

Aspects of magnetic growth & the turbulent dynamo

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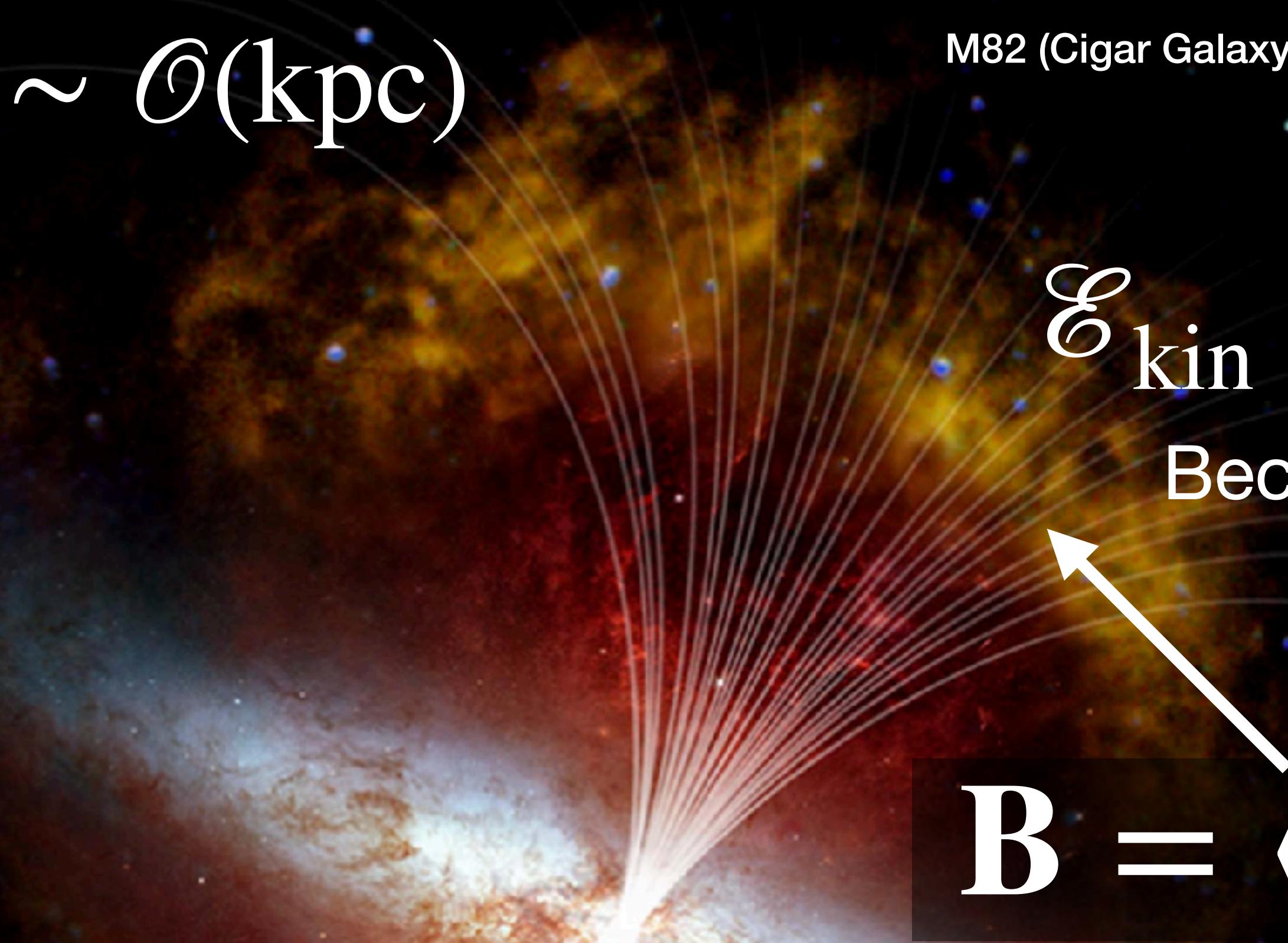
Canadian Institute for Theoretical Astrophysics

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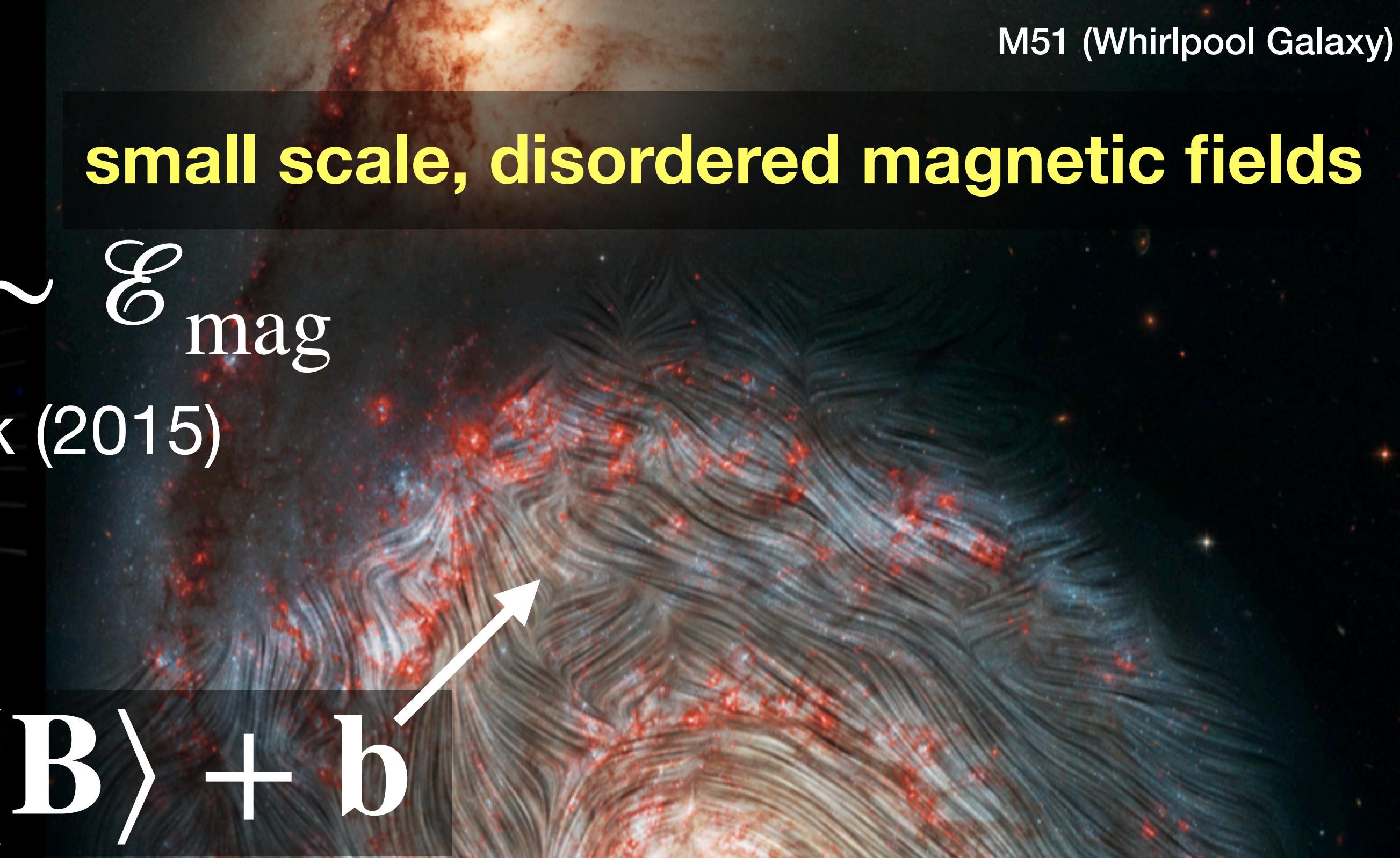
$L \sim \mathcal{O}(\text{kpc})$

M82 (Cigar Galaxy)

M51 (Whirlpool Galaxy)



large scale, ordered magnetic fields



small scale, disordered magnetic fields

$$\mathcal{E}_{\text{kin}} \sim \mathcal{E}_{\text{mag}}$$

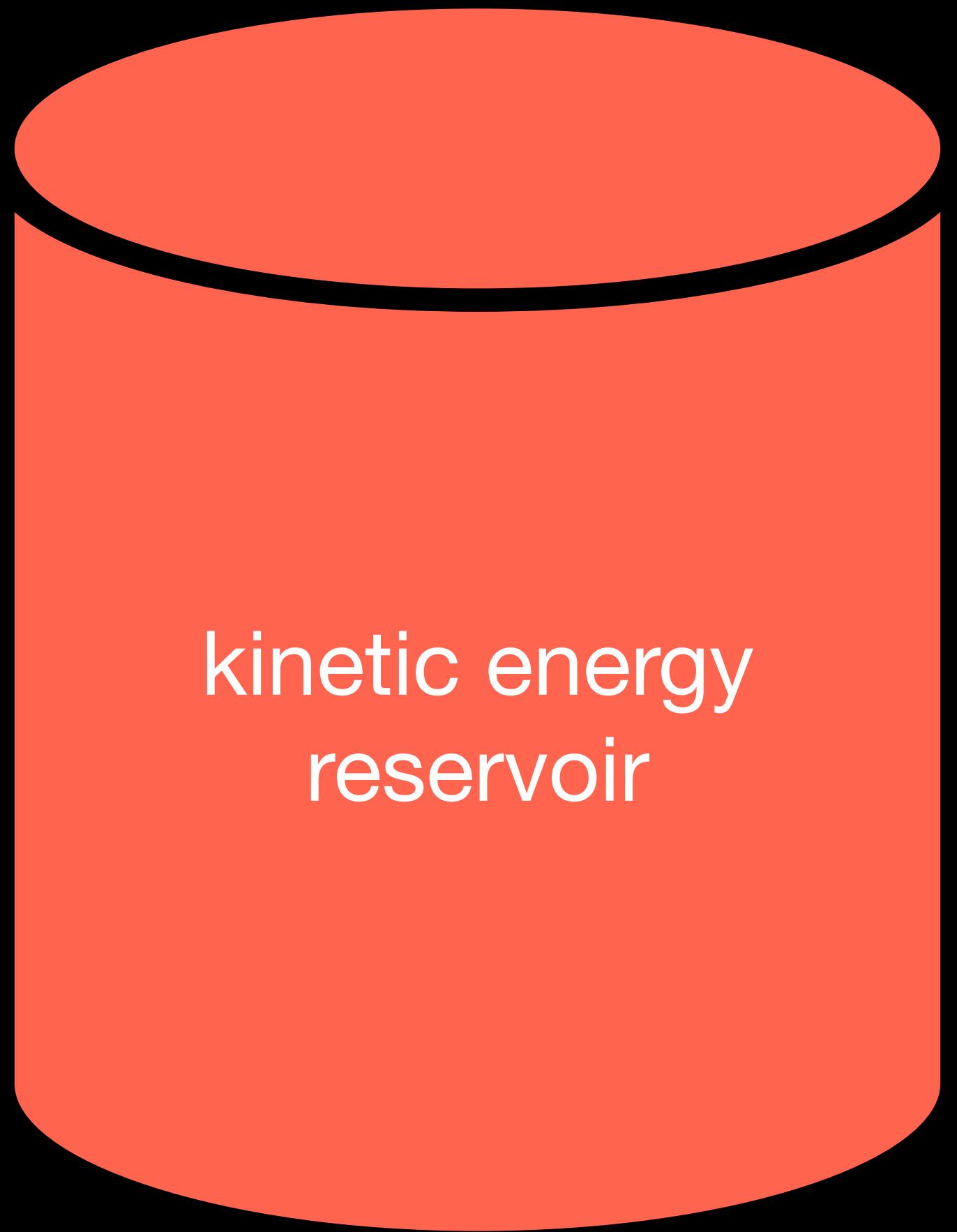
Beck (2015)

$$\mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{b}$$

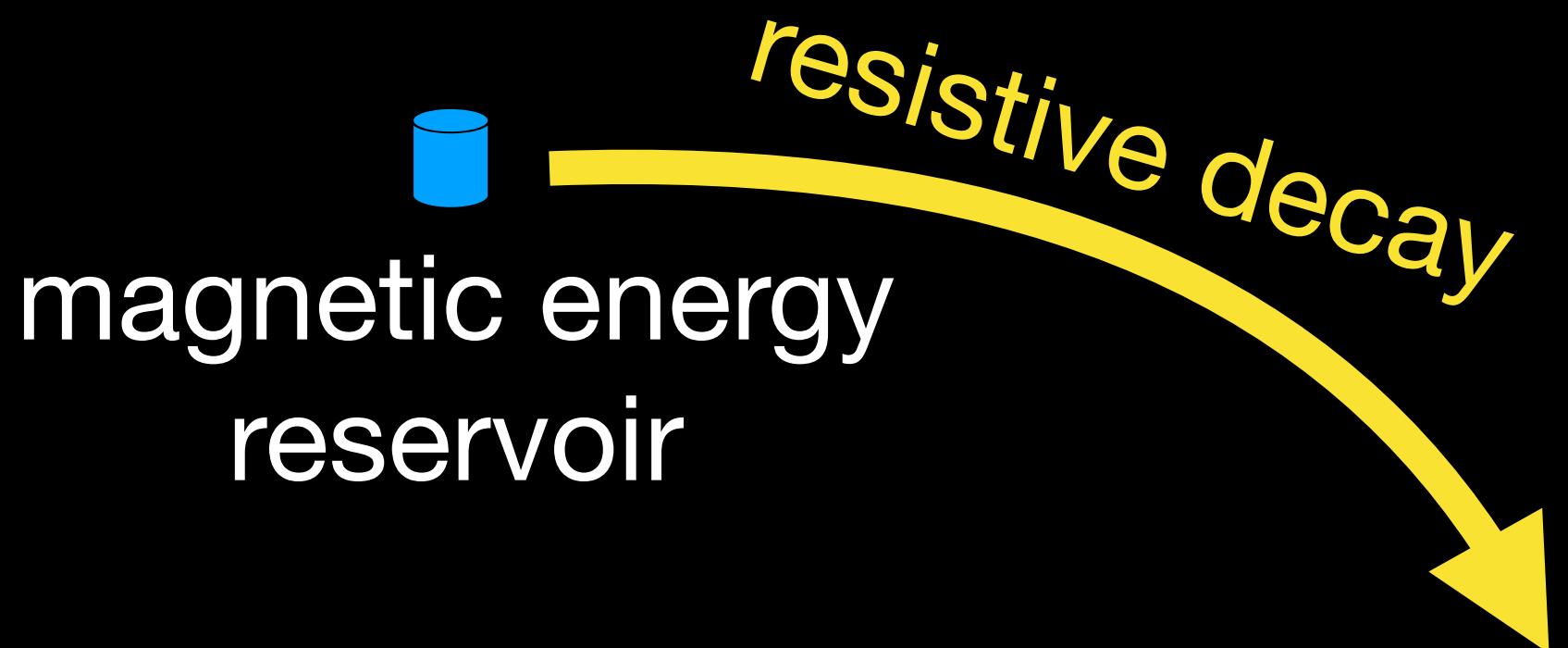
large scale, ordered magnetic fields

What is a magnetic dynamo?

