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An analysis of the large amplitude simple pendulum using Fourier series

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The motion of a pendulum is derived using Fourier series and perturbation analysis at levels appropriate for undergraduate physics students. Instead of using the elliptic integral of the first kind, higher order terms of the Taylor-expanded differential equation are considered, leading to increasingly accurate corrections to the period in terms of a single expansion parameter. The relation between the expansion parameter and the initial conditions is not fixed, allowing many solutions to the motion in terms of the expansion parameter but a unique solution in terms of the initial conditions.
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I. INTRODUCTION

Pendula are studied in a variety of physics classes because they provide a simple problem in which to apply Newton's laws, the Lagrangian formalism, and other techniques. For a simple pendulum of length ℓ that is affected by gravity g, any of these methods will lead to the equation of motion,

$$\ddot{\theta} + (g/\ell)\sin(\theta) = 0. \tag{1}$$

For small θ, the sine term is usually approximated as θ, the first term in its Taylor expansion, which is well known as the small-angle approximation. Apply the small-angle approximation to Eq. (1) and the general solution of Eq. (1) is

$$\theta = \theta_0 \cos\left(\sqrt{g/\ell}t + \phi\right). \tag{2}$$

This has a period

$$T_0 = 2\pi\sqrt{\ell/g}, \tag{3}$$

which is not dependent on the amplitude of the pendulum due to the small-angle approximation, which makes a pendulum useful for timekeeping. But what happens when the angle is not small? How does the period depend on the amplitude θ_0 when the motion goes beyond the small angle regime?

This query has interested many past and present physicists, as demonstrated by the extensive number of publications on the topic in this and similar journals. Daniel Bernoulli explored this in the late 1740s when he derived the exact period of motion for a pendulum with arbitrary amplitude as an integral. From this, Bernoulli found a remarkably accurate and simple approximation,[1]

$$T_{\mathrm{Ber}}(\theta_0) = T_0\left(1 + \frac{1}{16}\theta_0^2\right), \tag{4}$$

though there are other ways to achieve this result.[2,3] Since Bernoulli published Eq. (4), there have been countless improvements to the approximation over a wide range of conditions.[4] In Eq. (1), the term $\sin(\theta)$ can be approximated in the form $F(\theta_0)\theta$, where $F(\theta_0)$ is an amplitude-dependent coefficient. One can guess that $F(\theta_0) = (\sin(\theta_0)/\theta_0)^{\alpha}$, and under the condition that the main corrective term is equal to the first term in the power-series expansion of the integral, Molina found[5]

$$T_{\mathrm{M}}(\theta_0) = T_0\left(\frac{\theta_0}{\sin(\theta_0)}\right)^{3/8}. \tag{5}$$

Kidd and Fogg published a subsequent approximation using the same method of trial-and-error empirical fits.[6] This approach, geared more towards undergraduate levels of mathematical maturity, gave

$$T_{\mathrm{KF}}(\theta_0) = \frac{T_0}{\sqrt{\cos\left(\frac{\theta_0}{2}\right)}}, \tag{6}$$

where it's noted that the approximation is best for $\theta_0 \le \pi/2$. This solution was compared to Eq. (5) by Hite[7] to see which approximation was better and was reworked by Millet[8] to yield higher accuracy.

In recent years more mathematically rigorous and innovative practices have been employed to achieve approximations of the period of a simple pendulum, such as approximating the large-angle period as motion along the separatrix of the phase portrait,[9] which gives

$$T_{\mathrm{But}}(\theta_0) = \frac{2T_0}{\pi}\ln\left(\frac{8}{\pi - \theta_0}\right) \tag{7}$$

for initial angular displacements around 180°. While these approximations are useful, they do not make optimal use of the pedagogical possibilities of the problem. In this paper, we will present an approach to approximating the period of a nonlinear pendulum by using Fourier series and the differential equation, as opposed to using the elliptic integral for the exact solution. We will solve for the first 3 corrective terms to the period and the time dependence of the angular displacement. The mathematics will be within the reach of juniors and seniors, and will provide guided practice with Fourier series and perturbation expansions. Beyond simply providing an approximate solution to the nonlinear pendulum, this paper aims to equip physics professors and students alike with another avenue of exploring nonlinear oscillations.

