2024 Student Problems in Astro-plasma

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Introduction

Astrophysical plasma is a rich and rewarding subject, deserving of more attention than any single lifetime can provide. The reason is twofold: (1) most of the baryonic matter in the Universe is in a plasma state, from the surrounding magnetosheath around the Earth to the accreting gas around black holes at the centers of galaxies—these are all plasmas of different flavors and varieties; and (2) some aspects of plasma (for example, the nature of collisional plasma turbulence) may be relatively universal across vastly different length scales and astrophysical phenomena. This makes it a truly unique part of astrophysics, allowing us to probe very fundamental problems that apply to many different parts of the Universe. The language of astrophysical plasma theory, simulations, and laboratory experiments is the language of vector and tensor calculus. In the following two test problems, you will have the opportunity to showcase some of the skills you already possess in vector calculus manipulations and numerical derivatives. These skills are very important for working with us and making progress in an astro-plasma related project.

Problem 1: from momentum to the vorticity

The fluid vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, where $\boldsymbol{u}$ is the fluid velocity, is an important quantity for the generation of turbulent modes, transport and dissipation of angular momentum, and the mixing of metals and dust in our Galaxy. Consider the ideal momentum equation for an unmagnetized fluid (the Euler equations) written in conservative form,

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} - P\mathbb{I}) = \nabla \phi, \tag{1}$$

where $\rho \boldsymbol{u} \otimes \boldsymbol{u} = \rho u_i u_j$ is the Reynolds stress, and $P\mathbb{I} = P\delta_{ij}$ is the isotropic pressure tensor, and $\nabla\phi$ is a conservative external force, and the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{2}$$

where ρ is the plasma mass density. Using both of these equations, show that,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \otimes \boldsymbol{\omega} = -\boldsymbol{\omega}(\nabla \cdot \mathbf{u}) + \boldsymbol{\omega} \cdot \nabla \otimes \mathbf{u} + \frac{1}{\rho^2}\nabla P \times \nabla \rho, \tag{3}$$

where $\nabla \otimes \boldsymbol{u}$ is the velocity gradient tensor, $\partial_i u_j$. *Hint:* isolate the $\partial_t u_i$ derivative and use the continuity equation to get rid of the $\partial_t \rho$ term, then take the curl of the $\partial_t u_i$ equation.

Problem 2: computing the vorticity

Taking derivatives of quantities is at the heart of all of physics. Using the provided three-dimensional, periodic velocity field (`velocity.npy`), compute the curl $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, slice it in two-dimensions along the last axis, and visualise it. It should reveal a Kelvin Helmholtz instability roll-up. You can use your favourite coding language and visualisation software. For this, we recommend `python` since it will make reading the data quite easy (see `read_vel.py` here, for a `python` example to read the vector field in and the details of each of the coordinate axes in the data).