Bayesian Statistics (with M&Ms)

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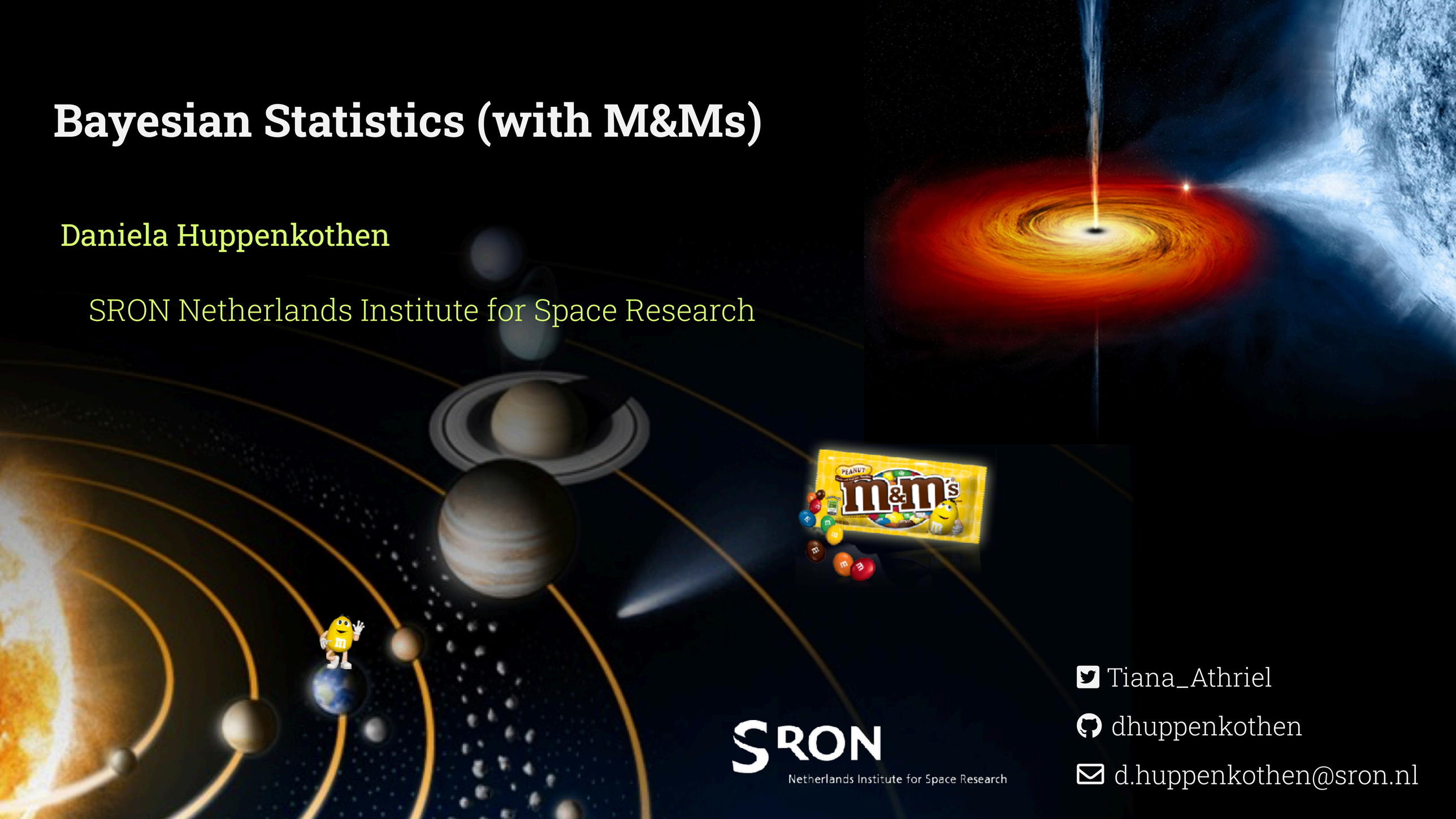

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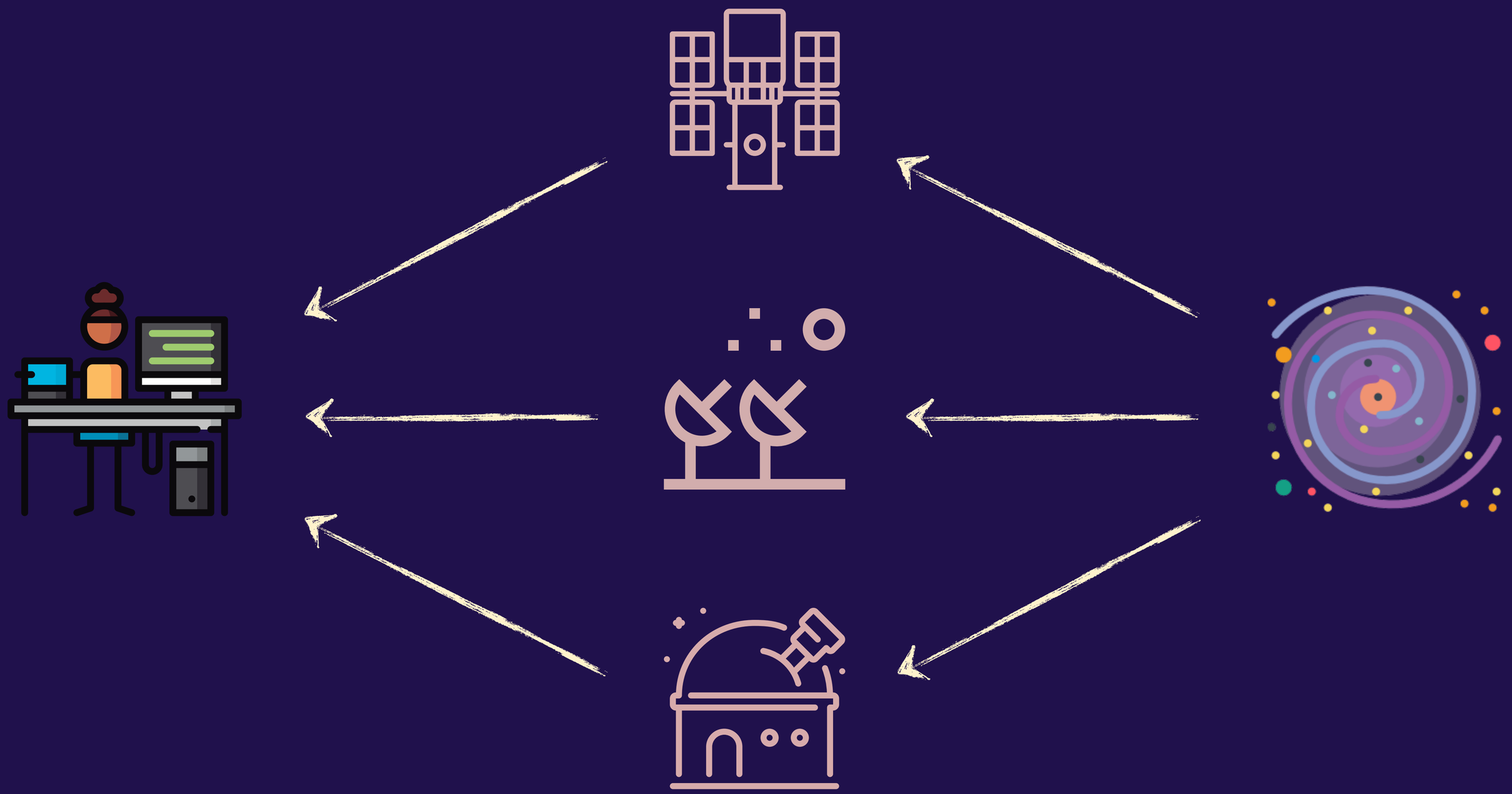
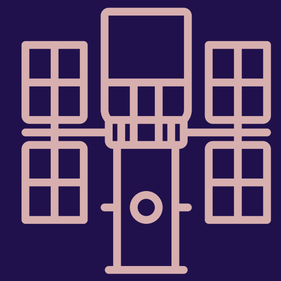
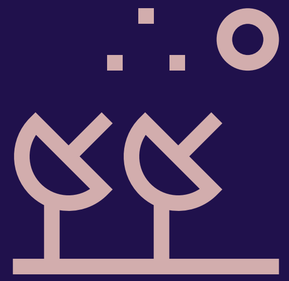
All models are **wrong**,
but some are **useful**

— George Box



Statistics!





Part 1: Probabilities

What snacks will we have for coffee break today?

What snacks will we have for coffee break today?

p()

p()

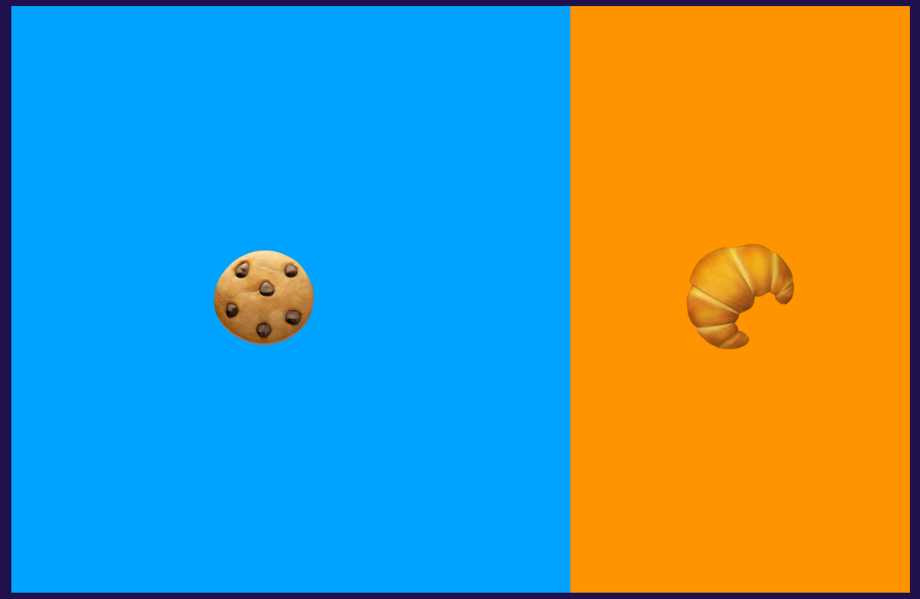
Basic rule of probability (1)

$$0 \leq p(\text{🍪}) \leq 1$$

If p(🍪) = 0.3, what is p(🥐)?