II. EXACT SOLUTION TO THE MOTION

To derive the exact solution, we start with

$$\ddot{\theta} + \omega_0^2 \sin(\theta) = 0, \tag{8}$$

in which

$$\omega_0^2 \equiv g/\ell. \tag{9}$$

Multiply both sides by the time derivative of the angular displacement,

$$\ddot{\theta}\dot{\theta} + \omega_0^2 \sin(\theta)\dot{\theta} = 0, \tag{10}$$

and recognize the left side as a time derivative via chain rule,[10] which gives the usual result for the time-independence of the pendulum's mechanical energy

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \omega_0^2 \cos(\theta)\right) = 0. \tag{11}$$

We can now integrate this expression, and under the initial conditions $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$, we get

$$\dot{\theta} = \pm\sqrt{2}\omega_0\sqrt{\cos(\theta) - \cos(\theta_0)} = d\theta/dt. \tag{12}$$

If we integrate over a quarter period, then the bounds for θ become $0 \leq \theta \leq \theta_0$ (for the positive root),[11] which yields

$$T(\theta_0) = \frac{2\sqrt{2}}{\omega_0}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos(\theta) - \cos(\theta_0)}} \tag{13}$$

as the exact period of motion. This is known as the incomplete elliptic integral of the first kind and can be numerically evaluated by using the Arithmetic-geometric mean,[12] or through the use of standard software packages. Furthermore, a final important approximation can be obtained for the period by simply calculating the power-series expansion of the elliptic integral,[13,14]

$$T_S(\theta_0) = T_0\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \cdots\right), \tag{14}$$

of which only the first corrective term was derived by Bernoulli.

III. FOURIER ANALYSIS, PERTURBATION THEORY, AND ERROR

A. Fourier and perturbation analysis

1. Third-order correction

A pendulum is periodic, so it is natural to question why we wouldn't use Fourier analysis, the go-to tool for periodic motion, to find the angular displacement $\theta(t)$. The answer is that we need the period of the motion to be able to employ a Fourier series, which is something that we can't immediately pull from the differential equation due to its non-linearity. However, instead of just scrapping this idea, we explore the possibility that the period and the angular displacement function can be found simultaneously by using perturbation theory in conjunction with Fourier analysis.

Perturbation theory involves starting from a well-known solution and adding small corrections whose exact forms are determined by the constraints on the system. For the non-linear pendulum, we can start with the solution under the small-angle approximation,

$$\theta(t) = \theta_0 \cos(\omega_0 t), \tag{15}$$

where we've chosen the initial conditions to be $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. This solution is fine under the small-angle approximation, but what if we were to use more than just the first term in the Taylor series for the sine function? Eq. (15) would break down and the differential equation would no longer be satisfied. This is where we use Fourier analysis: We write the exact solution of motion as a Fourier cosine series,[10,15–19]

$$\theta(\theta_0, t) = \sum_{n=0}^{\infty} \alpha_{2n+1}(\theta_0) \cos((2n+1)\Omega t), \tag{16}$$

where Ω is the corrected angular frequency—which we don't know yet—and α_{2n+1} are the amplitude-dependent coefficients. While a Fourier series generally contains both even and odd harmonics of the sine and cosine functions, the initial conditions for the system reduce it to a cosine series, and the anti-symmetry of the angular displacement about the $\theta = \pi/2$ position eliminates all the even harmonics of the cosine function.

At this stage, we should determine the constraints of the system that we will use in our perturbative analysis. First, the initial conditions tell us

$$\theta_0 = \sum_{n=0}^{\infty} \alpha_{2n+1}(\theta_0). \tag{17}$$

This isn't enough to completely determine the solution, though. Luckily, the use of Fourier series gives us another constraint: The amplitude-dependent Fourier coefficients must satisfy the differential equation. We can see through an example that these two conditions are enough to completely determine the exact solution to the differential equation. Let's start by just calculating a third-order correction to the period and angular displacement. We take a truncated version of our proposed exact solution which takes the form,

$$\theta(t) = \alpha_1 \cos(\Omega t) + \alpha_3 \cos(3\Omega t), \tag{18}$$

which gives us just enough free parameters to satisfy the differential equation. If we were to just use the small-angle approximation on Eq. (8), then we'd find the trivial solution with $\alpha_3 = 0$, which doesn't help us at all; we want to find some non-zero α_3 such that the differential equation is still satisfied. To do this, we can use the first two non-zero terms in the Taylor expansion of the sine function in the differential equation,

$$\sin(\theta) \approx \theta - \theta^3/3!, \tag{19}$$

meaning the differential equation that we must satisfy is

$$\ddot{\theta} + \omega_0^2\left(\theta - \frac{\theta^3}{3!}\right) = 0. \tag{20}$$

