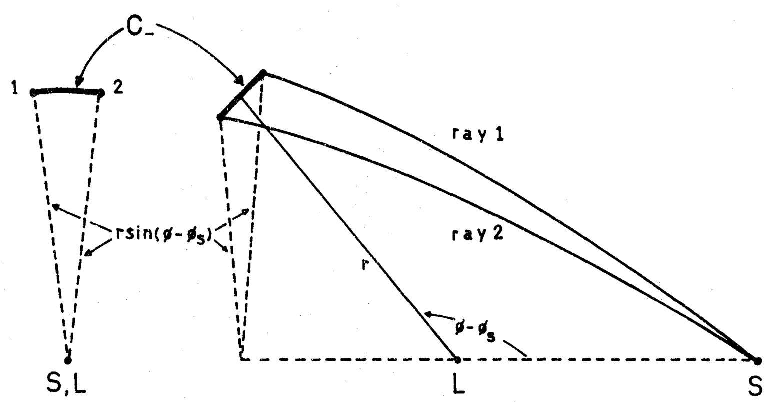

Figure 1. The determination of C_- for a source at S and a lens centred at L. The two rays are tangentially adjacent.

Having obtained C_+ and C_-, we can evaluate the beam area, $A \propto C_+ C_-$, and the axial ratio of the resulting image. Comparing this with the result obtained without a lens present, we then have the amplification factor. If we are interested in the amplification of apparent luminosities, then we must correct by the standard redshift factor, due to Etherington (1933).

This method of obtaining $C_\pm$ is just a full treatment of the bending approach used by Liebes (1964), Refsdal (1964), and others, where here we are calculating the bending completely, while their work used only the simple Einstein bending formula, which is valid only for m/h, h/r_s, and $h/r_o \ll 1$ (see Fig. 2). Consequently, this method enables one to treat more complex situations, where, for example, the source–lens distance is comparable to the impact parameter, as might arise in a bound system.

As an example, consider the situation where photons pass outside some spherical body, so that $\mathcal{R} = 0$ along the beam. It then follows from equation (24) that C_- is obtained from C_+ by changing the sign of F, that is by changing the sign of m. Take $\alpha = m/h \ll 1$ and expand C_+ in the form:

$$C_+/C_f = 1 + A\alpha + B\alpha^2 + \dots, \tag{42}$$

where C_f is the flat-space solution, i.e. for which $F = 0$. Changing the sign of α to obtain C_-, we obtain the amplification factor

$$Amp = \frac{C_f^2}{C_+ C_-} = 1 + (A^2 - 2B)\,\alpha^2 + 0(\alpha^4), \tag{43}$$

so that it will be necessary to calculate both A and B to obtain Amp to lower order.

Defining $u = r/h$, and using equation (42), we can write C_+ in the form

$$C_+ = \sqrt{u_o^2 - 1}\,\sqrt{1 + 2\alpha/u_o(u_o^2 - 1)}\;|I(u_o) + kI(u_s)|\,h, \tag{44}$$

where

$$I(u) = \frac{1}{h}\int_{-\infty}^{u} \left[1 + \frac{2\alpha}{u(u^2 - 1)}\right]^{-3/2} \frac{u\,du}{(u^2 - 1)^{3/2}}, \tag{45}$$

and subscripts 's' and 'o' will indicate evaluation at source and observer respectively. $k = +1$ if the lens lies between the source and observer and $k = -1$ otherwise. Expanding in powers of α and integrating, and defining $l = r_o r_s/h(r_o + r_s)$, we have:

$$A = 3l(\beta_o + \beta_s) + 1/u_o(u_o^2 - 1), \tag{46}$$

and

$$B = \frac{3l(\beta_o + \beta_s)}{u_o(u_o^2 - 1)} + \frac{15}{2} l(\gamma_o + \gamma_s) - \frac{1}{2u_o^2(u_o^2 - 1)^2}, \tag{47}$$

where:

$$\beta = \frac{u(2u^2 - 3)}{3(u^2 - 1)^{3/2}} = \frac{2}{3}\left\{1 - \frac{3}{8u^4} - \frac{5}{8u^6} + \ldots\right\}, \tag{48}$$

and

$$\gamma = (u^2 - 1)^{-1/2} - \frac{1}{3}(u^2 - 1)^{-3/2} + \frac{1}{5}(u^2 - 1)^{-5/2} - \sin^{-1}(1/u) = \frac{9}{22u^7} + 0\left(\frac{1}{u^9}\right). \tag{49}$$