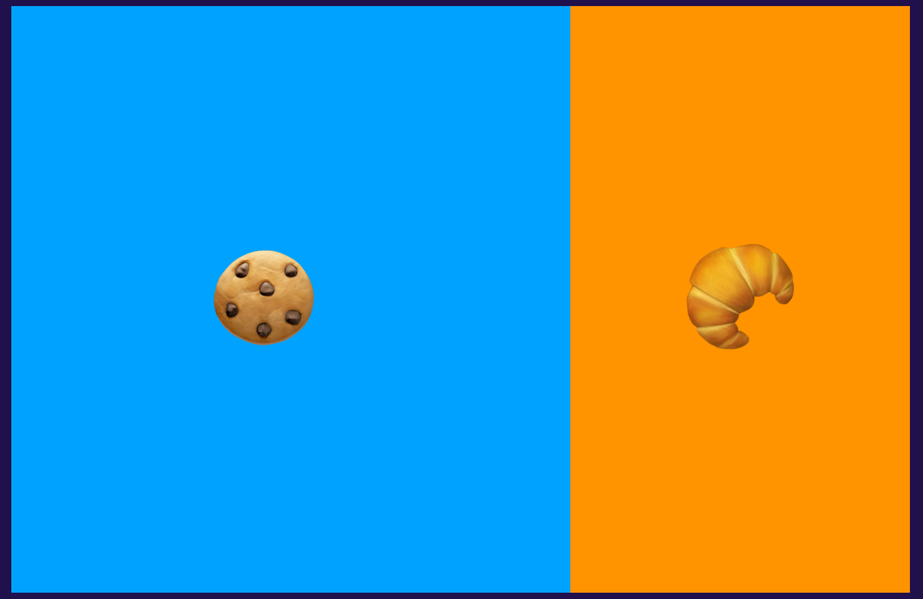
a) 0.3
b) 0.5
c) 0.01
d) 0.7

p(🍪) + p(🥐) = 1

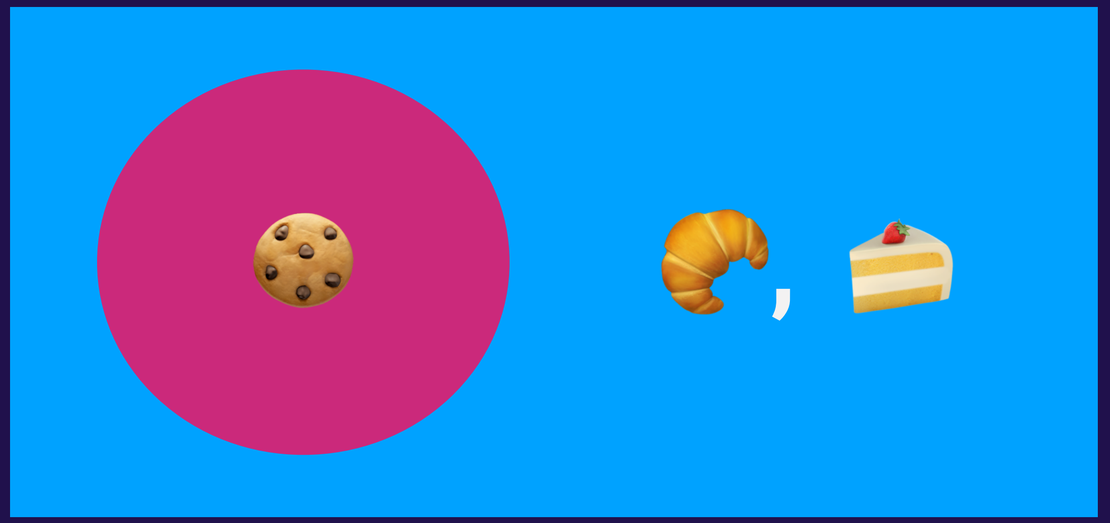


p(🍪) + p(🥐) = 1

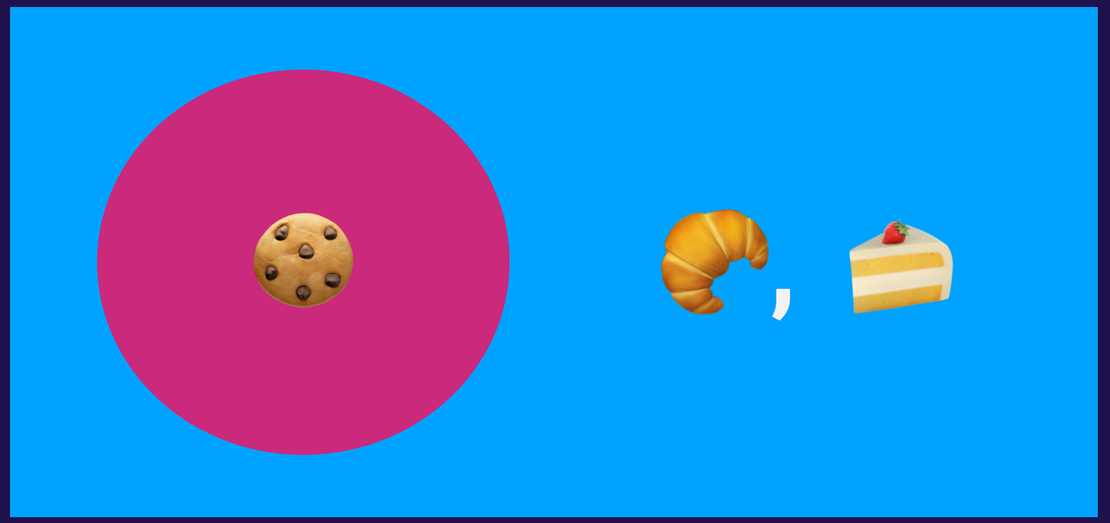
$$\sum_i P_i(x) = 1$$



p(🍪) + p(🥐) + p(🍰) = 1



p(🍪) + p(🥐) + p(🍰) = 1



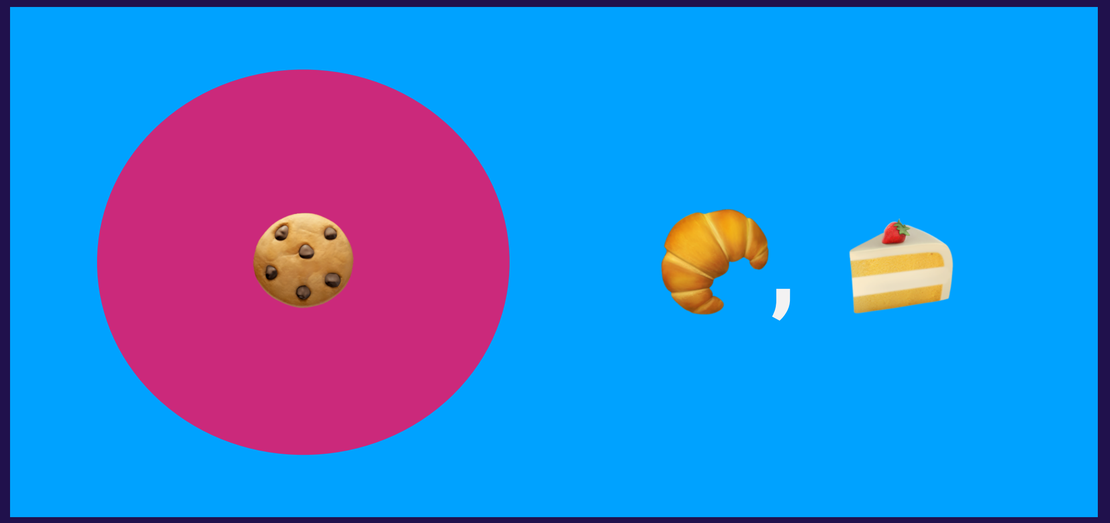
p(🥐) + p(🍰) = p(not 🍪)

“complement”

p(🍪) + p(🥐) + p(🍰) = 1

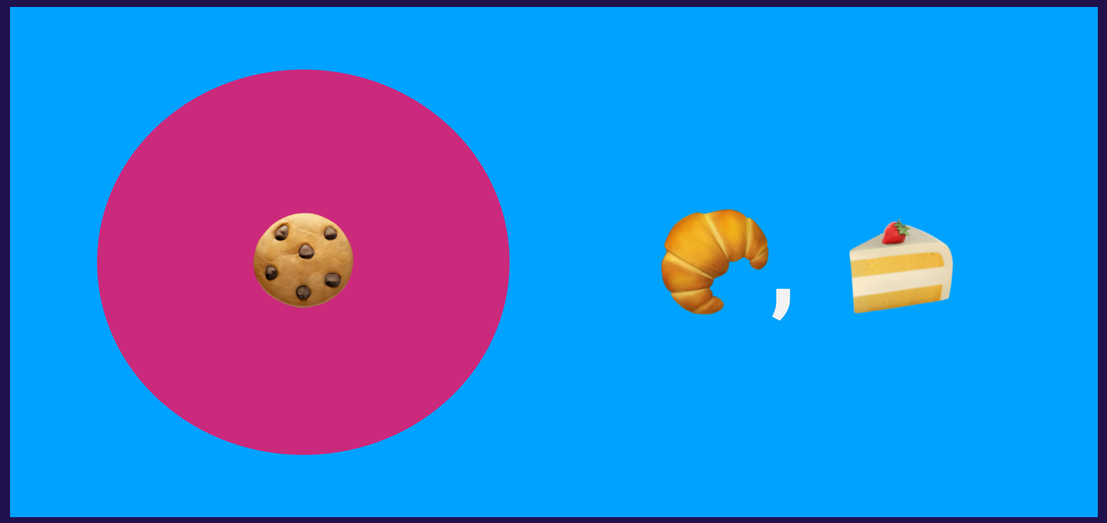
p(🥐) + p(🍰) = p(not 🍪)

p(🍪) =



“complement”

p(🍪) + p(🥐) + p(🍰) = 1



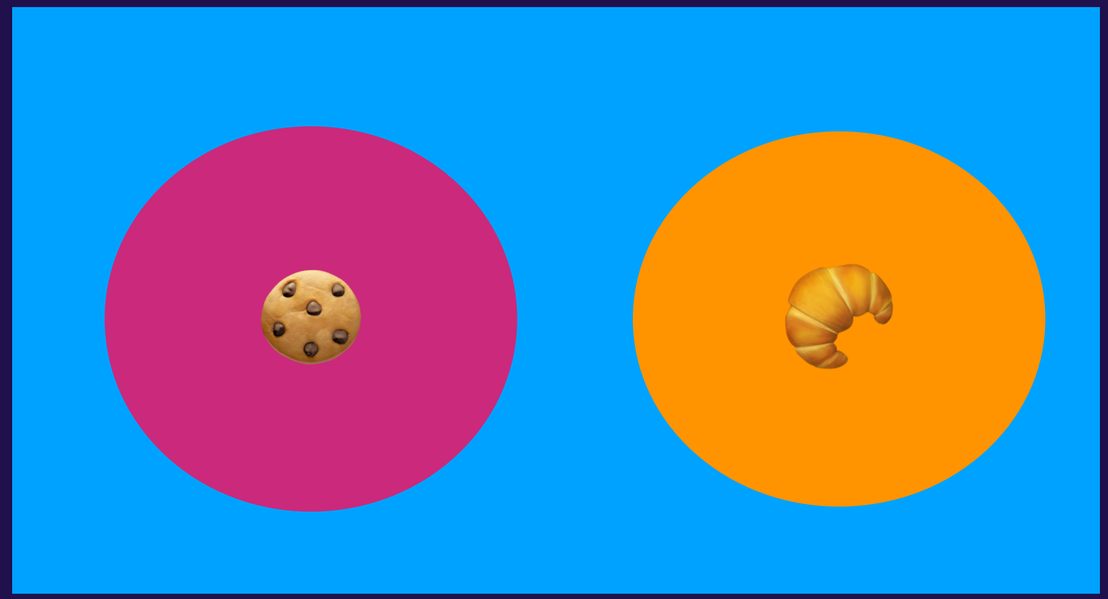
p(🥐) + p(🍰) = p(not 🍪)

“complement”

p(🍪) = 1 - p(not 🍪)

p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

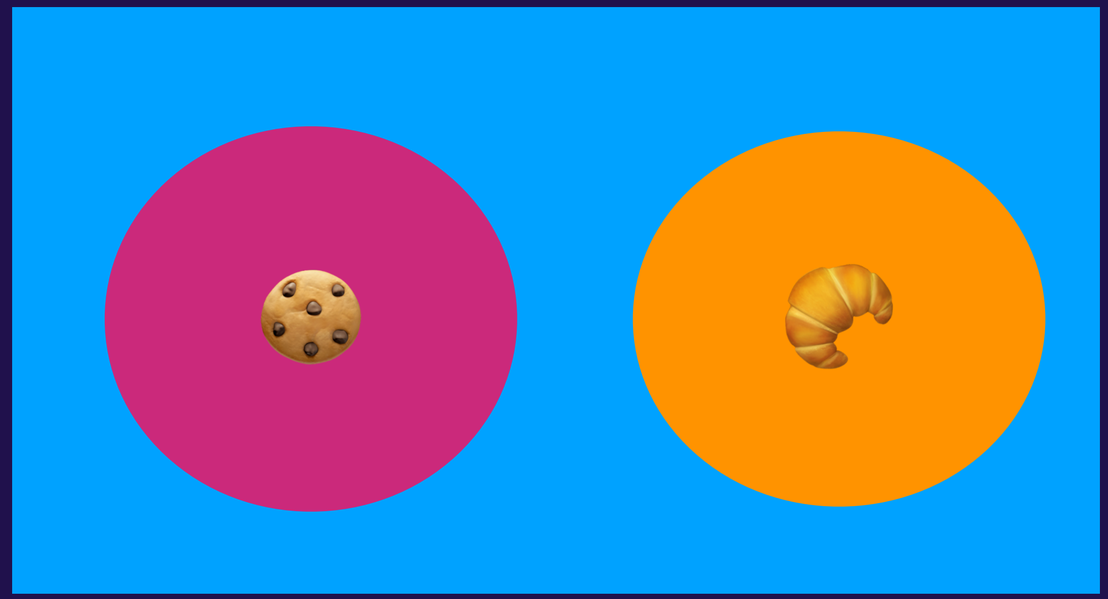
p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”



p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”

🍪 , 🥐 are independent:

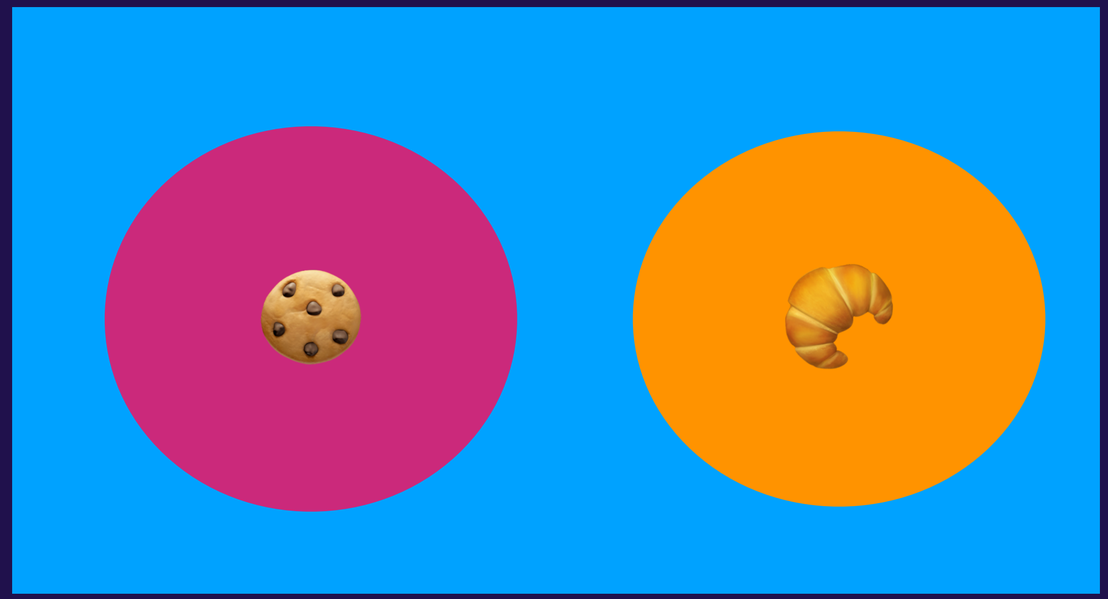


p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”

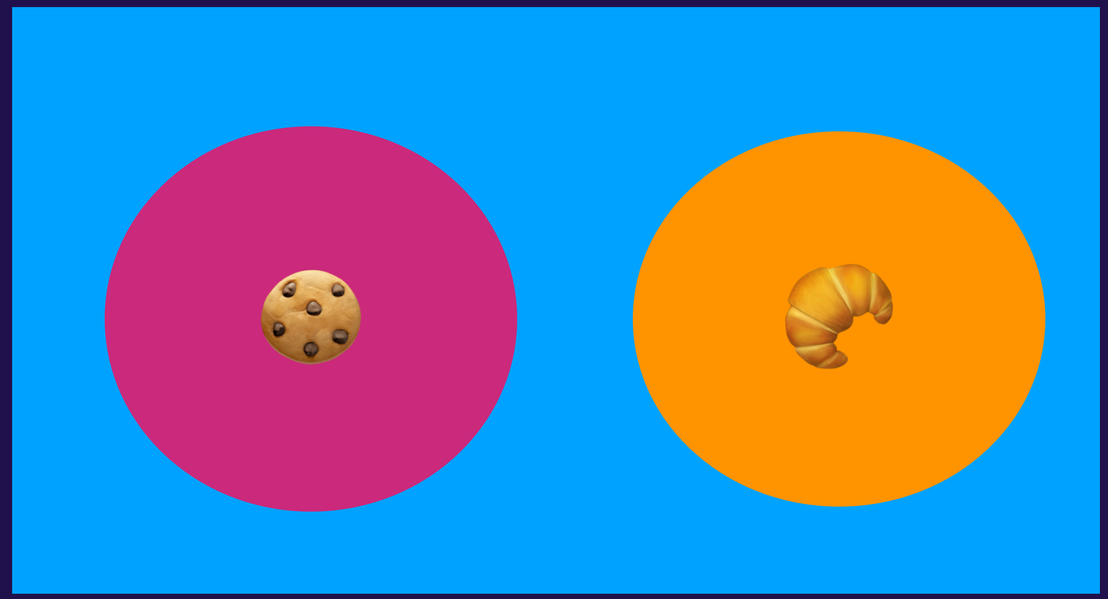
🍪 , 🥐 are independent:

p(🍪 ∩ 🥐) = 0



p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”



