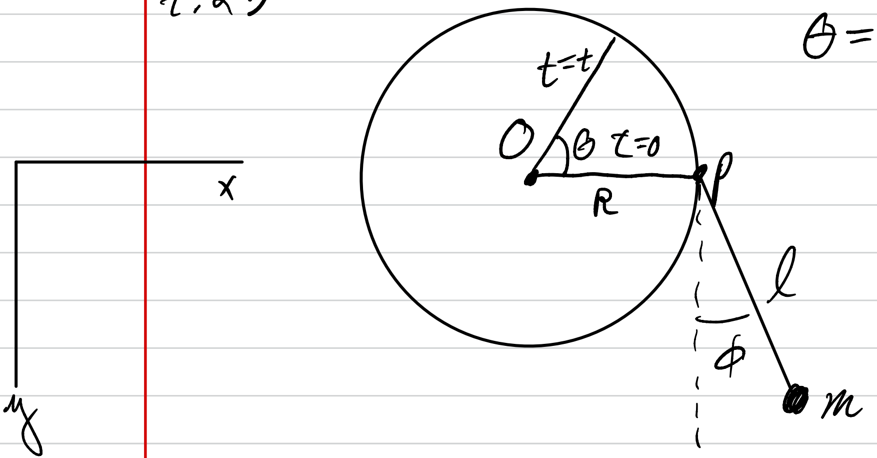
P441 HW8

7.29)



$\theta = \omega t$

$$x = R\cos\theta + l\sin\phi = R\cos\omega t + l\sin\phi$$
$$y = -R\sin\theta + l\cos\phi = -R\sin\omega t + l\cos\phi$$

$$\dot{x} = -R\omega\sin\omega t + l\dot{\phi}\cos\phi$$
$$\dot{y} = -R\omega\cos\omega t - l\dot{\phi}\sin\phi$$

$$v^2 = R^2\omega^2\sin^2\omega t - 2Rl\omega\dot{\phi}\sin\omega t\cos\phi + l^2\dot{\phi}^2\cos^2\phi + R^2\omega^2\cos^2\omega t + 2R\omega l\dot{\phi}\cos\omega t\sin\phi + l^2\dot{\phi}^2\sin^2\phi$$

$$= R^2\omega^2 + l^2\dot{\phi}^2 + 2Rl\omega\dot{\phi}(\cos\omega t\sin\phi - \sin\omega t\cos\phi)$$

$$T = \frac{1}{2}m\left(R^2\omega^2 + l^2\dot{\phi}^2 + 2Rl\omega\dot{\phi}(\cos\omega t\sin\phi - \sin\omega t\cos\phi)\right)$$

$$U = -mg(y - \underbrace{l}_{\text{so that } U=0 \text{ at } t=0}) = -mgy + \text{const.} = -mg(-R\sin\omega t + l\cos\phi)$$

$$\mathcal{L} = T - U = \frac{1}{2}m\left(R^2\omega^2 + l^2\dot{\phi}^2 + 2Rl\omega\dot{\phi}(\cos\omega t\sin\phi - \sin\omega t\cos\phi)\right) - mgR\sin\omega t + mgl\cos\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = mRl\omega\dot{\phi}(\cos\omega t\cos\phi + \sin\omega t\sin\phi) + mgl\sin\phi$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = ml^2\dot{\phi} + mRl\omega(\cos\omega t\sin\phi - \sin\omega t\cos\phi)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = ml^2\ddot{\phi} + mRl\omega\left(\dot{\phi}\cos\omega t\cos\phi - \omega\sin\omega t\sin\phi - (\dot{\phi}\sin\omega t\sin\phi + \omega\cos\omega t\cos\phi)\right)$$

$$= ml^2\ddot{\phi} + mRl\omega\left(\dot{\phi}\left(\cancel{\cos\omega t\cos\phi + \sin\omega t\sin\phi}\right) - \omega\left(\sin\omega t\sin\phi + \cos\omega t\cos\phi\right)\right)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} = mRl\omega\dot{\phi}\left(\cancel{\cos\omega t\cos\phi + \sin\omega t\sin\phi}\right) - mgl\sin\phi$$

$$\Rightarrow ml^2\ddot{\phi} = -mgl\sin\phi + mRl\omega^2\left(\sin\omega t\sin\phi + \cos\omega t\cos\phi\right)$$

