The Riemann Curvature Tensor

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Abstract

A tensor is a mathematical object that has applications in areas including physics, psychology, and artificial intelligence. The Riemann curvature tensor is a tool used to describe the curvature of n-dimensional spaces such as Riemannian manifolds in the field of differential geometry. The Riemann tensor plays an important role in the theories of general relativity and gravity as well as the curvature of spacetime. This paper will provide an overview of tensors and tensor operations. In particular, properties of the Riemann tensor will be examined. Calculations of the Riemann tensor for several two and three dimensional surfaces such as that of the sphere and torus will be demonstrated. The relationship between the Riemann tensor for the 2-sphere and 3-sphere will be studied, and it will be shown that these tensors satisfy the general equation of the Riemann tensor for an n-dimensional sphere. The connection between the Gaussian curvature and the Riemann curvature tensor will also be shown using Gauss's Theorem Egregium.

Keywords: tensor, tensors, Riemann tensor, Riemann curvature tensor, curvature

1 Introduction

Coordinate systems are the basis of analytic geometry and are necessary to solve geometric problems using algebraic methods. The introduction of coordinate systems allowed for the blending of algebraic and geometric methods that eventually led to the development of calculus. Reliance on coordinate systems, however, can result in a loss of geometric insight and an unnecessary increase in the complexity of relevant expressions. Tensor calculus is an effective framework that will avoid the cons of relying on coordinate systems. Tensor calculus acknowledges the need for coordinate systems but avoids implementing a coordinate system until needed. This allows for equations that are valid in all coordinate systems simultaneously [4].

Curvature describes how a geometric object such as a curve deviates from a straight line or a surface from a flat plane. Curvature can be expressed simply as a scalar that represents the magnitude of this deviation. Curvature can also be described as a vector that takes into account the direction of the curve along with the magnitude. For more complex objects such as surfaces or n-dimensional spaces, a more complex object is needed to describe the curvature [3]. One such object, which will be the primary focus of this paper, is the Riemann curvature tensor.

This paper will explore the use of tensor calculus in describing the curvature of manifolds. Prior to this, the underlying ideas and definitions related to tensors will be discussed. The focus of this paper is the calculations of the Riemann tensor for the surfaces of the 2-sphere, 2-torus, and 3-sphere. Results of these calculations are provided in the body of the paper and more detailed calculations are provided in an appendix.

2 Background

Consider a physical quantity such as a vector. A vector can be described in terms of its magnitude and direction and therefore does not require the application of a coordinate system. If we want to perform calculus with vectors, we need to be able to describe their components in terms of the surrounding space, which requires the use of a coordinate system. Because this choice of coordinate system is arbitrary, the vector is not dependent upon this choice. Once a coordinate system has been chosen, we can describe the vector in terms of the basis vectors of the chosen system. Under a change of coordinates, this description would differ although the vector itself is unchanged. Is there a way to describe the vector without using a particular coordinate system? This is the aim of tensor calculus. In order to construct expressions independent of the choice of coordinates, we must determine how individual elements change under a change of coordinates [4].

A variant is an object that can be calculated using a similar rule in different coordinate systems. As expected, the results vary when the same rule is applied to different coordinate systems. Tensors are a special type of variant that transform according to certain rules under a change of coordinates [8]. A tensor is a multi-dimensional array of mathematical objects, generally numbers or functions. Tensors are classified based on their number and type of components. The components of a tensor may be covariant or contravariant. Covariance and contravariance describe how quantities change with a change of basis. Covariant components change in the same way as changes to the basis vectors and are denoted with lower indices a_i. Contravariant components change inversely to changes in the basis vectors and are denoted with upper indices a^i [9].

The rank of a tensor is the total number of covariant and contravariant components. Tensors of rank 0 are scalars, tensors of rank 1 are vectors, and tensors of rank 2 are matrices. For example, the metric tensor, which has rank two, is a matrix. Higher-order tensors are multi-dimensional arrays. A visualization of a rank 3 tensor from [3] is shown in figure 1 below.