Starting with a weak seed magnetic field

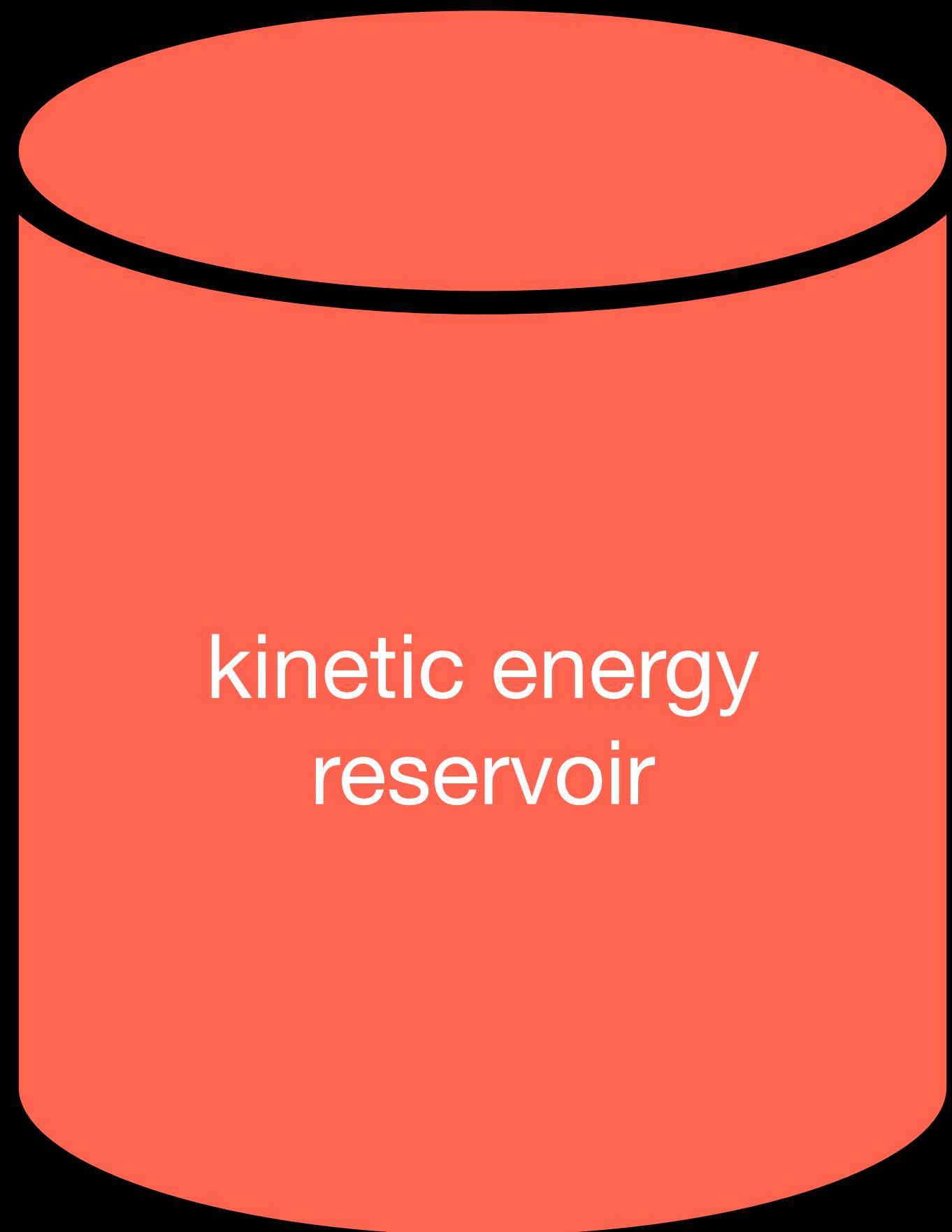


kinetic energy
reservoir



What is a magnetic dynamo?

Growth



kinetic energy
reservoir

energy flux in



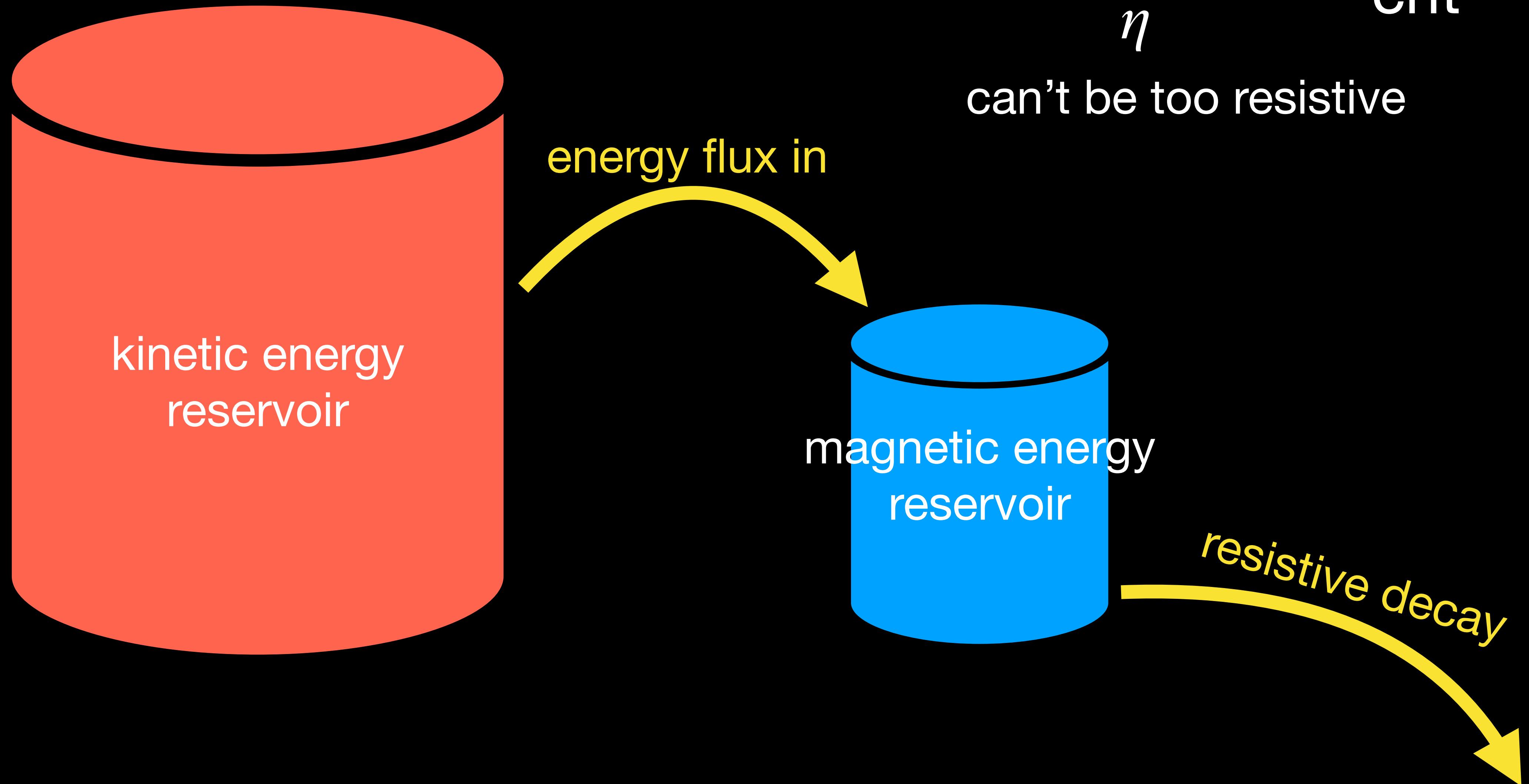
resistive decay
magnetic energy
reservoir

$$Rm \sim \frac{U_0 L}{\eta} > Rm_{crit}$$

can't be too resistive

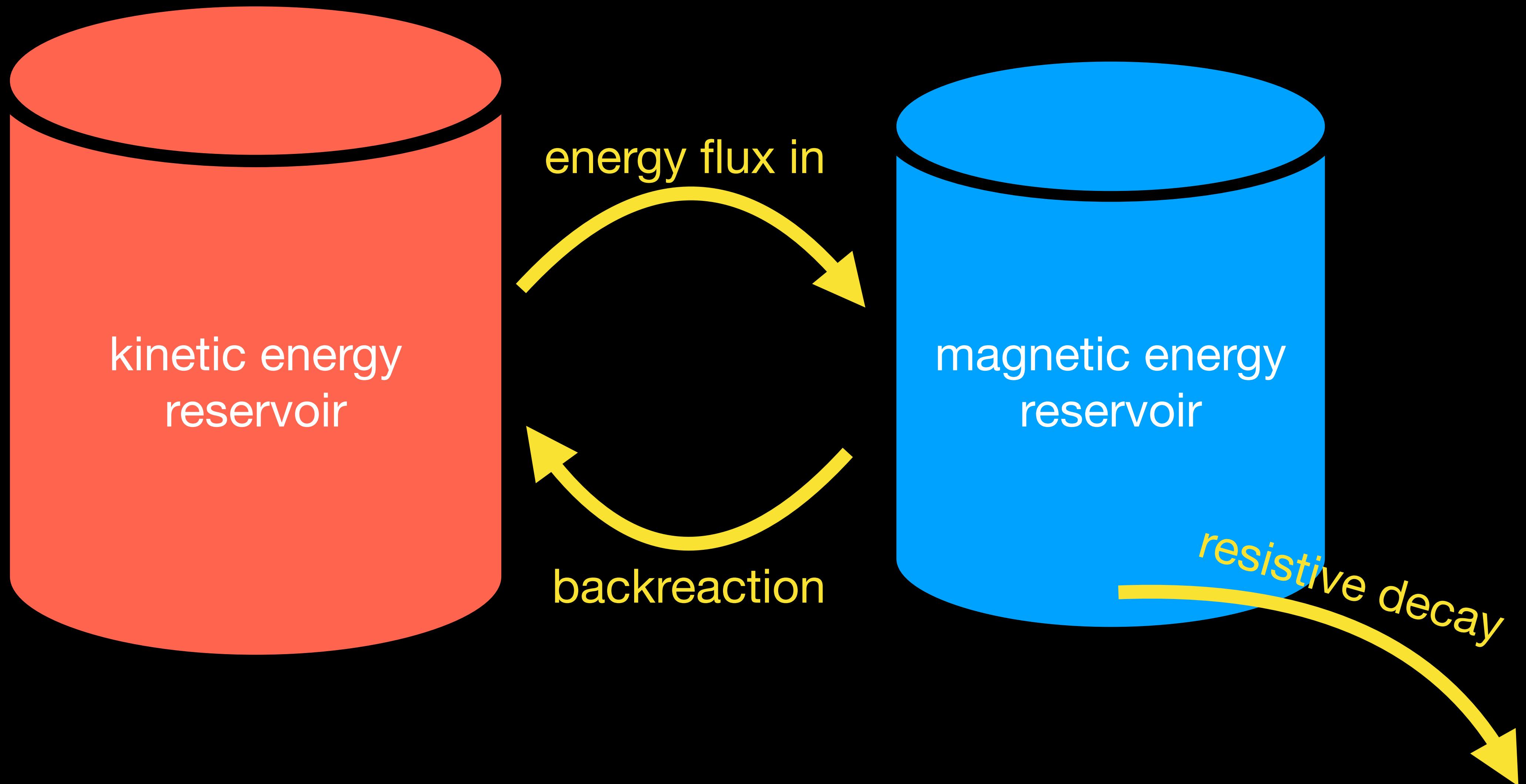
What is a magnetic dynamo?

Growth



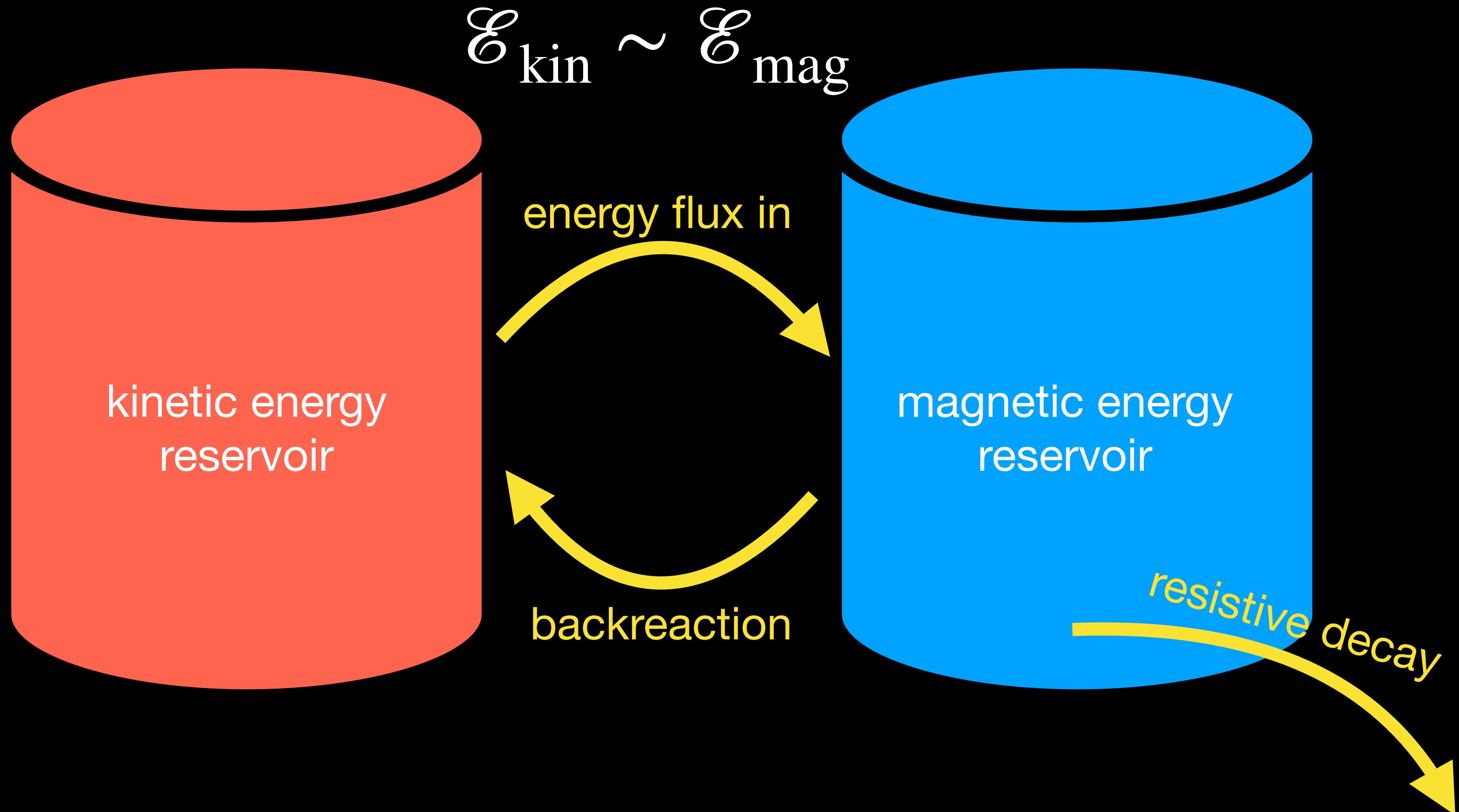
What is a magnetic dynamo?

Nonlinearities and backreaction



What is a magnetic dynamo?