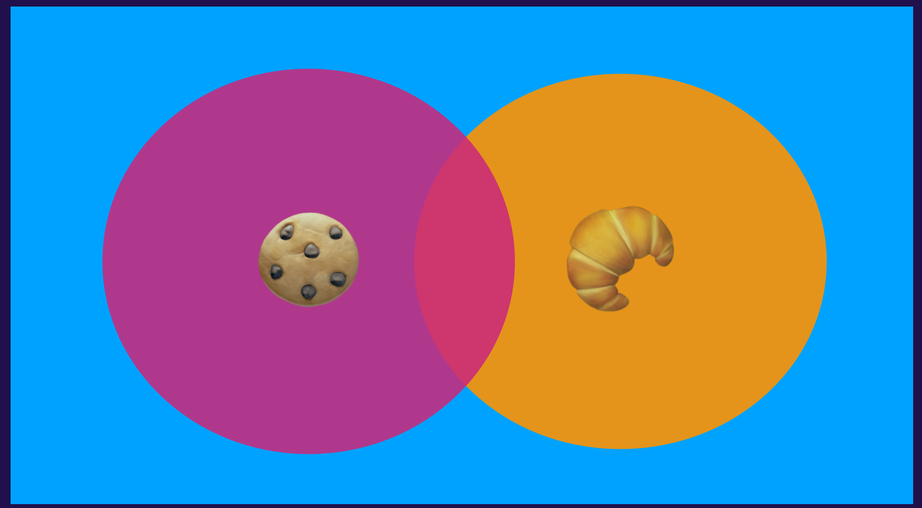
🍪 , 🥐 are independent:

p(🍪 ∩ 🥐) = 0

p(🍪 ∪ 🥐) = p(🍪) + p(🥐)

p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

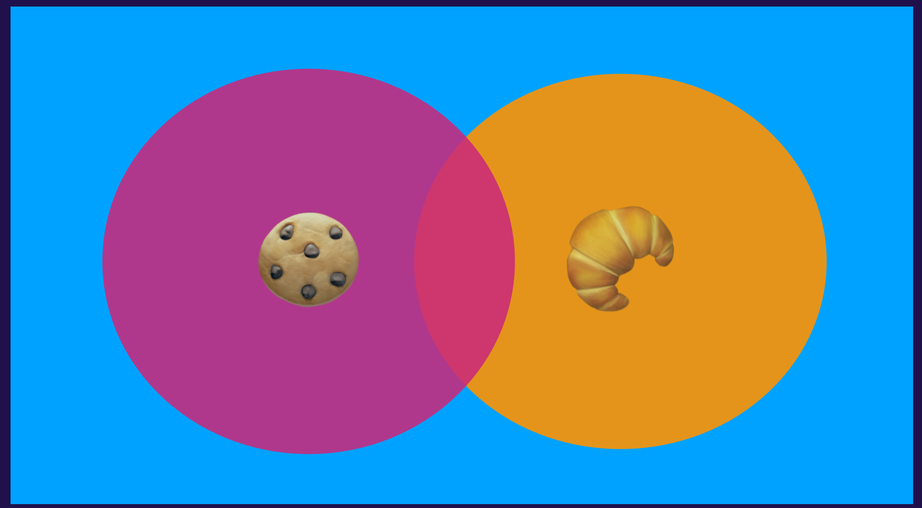
p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”



p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”

🍪 , 🥐 are **not** independent:



p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

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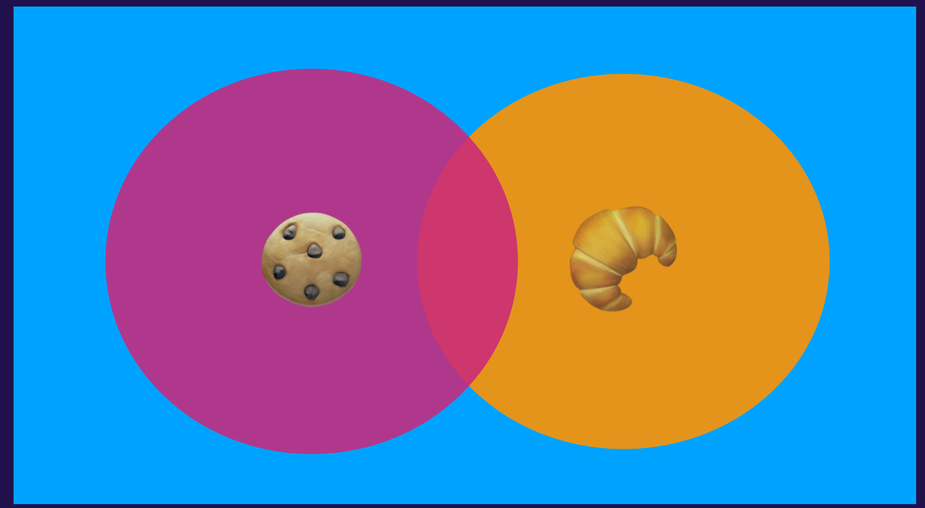
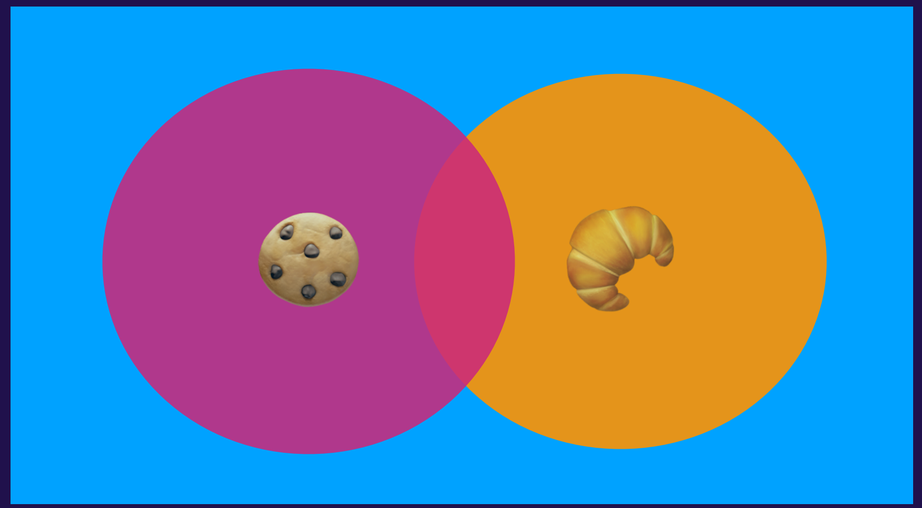


🍪 , 🥐 are not independent:

p(🍪 ∩ 🥐) = x

p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”



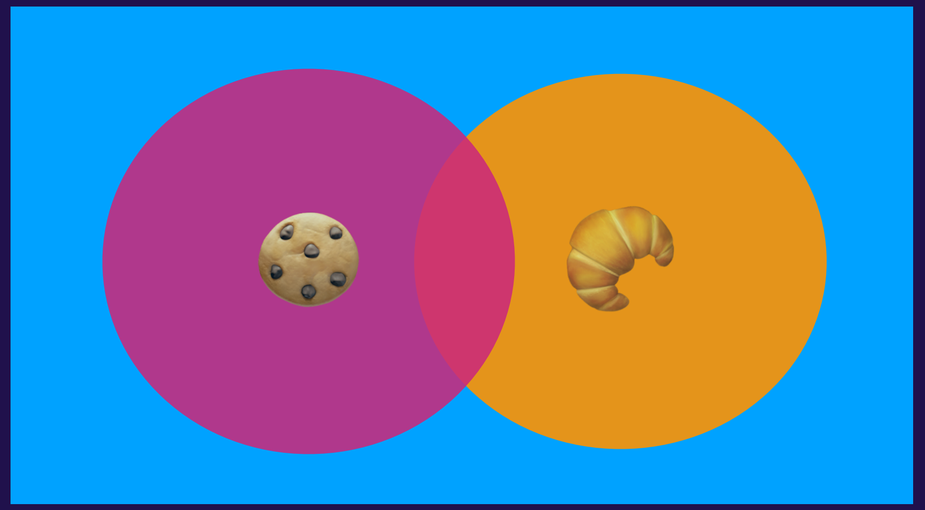
🍪 , 🥐 are **not** independent:

p(🍪 ∩ 🥐) = x

p(🍪 ∪ 🥐) = p(🍪) + p(🥐)

p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

p(🍪 ∪ 🥐) = “p(🍪 or 🥐)”



🍪 , 🥐 are not independent:

p(🍪 ∩ 🥐) = x

p(🍪 ∪ 🥐) = p(🍪) + p(🥐) - p(🍪 ∩ 🥐)

p(🍪 ∩ 🥐) = “p(🍪 and 🥐)”

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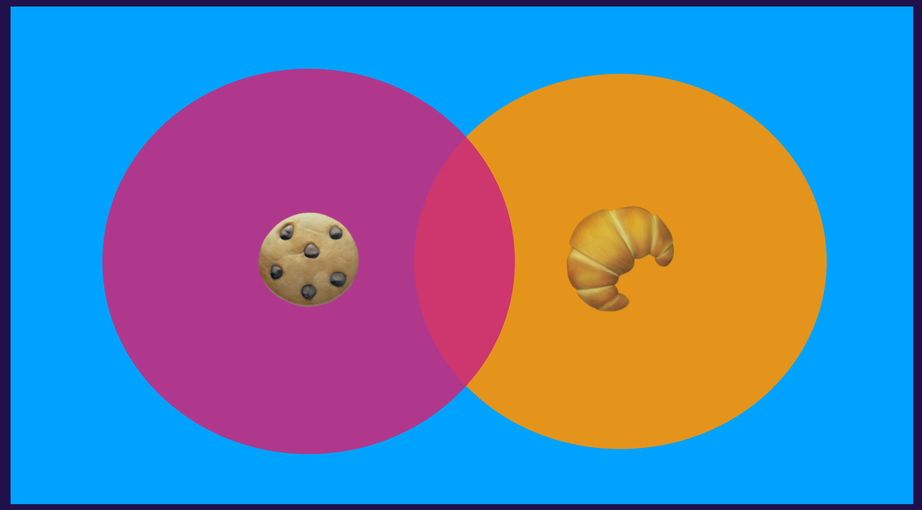


🍪 , 🥐 are not independent:

p(🍪 ∩ 🥐) = x

p(🍪 ∪ 🥐) = p(🍪) + p(🥐) - p(🍪 ∩ 🥐)

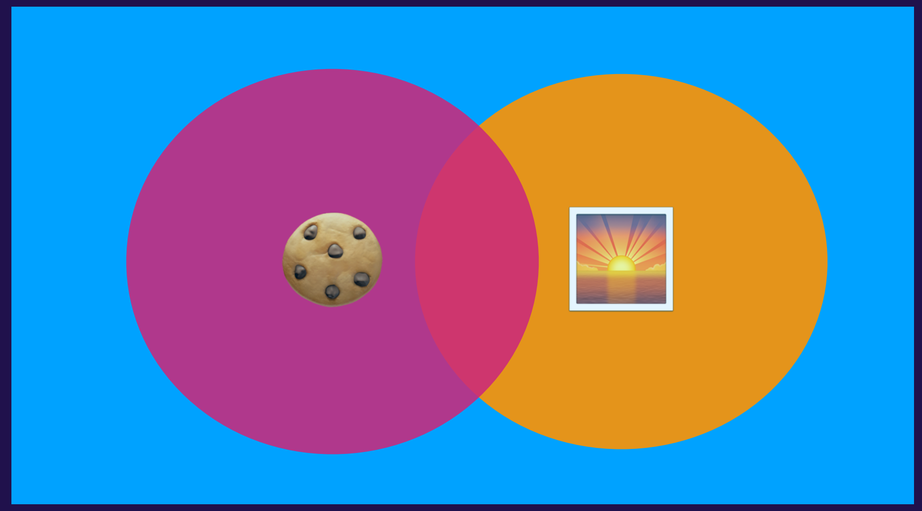
Let’s add another category

Is there a break with snacks in the morning or the afternoon?

p(🌅) = there are snacks in the morning
p(😴) = there are snacks in the afternoon

Do we get cookies more often in the morning or in the afternoon?

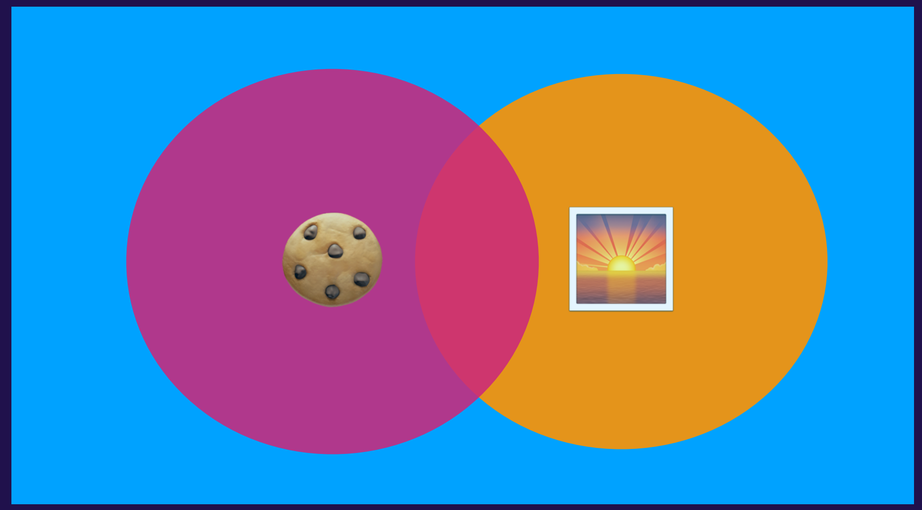
p(🍪 | 🌅) = “probability of 🍪 given 🌅 ”



Do we get cookies more often in the morning or in the afternoon?

p(🍪 | 🌅) = “probability of 🍪 given 🌅 ”

p(🍪 | 🌅) = p(🍪 ∩ 🌅) / p(🌅)

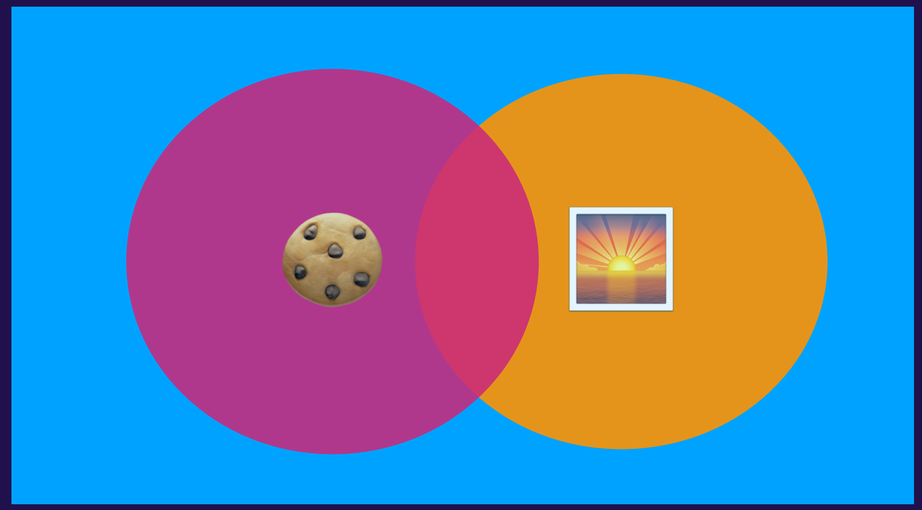


Do we get cookies more often in the morning or in the afternoon?

p(🍪 | 🌅) = “probability of 🍪 given 🌅 ”

p(🍪 | 🌅) = p(🍪 ∩ 🌅) / p(🌅)

p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)



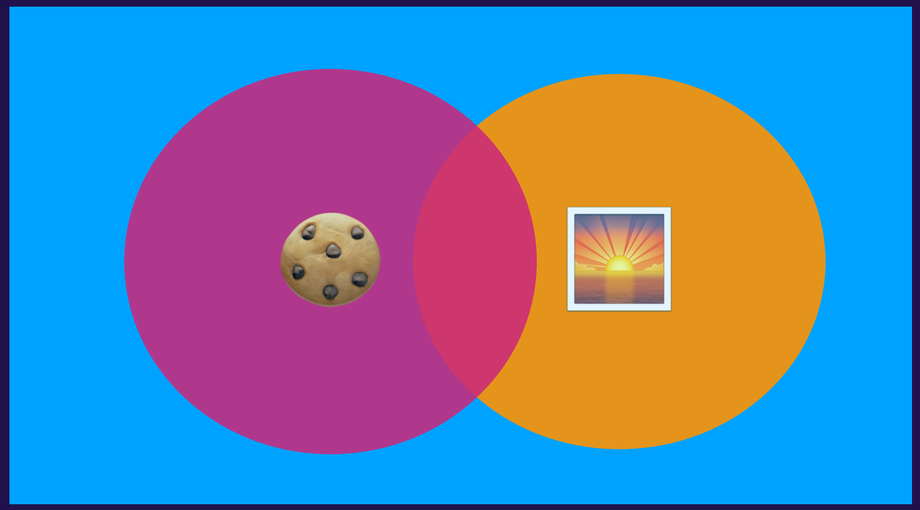
Which snacks do we get at which time of the day?