This equation is what we need to solve to obtain the third-order correction. If we insert Eq. (18) into Eq. (20), then we get

$$\begin{aligned}0 = &\Delta^2(-\alpha_1 \cos(\Omega t) - 9\alpha_3 \cos(3\Omega t))\\ &+\alpha_1 \cos(\Omega t) + \alpha_3 \cos(3\Omega t)\\ &-(\alpha_1 \cos(\Omega t) + \alpha_3 \cos(3\Omega t))^3/3!\end{aligned} \tag{21}$$

where we've defined $\Delta \equiv \Omega/\omega_0$ for convenience. Here we can utilize the condition on the Fourier coefficients, but with a slight twist. If we knew the solution to the motion $\theta(t)$, then to obtain the Fourier coefficients we would integrate $(\Omega/\pi)\theta(t)\cos(\Omega t)$ and $(\Omega/\pi)\theta(t)\cos(3\Omega t)$ with respect to Ωt from $-\pi$ to π. However, we don't know what the solution is yet, so this method won't produce any useful results. What we can do instead is use Eq. (21) in place of $\theta(t)$. By using Eq. (21), we know that each integral must independently equal zero. We therefore not only obtain information about the Fourier coefficients through the integration process, but we also guarantee that they satisfy the differential equation. In other words, instead of finding the Fourier coefficients directly, we are determining a system of equations that they must satisfy, and then solving that system to determine what the Fourier coefficients are, which in turn can be used to solve for Δ. Multiplying Eq. (21) by $\cos(\Omega t)$ and integrating from $-\pi$ to π yields the relationship

$$0 = -8 + \alpha_1^2 + \alpha_1\alpha_3 + 2\alpha_3^2 + 8\Delta^2 \tag{22}$$

and multiplying by $\cos(3\Omega t)$ and integrating yields

$$0 = \alpha_1^3 + 6\alpha_1^2\alpha_3 + 3\alpha_3(-8 + \alpha_3^2 + 72\Delta^2). \tag{23}$$

Remember that we are trying to solve for α_1, α_3 and Δ simultaneously. We currently only have two equations, so this system is unsolvable. However, the coefficients are all functions of the amplitude θ_0, so we combine Eqs. (22) and (23) with the appropriately truncated version of Eq. (17),

$$\theta_0 = \alpha_1 + \alpha_3 \tag{24}$$

to get a system of equations to solve for the three unknowns α_1, α_3 and Δ in terms of θ_0.

These equations are non-linear and thus there exist multiple solutions, many of them complex and of no physical interest. Therefore, we must make an ansatz: We recognize that sine is an odd function, and thus, we guess that our solution to Eq. (20) is going to be odd in θ_0. The way we structure this from a perturbative view is that we guess that α_1 will be of order θ_0, α_3 of order θ_0^3, and $\Delta - 1$ of order θ_0^2 (here, to be of order θ_0^n means the lowest degree of θ_0 will be n). How then might we use this ansatz to help us solve the equations? We start by making the following observation: The solution to the exact motion is still periodic, which means the magnitude of the corrections should decrease with the number of terms, and thus it is reasonable to assume that $\alpha_1 \gg \Delta - 1 \gg \alpha_3$. As well, since α_1 is of order θ_0, at the maximum possible displacement $\theta_0 = \pi$, α_1 is not much greater than 1, and thus we should have $\alpha_3 \ll 1$ for $0 < \theta_0 \leq \pi$. With these ideas together, it becomes clear that we can solve for Δ and α_3 in terms of α_1, which will give a good approximation to the Fourier coefficients and period.