Saturation

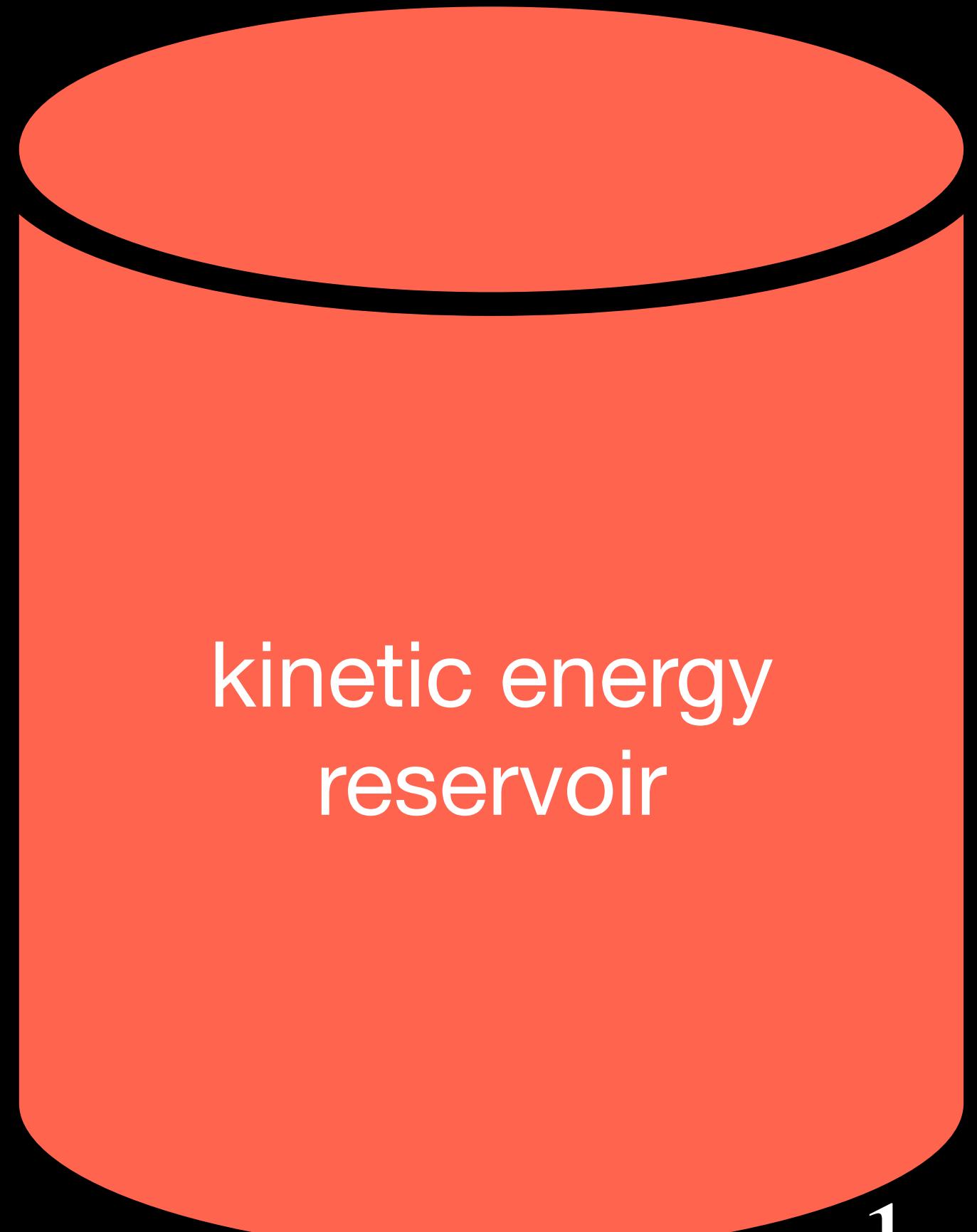


Again more quantitative: What is a magnetic dynamo?

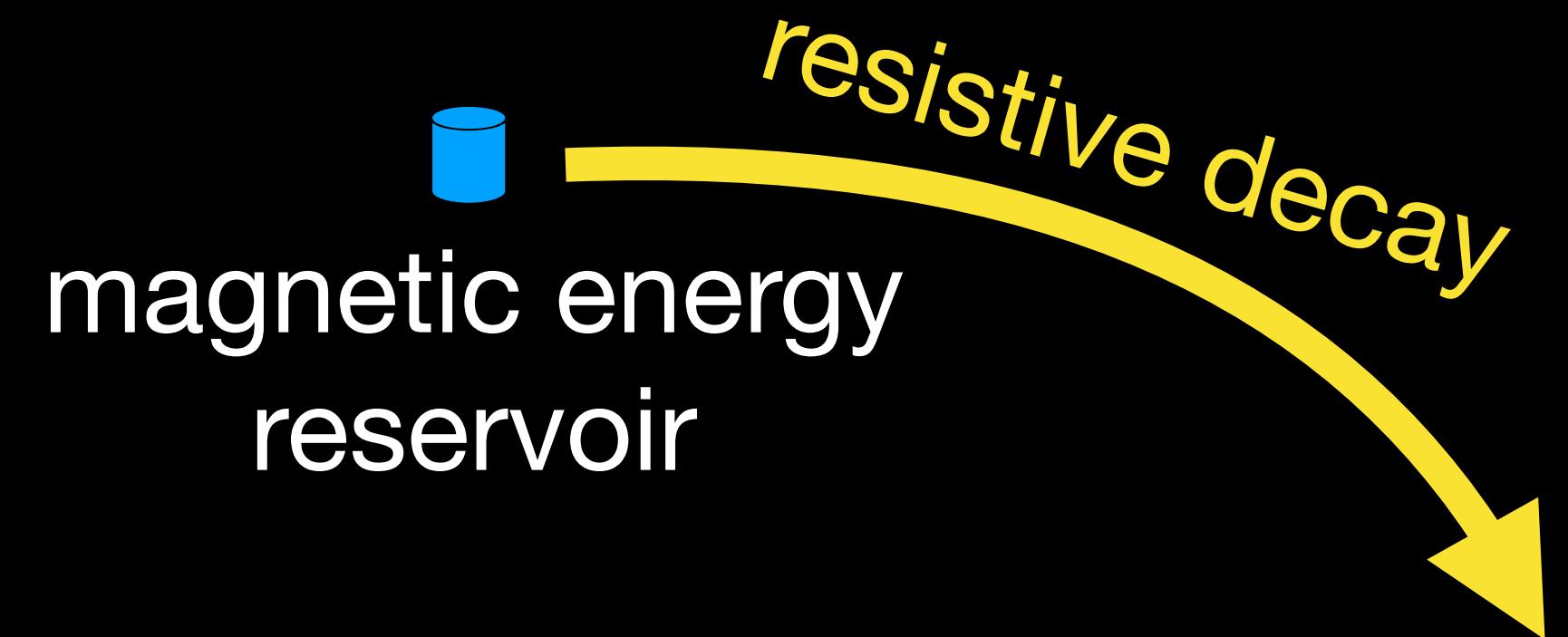
Starting with a weak seed magnetic field

$$\langle \mathbf{u} \cdot \nabla \cdot \mathbf{F}_u + \mathbf{u} \cdot \mathbf{f}_{\text{turb}} \rangle_t = \frac{1}{\text{Re}} \langle \mathbf{u} \cdot \mathbb{D}_\nu(\mathbf{u}) \rangle_t$$

$$\varepsilon_{\text{in}} = \varepsilon_{\text{out}}$$



$$\mathbf{b} \cdot \partial_t \mathbf{b} = \partial_t \mathcal{E}_{\text{mag}} = \frac{1}{\text{Rm}} \mathbf{b} \cdot \mathbb{D}_\eta(\mathbf{b})$$

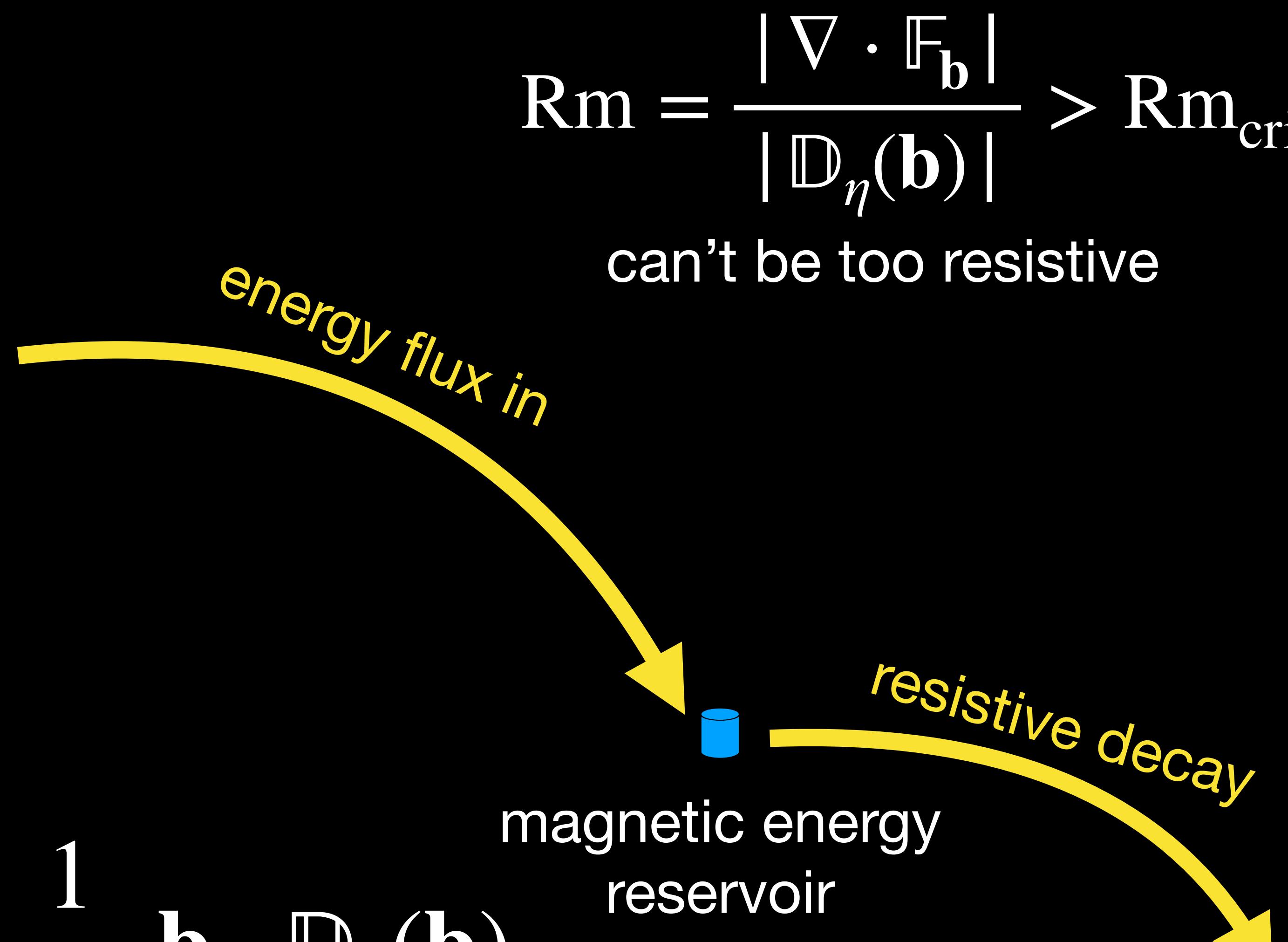
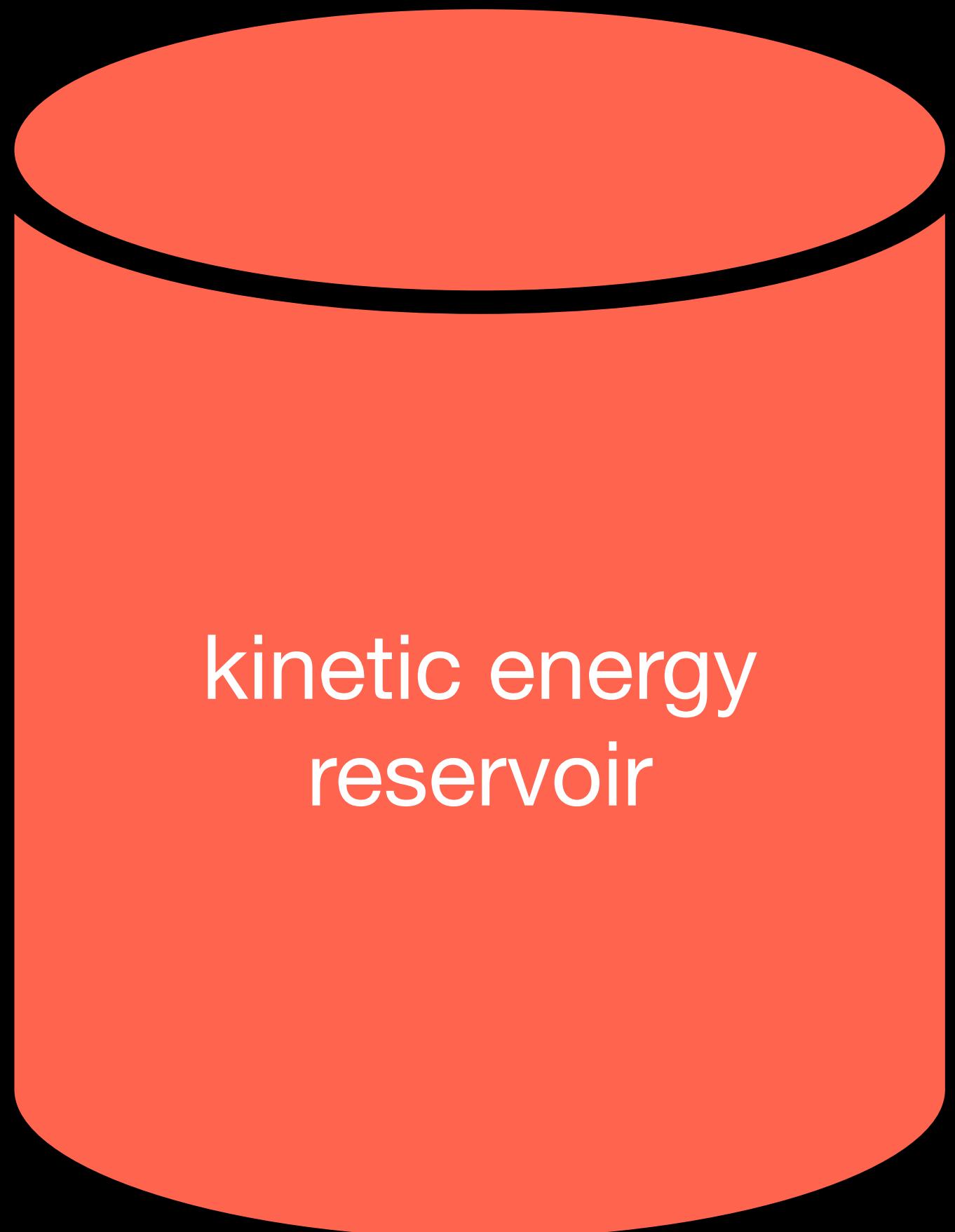


$$\text{Rm} \sim \frac{u_0 \ell_0}{\eta}$$

$$\text{Re} \sim \frac{u_0 \ell_0}{\nu}$$

Again more quantitative: What is a magnetic dynamo?