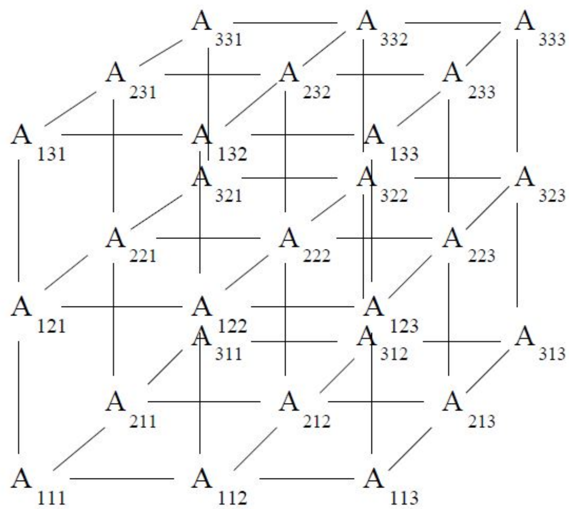

Figure 1: Visualization of a rank 3 tensor.

Intrinsic objects are those which can be obtained by measuring distances and computing the derivatives of those distances in some space. The metric tensor is an example of an intrinsic object. The metric tensor describes how to compute distances and lengths of curves in a given space [7]. In three dimensional Euclidean space, the distance ds between two points whose cartesian coordinates are (x, y, z) and $(x + dx, y + dy, z + dz)$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{1}$$

Riemann generalized this idea and extended it to spaces of n-dimensions. Consider two points whose coordinates in any system are x^i and $x^i + dx^i$ where i ranges from 1 to n. He defined the infinitesimal distance between the two points to be

$$ds^2 = g_{ij}dx^i dx^j \tag{2}$$

where the coefficients g_{ij} are functions of the coordinates x^i. These coefficients collectively make up the metric tensor g_{ij} [10]. For example, the metric tensor for three dimensional Euclidean space in cartesian coordinates is the collection of the coefficients from equation 1. The metric tensor is therefore the 3x3 identity matrix.

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Because the metric tensor is an intrinsic object, subsequent objects that can be described in terms of the metric tensor and its derivatives are also intrinsic. One object that can be derived from the metric tensor is the Christoffel symbol. The Christoffel symbol describes the variation in basis vectors from one point to another in curvilinear coordinate systems. The Christoffel symbols are the n^3 partial derivatives of each basis vector differentiated with respect to each coordinate.The Christoffel symbols measure the rate of change of the covariant basis with respect to the coordinate variables [1]. Christoffel symbols of the first kind (three lower indices) can be found directly from the metric tensor using the following formula

$$\Gamma_{ijk} = \frac{1}{2}(-g_{ijk} + g_{jki} + g_{kij}) \tag{4}$$

where the partial derivative of a tensor with respect to x^k is indicated by a final subscript k [5]. Christoffel symbols of the second kind (one upper and two lower indices) are obtained by taking the inner product (defined in methods section below) of the Christoffel symbols of the first kind with the metric tensor

$$\Gamma^i_{jk} = g^{ir}\Gamma_{jkr} \tag{5}$$

where g^{ir} is the contravariant metric tensor.

In general, partial derivatives of the components of a vector or a tensor are not components of a tensor. This is the case for Christoffel symbols which are partial derivatives of the metric tensor but are not tensors themselves. Because tensor properties, namely invariance, are desirable, a new differential operator ∇_i, the covariant derivative, arises. The covariant derivative produces tensors from tensors. The resulting tensors are one covariant order greater than the original tensor[4]. In affine coordinates, the covariant basis is the same at all points. Subsequently, the covariant derivative is commutative. However, this is not the case for curved surfaces.