Now we can continue with solving the system of equations. In Eq. (22), we can easily solve for Δ^2,

$$\Delta^2 = 1 - \frac{1}{8}(\alpha_1^2 + \alpha_1\alpha_3 + \alpha_3^2), \tag{25}$$

and then using $\alpha_1 \gg \alpha_3$, we find

$$\Delta \approx 1 - \alpha_1^2/16 \tag{26}$$

after using a first-order Taylor expansion to simplify the square root. We can now solve for the Fourier coefficients. Insert Eq. (26) into Eq. (23) to find

$$\alpha_3 = \frac{1}{192}(-\alpha_1^3 + 426\alpha_1^2\alpha_3 - 216\alpha_1^4\alpha_3 - 3\alpha_3^3). \tag{27}$$

We can easily see that $\alpha_1^3 \gg \alpha_1^2\alpha_3 \gg \alpha_3^3$, but how does α_1^3 relate to $\alpha_1^4\alpha_3$? This essentially comes down to determining how the magnitude of $\alpha_1\alpha_3$ compares to 1. We already showed $\alpha_3 \ll 1$ and α_1 is not much greater than 1 (α_1 is at most a bit greater than π), so their product will still be much less than 1, meaning $\alpha_1^3 \gg \alpha_1^4\alpha_3$. Thus, we find

$$\alpha_3 \approx -\alpha_1^3/192. \tag{28}$$

Finally, we solve Eq. (24) for α_1, plug the result into Eq. (28), and keep only terms proportional to θ_0^3 so that to correct order,

$$\alpha_3 = -\theta_0^3/192, \tag{29}$$

which we can then use to find that

$$\alpha_1 = \theta_0 + \theta_0^3/192, \tag{30}$$

$$\Delta = 1 - \theta_0^2/16, \tag{31}$$

both to appropriate order. We have thus found the first correction to the period and the corresponding Fourier coefficients.

2. *Fifth-order correction*

Let's now take on the task of calculating the second correction to the period. In principle, we could use the same approach as in Sec. III A 1 to solve for these higher-order corrections, but we'd quickly find that the algebra gets much too complex. We'll still take the approach of solving for everything in terms of the first Fourier coefficient, but to simplify the calculation, we introduce an expansion parameter ϵ that we will use to keep track of the relative magnitudes of coefficients. To illustrate its use, let's extend the form of the angular displacement to

$$\theta = \alpha_1 \cos(\Omega t) + \alpha_3 \cos(3\Omega t) + \alpha_5 \cos(5\Omega t). \tag{32}$$

Each Fourier coefficient will become a polynomial in ϵ (a so-called perturbative expansion), but the exact forms require some critical assumptions about $\Delta - 1$ and each α_{2n+1}. We assume that $\Delta - 1$ will only contain even powers of θ_0 and that each α_{2n+1} will only contain odd powers of θ_0. The constraints on the parity of the powers of θ_0 in $\Delta - 1$ and each α_{2n+1} are not immediately clear but can be deduced by explicit calculations, which are detailed in

Appendices A and B, respectively. We, therefore, write our Fourier coefficients as

$$\alpha_1 = \epsilon A_1 + \epsilon^3 A_3 + \epsilon^5 A_5, \tag{33}$$

$$\alpha_3 = \epsilon B_1 + \epsilon^3 B_3 + \epsilon^5 B_5, \tag{34}$$

and

$$\alpha_5 = \epsilon C_1 + \epsilon^3 C_3 + \epsilon^5 C_5, \tag{35}$$

and the angular frequency becomes

$$\Delta = 1 + \epsilon^2 \delta_2 + \epsilon^4 \delta_4, \tag{36}$$

where δ_2 and δ_4 are the second- and fourth-order corrections to the angular frequency, respectively. We now aim to solve for the everything in terms of A_1, A_3, and A_5 instead of α_1.

At this point, it is necessary to address an important question that may come up. That is, it is natural to ask why we don't use θ_0 as the expansion parameter in the larger calculations. By using ϵ as the expansion parameter in the perturbative expansion, that must imply that there is some relationship between ϵ and θ_0; why is it not just $\epsilon = \theta_0$? When we move to higher-order corrections, we want to refrain from assuming $\epsilon = \theta_0$ because there may be other relationships which lead to a far simpler system of equations to solve. For now, we will make this distinction and later explore the possible relationships between ϵ and θ_0.