The amplification factor is then

$$Amp = 1 + \alpha^2\left\{9l^2(\beta_o + \beta_s)^2 - 15l(\gamma_o + \gamma_s) + \frac{2}{u_o^2(u_o^2 - 1)^2}\right\} + 0(\alpha^4). \tag{50}$$

When both u_o and u_s are much larger than unity, so that the impact parameter is small relative to the lens–source and lens–observer distances, it follows from the expressions for β and γ that the usual amplification factor results, i.e.

$$Amp = 1 + \left(\frac{4mr_o r_s}{h^2(r_o + r_s)}\right)^2 + \ldots, \tag{51}$$

and results only from the term $A\alpha$ in the expansion for $C_\pm$. If the source lies near the lens, having $u_s \sim 1$, we must consider β_s and γ_s explicitly in equation (50) for they both contribute to a decrease in the amplification factor. Hence one must be rather careful about using equation (51) in this case. It is not surprising that there is less amplification when $u_s \sim 1$, for the simple result is identical to that obtained by Liebes (1964) and others using the Einstein bending formula, but if $u_s \sim 1$, not all the bending will have occurred, so one would expect less amplification.

4 Corrected distances

In many cases of astrophysical interest, we may know the large-scale distribution, etc. of a lensing object, while, due to 'clumpiness' in the lens, a typical beam of observation may pass through regions where, locally, the density deviates significantly from the smoothed density which dominates the large-scale structure of the lens. In effect, a beam of light, passing through a clumpy distribution of matter samples the inhomogeneity in a positive-definite fashion, while the large-scale dynamics depends only on the smoothed density distribution. Since astronomers tend to look away from foreground luminous or opaque objects when observing more distant objects, they usually selectively sample optically the less dense, 'interclump', regions of the universe. Hence there exists an observationally induced bias in their results, which has been discussed by various authors (Zel'dovich 1964; Bertotti 1966; Gunn 1967; Kantowski 1969; Dyer 1973; Dyer & Roeder 1972, 1973, 1974; and others). Of course, if one were to actually observe over the *complete* celestial sphere, there would be no net loss of luminosity, a result which is apparent from the very basis of the geometric optics approximation. This has been discussed recently by Weinberg (1976).

In view of the above discussion, it is useful to have a solution for $C_\pm$ which depends more directly on the matter distribution, through the forms of $\mathcal{R}$ and F given in equations (27) and (36, 37). The zero-order solution for $C_\pm$ is τ, where we take $\tau = 0$ at the source, which occurs for $\mathcal{R}$ and F equal to zero, i.e. no lens. If $\mathcal{R}$ and F do not vanish all along the beam, we can obtain successive additive corrections, $y_i^\pm(\tau), i = 1, 2, \ldots$, to the zero-order solution, $y_0^\pm(\tau) = \tau$. From equation (24), we have

$$y_i^\pm(\tau) = \int_0^\tau ds \int_0^s d\lambda \, \{\mathcal{R}(\lambda) \pm F(\lambda)\} \, y_{i-1}^\pm(\lambda). \tag{52}$$

This suggests a more convenient form of solution in terms of a series of 'corrected distances', $d_i^\pm(\tau)$, given by

$$d_i^\pm(\tau) = \tau + \int_0^\tau ds \int_0^s d\lambda \, \{\mathcal{R}(\lambda) \pm F(\lambda)\} \, d_{i-1}^\pm(\lambda), \tag{53}$$

where $d_0^\pm(\tau) = \tau$ is just the 'uncorrected' straight-line distance. It is then a straightforward procedure to build up a solution to the required level of accuracy. The beam area is then proportional to $d_k^+(\tau)\, d_k^-(\tau)$ where k is the order of highest correction.

The lowest order of correction to $C_\pm$ is

$$D_\pm = \tau \left\{ 1 + \frac{1}{\tau} \int_0^\tau ds \int_0^s \mathcal{R}\lambda \, d\lambda \pm \frac{1}{\tau} \int_0^\tau ds \int_0^s \mathcal{F}\lambda \, d\lambda + \ldots \right\}, \tag{54}$$

and the resulting amplification factor is, to lowest order:

$$Amp = 1 + \frac{2}{\tau} \int_0^\tau ds \int_0^s T\lambda \, d\lambda + \ldots, \tag{55}$$

where we have written T for $-\mathcal{R}$, and $T \geqslant 0$. This first-order correction contains only Ricci focusing due to matter within the beam. Weyl focusing first occurs in the second-order correction, though it can, in many cases, overwhelm the Ricci focusing.