	p(🍪)	p(🥐)	p(🍰)
p(🌅)	0,25	0,06	0,29
p(😴)	0,25	0,14	0,01

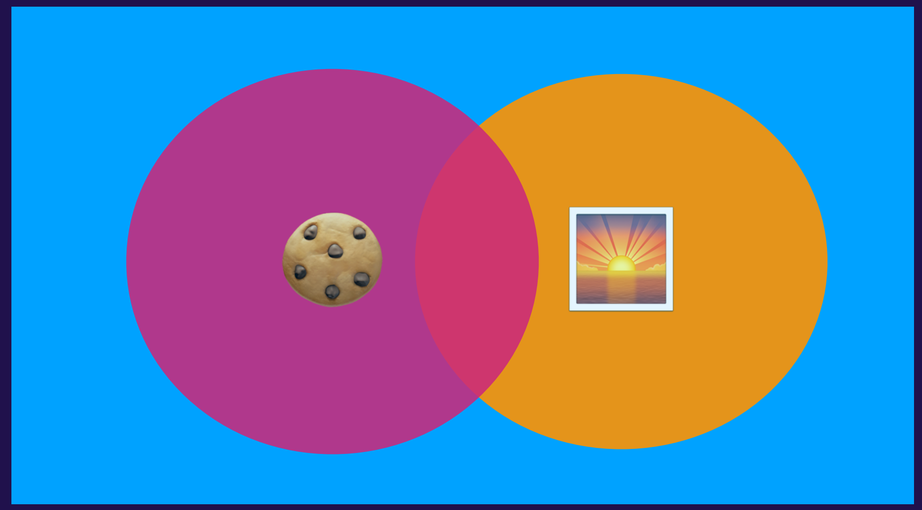
Which snacks do we get at which time of the day?

	p(🍪)	p(🥐)	p(🍰)
p(🌅)	0,25	0,06	0,29
p(😴)	0,25	0,14	0,01

- p(🍰) = ?
- p(🌅 | 🥐) = ?
- p(🍰 | 🌅) = ?

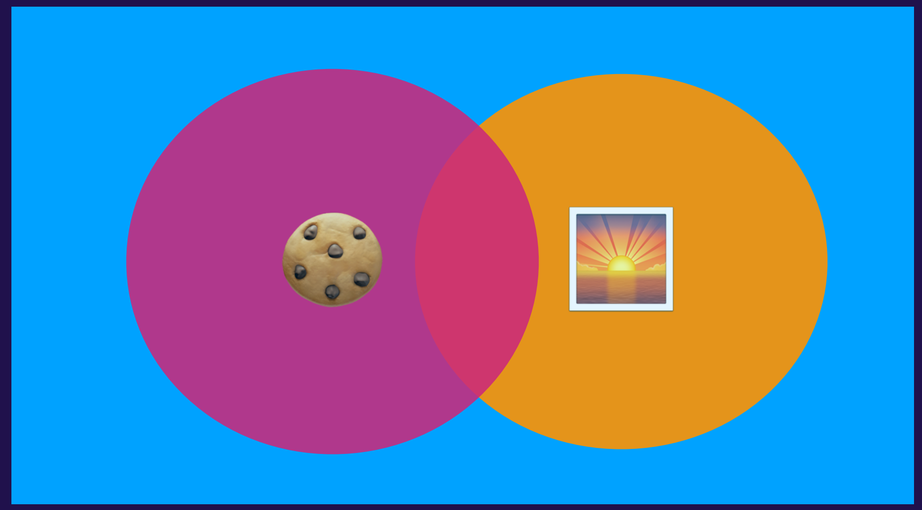


p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)



p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)

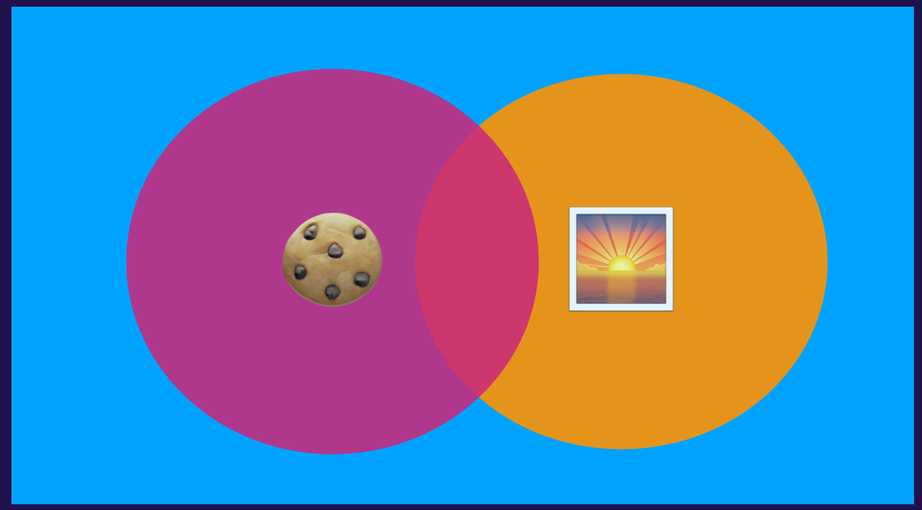
What is p(🌅 | 🍪) ?

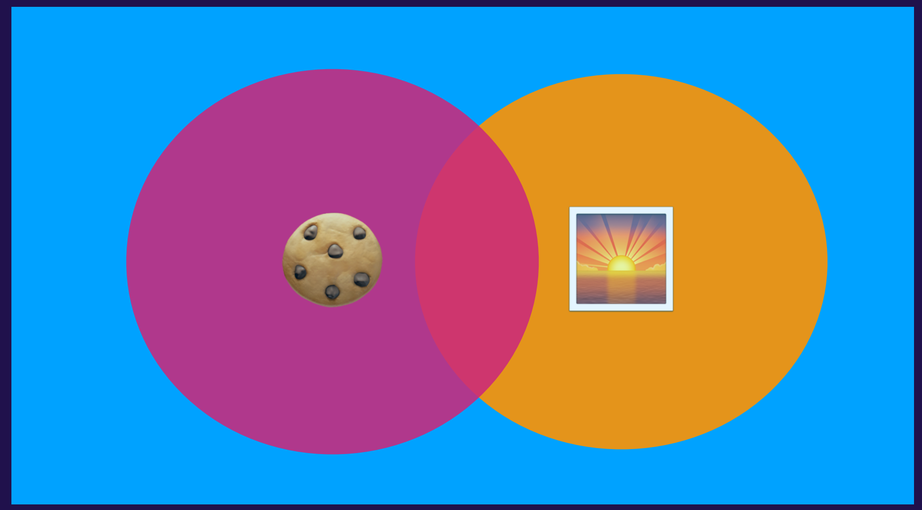


p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)

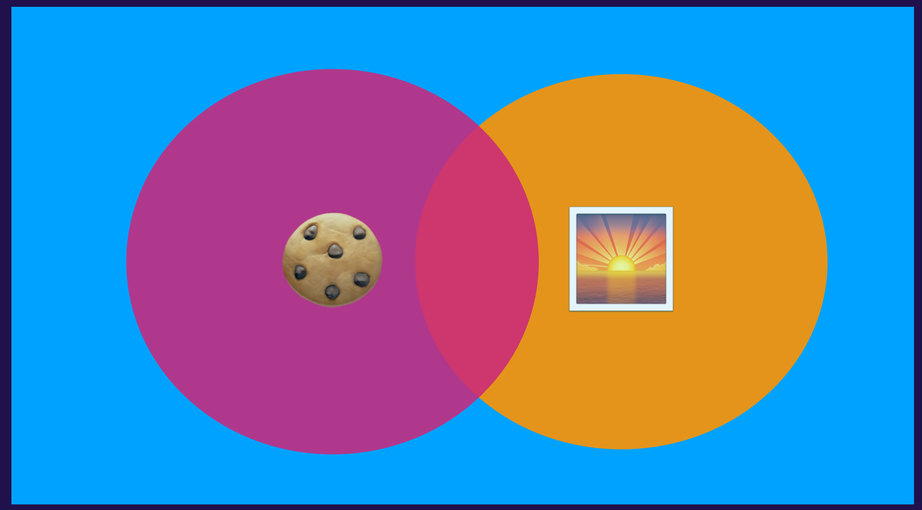
What is p(🌅 | 🍪) ?

p(🌅 | 🍪) = p(🍪 , 🌅) / p(🍪)





p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)



p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)

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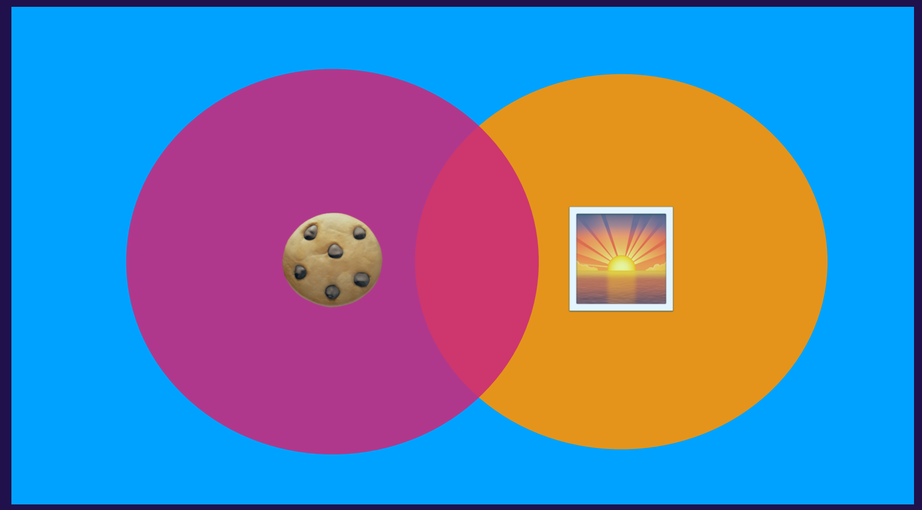
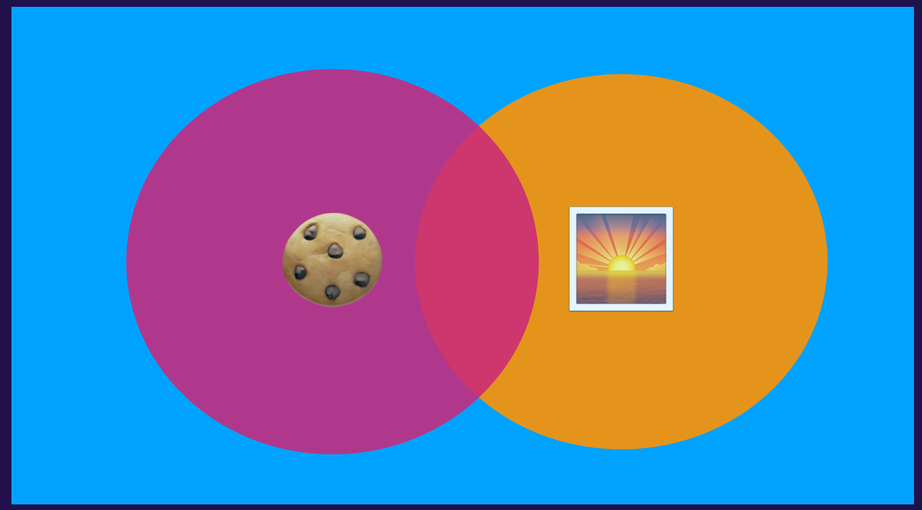


$$p(🍪 \mid 🌅) = p(🍪, 🌅) / p(🌅)$$

$$p(🌅 \mid 🍪) = p(🍪, 🌅) / p(🍪)$$

$$p(🌅 \mid 🍪)\, p(🍪) = p(🍪 \mid 🌅)\, p(🌅)$$

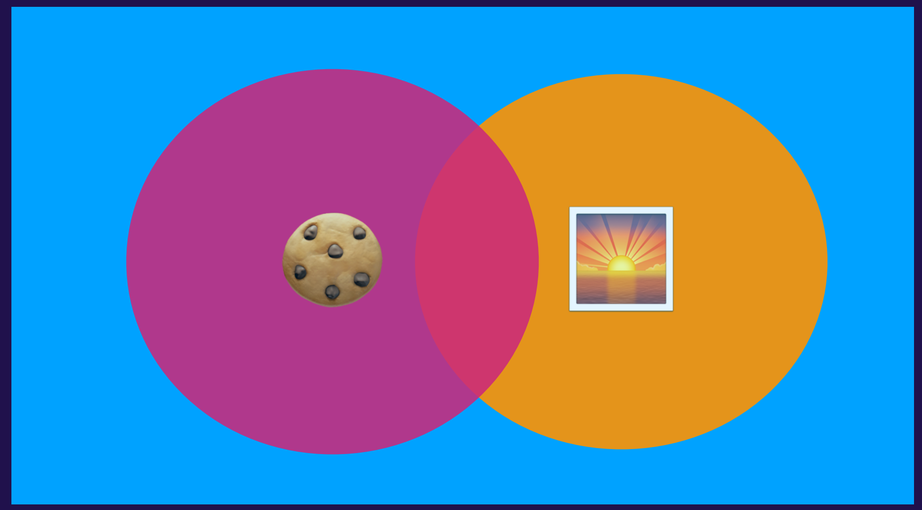


p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)

p(🌅 | 🍪) = p(🍪 , 🌅) / p(🍪)

p(🌅 | 🍪) p(🍪) = p(🍪 | 🌅) p(🌅)

Bayes rule

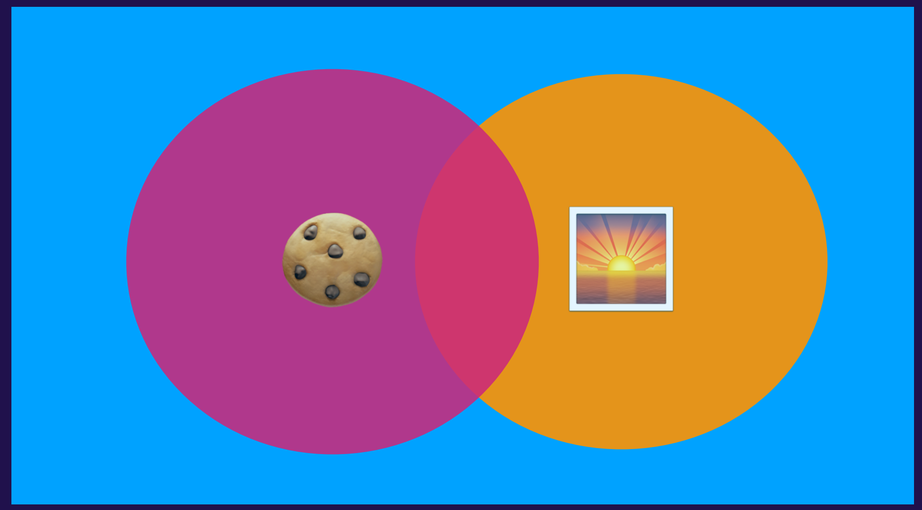


p(🍪 | 🌅) = p(🍪 , 🌅) / p(🌅)

p(🌅 | 🍪) = p(🍪 , 🌅) / p(🍪)

p(🌅 | 🍪) p(🍪) = p(🍪 | 🌅) p(🌅)