We now need to apply the perturbative constraints to get a system of equations for the Fourier coefficients and the corrections to the period. In the third-order correction we used a two-term Taylor expansion for the sine function; for the fifth-order correction, we then need to use a three-term approximation. The differential equation that we work with now is

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \right) = 0. \tag{37}$$

To determine the constraints on the Fourier coefficients, we substitute in Eqs. (32)–(36) into Eq. (37), multiply the result by $\cos(n\Omega t)$ and integrate with respect to Ωt from $-\pi$ to π for $n = 1$, 3, and 5. The result is three equations, each of which can be broken into three more equations which contain only terms proportional to ϵ, ϵ^3, and ϵ^5, respectively—we ignore anything of a higher degree than ϵ^5 since we only care about the next order correction. Our working equations are then the nine coefficients of these polynomials. To see an in depth derivation of the system of equations using Mathematica, see the supplementary material.[20] Also see the supplementary material for an explicit calculation of the solutions.[20] It is surprising and very interesting that these equations do not determine all the possible values of A_n, B_n and C_n. That is, we run into the problem that we have more unknowns than equations, which means the equations are under-determined. In this case, we have a bit of wiggle room since we used ϵ as the expansion parameter. This allows us to make a choice for one of the coefficients and use that to construct the remaining coefficients in a way that satisfies Eq. (37). This is what's known as a gauge choice: A choice regarding the mathematics that doesn't affect the physics. For simplicity, let's consider $A_3 = A_5 = 0$ and $A_1 = 1$. When we solve for the subsequent coefficients and correction terms, we find

$$\begin{aligned} &B_1 = C_1 = C_3 = 0, \\ &B_3 = -1/192, \\ &B_5 = -1/4096, \\ &C_5 = 1/20\,480, \\ &\delta_2 = -1/16, \\ &\delta_4 = 1/1024. \end{aligned} \tag{38}$$

From an appropriately truncated version of Eq. (17), we can solve for θ_0 as a function of ϵ,

$$\theta_0 = \epsilon - \epsilon^3/192 - \epsilon^5/5120. \tag{39}$$

This would allow us to write $\theta(t)$ and Ω as functions of ϵ, but this isn't helpful because ϵ has no physical significance. Instead, we look to get ϵ as a function of θ_0, but how can we do this? Here we introduce a useful mathematical tool called the method of iteration. Notice from Eq. (39) that we can write

$$\theta_0 = \epsilon - F(\epsilon), \tag{40}$$

where

$$F(z) = z^3/192 + z^5/5120. \tag{41}$$

We can write Eq. (40) as

$$\epsilon = \theta_0 + F(\epsilon), \tag{42}$$

and because $\epsilon \approx \theta_0$ (but not exactly equal!), this means we can approximate Eq. (42) as

$$\epsilon = \theta_0 + F(\theta_0). \tag{43}$$

We can then substitute Eq. (43) into Eq. (42) to get a better approximation

$$\epsilon = \theta_0 + F(\theta_0 + F(\theta_0)). \tag{44}$$

We can continue this improve-and-substitute procedure as many times as we need to get ϵ as a function of θ_0 to the correct order. To see how this is done using Mathematica, see the supplementary material.[20] Up to θ_0^5, we find

$$\epsilon = \theta_0 + \theta_0^3/192 + 17\theta_0^5/61\,440. \tag{45}$$

This wasn't the only choice we could make though. For example, what would happen if we instead chose $A_1 = -A_3 = 1$ and $A_5 = 0$? For this choice, the Fourier coefficients and correction terms are

$$\begin{aligned} &B_1 = C_1 = C_3 = 0, \\ &B_3 = -1/192, \\ &B_5 = 63/4096, \\ &C_5 = 1/20\,480, \\ &\delta_2 = -1/16, \\ &\delta_4 = 129/1024. \end{aligned} \tag{46}$$

We can also use the iteration method to calculate ϵ as a function of θ_0 for this gauge choice

$$\epsilon = \theta_0 + 193\theta_0^3/192 + 185\,297\theta_0^5/61\,440. \tag{47}$$

To our surprise, we find that Eqs. (45) and (47) are not the same! While this may seem alarming, it does not mean there are two separate solutions to this well-posed physical problem. When we use either of the solutions above to find the Fourier coefficients to the indicated degree of θ_0, we get

$$\alpha_1(\theta_0) = \theta_0 + \theta_0^3/192 + 17\theta_0^5/61\,440, \tag{48}$$

$$\alpha_3(\theta_0) = -\theta_0^3/192 - \theta_0^5/3072, \tag{49}$$

$$\alpha_5(\theta_0) = \theta_0^5/20\,480, \tag{50}$$

which agrees with the results of various other works.[10,17,19] The period up to θ_0^6 (see Appendix C) also comes out to be