Growth



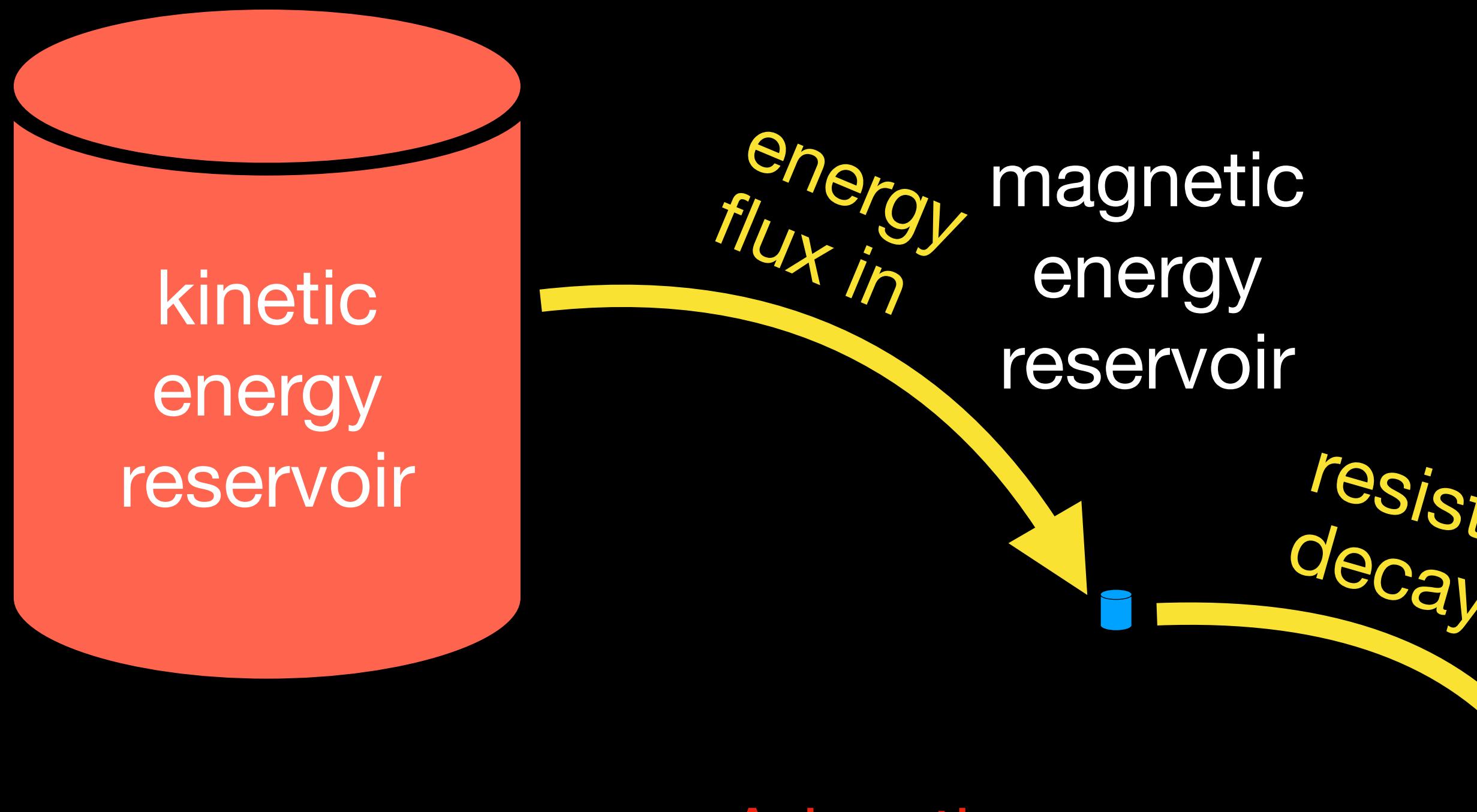
$$\partial_t \mathcal{E}_{\text{mag}} + \mathbf{b} \cdot \nabla \cdot \mathbb{F}_b = \frac{1}{Rm} \mathbf{b} \cdot \mathbb{D}_\eta(\mathbf{b})$$

$$Rm = \frac{|\nabla \cdot \mathbb{F}_b|}{|\mathbb{D}_\eta(\mathbf{b})|} > Rm_{\text{crit}}$$

can't be too resistive

Again more quantitative: What is a magnetic dynamo?

Flux terms



$$R_m = \frac{|\nabla \cdot F_b|}{|\mathbb{D}_\eta(b)|} > R_{m\text{crit}}$$

can't be too resistive

$$\mathbf{u} \otimes \mathbf{u} : \nabla \otimes \mathbf{u} = u_j u_i \partial_i u_j$$

$$\mathbf{b} \cdot \nabla \cdot F_b = \mathbf{b} \otimes \mathbf{u} : \nabla \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{b} : \nabla \otimes \mathbf{u} - \frac{1}{2} \mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u}) \mathbb{I}$$

Advection

compression

coupling to velocity gradients

Again more quantitative: What is a magnetic dynamo?

Gradient coupling

$$\mathbf{b} \cdot \nabla \cdot \mathcal{F}_b = \mathbf{b} \otimes \mathbf{u} : \nabla \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{b} : \nabla \otimes \mathbf{u} - \frac{1}{2} \mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u}) \mathbb{I}$$

advection

coupling to velocity gradients

compression

$$\nabla \otimes \mathbf{u} = \mathbb{A} + \mathbb{S} + \mathbb{B}$$

$$\mathbb{S} = \frac{1}{2} \left(\partial_i u_j + \partial_j u_i \right) - \frac{1}{3} \delta_{ij} \partial_k u_k$$

volume preserving

$$\mathbb{A}_{ij} = \frac{1}{2} \left(\partial_i u_j - \partial_j u_i \right) = -\frac{1}{2} \epsilon_{ijk} \omega_k$$

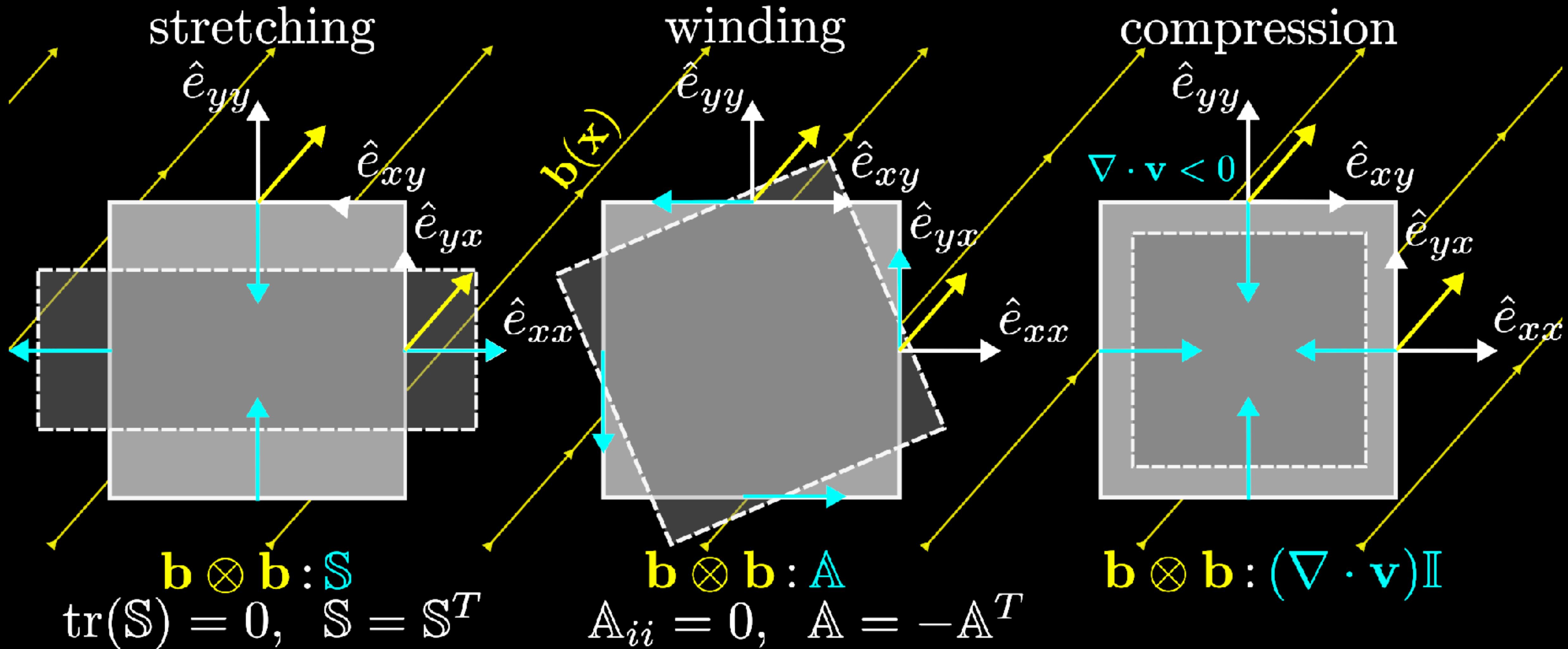
volume preserving

$$\mathbb{B}_{ij} = \frac{1}{3} \delta_{ij} \partial_k u_k$$

volume changing

Again more quantitative: What is a magnetic dynamo?

Gradient tensor decomp.



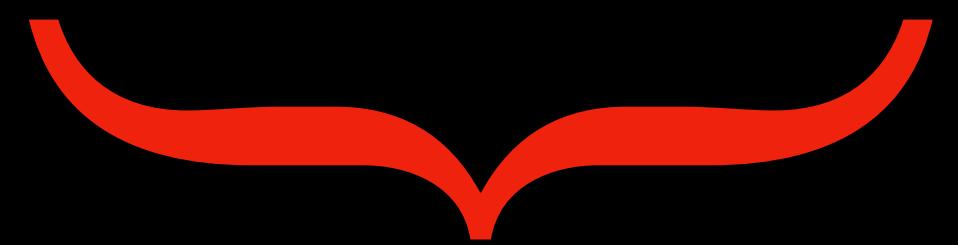
Again more quantitative: What is a magnetic dynamo? Gradient tensor decomp.

$$\mathbf{b} \cdot \nabla \cdot \mathbb{F}_b = \overbrace{\mathbf{b} \otimes \mathbf{u} : \nabla \otimes \mathbf{b}}^{\text{advection}} + \underbrace{\mathbf{b} \otimes \mathbf{b} : \mathbb{S}(\mathbf{u}) + \mathbf{b} \otimes \mathbf{b} : \mathbb{A}(\mathbf{u})}_{\text{stretching}} - \frac{1}{6} \mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u}) \mathbb{I} \underbrace{\phantom{\mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u}) \mathbb{I}}_{\text{compression}}}$$

Again more quantitative: What is a magnetic dynamo? Gradient tensor decomp.

$$\mathbf{b} \cdot \nabla \cdot \mathbb{F}_b = \mathbf{b} \otimes \mathbf{u} : \nabla \otimes \mathbf{b}$$

$$+ \mathbf{b} \otimes \mathbf{b} : \mathbb{S}(\mathbf{u}) + \mathbf{b} \otimes \mathbf{b} : \mathbb{A}(\mathbf{u}) - \frac{1}{6} \mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u}) \mathbb{I}$$



rotation (\mathbb{A} is actually a representation of $\mathfrak{SO}(3)$)

symmetric antisymmetric

$$\mathbf{b} \otimes \mathbf{b} : \mathbb{A} = 0$$

Always exactly orthogonal! You can never grow magnetic field flux with rotation operator!

Again more quantitative: What is a magnetic dynamo?

Energy flux

Remaining terms

$$\mathbf{b} \cdot \nabla \cdot \mathbb{F}_b = \mathbf{b} \otimes \mathbf{u} : \nabla \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{b} : \mathbb{S}(\mathbf{u}) - \frac{1}{6} \mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u}) \mathbb{I}$$

Each term could potentially describe an interaction between three difference modes (triad interactions)...

e.g., $\mathbf{b}(\mathbf{k}')$, $\mathbf{b}(\mathbf{k}'')$, $\mathbf{b}(\mathbf{k}''')$, $\mathbf{u}(\mathbf{k}')$, ...

$[\mathbf{b} \cdot \nabla \cdot \mathbb{F}_b] \sim U^3/L$ energy flux density

Again more quantitative: What is a magnetic dynamo? Cascade versus dynamo

Momentum conservation:

$$\begin{array}{ccc} \text{doner} & \text{receiver} \\ \mathbf{k}' + \mathbf{k}'' + \mathbf{k}''' = 0 & \text{or} \\ \text{mediator} & \end{array}$$

antisymmetry property:
(giveth = - taketh)

$$\begin{array}{ccc} \text{doner} & \xrightarrow{\mathbf{k}''} & \text{receiver} \\ \mathbf{k}' & \xrightarrow{\mathbf{k}''} & \mathbf{k}''' = -\mathbf{k}''' \xrightarrow{\mathbf{k}''} \mathbf{k}' \\ & \text{mediator} & \end{array}$$

Can extract these interactions directly from stochastic magnetic fields by constructing filtered vector fields

$$\mathbf{b}' = \mathbf{b}(\mathbf{r}') = \int \delta^3(\mathbf{k} - \mathbf{k}') \mathbf{b}(\mathbf{k}) \exp \{2\pi i \mathbf{k} \cdot \mathbf{r}'\}$$

Again more quantitative: What is a magnetic dynamo? Cascade versus dynamo

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}'''$$

Rewrite magnetic energy equation in terms of triad interactions:

$$\mathbf{b}''' \cdot \partial_t \mathbf{b}' = \partial_t \mathcal{E}_{\text{mag}} = -\mathbf{b}''' \cdot \nabla \cdot \mathbb{F}_{\mathbf{b}'} + \frac{1}{Rm} \mathbf{b}''' \cdot \mathbb{D}_\eta(\mathbf{b}')$$

where

$$\mathbf{b}''' \cdot \nabla \cdot \mathbb{F}_{\mathbf{b}'} = \mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}'$$

$$+ \mathbf{b}'' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' - \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}') \mathbb{I}$$

Again more quantitative: What is a magnetic dynamo? Cascade versus dynamo

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$$+ \mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' - \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}') \mathbb{I}$$

$$= \mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}'$$

$$+ \mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' - \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}') \mathbb{I}$$

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$$+ \mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' - \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}') \llbracket$$

magnetic cascade terms

$$= \mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}' - \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}') \llbracket$$

$$+ \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}'}_{\text{kinetic to magnetic energy transfer}}$$

looks like flux generation
via compression... it's not

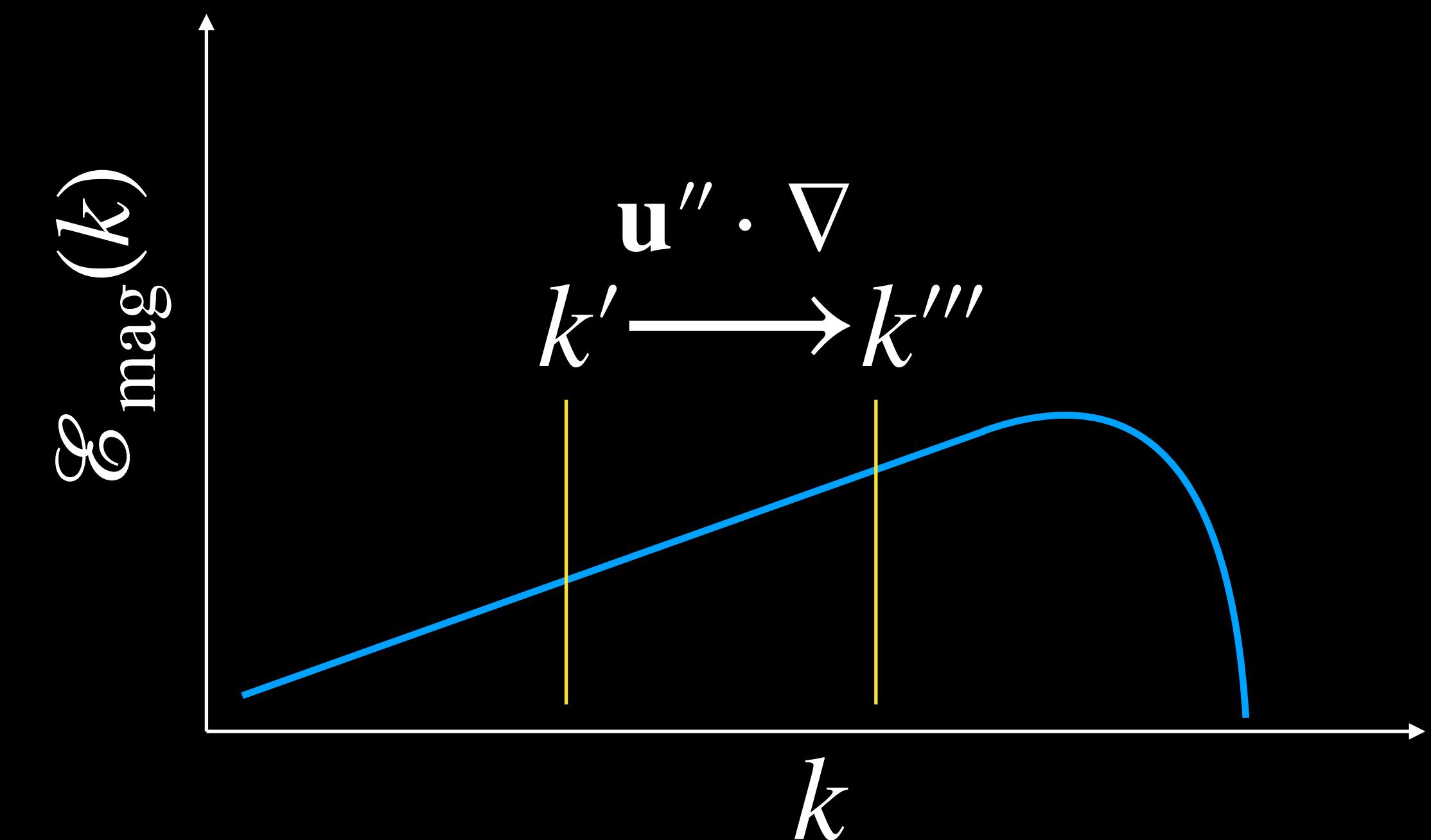
Again more quantitative: What is a magnetic dynamo?