At least one case where the Ricci focusing can be very important occurs when both the source and observer are located inside the lens. An important example of this is the universe itself, as discussed by Dyer (1973) and Dyer & Roeder (1972, 1973) for the 'zero-shear' models of locally inhomogeneous universes.

In the case of a very diffuse distribution, or in the vacuum region surrounding an isolated lens, the Weyl terms often begin to overwhelm the Ricci terms, so that equation (55) is no longer applicable. Suppose now that $\mathcal{R}$ is so small that we can ignore it along the beam, as might be the case when looking through a cluster of stars or galaxies at a more distant object, and supposing there to be little interstellar or intergalactic medium. In that case, the amplification factor is, to lowest order

$$Amp = 1 - 2y_2^+/\tau + (y_1^+/\tau)^2 + \ldots, \tag{56}$$

where here y_1^+ and y_2^+ are given by equation (52) with $\mathcal{R} = 0$. It is again clear that we must calculate the first two orders of correction to $C_\pm$ to obtain the lowest order correction to Amp. In most cases, such as when the source and object are at distances from the lens that are significantly larger than the impact parameter, the term in y_2 becomes insignificant compared to the other term.

As a simple example, consider a lens, of radius b, and density law $\rho(r) = \rho_c\{1 - (r/a)^2\}$, surrounded by vacuum. Inside this sphere, the Weyl driving term is constant along any ray, given by

$$F = \frac{3mh^2}{L^5}, \quad \text{where} \quad L^5 = \tfrac{1}{2} b^3 (5a^2 - 3b^2). \tag{57}$$

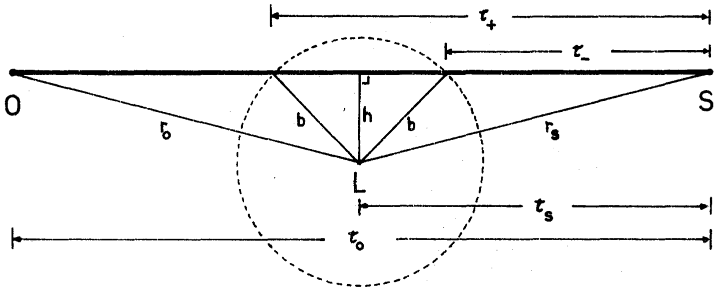

Figure 2. A gravitational lens, of radius b, is centred at L, with source S and observer O.

Suppose that such a lens is situated relative to an observer and a source as shown in Fig. 2. Outside the sphere, F is just $3mh^2/r^5$. We shall take r_o and r_s to be much larger than h, in which case we are justified in dropping the y_2 term in equation (56). This also allows us to use the straight line approximation for the $r(\tau)$ relation in evaluating the integrals. We then find the amplification

$$Amp = 1 + \left\{\frac{4mr_o r_s}{h^2(r_o + r_s)}\left(1 - \sqrt{1 - \frac{h^2}{b^2}}\left[1 + \frac{1}{2}\frac{h^2}{b^2} - \frac{6h^4 b}{L^5}\right]\right)\right\}^2 + \ldots, \tag{58}$$

which reduces to equation (51) when the ray does not pass through the sphere, i.e. when $b \to h$. We then have the amplification due to Weyl focusing in spherical leneses ranging from the homogeneous sphere, when $L \to \infty$, to centrally condensed spheres, until, when $L = a = b$, we have the sphere whose density vanishes at the surface.

If we are observing a distant source through an intervening spherical cluster of 'clumps' with little or no interclump medium, with the line of sight not passing too close to any clump, equation (58) yields the expected amplification. For rays passing near the centre of the lens, i.e. $h \ll b$, the second term in equation (58) becomes proportional to h^4, so that the Weyl focusing vanishes as $h \to 0$, as we would expect from symmetry. If we hold m, b, and h fixed, and decrease L from ∞ (the homogeneous sphere), we note that the amplification increases. This arises from our moving a larger fraction of the total mass into the cylinder defined by rotating the ray about the axis of symmetry, where the mass can become effective in Weyl focusing.

