Tangent planes

Consider the surface in $\mathbb{R}^3$ defined by the equation

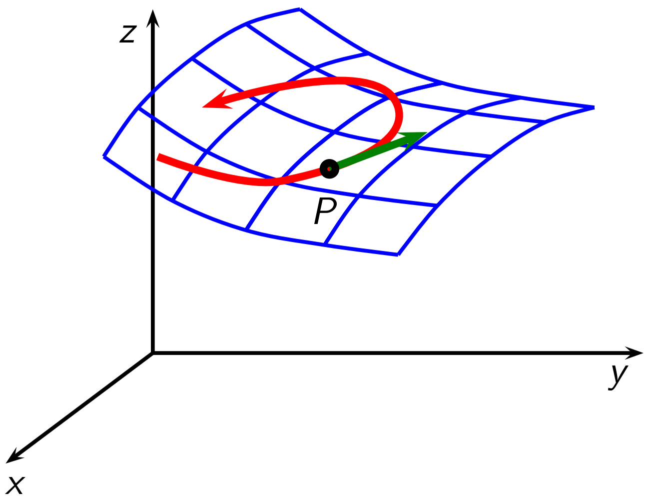
$$\phi(x,y,z) = c$$

for some constant c and differentiable function ϕ and let

$$\mathbf{r}(t) = \big(f(t), g(t), h(t)\big)$$

be a differentiable curve lying in the surface with tangent vector given by

$$\mathbf{r}'(t) = \big(f'(t), g'(t), h'(t)\big)$$



Since all points along $\mathbf{r}(t)$ lie in the surface,

$$\phi\big(f(t),g(t),h(t)\big) = c \;\Rightarrow\; \big(\phi\circ\mathbf{r}\big)(t) = c \;\Rightarrow\; D_{\mathbf{r}(t)}\phi\, D_t\mathbf{r} = 0 \;\Rightarrow\; \nabla\phi\cdot\mathbf{r}'(t) = 0.$$

Hence all curves passing through a point P on the surface have tangent vector normal to $\nabla\phi$ and so they all lie in a common plane called the tangent plane at P.

Tangent planes

Find the tangent plane to the surface

$$x^2 + y^2 + z^2 = 6$$

at the point $(1, 2, -1)$.

The surface is

$$\phi(x,y,z) = 6$$

where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is the differentiable function given by

$$\phi(x,y,z) = x^2 + y^2 + z^2.$$

So a normal to the tangent plane at (x,y,z) on the surface is

$$\nabla\phi = (2x, 2y, 2z).$$

At $(1,2,-1)$ the normal is

$$\nabla\phi(1,2,-1) = (2,4,-2)$$

and hence an equation for the tangent plane at $(1,2,-1)$ is

$$2x + 4y - 2z = 12.$$

Tangent lines

Find the tangent line to the curve

$$3x^2 + 2y^2 = 14$$

at the point $(2, 1)$.

The curve is $\phi(x, y) = 14$ with $\phi : \mathbb{R}^2 \to \mathbb{R}$ a differentiable function given by

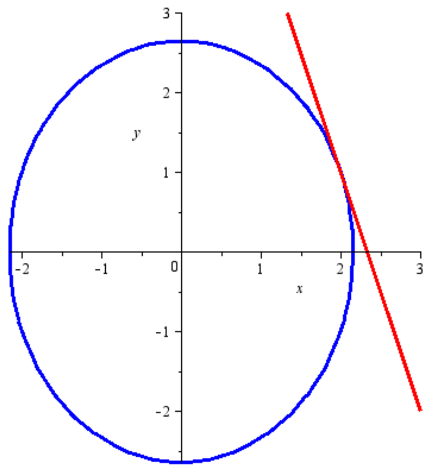
$$\phi(x, y) = 3x^2 + 2y^2.$$

A normal at (x, y) on the curve is $\nabla\phi = (6x, 4y)$ and at $(2, 1)$,

$$\nabla\phi(2, 1) = (12, 4).$$

Hence a Cartesian equation for the tangent line is

$$12x + 4y = 28.$$



Note that we don't need to solve for y to find the tangent line.

[Exercise: check using a 'first year' method with $y = \sqrt{7 - \frac{3}{2}x^2}$.]

Tangent planes

Consider the surface S_1 in $\mathbb{R}^3$ defined by

$$S_1 = \{(x, y, z) : x^3 + 2y^2 - z = 0\}.$$

At the point $(2, 1, 10)$ find

(i) a parametric equation of the normal line and
(ii) a Cartesian equation of the tangent plane.

The surface is the 0 level set of the differentiable function $\phi : \mathbb{R}^3 \to \mathbb{R}$ given by $\phi(x, y, z) = x^3 + 2y^2 - z$.

So a normal to the surface at (x, y, z) is given by $\nabla\phi = (3x^2, 4y, -1)$ and at $(2, 1, 10)$ by $\nabla\phi(2, 1, 10) = (12, 4, -1)$.

(i) $\mathbf{r}(t) = (2, 1, 10) + t(12, 4, -1), \quad t \in \mathbb{R}.$
(ii) $12x + 4y - z = 18.$

Tangent planes

Find the best affine approximation to $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y) = x^3 + 2y^2$ at the point $(2, 1)$ and compare this with the equation of the tangent plane to S_1.