Growth

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}'''$$

Rewrite magnetic energy equation in terms of triad interactions:

$$\begin{aligned} & \underbrace{\mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}' - \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}') \mathbb{I}}_{\text{magnetic cascade terms}} \\ & + \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}'}_{\text{kinetic to magnetic energy transfer}} \end{aligned}$$



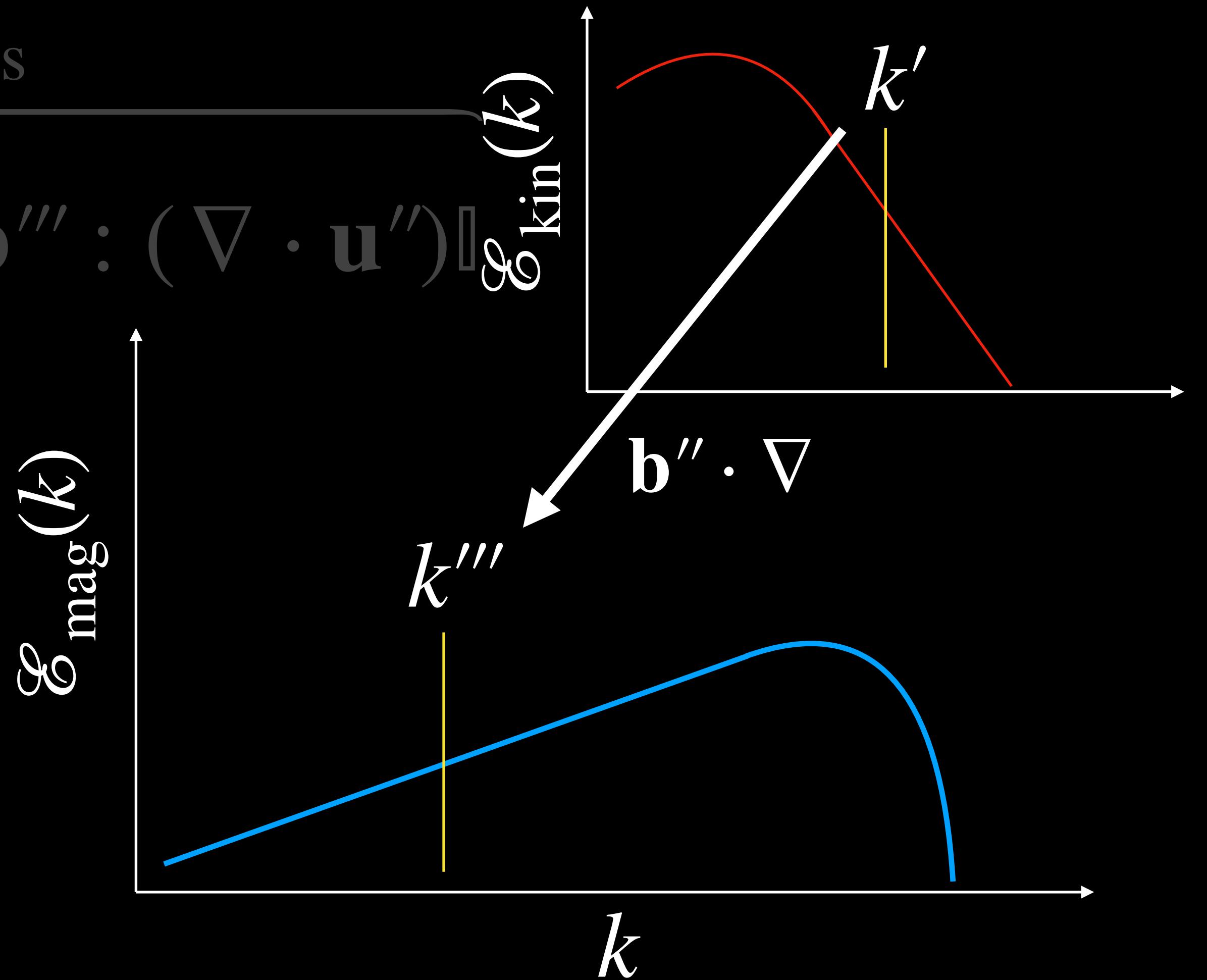
Again more quantitative: What is a magnetic dynamo?

Cascade versus dynamo

$$\mathbf{k}' \xrightarrow{\quad} \mathbf{k}'' \xrightarrow{\quad} \mathbf{k}'''$$

Rewrite magnetic energy equation in terms of triad interactions:

$$= \underbrace{\mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}'}_{\text{kinetic to magnetic energy transfer}} - \frac{1}{2} \underbrace{\mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}') \mathcal{E}_{\text{kin}}(k)}_{\text{magnetic cascade terms}}$$



Again more quantitative: What is a magnetic dynamo?

Dynamo and compression

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}'''$$

Kinetic to magnetic energy flux

$$\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' = \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(\mathbf{u}')}_{\text{dynamo}} + \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(\mathbf{u}')}_{\text{flux compression}}$$

antisymmetric property

$$\begin{aligned} \mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(\mathbf{u}') &= -\mathbf{u}''' \otimes \mathbf{b}'' : \mathbb{S}(\mathbf{b}') = -\mathbf{u}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{b}' \\ \mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(\mathbf{u}') &= -\frac{1}{2} \mathbf{u}''' \cdot \underbrace{\nabla \otimes (\mathbf{b}' \cdot \mathbf{b}')}_{\text{magnetic pressure}} \end{aligned}$$

flux compression

magnetic tension

magnetic pressure

Again more quantitative: What is a magnetic dynamo?

Dynamo and compression

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}'''$$

Kinetic to magnetic energy flux

$$\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' = \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(\mathbf{u}')}_{\text{dynamo}} + \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(\mathbf{u}')}_{\text{flux compression}}$$

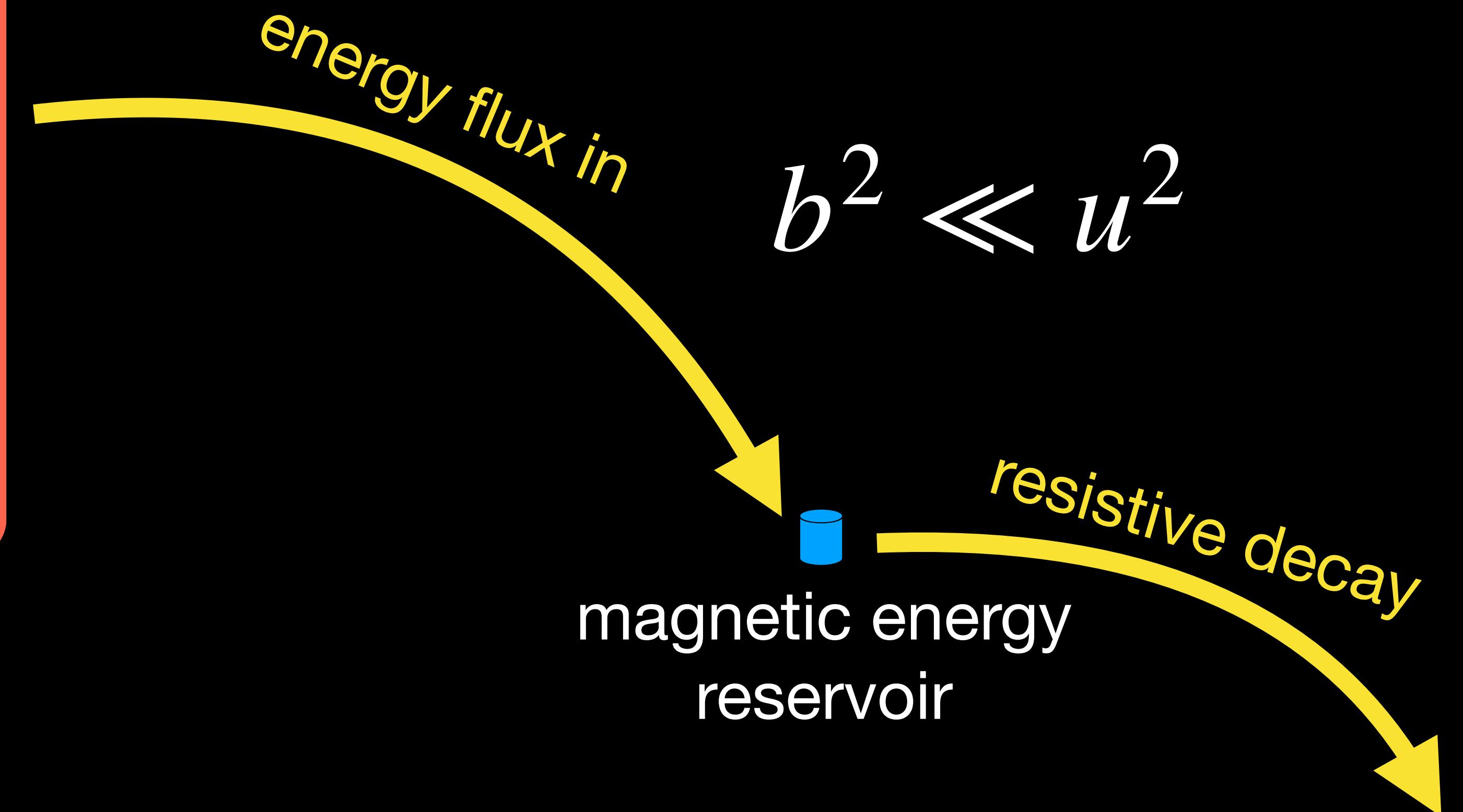
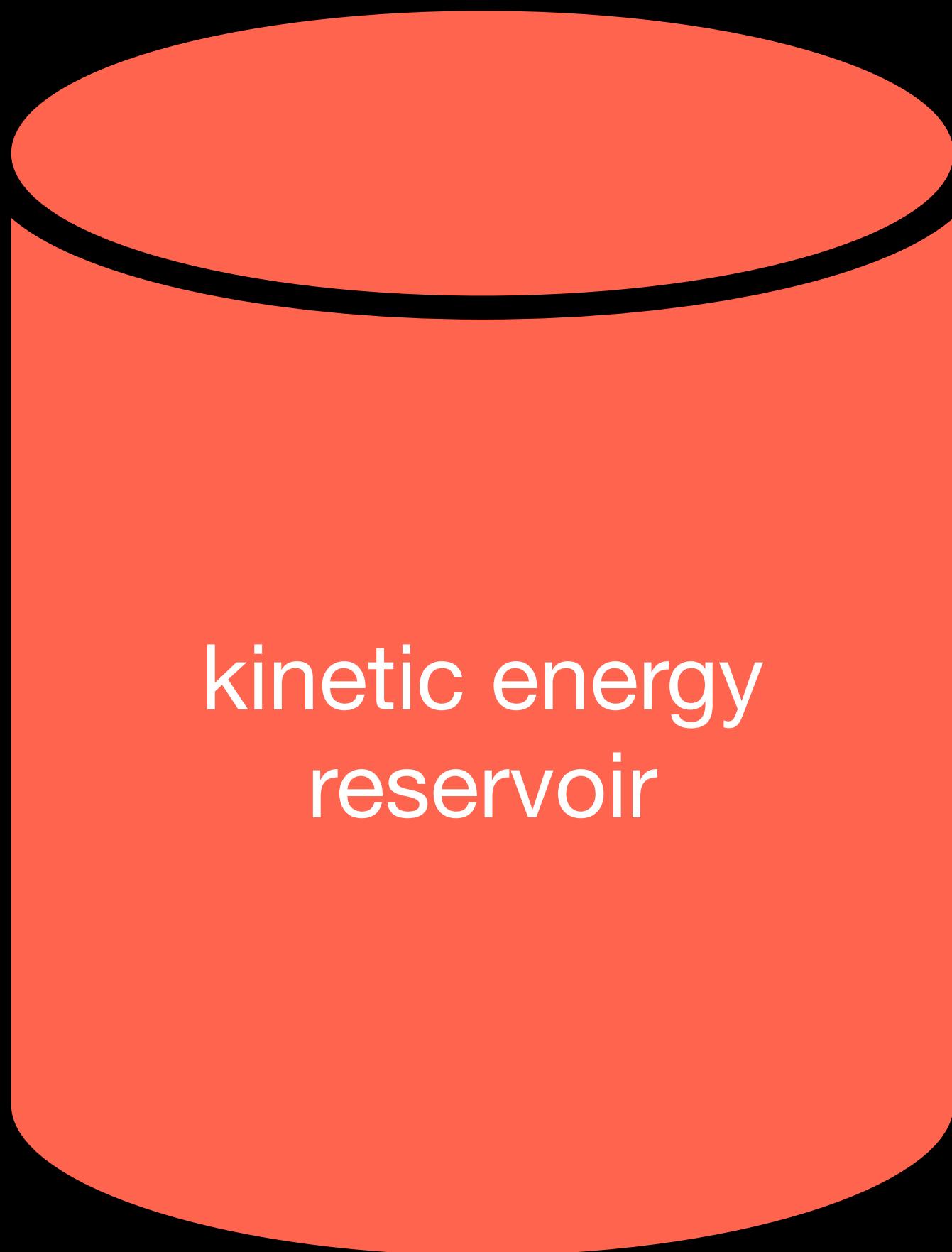
antisymmetric property

$$\underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(\mathbf{u}')}_{\text{dynamo}} = -\mathbf{u}''' \otimes \mathbf{b}'' : \mathbb{S}(\mathbf{b}') = -\mathbf{u}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{b}'$$

- We learn:**
- which term gives rise to dynamo (cascade, flux compression)
 - tension always balances with stretching from dynamo,
 - and pressure always balances flux compression + flux freezing