$$T = T_0/(1 - \theta_0^2/16 + \theta_0^4/3072 - 23\theta_0^6/737\,280), \tag{51}$$

where $T_0 = 2\pi/\omega_0$, for both gauge choices. Now we can return to answer our question of why we don't use $\epsilon = \theta_0$: at higher orders, we can significantly simplify the system of equations by setting certain coefficients equal to zero and allowing ϵ to be a polynomial in θ_0. In fact, there exist infinitely many results for ϵ as a function of θ_0, which correspond to the infinitely many gauges we could choose. An interesting consequence of this is that there is a gauge where $\epsilon = \theta_0$ is valid, and the exact coefficients can be determined by fixing $\epsilon = \theta_0$, then requiring that the initial conditions be satisfied and the solution $\theta(t)$ be the same under the new coefficients.

3. Discussion of methodology

While Secs. III A 1 and 2 present a novel method to analyzing the non-linear pendulum, there must be some justification as to the pertinence of this method considering the extensive research that has already been put into such a popular and applicable system. The idea of solving the motion as a Fourier series could be used as an illustrative example in an undergraduate mechanics course, but more importantly would fit well in a course in mathematical methods at the junior and senior level. Such a course is taught at many institutions, and the mathematical techniques used in this paper are widely applicable to many fields of physics. After all, our method can be generalized for almost any non-linear oscillator, not just the non-linear pendulum. While they are multi-step, the derivations detailed in this paper involve only single-variable calculus and algebra. Thus, students and researchers can find great use of this paper via the detailed calculations, which can serve as both a supplemental aid to their own research and as content which provides practice with Fourier and perturbative analyses.

B. Approximation error

It is helpful to look at the fractional error of the approximations to fully understand the extent to which they get close to the exact solutions. Figure 1 shows the fractional error of a one-term, two-term, and three-term correction of the period compared to the exact period, along with that of the previously mentioned published approximations.

We can briefly analyze this plot in two regimes: $0 \le \theta_0 \le \pi/2$ and $\pi/2 < \theta_0 \le \pi$. In the former regime, we find the expected result that the 3-term approximation presented in this paper is comparatively the most accurate. This is likely the regime that an undergraduate lab would work in as it fits a simple string and bob setup. The latter regime is where we notice some interesting changes, however. Here, the error of the 1-term approximation decreases noticeably and becomes more accurate than the 2-term and 3-term approximations. This implies some underlying interplay between the elliptic integral of the first kind and the 1-term approximation, which will not be explored in this paper.

It is important to understand that, though these approximations are on-par with the works mentioned in the introduction, they are not the best approximations that have been published. For example, the use of the arithmetic-geometric mean by Carvalhaes and Suppes[12] produced

$$T_{\mathrm{CS}}(\theta_0) = \frac{4T_0}{\left(1 + \sqrt{\cos\left(\frac{\theta_0}{2}\right)}\right)^2}, \tag{52}$$

as a result, which turns out to be about 1 order of magnitude more accurate than our 3-term approximation. Belendez *et al.*[21] then combined the results of Eqs. (6) and (52) to determine and even better two term approximation,

$$T_{\mathrm{Bel}}(\theta_0) = \frac{64T_0}{\left(16\left(1 + \sqrt{1 + \cos\left(\frac{\theta_0}{2}\right)}\right)^2 - \left(1 - \sqrt{1 + \cos\left(\frac{\theta_0}{2}\right)}\right)^4\right)}, \tag{53}$$

which is two orders of magnitude more accurate. However in the context of, say, undergraduate labs, the measurement error is usually about 1 part in 1000, so it suffices to just use the 1-term approximation for simplicity. In the context of theoretical study, it is arguable that those who are less familiar with advanced mathematics would find the methods outlined in this paper more approachable, especially in the case of self-study.