We have neglected the Ricci focusing of the interclump medium within the sphere, which we now suppose to have density equal to a fixed fraction, α, of the total local mass density. Evaluating T from equation, and taking $\exp(2C)$ (a redshift term) to be unity and $p \simeq 0$ for this lowest order correction, the amplification due to Ricci focusing alone is

$$Amp = 1 + \frac{24\alpha m\sqrt{1 - h^2/b^2}}{b^2} \cdot \frac{r_o r_s}{r_o + r_s} + \ldots, \tag{59}$$

treating only the uniform sphere, though more complex cases are a simple extension. This result can be compared with equation (58), though a proper comparison would require a detailed calculation, including cross-terms in $\mathcal{R}$ and F integrals. Generally speaking the Ricci focusing becomes more important when the size of the lens is comparable to the lens–source or lens–observer distances. This condition is equivalent to requiring that either the source or observer lie well within the focal distance of the lens, as discussed by Dyer & Roeder (1976), which may be satisfied by such objects as the Coma cluster of galaxies.

Conclusion

We have discussed two methods of solution for $C_\pm$, and observed that while the first method gives exact results, the second method is much more flexible and applicable to lenses that might be of astrophysical interest. Extending the solutions given, one can solve, at least numerically, the lens effect of any static spherical lens.

The second method of solution is relevant to the controversy between Barnothy & Barnothy (1968, 1972) and de Silva (1970, 1972, 1974) and Sanitt (1971) concerning the possibility of the observed quasars being the gravitational lens images of distant Seyfert galaxy nuclei. de Silva and Sanitt each calculate the effective bending mass of a distributed-mass lens as a function of impact parameter (see also, Clark 1972). Since this effective bending mass is less than the total mass, they decrease their probability of a given intensification by the ratio of effective bending mass to total mass. Barnothy & Barnothy also calculated the effective bending mass, but correctly noted that this could differ significantly from the effective intensification mass, which they proceeded to obtain. Though the particular situation of galaxies as lenses for distant Seyfert galaxy nuclei is such that the distributed-mass lens does not appear to save the Barnothy (1965) hypothesis on the nature of quasars, the possibility of large intensifications by distributed-mass lenses remains in certain circumstances.

A null ray, with impact parameter h, is bent by the angle (Clark 1972)

$$\theta(h) = 4\hat{m}(h)/h, \tag{60}$$

where $\hat{m}(h)$ denotes the mass contained within the cylinder, of radius h, whose axis is parallel to the ray and passes through the centre of the lens. By considering radially adjacent rays, one can define a focal length for the lens, as a function of h

$$f(h) = \left(\frac{d\theta}{dh}\right)^{-1} = \frac{4\hat{m}(h)}{h^2}\left\{\frac{d(\log \hat{m})}{d(\log h)} - 1\right\}. \tag{61}$$

The second term in the brackets arises from the bending due to the matter between the beam and the axis of symmetry, and corresponds to Weyl focusing. The first term arises from the mass within the beam, and hence represents Ricci focusing of local matter. While de Silva did include the Ricci-type term in the calculation of the intensification due to a distributed-mass lens (de Silva 1972, equation 4.8), he chose effectively to ignore it when he modified his probability calculation for point-mass lenses. On the other hand, Barnothy & Barnothy chose to include Ricci focusing at its maximum value.

If one considers the lens galaxy to be composed mostly of stars, with little interstellar medium, then it becomes a difficult decision as to how much Ricci focusing to include. One must consider, for example, the size of the cone of observation as it passes through the lens galaxy relative to the average separation distances of stars in the galaxy. If the beam is large as it passes through the galaxy, it will probably include so many stars that the average density of matter in the beam is comparable to the average density, while if the beam size is very small, and is likely to contain no stars, then the average density within the beam could be significantly lower than the average density, so that the Ricci focusing might effectively vanish. Hence one must consider in some detail the relative locations of the lens galaxy and the Seyfert nucleus, etc. before making a firm decision. Of course, in statistical applications, such as the Barnothy (1965) hypothesis, one must also take account of observational selection procedures, etc.

The present discussion has been restricted to static lenses, while most lenses of astrophysical interest will be expanding or contracting. Rees & Sciama (1968) and Dyer (1976) have considered the simple case of an expanding uniform dust sphere, but with regard to the redshift of photons crossing such a sphere to study perturbations of the 3 K cosmic background radiation. In any more realistic situation, the problem becomes quite complicated, though for reasonable expansion rates, so that the expansion timescale is much larger than the photon crossing time, the effects can be expected to be very small. Hence in most astrophysical applications, such as the lens effect of the Coma cluster of galaxies (see Dyer & Roeder 1976), we can safely ignore the expansion of the lens.

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