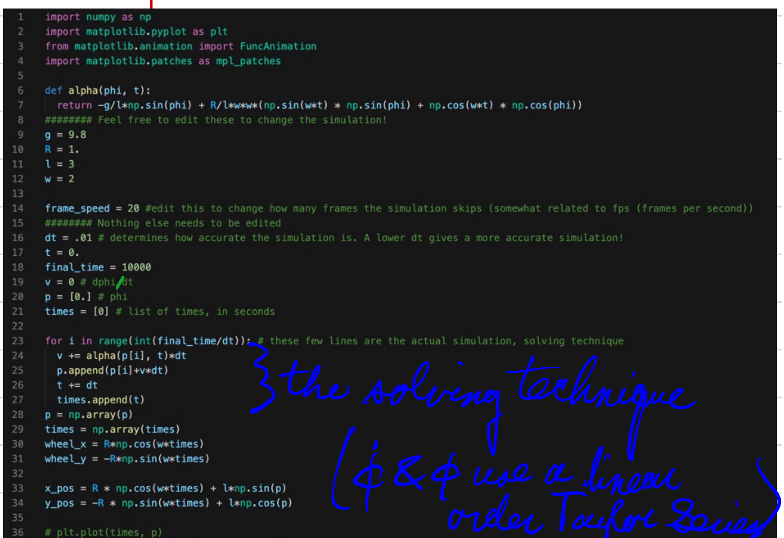
$$\boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi + \frac{R}{l}\omega^2\left(\sin(\omega t)\sin(\phi) + \cos(\omega t)\cos(\phi)\right)}$$

For $\omega = 0$: $\ddot{\phi} = -\frac{g}{l}\sin\phi \approx -\frac{g}{l}\phi$

$\phi \approx A\cos\left(t\sqrt{g/l} - \delta\right)$ for small ϕ & $\omega = 0$

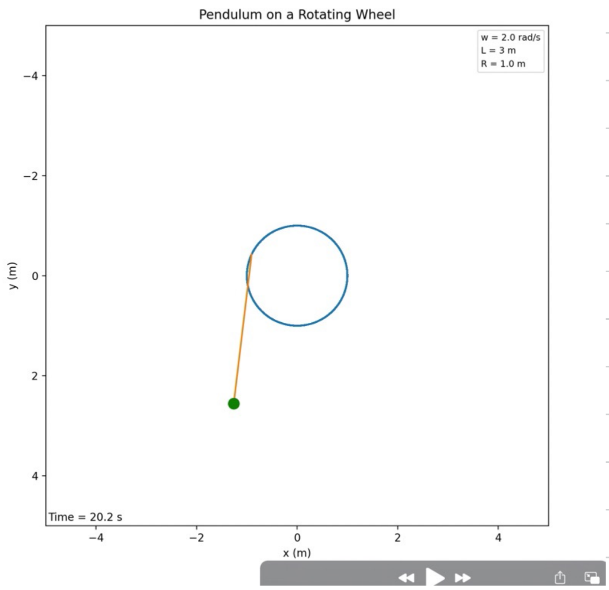
I made a simulation to numerically solve for $\phi(t)$ (therefore solving for $x(t)$ & $y(t)$) and animated.

Here are some snapshots but I submitted the code as well, with some example videos.



```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
import matplotlib.patches as mpl_patches

def alpha(phi, t):
    return -g/l*np.sin(phi) + R/l*w*w*(np.sin(w*t) * np.sin(phi) + np.cos(w*t) * np.cos(phi))
######## Feel free to edit these to change the simulation!
g = 9.8
R = 1.
l = 3
w = 2

frame_speed = 20 #edit this to change how many frames the simulation skips (somewhat related to fps (frames per second))
######## Nothing else needs to be edited
dt = .01 # determines how accurate the simulation is. A lower dt gives a more accurate simulation!
t = 0.
final_time = 10000
v = 0 # dphi/dt
p = [0.] # phi
times = [0] # list of times, in seconds

for i in range(int(final_time/dt)): # these few lines are the actual simulation, solving technique
    v += alpha(p[i], t)*dt
    p.append(p[i]+v*dt)
    t += dt
    times.append(t)
p = np.array(p)
times = np.array(times)
wheel_x = R*np.cos(w*times)
wheel_y = -R*np.sin(w*times)

x_pos = R * np.cos(w*times) + l*np.sin(p)
y_pos = -R * np.sin(w*times) + l*np.cos(p)

# plt.plot(times, p)
```

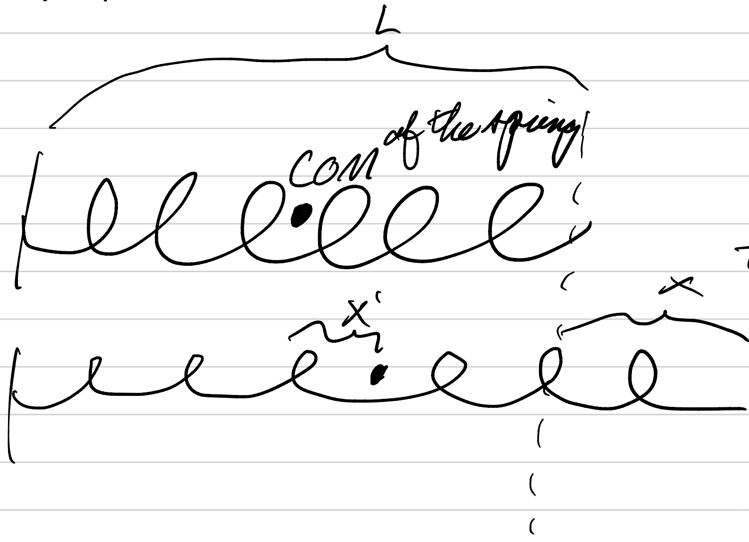
} the solving technique ($\dot{\phi}$ & ϕ use a linear order Taylor Series)



Feel free to go into the code, edit the parameters, and run the simulation to see what happens.