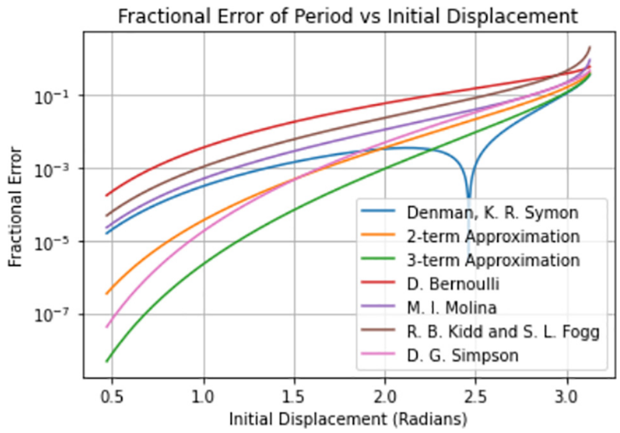

Fig. 1. The fractional error of one-, two-, and three-term corrections to the period as a function of the initial displacement compared to previously mentioned works, plotted on a log scale. Any multi-term series is truncated to 3 terms at most unless otherwise specified.

IV. CONCLUSION

We presented a novel method to derive increasingly accurate expressions for the period of a pendulum using a Fourier series, then perturbation analysis. Many published approaches use the elliptic integral, while our approach works directly with the differential equation which can seem more approachable to students who have yet to engage in the study of more advanced integral calculus. The application of epsilon tags (expansion parameters), Fourier series, and perturbations are useful for many physics and engineering problems where exact solutions are unwieldy or have not been found. This paper is an example of all three and can thus serve as a useful teaching aid. By going through each of the problem-solving steps explicitly, we provide general physics students with an effective outlet for self-study, with perhaps more benefit to those studying mathematical physics. The methods presented are especially useful in that they can be applied to a multitude of nonlinear oscillators, not just the nonlinear pendulum.

There are many avenues to expand the modeled system from a simple pendulum. For example, modeling a pendulum with damping would add a $\dot{\theta}$ term

$$\ddot{\theta} + \gamma\dot{\theta}^n + \omega_0^2 \sin(\theta) = 0, \tag{54}$$

which has previously been explored using different methods.[22] Two-coordinate systems such as a pendulum with a spring replacing the massless string, or even a double pendulum under small displacements could be studied. As well, the number of terms to keep track of grows very fast with the number of corrections, so future work should include finding a more efficient method for computing corrections.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts of interest to disclose.

APPENDIX A: PERTURBATION SERIES

The perturbation series introduced in Sec. III A will only include even powers of ϵ. In this appendix, we will show why that must be true. For the form of $\theta(t)$, we can use

$$\theta(t) = (\epsilon A_1 + \epsilon^3 A_3)\cos(\Omega t) + (\epsilon B_1 + \epsilon^3 B_3)\cos(3\Omega t), \tag{A1}$$

and for the angular frequency,

$$\Omega = \omega_0(1 + \epsilon\delta_1 + \epsilon^2\delta_2). \tag{A2}$$

In general, to determine the value of δ_n, one must eliminate all terms not proportional to ϵ^{n+1}, multiply the result by $\cos(\Omega t)$ and integrate with respect to Ωt from $-\pi$ to π; this produces an equation that we can use to solve for δ_n. We specifically need ϵ^{n+1} because any larger degree of ϵ will include other contributions from corrective terms which are yet unknown, making the equation unsolvable. We can start, for example, by solving for δ_1. Insert Eqs. (A1) and (A2) into Eq. (8), eliminate all terms not proportional to ϵ^2, multiply this result by $\cos(\Omega t)$ and integrate Ωt from $-\pi$ to π to find

$$0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\Omega t) \times (-2A_1\delta_1\cos(\Omega t))d(\Omega t). \tag{A3}$$

Using the identity $\cos(x)^2 = (1/2)(1 + \cos(2x))$, we see that the integral does not immediately evaluate to zero. However, since we know it must be equal to zero by Eq. (8), and given that A_1 is not zero, it must be that $\delta_1 = 0$. Generally, when finding δ_n, the collection of terms proportional to ϵ^{n+1} yields $-2A_1\delta_n\cos(\Omega t)$ when n is odd, which will always give $\delta_n = 0$. This is not the case for when n is even because there are terms proportional to ϵ^{n+1} that don't include any corrective terms. That is, because $n+1$ will be odd when n is even, terms proportional to ϵ^{n+1} which don't include any δ_n exist as byproducts of the linearization of $\sin(\theta)$ into a truncated odd-degree power series.