Bayes rule



$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Bayes rule

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Bayes rule

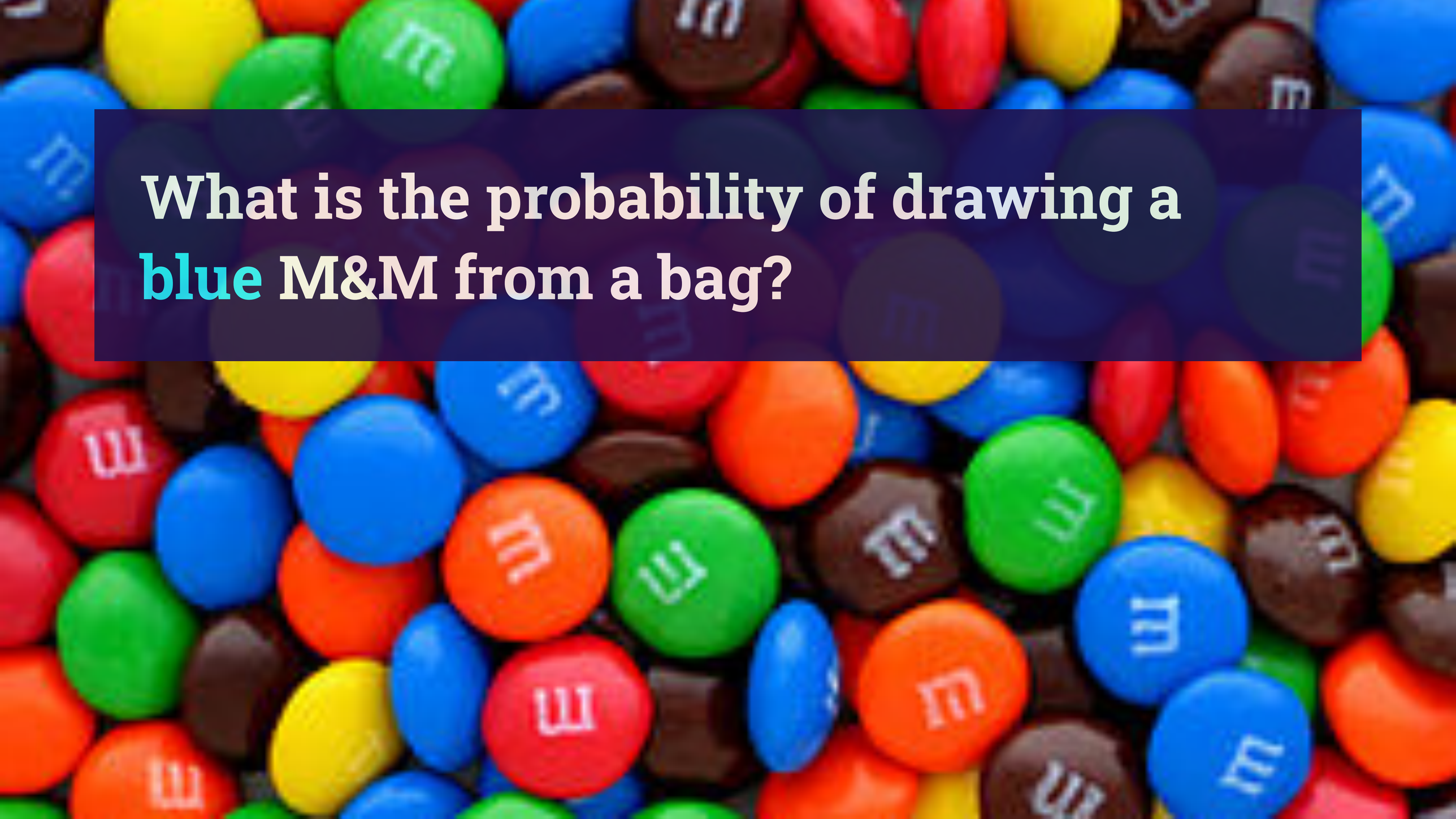
Important: $P(A \mid B) \neq P(B \mid A)$

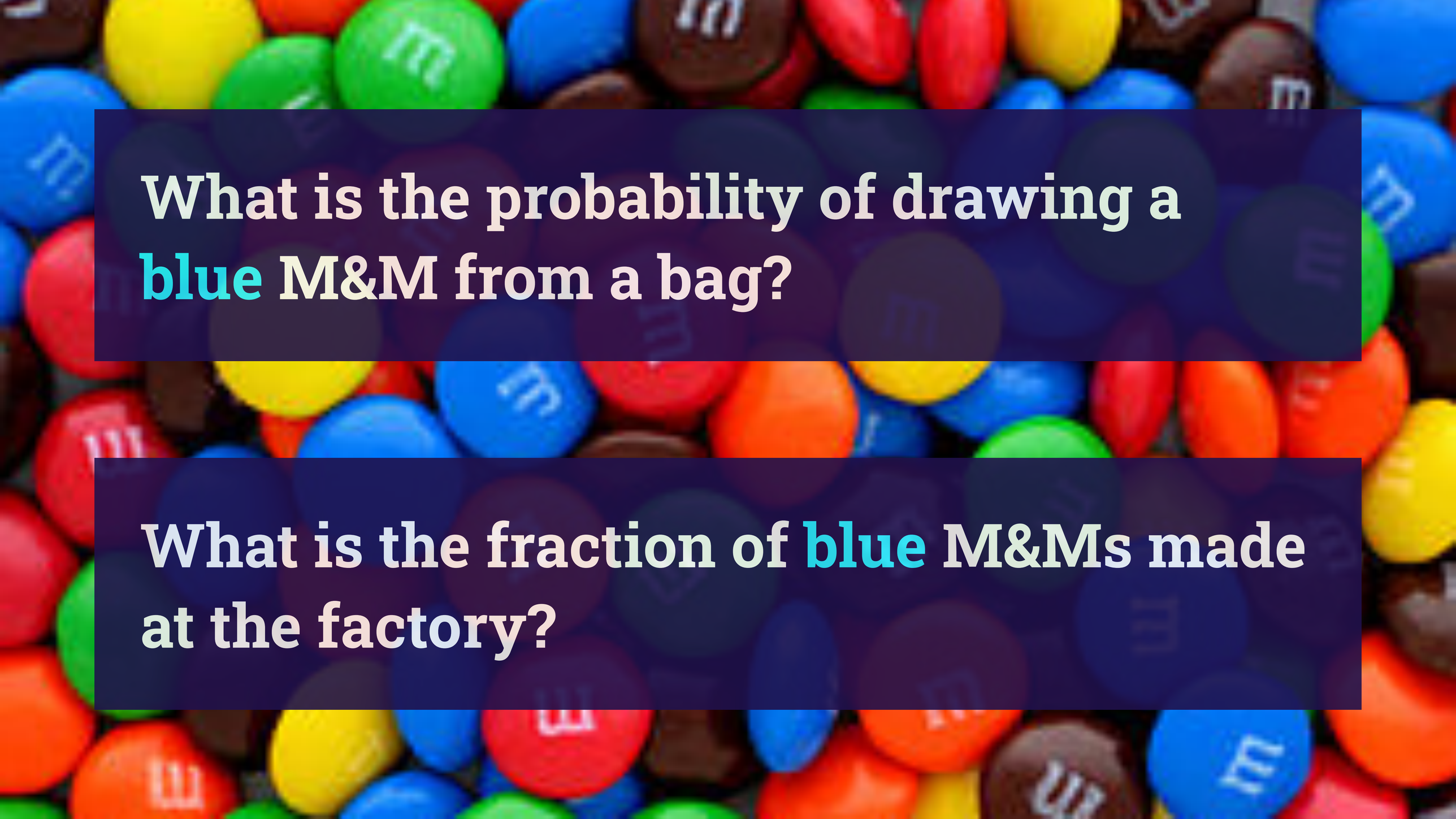
$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Bayes rule

Important: $P(A \mid B) \neq P(B \mid A)$

p(☔| ☁️) ≠ p(☁️ | ☔)


What is the probability of drawing a
blue M&M from a bag?


What is the probability of drawing a blue M&M from a bag?
What is the fraction of blue M&Ms made at the factory?


M&M's
FUN SIZE

1. Blue M&Ms are the best.

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2. Why are there sometimes more blue M&Ms in a package and sometimes fewer?

1. Blue M&Ms are the best.

2. Why are there sometimes more blue M&Ms in a package and sometimes fewer?

3. What’s the distribution of colours in a package of M&Ms? Are all colours equally distributed?

Why are there sometimes more blue smarties in a package and sometimes fewer?

Why are there sometimes more blue smarties in a package and sometimes fewer?

Why are there sometimes more blue smarties in a package and sometimes fewer?

posterior

likelihood

prior

Find the posterior distribution for the percentage of blue m&ms made at the factory

likelihood

$$P(q \mid N, k) \propto P(k \mid N, q) P(q)$$

likelihood

$$P(q \mid N, k) \propto P(k \mid N, q) P(q)$$

$k =$

likelihood

$$P(q \mid N, k) \propto P(k \mid N, q) P(q)$$

$k =$ **number of successes (blue m&ms)**

likelihood

$$P(q \mid N, k) \propto P(k \mid N, q) P(q)$$

$k =$ **number of successes (blue m&ms)**

$N - k =$

likelihood

$$P(q \mid N, k) \propto P(k \mid N, q) P(q)$$

$k =$ **number of successes (blue m&ms)**

$N - k =$ **number of failures (not-blue m&ms)**

likelihood

$$P(q \mid N, k) \propto P(k \mid N, q) P(q)$$

$k =$ **number of successes (blue m&ms)**

$N - k =$ **number of failures (not-blue m&ms)**

$q =$

likelihood

$$P(q \mid N, k) \propto P(k \mid N, q) P(q)$$

$k =$ **number of successes (blue m&ms)**

$N - k =$ **number of failures (not-blue m&ms)**

$q =$ **probability of drawing a blue m&m**

I have drawn the following sequence of blue (b) and not-blue (t) m&ms:

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$$S = b\ b\ t\ b\ t\ t\ t\ b\ t\ b$$

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What's the probability of drawing exactly that sequence?

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$$p(S) = qq(1-q)q(1-q)(1-q)(1-q)q(1-q)q$$

I have drawn the following sequence of blue (b) and not-blue (t) m&ms:

$$S = b\ b\ t\ b\ t\ t\ t\ b\ t\ b$$

What's the probability of drawing exactly that sequence?

$$\begin{aligned} p(S) &= qq(1-q)q(1-q)(1-q)(1-q)q(1-q)q \\ &= q^5(1-q)^5 \end{aligned}$$

The Binomial Distribution

Observed fraction of blue M&Ms in a single bag

True fraction of blue M&Ms produced at the factory

$$p(\text{🍬} \mid \text{🏭}) = p(\, k \mid N, q\,) \sim \text{Binom}(N, k, q)$$

“The probability of observing k blue M&Ms out of N given that the true fraction of M&Ms is q”

The Binomial Distribution

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True fraction of blue M&Ms produced at the factory

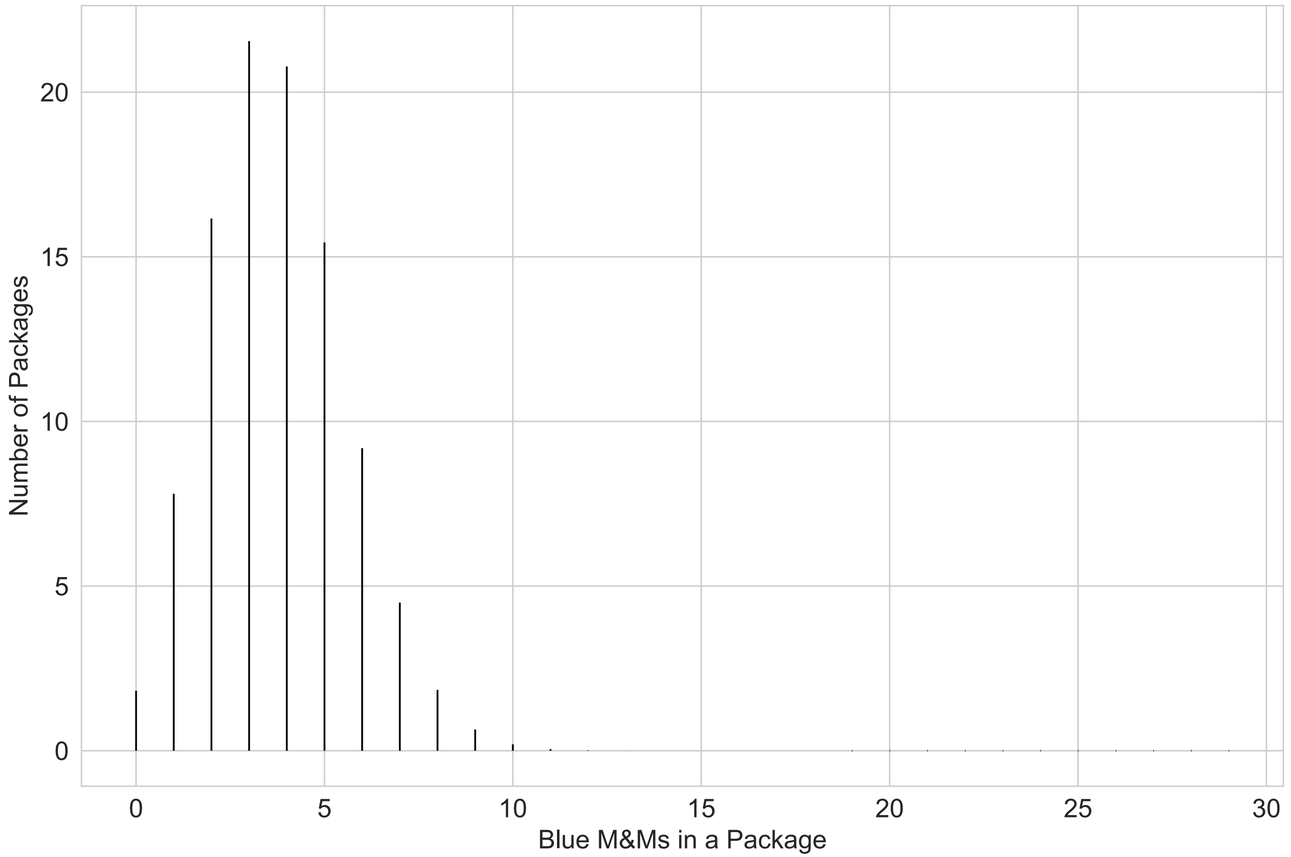
$$p(\text{🍬} \mid \text{🏭}) = p(\, k \mid N, q\,) \sim \text{Binom}(N, k, q)$$

$$= \binom{N}{k} q^k (1-q)^{(N-k)}$$

“The probability of observing k blue M&Ms out of N given that the true fraction of M&Ms is q”

First Assumption: The M&Ms are mixed together at the factory and then distributed into packages.