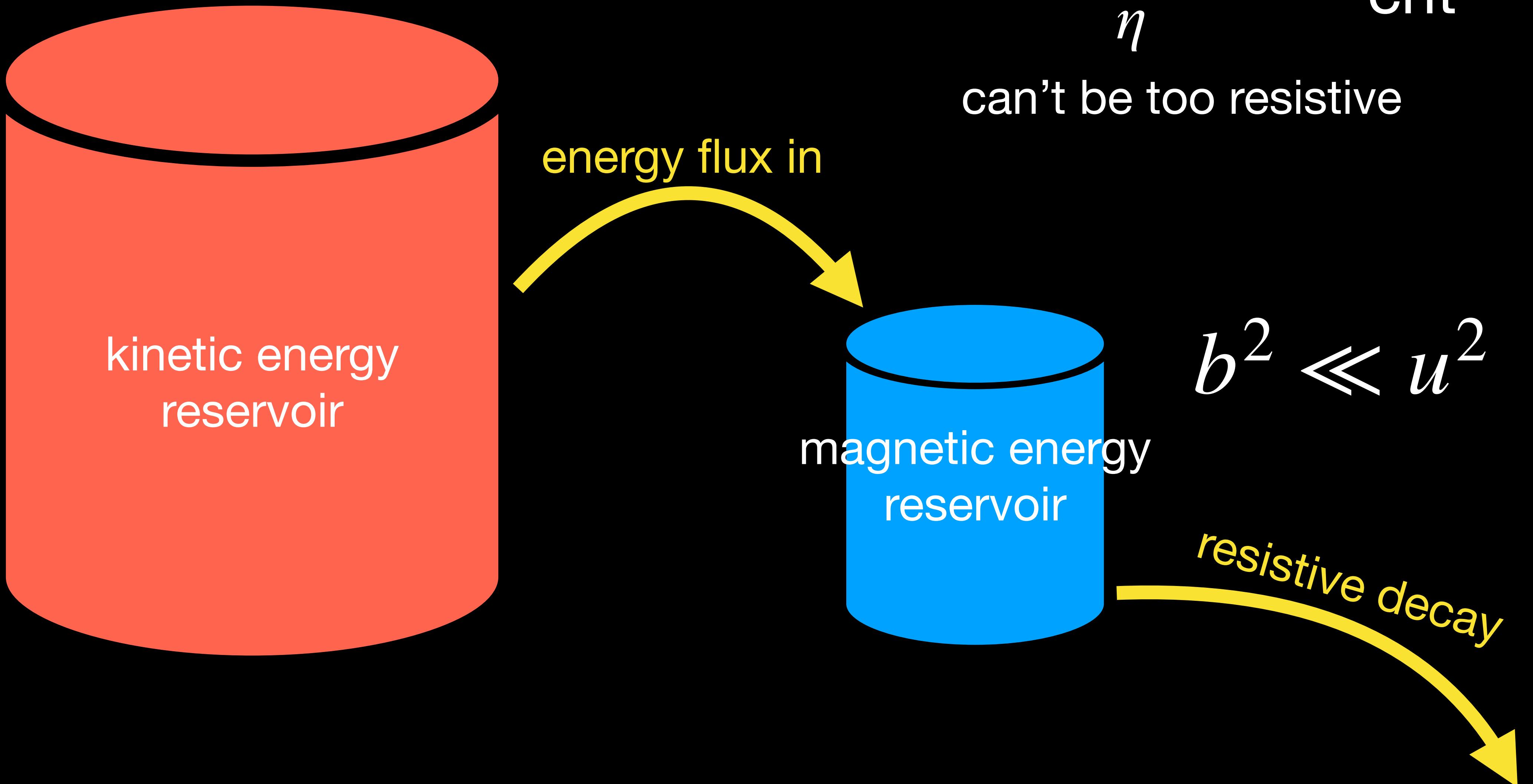
Again more quantitative: What is a magnetic dynamo?

Fast growth stage



Again more quantitative: What is a magnetic dynamo?

Fast growth stage



Again more quantitative: What is a magnetic dynamo?

Fast growth stage

$$\underbrace{\mathbf{b}''' \otimes \mathbf{b}'': \mathbb{S}(u')}_{\text{flux compression}} = -\mathbf{u}''' \otimes \mathbf{b}'': \mathbb{S}(b') = -\mathbf{u}''' \otimes \underbrace{\mathbf{b}'': \nabla \otimes \mathbf{b}'}_{\text{magnetic tension}}$$
$$\underbrace{\mathbf{b}''' \otimes \mathbf{b}'': \mathbb{B}(u')}_{\text{antisymmetric property}} = -\frac{1}{2}\mathbf{u}'''. \underbrace{\nabla \otimes (\mathbf{b}' \cdot \mathbf{b}')}_{\text{magnetic pressure}}$$

- \mathbf{b} coupled to \mathbf{u} via $\sim b^2/\ell$ in momentum equation
- For $b^2 \ll u^2$, \mathbf{u} only a very weak function of \mathbf{b}

No $\mathbf{u}(\mathbf{b})$
back
reaction!

$$\partial_t \mathbf{b} + \nabla \cdot (\mathbf{u} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{u}) = \frac{1}{Rm} \mathbb{D}_\eta(\mathbf{b})$$

independent of \mathbf{b}
linear in \mathbf{b}

Again more quantitative: What is a magnetic dynamo?

Fast growth stage – integral energy

Rewrite magnetic energy eq. and take 1st moments

$$d_t \langle \mathcal{E}_{\text{mag}} \rangle = 2 (\gamma - \eta k_{\text{rms}}^2) \langle \mathcal{E}_{\text{mag}} \rangle$$

where

$$\gamma = \langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \rangle, k_{\text{rms}}^2 = \langle \nabla \otimes \hat{\mathbf{b}} : \nabla \otimes \hat{\mathbf{b}} \rangle$$

Growth sourced via $\nabla \otimes \mathbf{u}$... which comes from the (hydro) turbulence

$$\ell_\nu \sim \text{Re}^{-3/4} \ell_0,$$

cascade

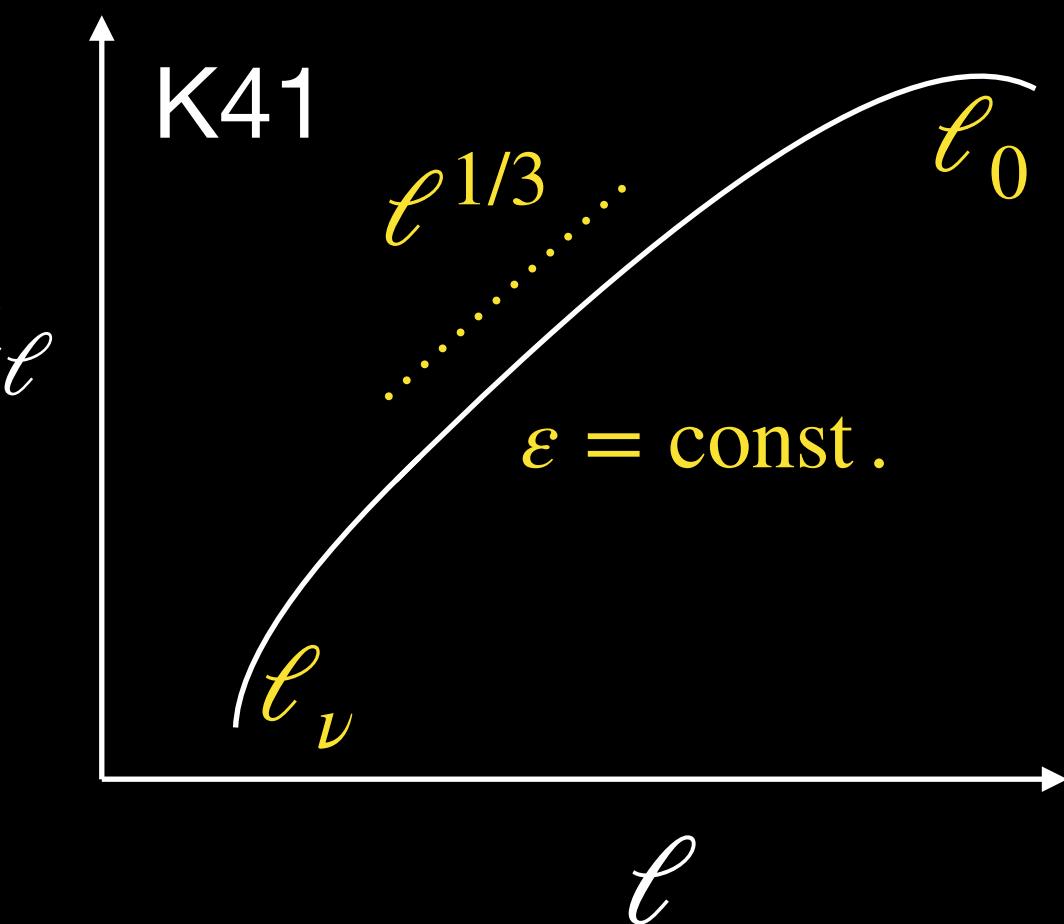
$$u_\ell \sim (\varepsilon \ell)^{1/3},$$

velocity scaling

$$\varepsilon \sim u^3 / \ell$$

constant energy flux

*assuming homogenous, isotropic, incompressible Kolmogorov-type turbulence



Again more quantitative: What is a magnetic dynamo? Fast growth stage – dynamo engine is the viscous scale

Growth sourced via $\nabla \otimes \mathbf{u}$... which comes from the (hydro) turbulence

$$u_\ell/\ell \sim \varepsilon^{1/3} \ell^{-2/3}, \quad t_\ell \sim \ell^{2/3}.$$

velocity gradients strongest at
small scales

Dynamical timescales smallest at
small scales

hence growth rate dominated by the smallest possible scales of
the flow gradients

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim u_\nu/\ell_\nu \sim 1/t_\nu$$

i.e., the viscous eddy scale ℓ_ν (on $\ell < \ell_\nu$ flow is diffusive).

Again more quantitative: What is a magnetic dynamo? Fast growth stage – integral energy

hence growth rate dominated by the smallest possible scales of the flow gradients

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim u_\nu / \ell_\nu \sim 1/t_\nu,$$

put in units of outer scale turnover time $t_0 = \ell_0 / u_0$

$$t_0 \gamma \sim t_0 / t_\nu, \quad t_0 / t_\nu \sim (\ell_0 / \ell_\nu)^{2/3} \sim (\text{Re}^{3/4})^{2/3} \sim \text{Re}^{1/2},$$

and to summarise,

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim 1/t_\nu \sim \text{Re}^{1/2} / t_0.$$

Again more quantitative: What is a magnetic dynamo? Fast growth stage – integral energy

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim 1/t_\nu \sim \text{Re}^{1/2}/t_0.$$

Consider the cold neutral medium in ISM. $T = 80$ K.

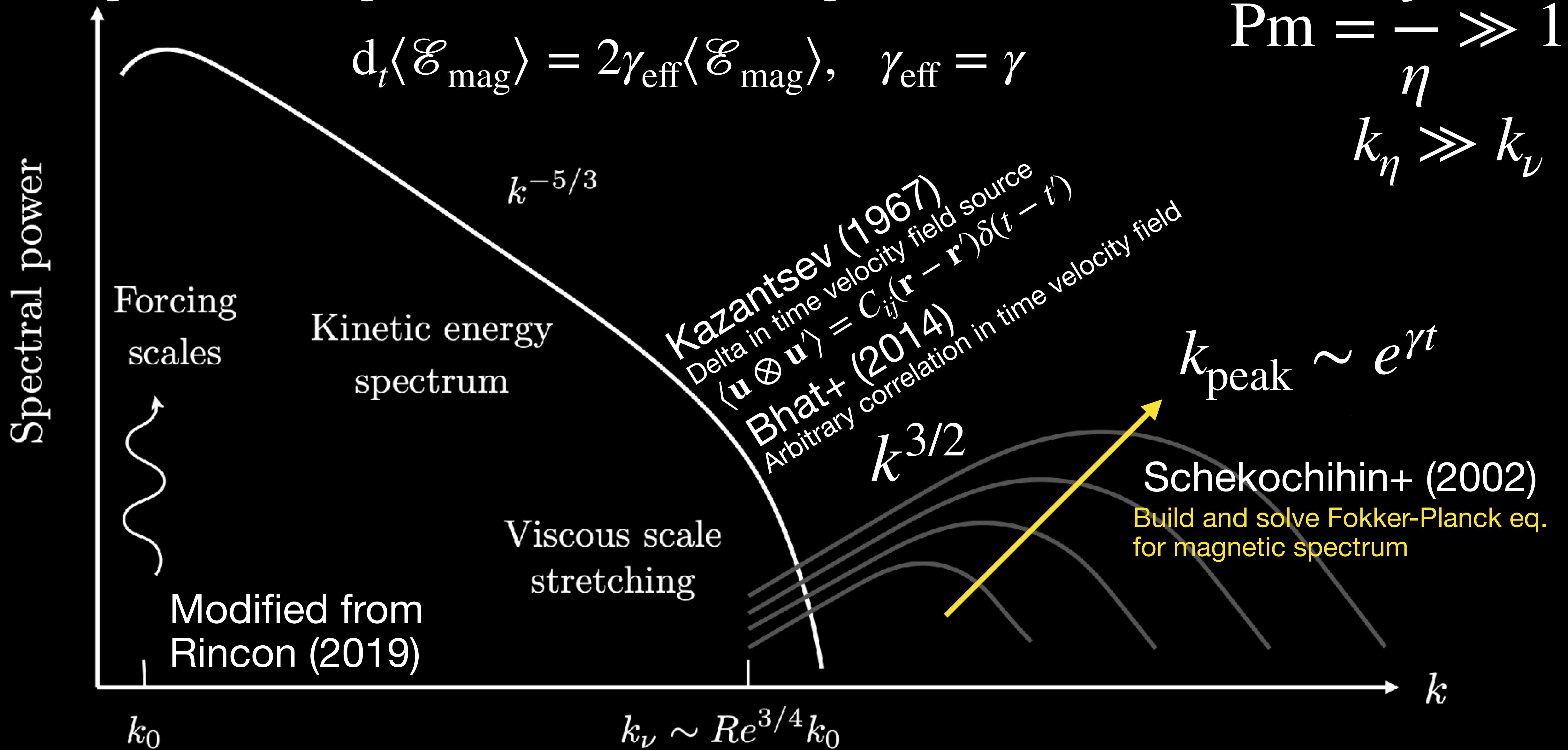
$$\text{Re} \sim 4 \times 10^{10}, \quad t_0 \sim 3 \text{ Myr}, \quad t_0/\text{Re}^{1/2} \sim 10 \text{ years}.$$

Every 10 years the cold phase of the Galaxy can increase its field by a factor of $e \approx 3$.
This is a diffusion-free turbulent dynamo because we assumed

$$d_t \langle \mathcal{E}_{\text{mag}} \rangle = 2\gamma_{\text{eff}} \langle \mathcal{E}_{\text{mag}} \rangle, \quad \gamma_{\text{eff}} = \gamma - \eta k_{\text{rms}}^2, \quad \eta k_{\text{rms}}^2 \rightarrow 0$$

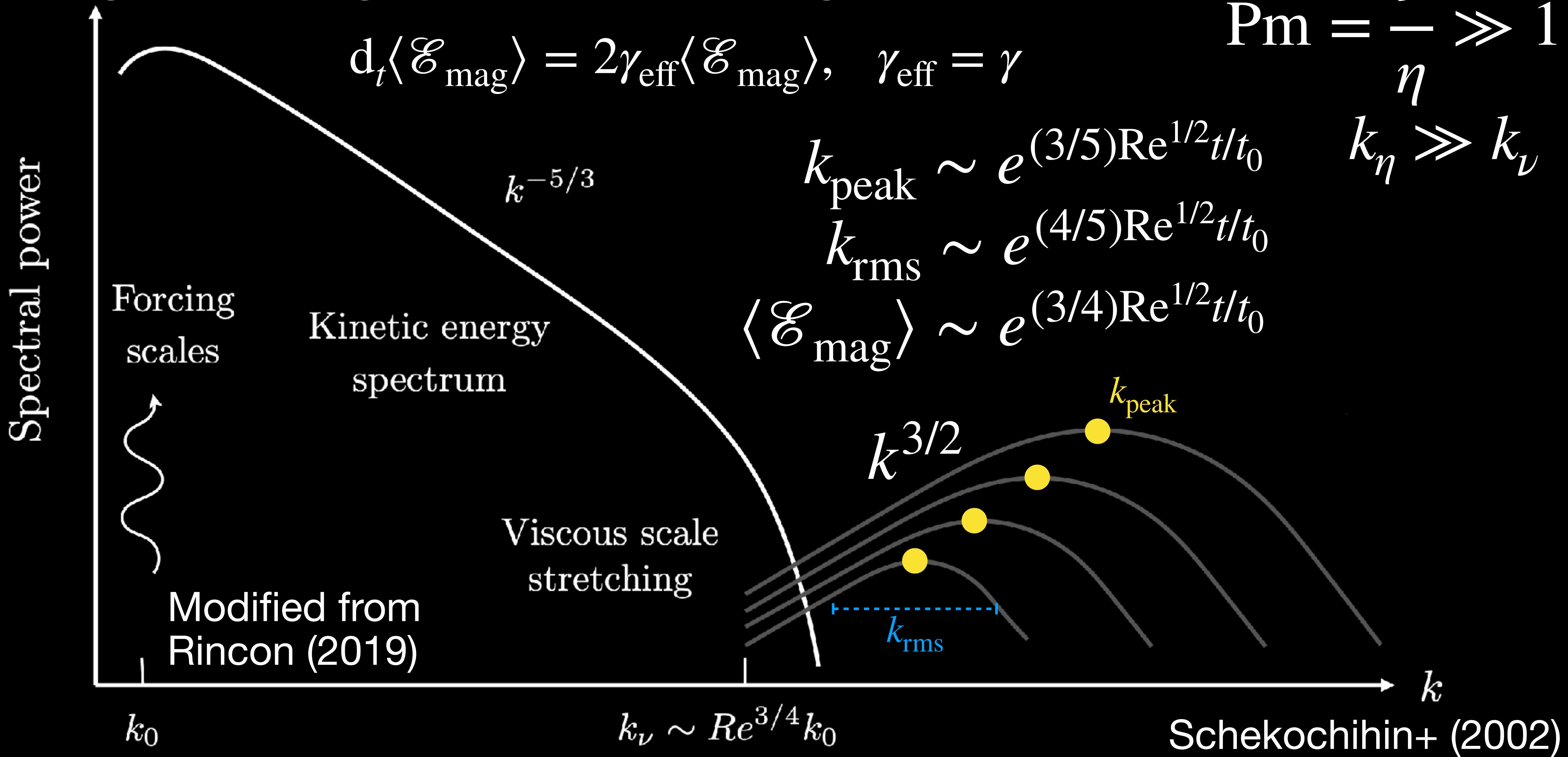
Again more quantitative: What is a magnetic dynamo?