APPENDIX B: FOURIER COEFFICIENTS

Similar to the perturbation series, it is also introduced in Sec. III A that the Fourier coefficients must have odd powers of ϵ, and here we will show this. First, we will define a function $\gamma_n(\epsilon)$ for mathematical convenience that will leave only terms proportional to ϵ^n when evaluating $\gamma_n(0)$,

$$\gamma_n(\epsilon) = \frac{1}{n!}\frac{d^n}{d\epsilon^n}(\ddot{\theta}/\omega_0^2 + \theta - \theta^3/3! + \theta^5/5!). \tag{B1}$$

Let's write the first coefficient as

$$\alpha_1(\theta_0) = \epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3 + \epsilon^4 A_4 + \epsilon^5 A_5 \tag{B2}$$

and do it the same way for $\alpha_3(\theta_0)$ and $\alpha_5(\theta_0)$ and their respective terms. Then, we need to look at any odd harmonic of cosine except for $\cos(\Omega t)$. From the previous work in

Sec. III A, $B_1 = C_1 = C_3 = 0$ still holds true. To determine the coefficients B_2 and C_2, for example, we need to multiply the harmonics of cosine with $\gamma_2(0) = (1/2)(-16B_2\cos(3\Omega t) -48C_2\cos(5\Omega t))$. To get B_2, we evaluate

$$0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(3\Omega t)\gamma_2(0)d(\Omega t). \tag{B3}$$

Integrating the product of different harmonics over a full period will always yield zero, but because $\cos(3x)^2 = (1/2)(1+\cos(6x))$, we will end up with $0 = -8B_2$ which clearly means $B_2 = 0$. If we multiplied instead by $\cos(5\Omega t)$, we can see we should also find $C_2 = 0$. In general, for any Fourier coefficient defined by $\alpha_n(\theta_0) \equiv \sum_{m=1}^{k}\epsilon^m\beta_m^{(n)}$, where n is odd, if m is even then the integral of $\cos(n\Omega t)\gamma_m(0)$ gives the equation

$$0 = (1-n^2)\beta_m^{(n)}, \tag{B4}$$

which is only satisfied when $\beta_m^{(n)} = 0$. To solve for some $\beta_m^{(n)}$, we aim to look for equations that don't contain any corrective terms: We must collect every term with the same degree of ϵ that the desired coefficient is multiplied to, which also brings along the harmonic it is multiplied to. By multiplying by that specific harmonic under the integral, we can further determine only the terms that are relevant to the coefficient, and in the case of an even power of ϵ, it can only be achieved through the θ and $\ddot{\theta}$ terms in Eq. (8); any other term produces an odd power of ϵ. When m is odd, there become other ways to achieve the desired power of ϵ which will influence the result of the integral.

APPENDIX C: CALCULATION OF Δ_6

At the end of Sec. III A we introduced the corrected period up to θ_0^6, but we didn't derive the third corrective term; we will do that here. We are going to extend each Fourier coefficient to include a term proportional to ϵ^7, and we will slightly modify the gauge in that $A_1 = 1$ but $A_3 = A_5 = A_7 = 0$. We must rewrite Eq. (8) as

$$0 = \Delta^2\ddot{\theta} + \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! \tag{C1}$$

After appropriately extending the representation of $\theta(t)$ to include a $\cos(7\Omega t)$ term, we perform the same integration used to Sec. III A to find

$$\begin{aligned}
&B_1 = C_1 = C_3 = D_1 = D_3 = D_5 = 0,\\
&B_3 = -1/192,\\
&B_5 = -1/4096,\\
&B_7 = -3/262\,144,\\
&C_5 = 1/20\,480,\\
&C_7 = 1/262\,144,\\
&D_7 = -1/1\,835\,008,\\
&\delta_2 = -1/16,\\
&\delta_4 = 1/1024,\\
&t\delta_6 = -1/65\,536.
\end{aligned} \tag{C2}$$

Using the above values, the relationship between ϵ and θ_0 up to θ_0^7 is

$$\epsilon = \theta_0 + \theta_0^3/192 + 17\theta_0^5/61\,440 + 1487\theta_0^7/82\,575\,360. \tag{C3}$$

Finally, we plug Eqs. (C2) and (C3) into the appropriate definition of the angular frequency

$$\Omega = \omega_0(1 + \epsilon^2\delta_2 + \epsilon^4\delta_4 + \epsilon^6\delta_6) \tag{C4}$$

and find that, up to θ_0^6, the period is

$$T = T_0/(1 - \theta_0^2/16 + \theta_0^4/3072 - 23\theta_0^6/737\,280) \tag{C5}$$

as previously stated.

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