The non-commutativity of the covariant surface derivative is measured with the Riemann tensor. The Riemann tensor is a four-index tensor that provides an intrinsic way of describing the curvature of a surface. The Riemann tensor of the second kind can be represented independently from the formula

$$R^i_{jkm} = \frac{\partial \Gamma^i_{jm}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^m} + \Gamma^i_{rk}\Gamma^r_{jm} - \Gamma^i_{rm}\Gamma^r_{jk} \tag{6}$$

The Riemann tensor of the first kind is represented similarly, using Christoffel symbols of the first kind

$$R_{ijkm} = \frac{\partial \Gamma_{jmi}}{\partial x^k} - \frac{\partial \Gamma_{jki}}{\partial x^m} + \Gamma_{imr}\Gamma^r_{jk} - \Gamma_{ikr}\Gamma^r_{jm} \tag{7}$$

Alternatively, the Riemann tensor of the first kind can be obtained by lowering the contravariant index. This is done by taking the inner product of the Riemann tensor of the second kind with the metric tensor as follows:

$$R_{ijkm} = g_{ir}R^r_{jkm} \tag{8}$$

For a sphere of radius r, components of the Riemann tensor of the first kind can be calculated directly from the metric, without first calculating the Christoffel symbols, using the following equation from [10].

$$R_{ijkm} = \frac{1}{r^2}(g_{ik}g_{jm} - g_{im}g_{jk}) \tag{9}$$

Tensor calculations are significantly simplified when symmetric properties of tensors are used. The Riemann tensor has several symmetric properties that will help to simplify calculations

$$\begin{array}{cc} \text{first skew symmetry} & R_{ijkl} = -R_{jikl} \\ \text{second skew symmetry} & R_{ijkl} = -R_{ijlk} \\ \text{block symmetry} & R_{ijkl} = R_{klij} \end{array} \tag{10}$$

In general, the number of independent components C of the Riemann tensor in n-dimensions is given by the equation

$$C = \frac{1}{12}n^2(n^2 - 1) \tag{11}$$

from [5].

A manifold is a space that can be curved but is locally flat. That is, near each point the space resembles Euclidean space. One example of a curved manifold is the surface of a sphere. The Riemann tensor is the most common tool used to describe the curvature of a Riemannian manifold. Contraction, a tensor operation defined later in the paper, of the Riemann tensor produces the Ricci tensor. The Ricci tensor provides a way measure the degree to which a space differs from Euclidean space. Contraction of the Ricci tensor produces the scalar curvature or Ricci scalar. The Ricci scalar is the simplest curvature invariant of a manifold.[4]

The Riemann tensor, Ricci tensor, and Ricci scalar are all derived from the metric tensor and are therefore intrinsic measures of curvature. There is another tensor, simply called the curvature tensor, that can be used to express the curvature of a surface. This tensor depends on the way in which the surface is embedded in the surrounding space. The curvature tensor is therefore an extrinsic object because it is not innate to the surface itself. Gauss's Theorem Egregium relates the Riemann tensor and the curvature tensor. This is a powerful result that links the intrinsic and extrinsic perspectives of curvature [4].

3 Methods

3.1 Summation Notation

Computations with tensors often involve the use of Einstein summation notation. This notation uses repeated indices rather than the traditional sigma notation to represent a sum:

$$a_i x_i = a_1 x_1 + a_2 x_2 + ... + a_n x_n \tag{12}$$

where $1 \leq i \leq n$ is the range for summation. Therefore the expressions $a_{ii}x_k$ and $a_{ik}x_k$ represent a summation while $a_{ij}x_k$ does not. Because the choice of summation index is arbitrary ($a_{ik}x_k$ and $a_{im}x_m$ represent the same summation) this index is called a dummy index. The non-repeated index has the same range as the dummy index and is called a free index.

3.2 Tensor Contraction and Inner Product

Another operation that can be performed on tensors is contraction. The contraction of a tensor is produced by summing over a pair of indices and is a generalization of the trace operation for matrices. Contraction can occur in a single mixed tensor or on a linear combination or product of tensors. Contraction on a single tensor reduces the rank of a tensor by order 2. Contraction can be repeated until all indices are gone, resulting in a tensor of order 0, which is an invariant. This is a powerful result because it allows for forming invariants using linear combinations, products, and contraction of tensors. [6] The inner product between two tensors is the contraction of the inner indices (the last index of the first tensor and the first index of the last tensor). Essentially, the inner product is equating an upper index of one tensor with a lower index of another tensor and summing over the repeated index [2].