The partial derivatives of f exist and are continuous everywhere. So f is differentiable and

$$Df = Jf = \begin{pmatrix} 3x^2 & 4y \end{pmatrix} \quad \text{or} \quad \nabla f = (3x^2, 4y).$$

The best affine approximation at $(2, 1)$ is

$$\begin{aligned} T(x, y) &= f(2, 1) + \nabla f(2, 1) \cdot (x - 2, y - 1) \\ &= 10 + (12, 4) \cdot (x - 2, y - 1) \\ &= 10 + 12(x - 2) + 4(y - 1) \\ &= -18 + 12x + 4y. \end{aligned}$$

Note that the graph of T give by $z = T(x, y)$ is

$$z = -18 + 12x + 4y \quad \Rightarrow \quad 12x + 4y - z = 18$$

Tangent planes

Find the curves obtained by the intersection of $S_1 = \{(x, y, z) : x^3 + 2y^2 - z = 0\}$ with the planes (i) $x = 2$, and (ii) $y = 1$.

Find the tangent vectors to these curves at the point $(2, 1, 10)$ and hence give a parametric equation for the tangent plane to S_1 at $(2, 1, 10)$.

(i) $\mathbf{r}_1(t) = (2, t, 8 + 2t^2)$, $\quad \mathbf{r}_1 : \mathbb{R} \to \mathbb{R}^3$.
(ii) $\mathbf{r}_2(t) = (t, 1, t^3 + 2)$, $\quad \mathbf{r}_2 : \mathbb{R} \to \mathbb{R}^3$.

Tangent vectors to the curves are

$$\mathbf{r}_1'(t) = (0, 1, 4t), \qquad \text{and} \qquad \mathbf{r}_2'(t) = (1, 0, 3t^2)$$

and at $(2, 1, 10)$ these are

$$\mathbf{r}_1'(1) = (0, 1, 4), \qquad \text{and} \qquad \mathbf{r}_2'(2) = (1, 0, 12).$$

So the tangent plane is given by

$$\mathbf{r}(s, t) = (2, 1, 10) + t(0, 1, 4) + s(1, 0, 12).$$

Tangent planes

Consider $g : \mathbb{R}^3 \to \mathbb{R}$ with

$$g(x, y, z) = 3x^2 - 24x + 3y^2 - 10y + 3z^2 - 59z + 333$$

and the surface S_2 defined as the 0 level set of g, that is,

$$S_2 = \{(x, y, z) : g(x, y, z) = 0\}.$$

(i) Describe S_2.

(ii) Show that S_2 touches S_1 tangentially at $(2, 1, 10)$.

(iii) Solve $g(x, y, z) = 0$ for z in terms of x and y for (x, y) "near" $(2, 1)$.
[That is find $f : \mathbb{R}^2 \to \mathbb{R}$ with $z = f(x, y)$ near $(2, 1)$.]

(iv) Find the best affine approximation to f near $(2, 1)$.

(v) What fact involving ∇g makes it possible to find f?

Tangent planes

(i)

$$g(x, y, z) = 3x^2 - 24x + 3y^2 - 10y + 3z^2 - 59z + 333$$

and

$$S_2 = \{(x, y, z) : 3x^2 - 24x + 3y^2 - 10y + 3z^2 - 59z + 333 = 0\}.$$

Completing the squares x, y and z,

$$g(x, y, z) = 3(x - 4)^2 + 3(y - \frac{5}{3})^2 + 3(z - \frac{59}{6})^2 - \frac{143}{6}.$$

So S_2 is implicitly defined by the equation

$$3(x - 4)^2 + 3(y - \frac{5}{2})^2 + 3(z - \frac{59}{6})^2 = \frac{143}{6}$$

which is a sphere of radius $\sqrt{\frac{143}{18}}$ centred at $(4, \frac{5}{2}, \frac{59}{6})$.

Tangent planes

(ii)

$$g(x,y,z) = 3x^2 - 24x + 3y^2 - 10y + 3z^2 - 59z + 333$$

and

$$S_2 = \{(x,y,z) : 3x^2 - 24x + 3y^2 - 10y + 3z^2 - 59z + 333 = 0\}.$$

First check that $g(2,1,10) = 0$ so that $(2,1,10)$ lies on S_2.

[We previously found that a normal to the tangent plane of S_1 at $(2,1,10)$ was $\nabla\phi(2,1,10) = (12,4,-1)$.]

Now, a normal to the tangent plane of S_2 is given by

$$\nabla g = \left(6(x-4), 6(y-\frac{5}{3}), 6(z-\frac{59}{6})\right) \quad \Rightarrow \quad \nabla g(2,1,10) = (-12,-4,1).$$

Since one normal is a multiple of the other, the two tangent planes are parallel.

Tangent planes

(iii)

$$g(x,y,z) = 3x^2 - 24x + 3y^2 - 10y + 3z^2 - 59z + 333$$

and

$$S_2 = \{(x,y,z) : 3x^2 - 24x + 3y^2 - 10y + 3z^2 - 59z + 333 = 0\}.$$

$$\begin{aligned} 3(x-4)^2 + 3(y-\frac{5}{2})^2 + 3(z-\frac{59}{6})^2 - \frac{143}{6} &= 0 \\ \Rightarrow \quad 3(z-\frac{59}{6})^2 &= \frac{143}{6} - 3(x-4)^2 - 3(y-\frac{5}{2})^2 \\ \Rightarrow \quad z &= \frac{59}{6} + \sqrt{\frac{143}{18} - (x-4)^2 - (y-\frac{5}{2})^2} \end{aligned}$$

(iv)

The best affine approximation is given by the tangent plane that has already been found.

$$T(x,y) = 10 + 12(x-2) + 4(y-1).$$