Second Assumption: All colours are equally distributed



Binomial distribution

Blue Fraction: 0.125

First Assumption: The M&Ms are mixed together at the factory and then distributed into packages.

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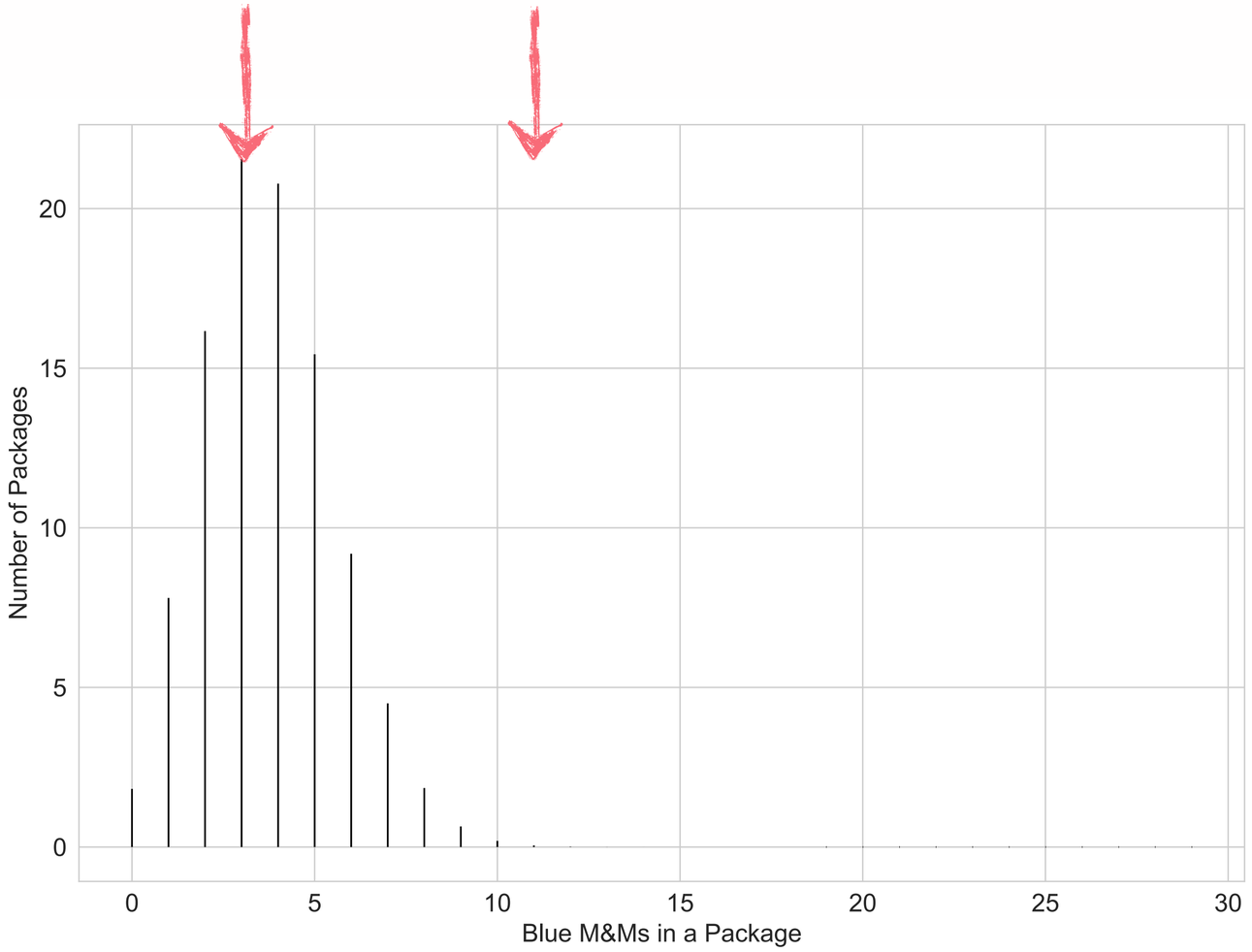

Binomial distribution

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First Assumption: The M&Ms are mixed together at the factory and then distributed into packages.

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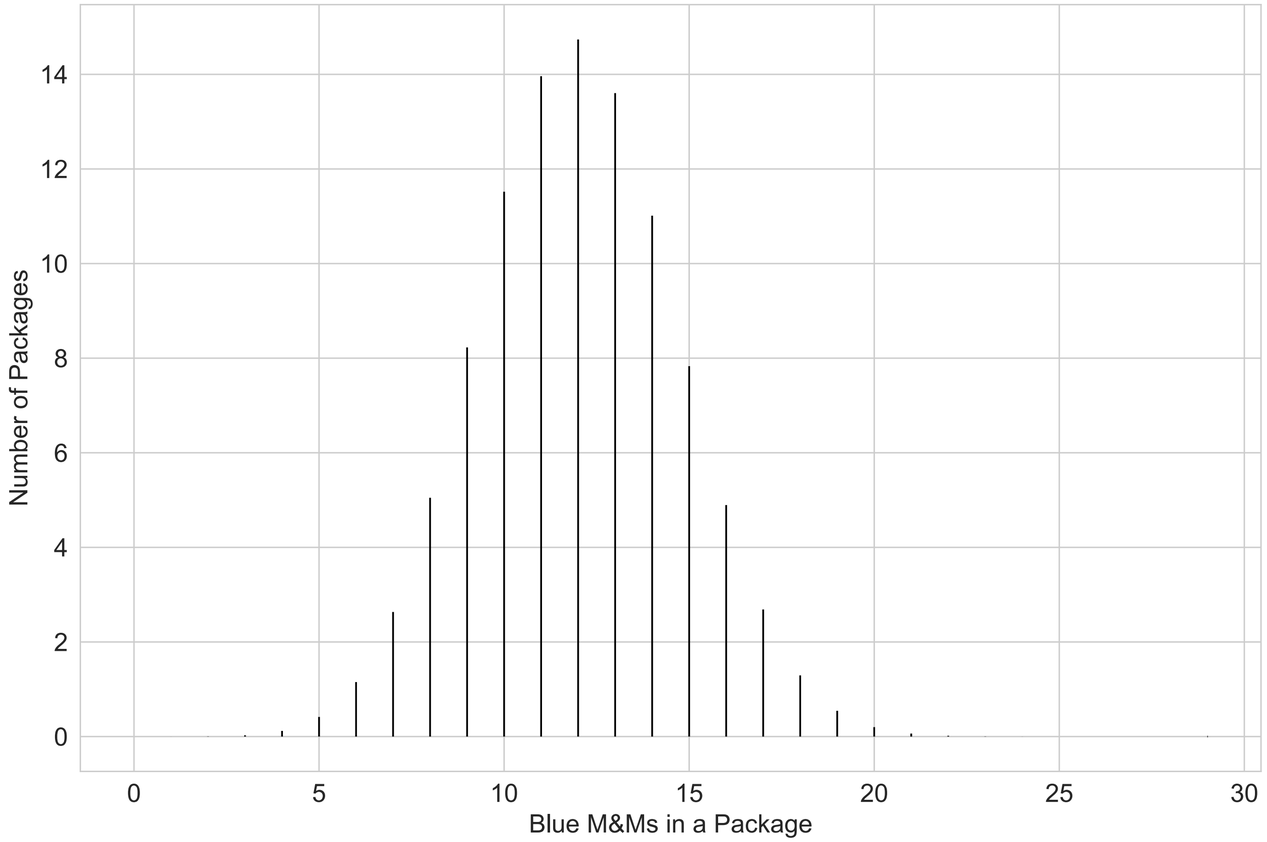
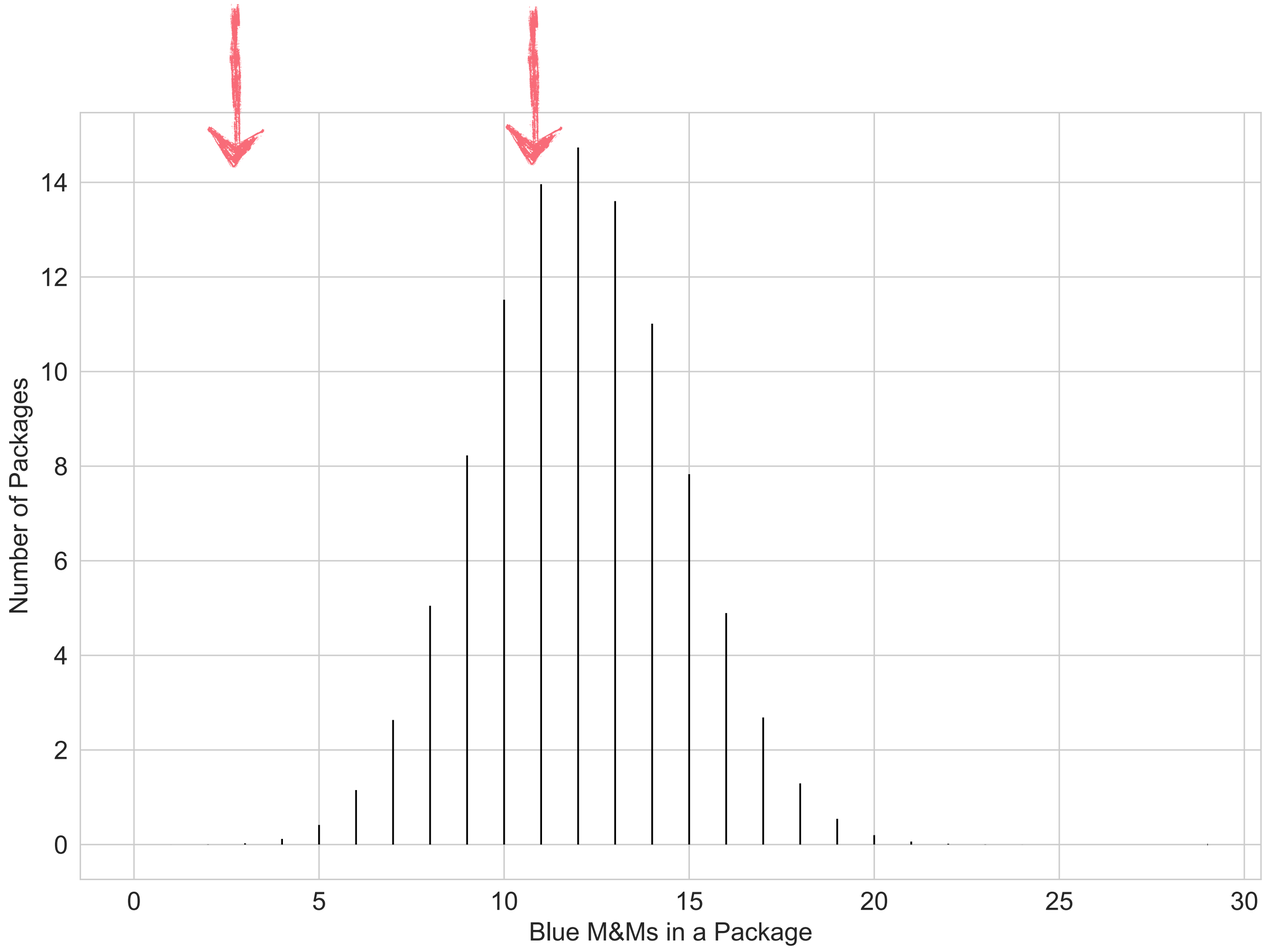


Binomial distribution

Blue Fraction: 0.4

First Assumption: The M&Ms are mixed together at the factory and then distributed into packages.

~~Second Assumption: All colours are equally distributed~~



Binomial distribution

Blue Fraction: 0.4

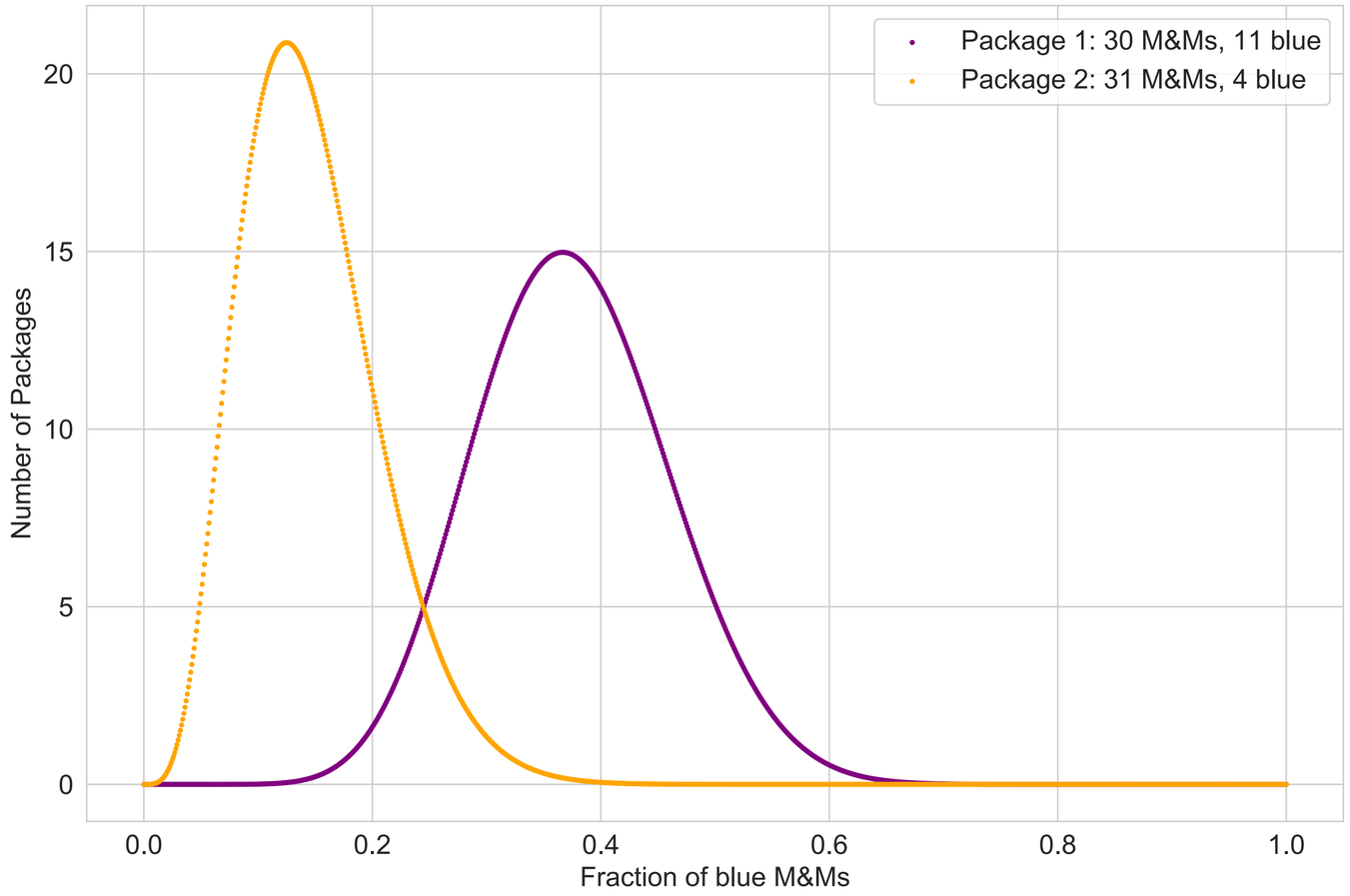
Question: Can we use this somehow to estimate the fraction of blue M&Ms?