7.34)

7.34) a)



x' is the position of a particular dm with the origin not mattering because constants don't change the Langrangian.

Each element of the spring is moving at a different speed:

$dT_{spring} = \frac{1}{2}\dot{x}'^2 dM$ $\qquad \lambda = M/L$

$dM = \lambda\, dx = \frac{M}{L} dx$

$$T_{spring} = \frac{1}{2}\int \dot{x}'^2 dM = \frac{1}{2}\int_0^L \dot{x}'^2 \frac{M}{L} dx' = \frac{M}{2L}\int_0^L \frac{x'^2}{L^2}\dot{x}^2 dx = \frac{M\dot{x}^2}{2L^3}\left.\frac{x'^3}{3}\right|_0^L$$

$$= \frac{M}{6}\dot{x}^2$$

$\dot{x}' = \frac{x}{L}\dot{x}$

$\dot{x}$ → velocity at the end (ind. of x')

$$\boxed{T_{spring} = \frac{1}{6} M\dot{x}^2}$$

$$\mathcal{L} = \frac{1}{6} M\dot{x}^2 + \frac{1}{2} m\dot{x}^2 - \frac{1}{2}kx^2 = \dot{x}^2\left(\frac{M}{6} + \frac{m}{2}\right) - \frac{1}{2}kx^2$$

b)

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = -kx \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x}(M/3 + m)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \ddot{x}(M/3 + m)$$

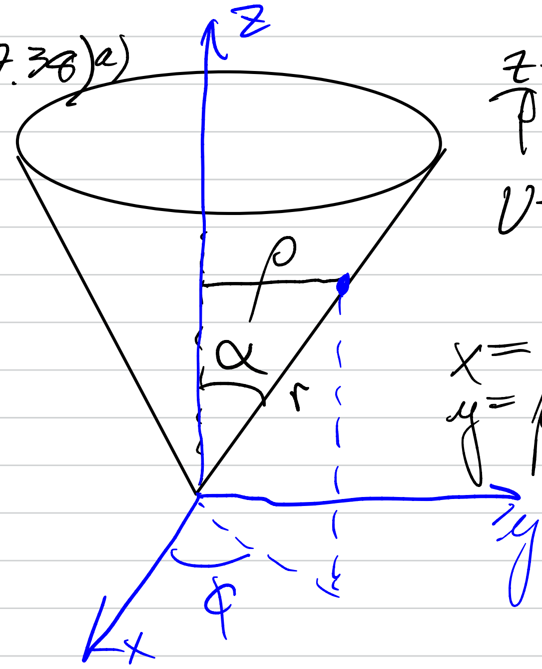
$$\ddot{x}(M/3 + m) + kx = 0$$

$$\ddot{x} + \frac{k}{M/3 + m}x = 0 \Rightarrow x = A\cos\left(t\underline{\sqrt{\frac{k}{M/3+m}}} - \delta\right)$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{M/3+m}}}$$

Thus, $M/3$ is the effective mass of the spring.

3) 7.36) a)



$$\frac{z}{\rho} = \cot\alpha \qquad z = r\cos\alpha$$
$$\rho = r\sin\alpha$$

$$U = mgz = mgr\cos\alpha$$

$$x = \rho\cos\phi = r\sin\alpha\cos\phi$$
$$y = \rho\sin\phi = r\cos\alpha\sin\phi$$

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \dot{\vec{r}} = \frac{d}{dt}\vec{r} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r(\hat{\alpha}\dot{\alpha} + \hat{\phi}\sin\alpha\,\dot{\phi})$$

$$= \dot{r}\hat{r} + r\sin\alpha\,\dot{\phi}\,\hat{\phi} \qquad (\dot{\alpha} = \dot{\theta} = 0)$$

$$\vec{v}^2 = \dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2$$

$$\boxed{\mathcal{L} = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2\right) - mgr\cos\alpha}$$

b)

$$l_z = m\rho^2\omega = m r^2\sin^2\alpha\,\omega = m r^2\sin^2\alpha\,\dot{\phi}$$

$$r: \frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = 0$$

$$\frac{\partial\mathcal{L}}{\partial r} = mr\sin^2\alpha\,\dot{\phi}^2 - mg\cos\alpha \qquad \frac{\partial\mathcal{L}}{\partial\dot{r}} = m\dot{r}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = m\ddot{r}$$

$$\Rightarrow \boxed{\ddot{r} - r\sin^2\alpha\,\dot{\phi}^2 + g\cos\alpha = 0}$$

$$\phi: \frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \text{const.}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m r^2\sin^2\alpha\,\dot{\phi} = \text{const.}$$

$$m r^2\sin^2\alpha\,\dot{\phi} = l_z = \text{const.}$$

$\Rightarrow l_z$, angular momentum, is conserved!