3.3 Raising and Lowering of Indices

The tensor type can be changed by taking the inner product with the metric tensor. Covariant tensors can be converted to contravariant tensors and vice versa using the covariant or contravariant metric tensor. This process is referred to as the raising and lowering of indices. This process does not change the rank of a tensor, but does change the type. In Cartesian coordinates the metric is the identity matrix and therefore raising and lowering indices does not change the value of the tensor components[6].

3.4 Metric Tensors

To calculate the Riemann tensor, we need only the metric tensor for the surface. The metric tensors for the surfaces of the 2-sphere, 2-torus, and 3-sphere are given below.
2-Sphere
$\theta = 1, \phi = 2$

$$g_{ij} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix} \tag{13}$$

2-Torus
$\theta = 1, \phi = 2$

$$g_{ij} = \begin{bmatrix} (R + r\cos\phi)^2 & 0 \\ 0 & r^2 \end{bmatrix} \tag{14}$$

3-Sphere
$\psi = 1, \theta = 2, \phi = 3$

$$g_{ij} = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & r^2 \sin^2\psi & 0 \\ 0 & 0 & r^2 \sin^2\psi \sin^2\theta \end{bmatrix} \tag{15}$$

4 Results

4.1 2-Sphere

Riemann Curvature Tensor

$$\begin{array}{ll} \text{First Kind} & \text{Second Kind} \\ R_{1212} = r^2 \sin^2\theta & R^1_{212} = \sin^2\theta \\ R_{1221} = -r^2 \sin^2\theta & R^1_{221} = -\sin^2\theta \\ R_{2121} = r^2 \sin^2\theta & R^2_{121} = 1 \\ R_{2112} = -r^2 \sin^2\theta & R^2_{112} = -1 \end{array} \tag{16}$$

Ricci Tensor

$$R_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{17}$$

Scalar Curvature

$$S = \frac{2}{r^2} \tag{18}$$

4.2 2-Torus

Riemann Curvature Tensor

$$\begin{array}{ll} \text{First Kind} & \text{Second Kind} \\ R_{1212} = r\cos\phi(R + r\cos\phi) & R^1_{212} = \frac{r\cos\phi}{R+r\cos\phi} \\ R_{1221} = -r\cos\phi(R + r\cos\phi) & R^1_{221} = -\frac{r\cos\phi}{R+r\cos\phi} \\ R_{2121} = r\cos\phi(R + r\cos\phi) & R^2_{121} = \frac{1}{r}\cos\phi(R + r\cos\phi) \\ R_{2112} = -r\cos\phi(R + r\cos\phi) & R^2_{112} = -\frac{1}{r}\cos\phi(R + r\cos\phi) \end{array} \tag{19}$$

Ricci Tensor

$$R_{ij} = \begin{bmatrix} \frac{1}{r}\cos\phi(R + r\cos\phi) & 0 \\ 0 & \frac{r\cos\phi}{R+r\cos\phi} \end{bmatrix} \tag{20}$$

Scalar Curvature

$$S = \frac{2\cos\phi}{r(R + r\cos\phi)} \tag{21}$$

4.3 3-Sphere

Riemann Curvature Tensor

$$\begin{array}{ll} \text{First Kind} & \text{Second Kind} \\ R_{1212} = r^2\sin^2\psi & R^1_{212} = \sin^2\psi \\ R_{1313} = r^2\sin^2\psi\sin^2\theta & R^1_{313} = \sin^2\psi\sin^2\theta \\ R_{2323} = r^2\sin^2\psi\sin^2\theta & R^2_{323} = \sin^2\psi\sin^2\theta \end{array} \tag{22}$$

Ricci Tensor

$$R_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2\sin^2\psi & 0 \\ 0 & 0 & 2\sin^2\psi\sin^2\theta \end{bmatrix} \tag{23}$$