Exercises 1, 2, 3

mybinder.org [ADD LINK!]

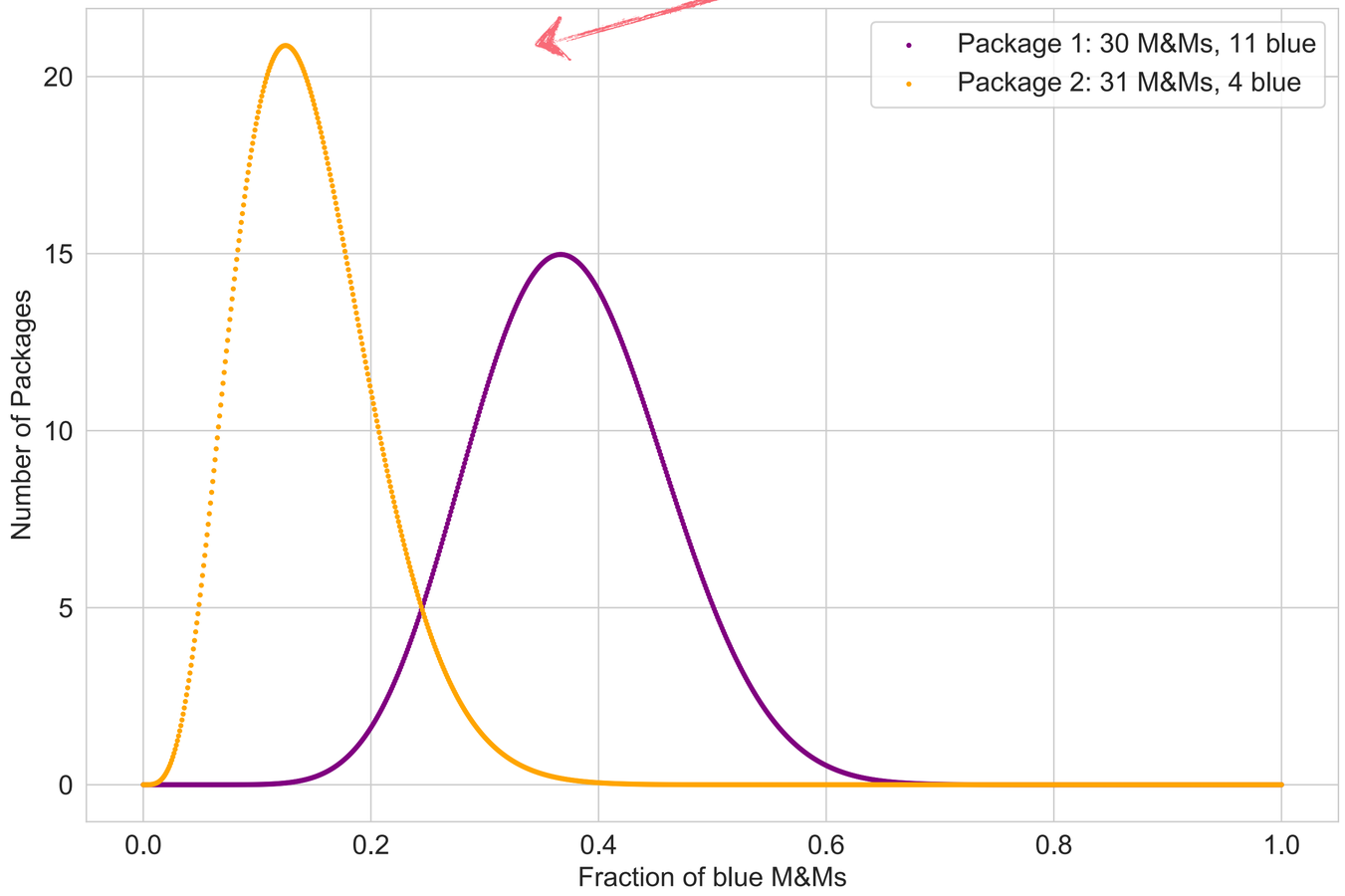
Exercises 1, 2, 3

The Likelihood Function



The Likelihood Function

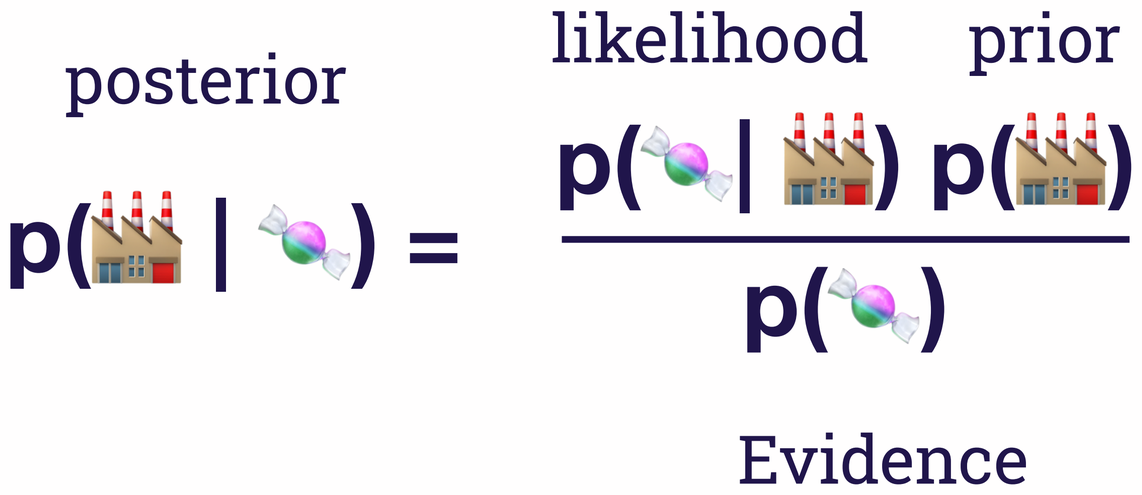
use optimization routines to find the maximum



Problem: The Likelihood assumes we know the **true underlying fraction** of M&Ms.

We're interested in p(🏭 | 🍬), not p(🍬| 🏭)

Conditional Probability Magic



Conditional Probability Magic

$$\underbrace{p(🏭 \mid 🍬)}_{\text{posterior}} = \frac{\overbrace{p(🍬 \mid 🏭)}^{\text{likelihood}} \overbrace{p(🏭)}^{\text{prior}}}{\underbrace{p(🍬)}_{\text{Evidence}}}$$

$$p(🏭 \mid 🍬) \propto p(🍬 \mid 🏭)\, p(🏭)$$

Conditional Probability Magic

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$$p(🏭 \mid 🍬) \propto p(🍬 \mid 🏭)\, p(🏭)$$

Bayes theorem

Question

What prior information do we have about p(🏭) ?

How many different colours of M&Ms are there?

What percentage of blue m&ms are made at the factory?

Do you think every bag of m&ms will have the same percentage of blue m&ms?

Sketch your prior

prior

$$P(\theta \mid y) \propto P(y \mid \theta) P(\theta)$$

prior

$$P(\theta|y) \propto P(y|\theta)P(\theta)$$

$$P(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

prior

$$P(\theta \mid y) \propto P(y \mid \theta) P(\theta)$$

beta distribution

$$P(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

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“conjugate prior”

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“conjugate prior”

$$\alpha = ?$$

prior

$$P(\theta \mid y) \propto P(y \mid \theta) P(\theta)$$

beta distribution

$$P(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

“conjugate prior”

$\alpha = ?$ $\beta = ?$

Write a function for the beta distribution and try out different values for alpha and beta

Some values to try:

$$\alpha = \beta$$
$$\alpha = \beta = 1$$
$$\alpha, \beta \quad \text{very large}$$
$$\alpha, \beta \quad \text{very small}$$

Write a function for the beta distribution and try out different values for alpha and beta

Some values to try:

$$\alpha = \beta$$
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$$\alpha, \beta \quad \text{very small}$$

mybinder.org [ADD LINK!], Exercise 5 + 6

$$P(\theta \mid y) \propto P(y \mid \theta) P(\theta)$$

likelihood

$$p(y) \propto \theta^{y}(1-\theta)^{n-y}$$

$$P(\theta \mid y) \propto P(y \mid \theta) P(\theta)$$

likelihood

$$p(y) \propto \theta^{y}(1-\theta)^{n-y}$$

prior

$$P(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$P(\theta \mid y) \propto P(y \mid \theta) P(\theta)$$

likelihood

$$p(y) \propto \theta^{y}(1-\theta)^{n-y}$$

prior

$$P(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$P(\theta \mid y) \propto P(y \mid \theta) P(\theta)$$

$$p(\theta) \propto \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}$$

Gather some data!

In groups of 3, pool your M&Ms

record the counts for all colours

Use your function for the beta distribution to plot both the prior and the posterior in the same plot

- Is the posterior distribution what you expected?
- Compare the posterior distribution to the prior distribution: Is this the result you expected, given six different colours?
- How sensitive is the posterior to the prior?

Use your function for the beta distribution to plot both the prior and the posterior in the same plot

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- Compare the posterior distribution to the prior distribution: Is this the result you expected, given six different colours?
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mybinder.org [ADD LINK!], Exercise 7+8

How would you expect the posterior to change with more data?

Let's pool all the data and find out!

https://forms.gle/zHy8LSbJsB8Pm9XA9

How would you expect the posterior to change with more data?

Let's pool all the data and find out!

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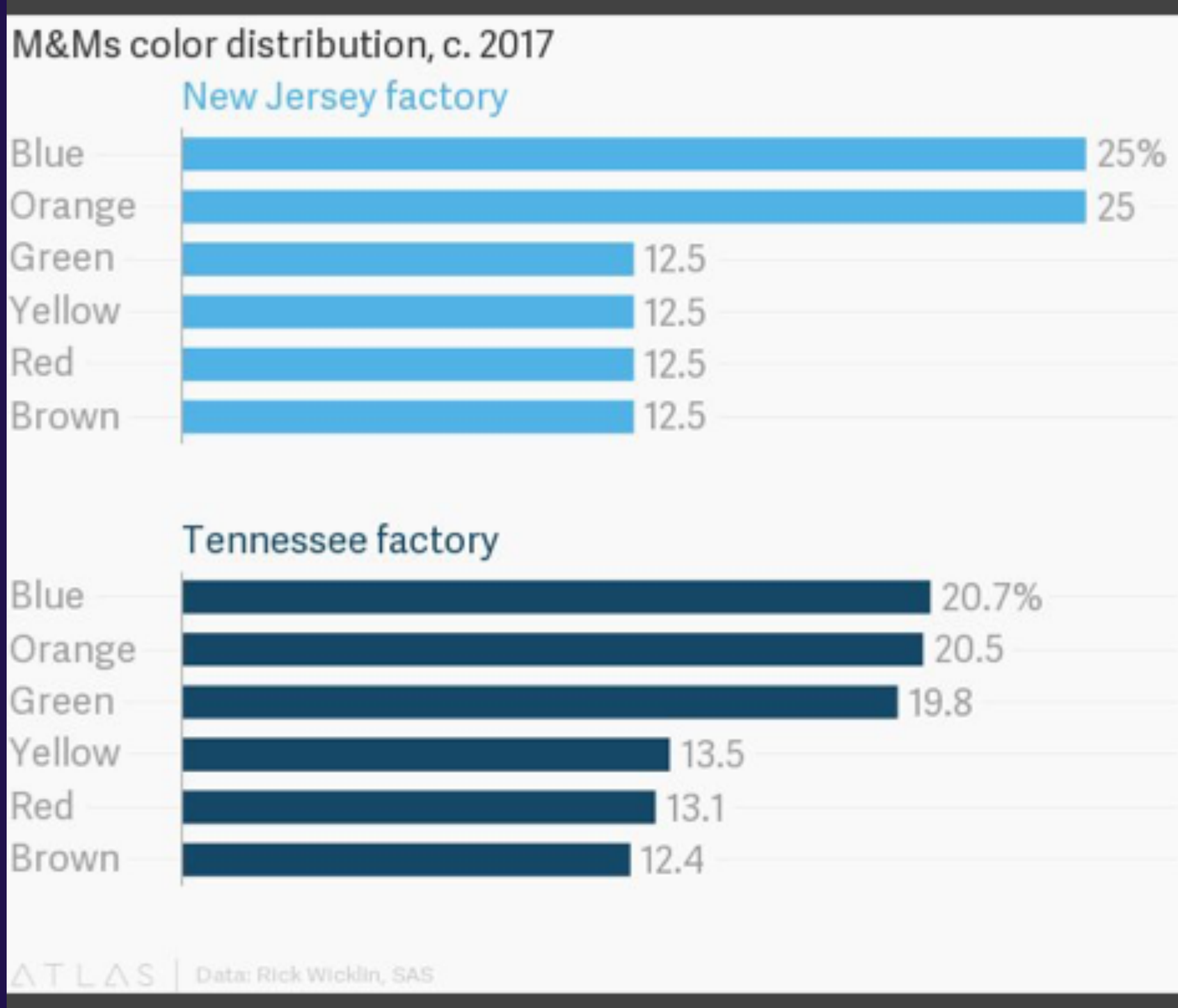
mybinder.org [ADD LINK!], Exercise 9

Priors are useful when data **isn't** very **informative**.

When the **data is informative**, the choice of prior **doesn't matter** much.