First growth stage: diffusion-free regime



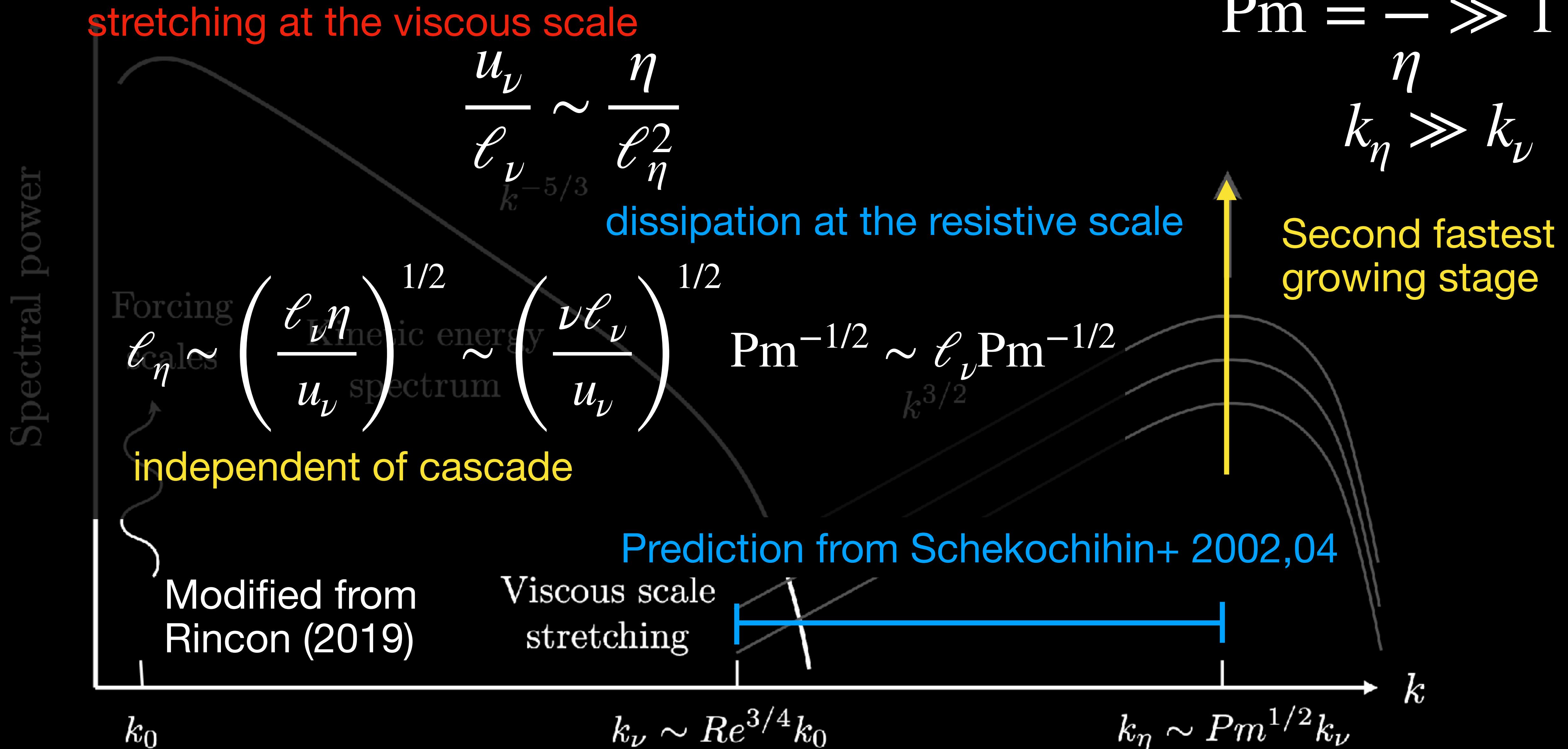
Again more quantitative: What is a magnetic dynamo?

First growth stage: diffusion-free regime



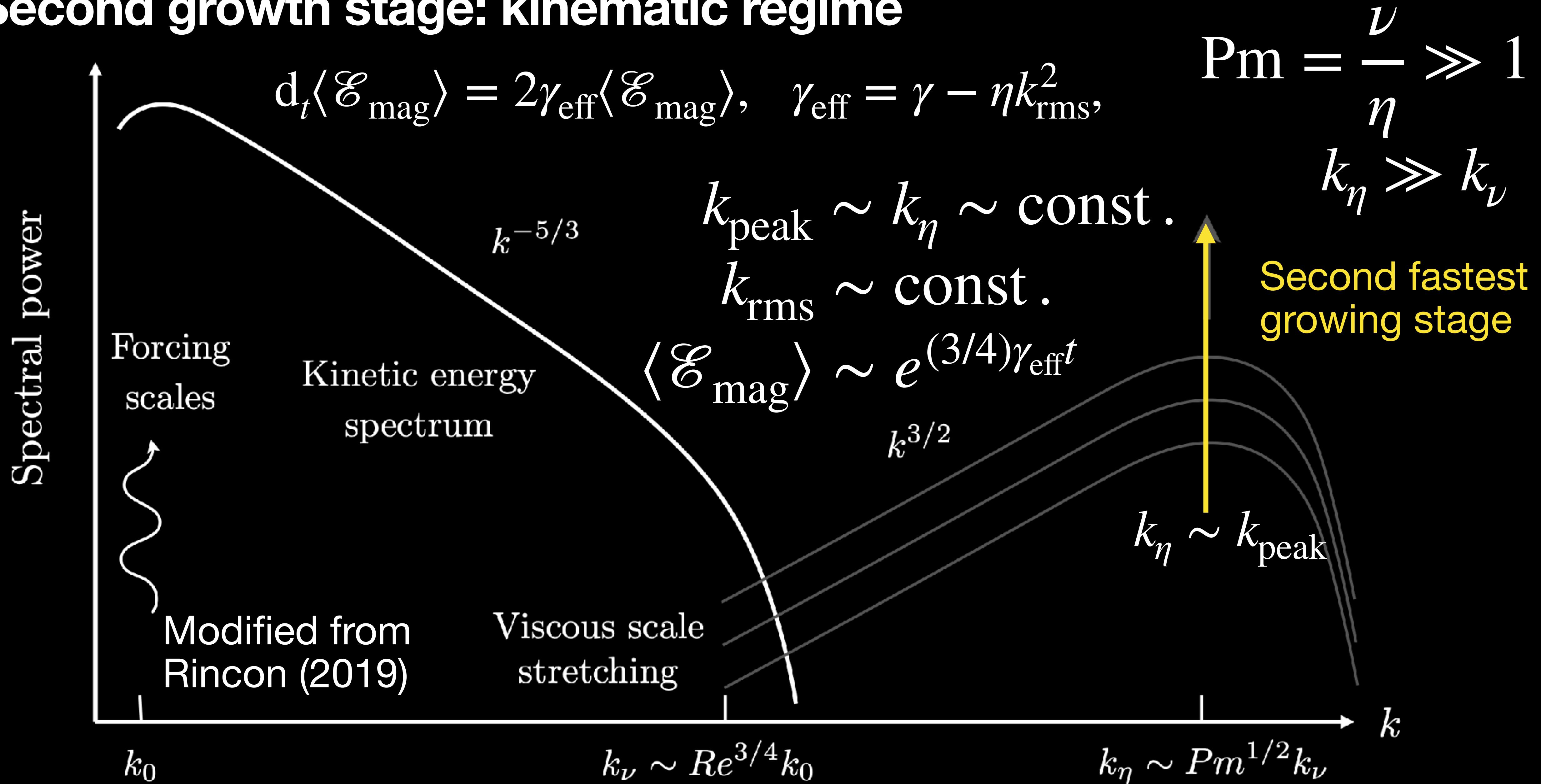
Again more quantitative: What is a magnetic dynamo?

Second growth stage: kinematic regime



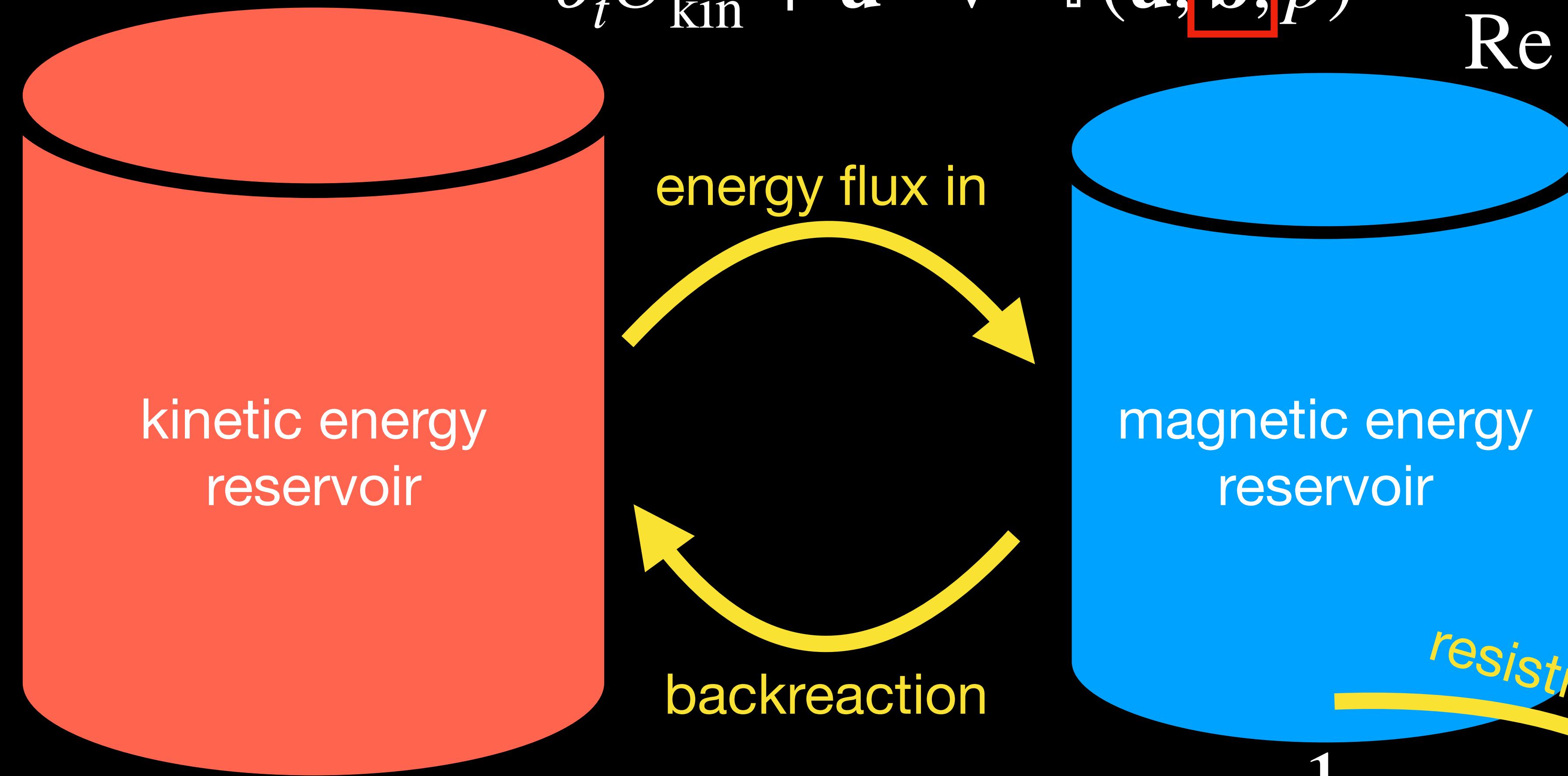
Again more quantitative: What is a magnetic dynamo?

Second growth stage: kinematic regime



Again more quantitative: What is a magnetic dynamo?