Scalar Curvature

$$S = \frac{6}{r^2} \tag{24}$$

5 Conclusion

Tensors are intriguing mathematical objects that possess the desirable property of invariance. The metric tensor is of particular importance, allowing for the concept of distances and maintaing the invariance of distance in different coordinate systems. Calculations involving tensors are systematic and are simplified by the use of summation notation. Tensors and tensor calculus provide a powerful framework for solving problems in areas of physics such as general relativity. Representing fundamental physical laws in a tensor form ensures that they are invariant under a change of coordinates. Complex objects like tensors are needed to represent characteristics, such as curvature, of higher-dimensional spaces. The Riemann tensor is one such object used to express the curvature of manifolds.

6 Appendix A: Calculations

This appendix contains calculations of the Christoffel symbols, Riemann tensor, Ricci tensor, and Ricci scalar for the surface of a 2-sphere. Quantities that evaluate to zero have been excluded for conciseness. The calculations of these objects for the surfaces of the 2-torus and 3-sphere are similar and use each surface's corresponding metric tensor as listed in section 3.4.
Christoffel Symbols of the Second Kind

$$\Gamma^i_{jk} = \frac{g^{ir}}{2}(g_{ijk} - g_{jki} + g_{kij})$$

$$\Gamma^1_{22} = \frac{g^{11}}{2}(g_{122} - g_{221} + g_{212})$$

$$\Gamma^1_{22} = \frac{1}{2r^2}(0 - 2r^2 \sin\theta\cos\theta + 0)$$

$$\Gamma^1_{22} = -\sin\theta\cos\theta \tag{25}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{g^{22}}{2}(-g_{212} + g_{122} + g_{221})$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2r^2\sin^2\theta}(-0 + 0 + 2r^2\sin\theta\cos\theta)$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{\cos\theta}{\sin\theta}$$

Riemann Tensor of the Second Kind

$$R^i_{jkm} = \frac{\partial \Gamma^i_{jm}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^m} + \Gamma^i_{nk}\Gamma^n_{jm} - \Gamma^i_{nm}\Gamma^n_{jk}$$

$$R^1_{212} = \frac{\partial \Gamma^1_{22}}{\partial x^1} - \frac{\partial \Gamma^1_{21}}{\partial x^2} + [\Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{21}\Gamma^2_{22}] - [\Gamma^1_{12}\Gamma^1_{21} + \Gamma^1_{22}\Gamma^2_{21}] \tag{26}$$

$$R^1_{212} = (\sin^2\theta - \cos^2\theta) - (-\cos^2\theta)$$

$$R^1_{212} = \sin^2\theta$$

Riemann Tensor of the First Kind (using eq. 9)

$$R_{ijkm} = \frac{1}{r^2}(g_{ik}g_{jm} - g_{im}g_{jk})$$

$$R_{1212} = \frac{1}{r^2}(g_{11}g_{22} - g_{12}g_{21})$$

$$R_{1212} = \frac{1}{r^2}(r^2(r^2\sin^2\theta) - 0) \tag{27}$$

$$R_{1212} = r^2\sin^2\theta$$

Ricci Tensor

$$R_{hk} = g^{ab}R_{ahbk}$$

$$R_{11} = g^{11}R_{1111} + g^{22}R_{2121}$$

$$R_{11} = \frac{1}{R^2\sin^2\theta}(R^2\sin^2\theta)$$

$$R_{11} = 1 \tag{28}$$

$$R_{22} = g^{11}R_{1212} + g^{22}R_{2222}$$

$$R_{22} = \frac{1}{R^2}(R^2\sin^2\theta)$$

$$R_{22} = \sin^2\theta$$

Ricci Scalar

$$\begin{aligned}
S &= g^{ij}R_{ij} \\
S &= g^{11}R_{11} + g^{22}R_{22} \\
S &= \frac{1}{R^2}(1) + \frac{1}{R^2\sin^2\theta}(\sin^2\theta) \\
S &= \frac{1}{R^2} + \frac{1}{R^2} \\
S &= \frac{2}{R^2}
\end{aligned} \tag{29}$$

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