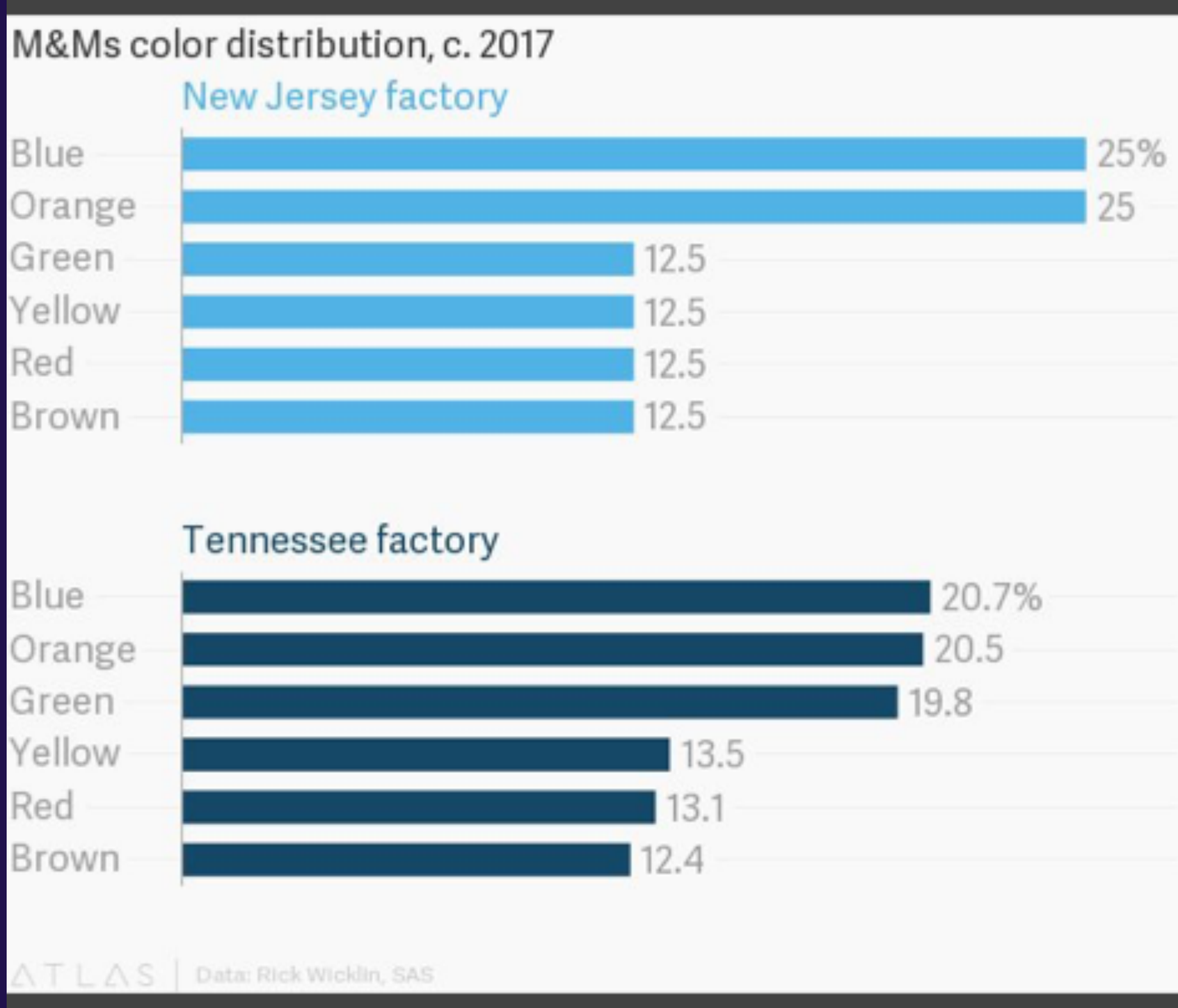
Surprise Twist!

Different factories make different distributions of m&ms!



Surprise Twist!

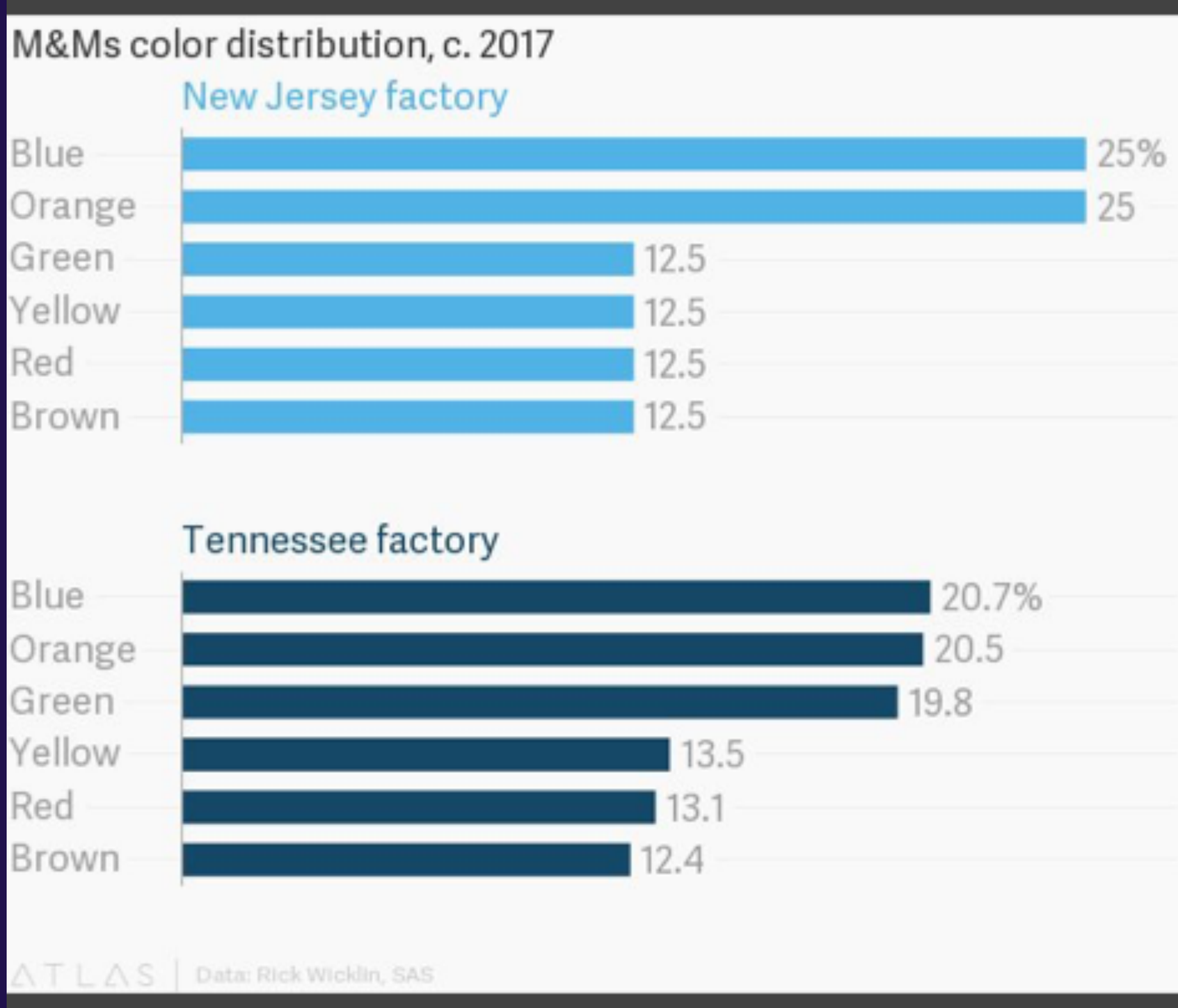
Which factory did your m&ms come from?

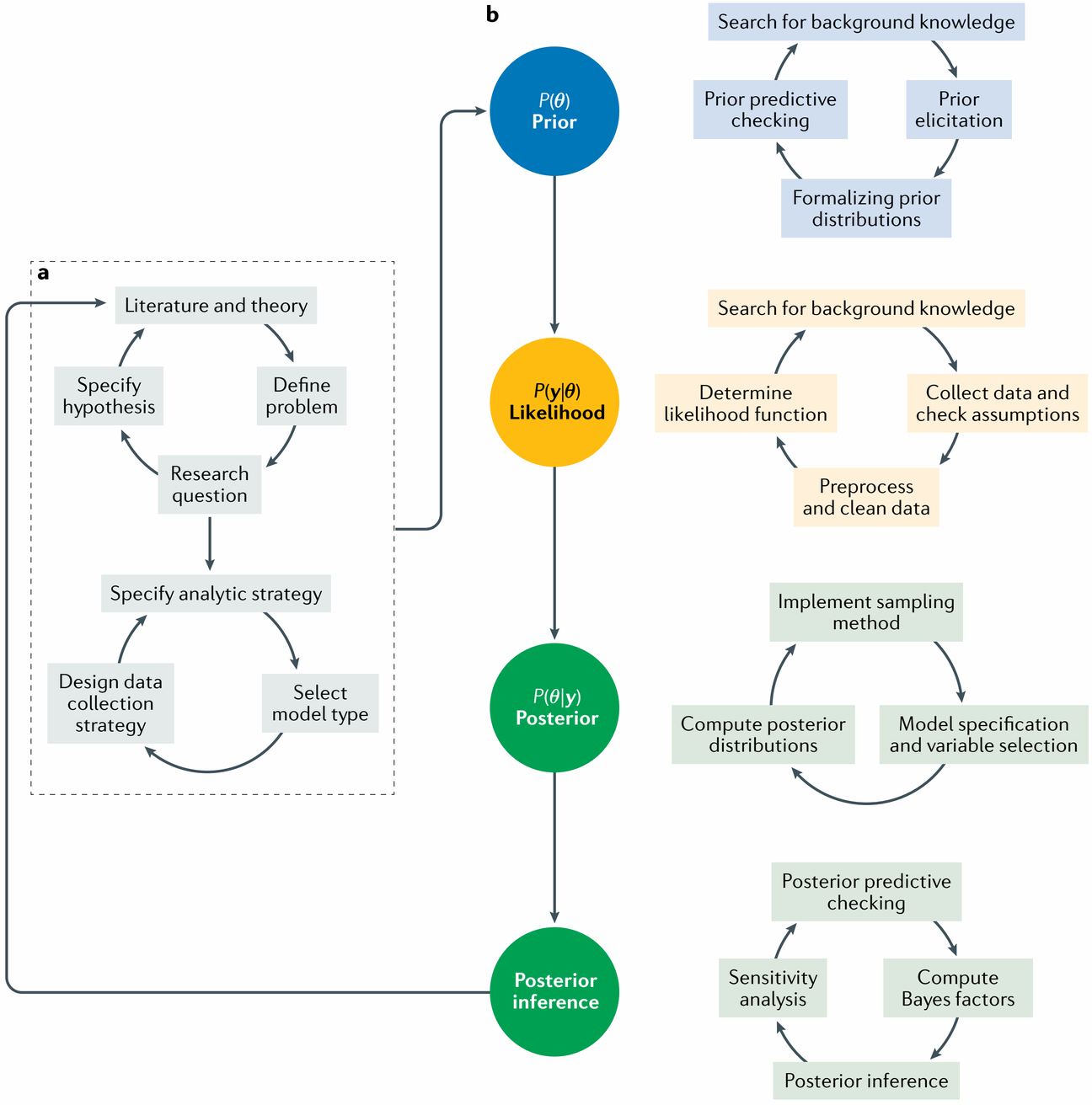


Surprise Twist!

New Jersey = HKL

Tennessee = CLV



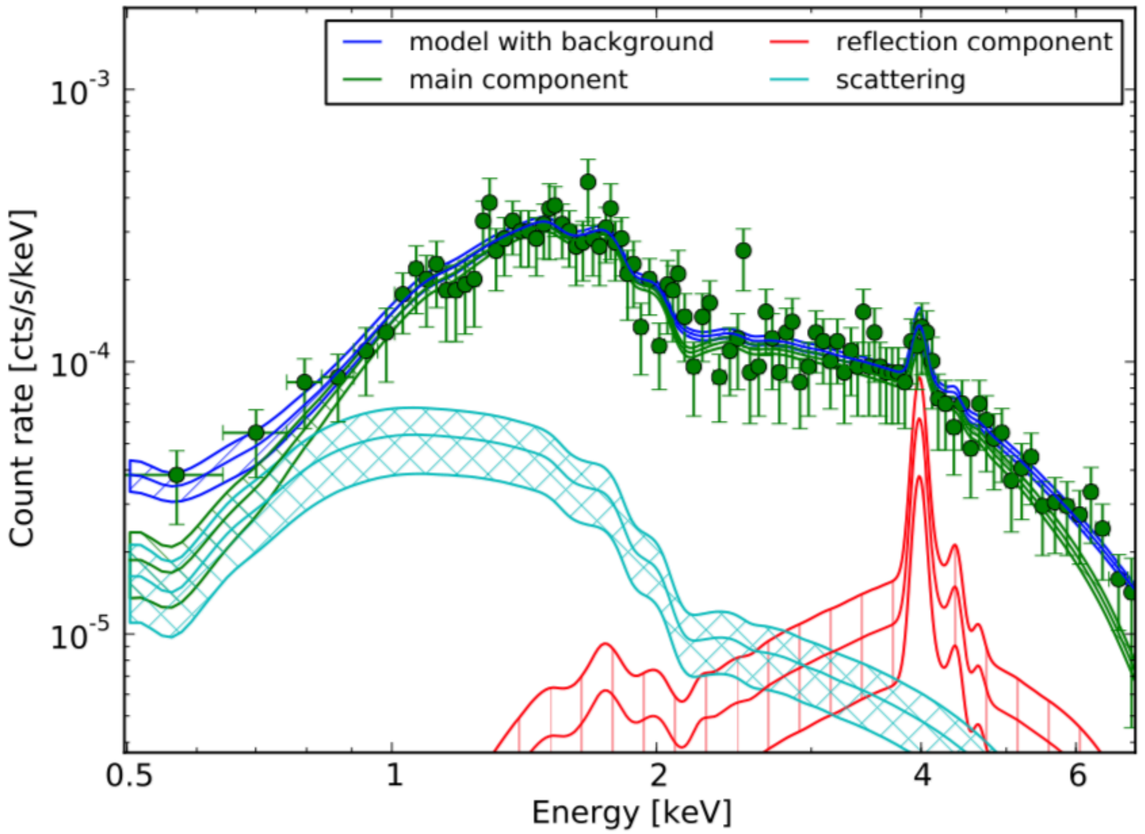


Have other people calculated the distribution of colours? What can we reason about p(🏭) ?

Assumption that all colours are mixed together at the factory and then distributed into packages: binomial likelihood

What's the probability of a certain fraction of blue smarties given our observations, p(🏭 | 🍬) ?

What about astronomy?



Data: X-ray spectrum

Model: absorption + scattering + reflection

Likelihood: Poisson

Buchner et al (2014)

What if we don’t have conjugate priors?

Markov Chain Monte Carlo*

* pick a parameter vector θ_1
* Compute the posterior $p(\theta_1|D)$
* Pick another parameter vector θ_2 according to $p(\theta_2|\ \theta_1)$
* Compute the posterior $p(\theta_2|D)$
* Compute $p_{accept} = p(\theta_2|D)/p(\theta_1|D)$
* Draw a random, uniform number r
* Keep θ_2 if $p_{accept} > r$, otherwise keep θ_2 with probability r

Hogg + Foreman-Mackey (2017): Data Analysis Recipes: Markov Chain Monte Carlo

* Metropolis-Hastings Algorithm

Ethical considerations in statistics

data and algorithms are social constructs

data and algorithms are social constructs

... are created by humans

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... are created by humans

... encode biases

data and algorithms are social constructs

... are created by humans

... encode biases

... can be (mis-)used to serve an agenda

Based on the M&Ms exercise, can you think of ways our statistical procedure could be mis-used or be made to be misleading?

Opinion: Is science really facing a reproducibility crisis, and do we need it to?

Daniele Fanelli

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Abstract

Efforts to improve the reproducibility and integrity of science are typically justified by a narrative of crisis, according to which most published results are unreliable due to growing problems with research and publication practices. This article provides an overview of recent evidence suggesting that this narrative is mistaken, and argues that a narrative of epochal changes and empowerment of scientists would be more accurate, inspiring, and compelling.

nature International weekly journal of science

NATURE | RESEARCH HIGHLIGHTS: SOCIAL SELECTION

Psychology journal bans *P* values

Test for reliability of results 'too easy to pass', say editors.

Chris Woolston

26 February 2015 | Clarified: 09 March 2015

A controversial statistical test has finally met its end, at least in one journal. Earlier this month, the editors of *Basic and Applied Social Psychology* (*BASP*) announced that the journal would no longer publish papers containing *P* values because the statistics were too often used to support lower-quality research [1].

Most mistakes made in statistical procedures are either mis-applications of methods or mis-interpretation of the results

Questions?