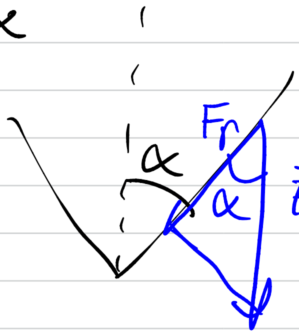
$$\dot\phi = \frac{l_z}{mr^2\sin^2\alpha} \Rightarrow \dot\phi^2 = \frac{l_z^2}{m^2r^4\sin^4\alpha}$$

$$\ddot r - r\sin^2\alpha\,\dot\phi^2 + g\cos\alpha = \ddot r - \left(\frac{l_z}{m\sin\alpha}\right)^2\frac{1}{r^3} + g\cos\alpha = 0$$

$$\boxed{\ddot r - \left(\frac{l_z}{m\sin\alpha}\right)^2\frac{1}{r^3} + g\cos\alpha = 0}$$

if $l_z = 0$: $\ddot r = -g\cos\alpha$

This makes sense because this is the r component of F_g/m



$m\ddot r = F_r = \vec F_g\cos\alpha$

$= -mg\cos\alpha$

Horizontal circular path when $\ddot r = 0$:

$$\left(\frac{l_z}{m\sin\alpha}\right)^2\frac{1}{r_0^3} = g\cos\alpha$$

$$\boxed{r_0 = \left[\left(\frac{l_z}{m\sin\alpha}\right)^2\frac{1}{g\cos\alpha}\right]^{1/3}}$$

c) $r(t) = r_0 + \epsilon(t)$ (let $\epsilon = \epsilon(t)$)

$\ddot r = \ddot\epsilon$

$$\ddot r - \left(\frac{l_z}{m\sin\alpha}\right)^2\frac{1}{r^3} + g\cos\alpha = 0$$

(*) $$\ddot\epsilon - \left(\frac{l_z}{m\sin\alpha}\right)^2\frac{1}{(r_0+\epsilon)^3} + g\cos\alpha = 0$$

$$\frac{1}{(r_0+\epsilon)^3} = \frac{1}{r_0^3+3r_0^2\epsilon+3r_0\epsilon^2+\epsilon^3}$$

$$\approx \frac{1}{r_0^3+3r_0^2\,\epsilon} = \frac{1}{r_0^3}\,\frac{1}{1+\frac{3\epsilon}{r_0^2}} \approx \frac{1}{r_0^3}\left(1-3\epsilon/r_0^2\right)$$

$$\left(\text{note: } (1+a)^b \approx 1+ab\right)$$

$$(*) \approx \ddot{\epsilon} - \left(\frac{l_z}{m\sin\alpha}\right)^2 \frac{1}{r_0^3}\left(1-3\epsilon/r_0^2\right) + g\cos\alpha$$

$$= \ddot{\epsilon} - \cancel{\left(\frac{l_z}{m\sin\alpha}\right)^2 \frac{1}{r_0^3}} + 3\epsilon\left(\frac{l_z}{m\sin\alpha}\right)^2 \frac{1}{r_0^2} + \cancel{g\cos\alpha}$$

cancel by def. of r_0

$$= \ddot{\epsilon} + 3\epsilon\left(\frac{l_z}{m\sin\alpha}\right)^2 = 0$$

$$\Rightarrow \epsilon = A\cos\left(\frac{l_z}{m\sin\alpha}\sqrt{3}\,t\right) + B\sin\left(\frac{l_z}{m\sin\alpha}\sqrt{3}\,t\right)$$

$$\Rightarrow r(t) = r_0 + A\cos\left(\frac{l_z}{m\sin\alpha}\sqrt{3}\,t\right) + B\sin\left(\frac{l_z}{m\sin\alpha}\sqrt{3}\,t\right) \quad (**)$$

Thus, this circular path is stable because it oscillates with a period of $\frac{2\pi}{\sqrt{3}}\frac{m\sin\alpha}{l_z}$ around r_0

It's stable because $(**)$ is the solution but qualitatively because for $\pm\epsilon$, $\ddot{\epsilon}$ is $\mp$, meaning there is a restoring effect.