Linear growth and backreaction



$$\partial_t \mathcal{E}_{\text{kin}} + \mathbf{u} \cdot \nabla \cdot \mathbb{F}(\mathbf{u}, \mathbf{b}, p) = \frac{1}{\text{Re}} \mathbf{u} \cdot \mathbb{D}_\nu(\mathbf{u})$$
$$\partial_t \mathbf{b} + \nabla \cdot (\mathbf{u}(\mathbf{b}) \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{u}(\mathbf{b})) = \frac{1}{\text{Rm}} \mathbb{D}_\eta(\mathbf{b})$$

induction equation now nonlinear in \mathbf{b}

Again more quantitative: What is a magnetic dynamo? Linear growth and backreaction

The backreaction is directly encoded in the antisymmetric property

$$\begin{aligned} \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(u')}_{\text{flux compression}} &= -\mathbf{u}''' \otimes \mathbf{b}'' : \mathbb{S}(b') = -\mathbf{u}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{b}' \\ \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(u')}_{\text{dynamo}} &= \frac{1}{2} \mathbf{u}''' \cdot \nabla \otimes (\mathbf{b}' \cdot \mathbf{b}'') \end{aligned}$$

$$\mathcal{T}_{ub}(k', k'', k''') - \mathcal{T}_{bu}(k', k'', k''')$$

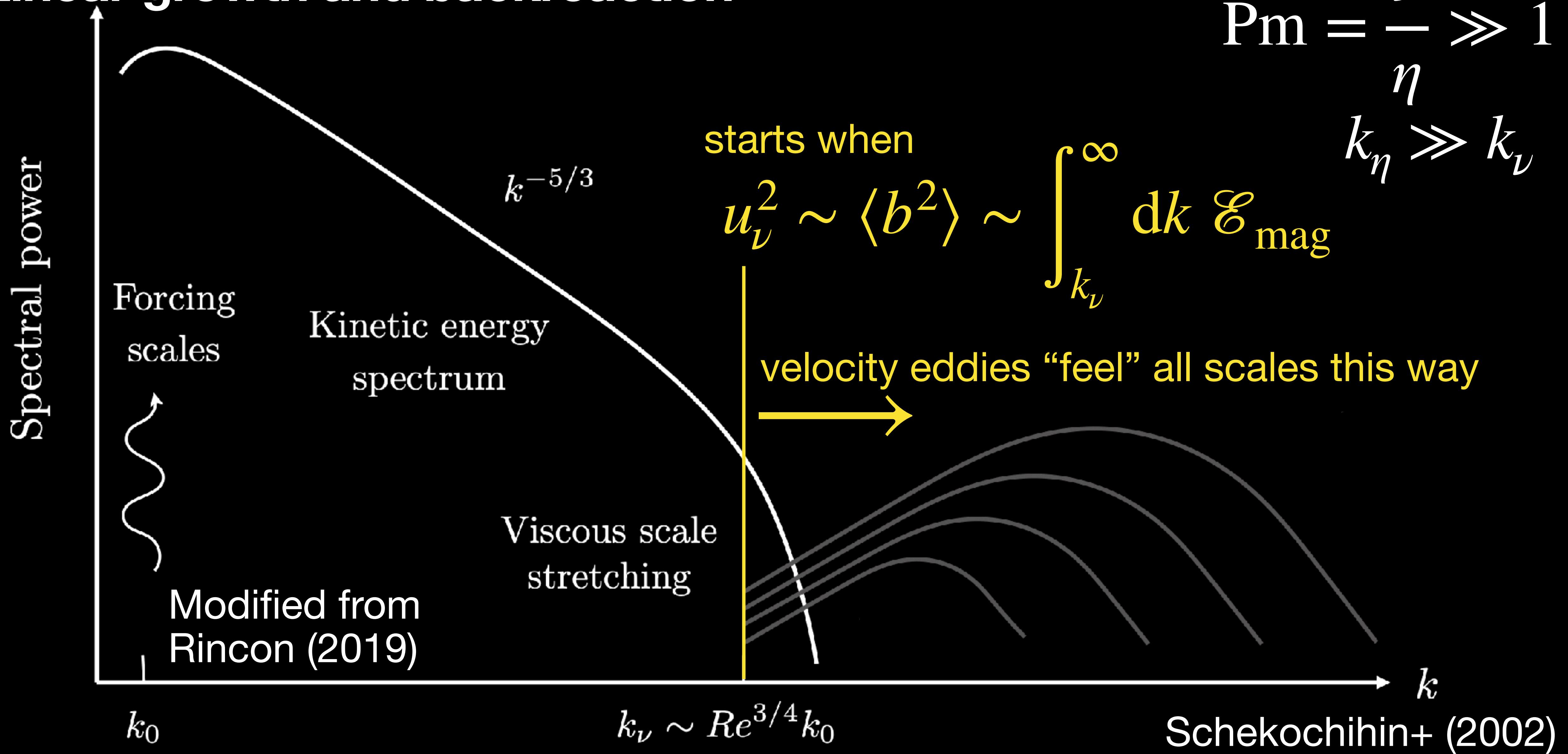
Energy flux from kinetic to magnetic energy densities

Energy flux from magnetic to kinetic energy densities

Always present, always balancing \mathcal{T}_{ub} , but now non-negligible.

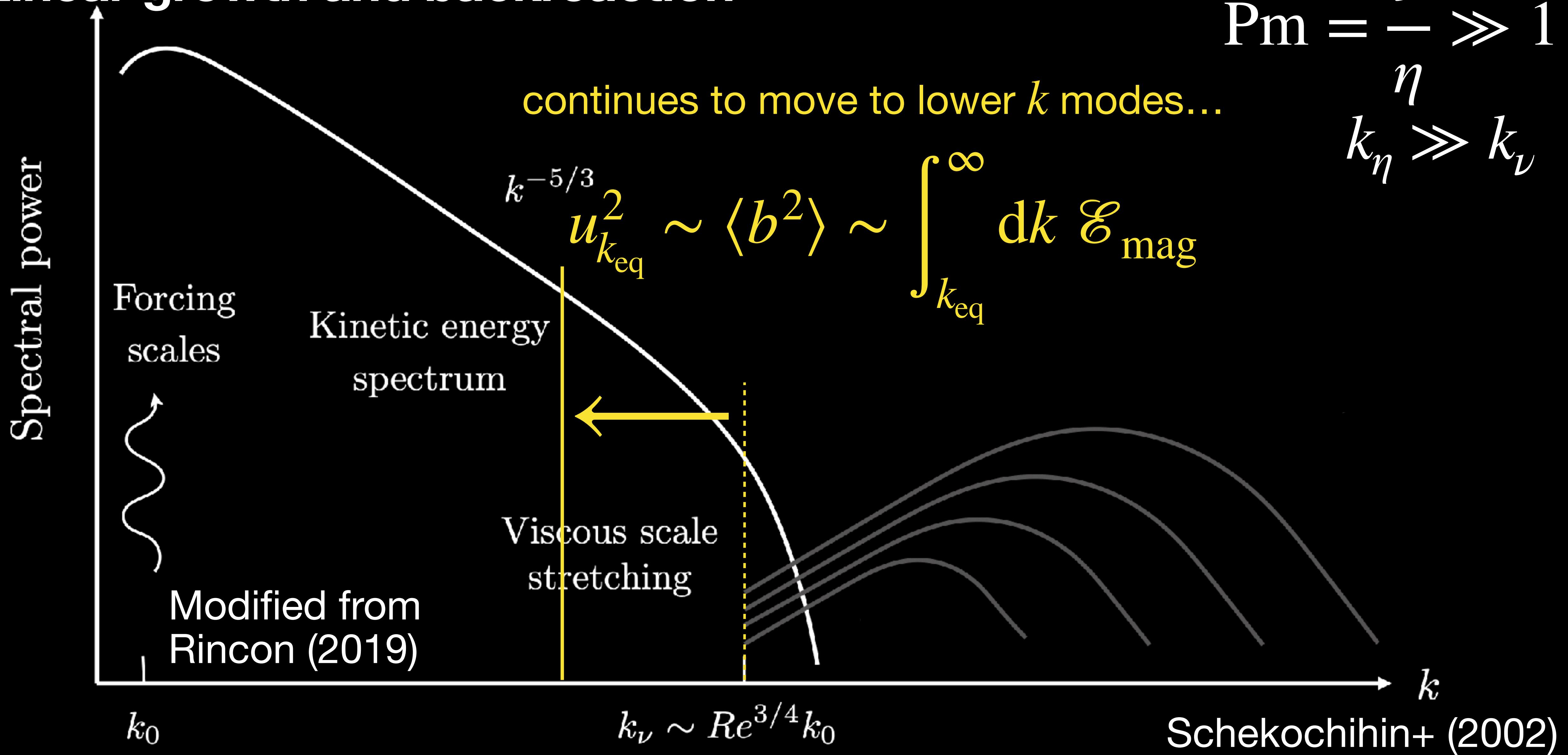
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Linear growth and backreaction



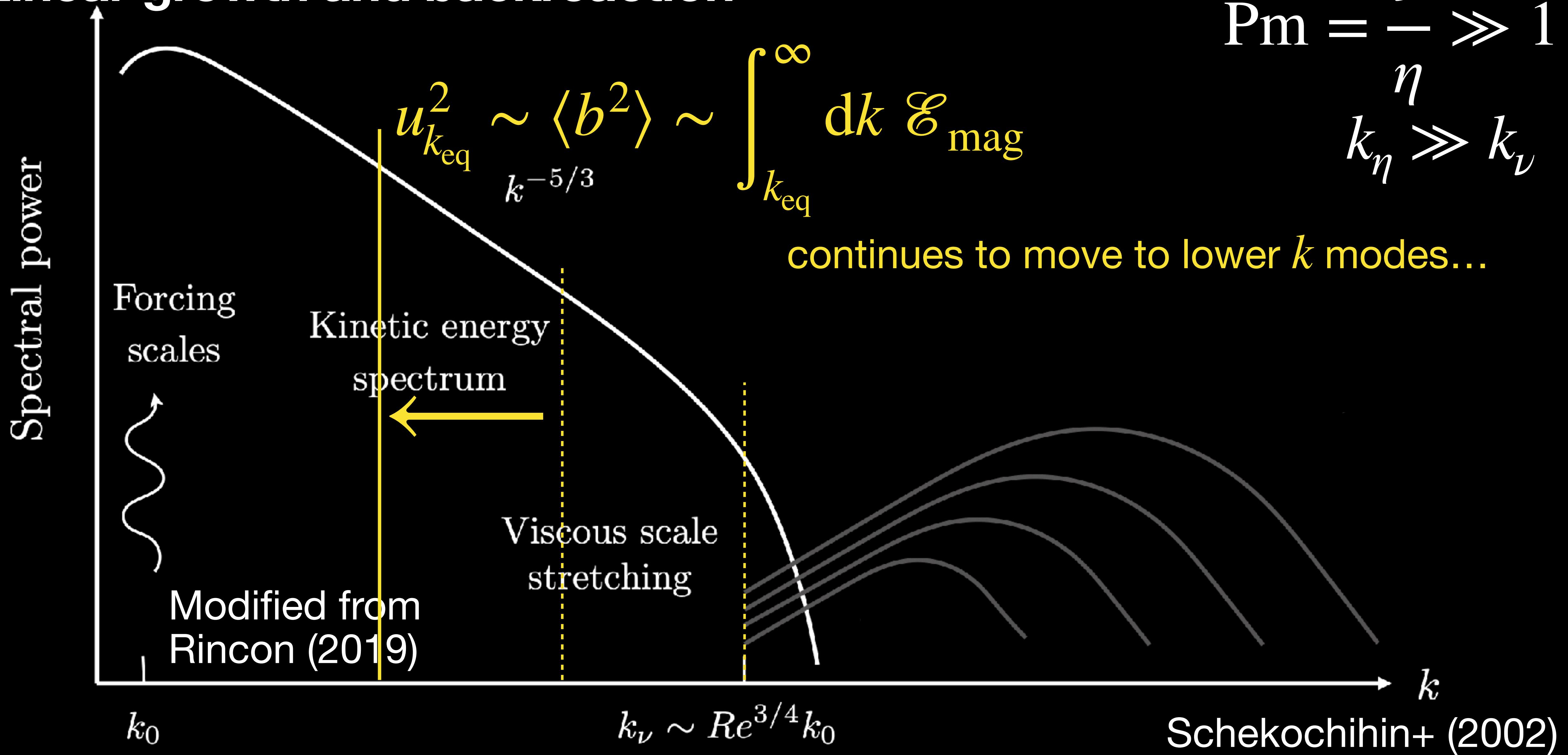
Again more quantitative: What is a magnetic dynamo?

Linear growth and backreaction



Again more quantitative: What is a magnetic dynamo?

Linear growth and backreaction



Again more quantitative: What is a magnetic dynamo? Linear growth and backreaction

Stretching has to move to larger scales, i.e.,

$$d_t \langle \mathcal{E}_{\text{mag}} \rangle = 2\gamma_{\text{eff}} \langle \mathcal{E}_{\text{mag}} \rangle, \quad \gamma_{\text{eff}} \sim \frac{u_{\ell^{\text{eq}}}}{\ell^{\text{eq}}},$$

hence,

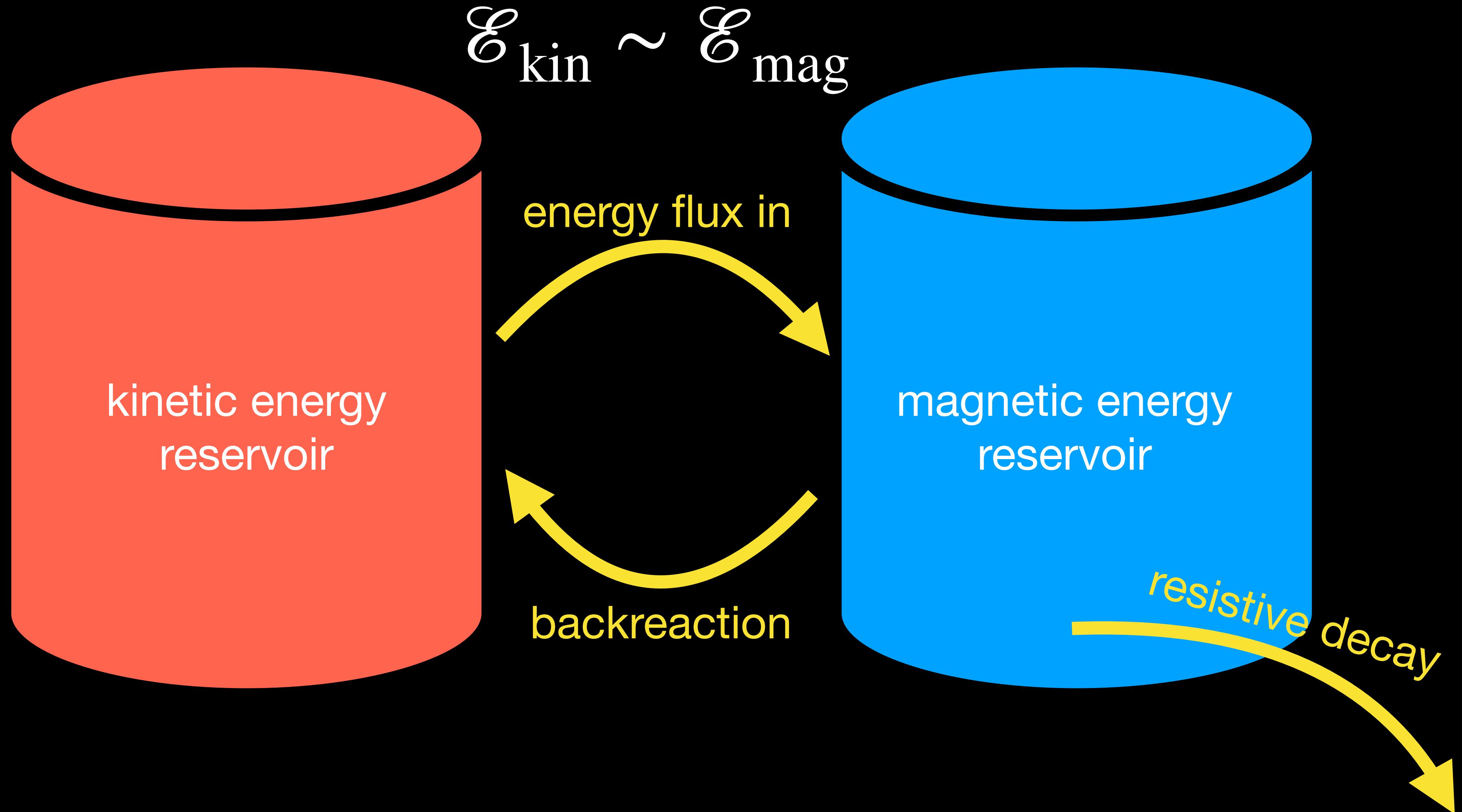
$$d_t \langle \mathcal{E}_{\text{mag}} \rangle \sim \frac{u_{\ell^{\text{eq}}}}{\ell^{\text{eq}}} \langle \mathcal{E}_{\text{mag}} \rangle \sim \frac{u_{\ell^{\text{eq}}}}{\ell^{\text{eq}}} u_{\ell^{\text{eq}}}^2 \sim \varepsilon_{\text{eq}} \sim \varepsilon,$$

Using simply $\langle \mathcal{E}_{\text{mag}} \rangle \sim u_{\ell^{\text{eq}}}^2$, by definition, and ε is constant everywhere in the cascade

$$\langle \mathcal{E}_{\text{mag}} \rangle \sim \varepsilon t.$$

Again more quantitative: What is a magnetic dynamo?

Saturation



More soon....