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Secular terms of classical planetary theories using the results of general theory

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Summary. Using the methods of the general theory given in (Laskar, 1985), we have analytically computed the differential system giving the secular variations of the orbital elements for the 8 major planets, at the order 2 with respect to the masses and up to degree 5 in the eccentricity-inclination variables with a relative precision of 10^{-6}. Relativistic and lunar perturbations are included. The entire system is integrated numerically over 10000 years and then developed in Taylor expansion around J2000.

We obtain new polynomial secular terms for the inner planets up to the power 10 of the time. Comparisons are made with Bretagon's theory VSOP82 and with the numerically integrated JPL ephemeris, DE102 (Newhall et al., 1983). The global accuracy is approximately 0".042 02/1000 yr for the inclination of the Earth. Using the theory of the rotation of the rigid Earth of Kinoshita (1977), we derive new formulas for the precessional quantities, up to t^{10}, and valid over 10000 years.

Key words: celestial mechanics – planetary theory – secular perturbations – precession

1. Introduction

In an earlier publication (Laskar, 1985) we described our methods for the construction of a general planetary theory based on the work of Duriez (1977, 1979). We have shown that it was possible to obtain accurate results in the computation of the differential system giving the secular variations of the orbital elements of the 8 major planets, at the order 2 with respect to the masses and up to degree 5 in the variables eccentricity-inclination.

In the present paper we integrate this differential system numerically and obtain the secular terms which appear in semi-analytical planetary theories in a polynomial form. The relativistic perturbations are included, along with the effect of the Moon on the motion of the Earth-Moon center of mass (Sect. 2)

To estimate the precision of our theory we make direct comparisons with the semi-analytical theory VSOP82 developed by Bretagnon (1982) at the Bureau des Longitudes. We also make comparisons with the numerically integrated JPL ephemeris, DE102 (Newhall et al., 1983) (Sect. 3).

In Sect. 4 we use both the secular motion of the ecliptic computed in Sect. 3 and the theory of the rotation of the rigid Earth of Kinoshita (1977) to derive new formulas for the precessional quantities, developed up to the power 10 of the time.

2. Equations for the planetary secular terms

2.1. Notation

We employ classical notation for the elliptic elements of a planet:

- a: semi-major axis
- e: eccentricity
- i: inclination
- Ω: longitude of the ascending node
- ϖ: longitude of perihelion
- ε: mean longitude at the initial epoch

The variable n denotes the mean motion, N the mean mean motion, A the semi-axis of reference (Laskar, 1985), and λ the mean longitude. We have:

$$\lambda = \int n\,dt + \varepsilon \tag{2}$$

and the Kepler relation:

$$n^2 a^3 = N^2 A^3 = k^2(1 + m/m_\odot) \tag{3}$$

where k is the Gaussian gravitational constant, m the mass of the designated planet, and $m_\odot$ the solar mass.

We use also the variables p, z, ζ defined by

$$\begin{aligned} n &= N(1+p) \Leftrightarrow a = A(1+p)^{-2/3} \\ z &= e \exp\sqrt{-1}\varpi \\ \zeta &= \sin\frac{i}{2}\exp\sqrt{-1}\Omega \end{aligned} \tag{4}$$

and the conjugates $\bar{z}\ \bar{\zeta}$. The variables z, $\bar{z}$, ζ, $\bar{\zeta}$ are called the eccentricity-inclination variables, as well as their real and imaginary parts, $\mathbf{k}, \mathbf{h}, \mathbf{q}, \mathbf{p}$:

$$\begin{aligned} z &= \mathbf{k} + \sqrt{-1}\mathbf{h} \\ \zeta &= \mathbf{q} + \sqrt{-1}\mathbf{p} \end{aligned} \tag{5}$$

In all this paper, except when explicitly specified, all the variables are mean variables.

2.2. *The autonomous system*

With the methods developed in a previous paper (Laskar, 1985), we have computed the autonomous system of order 2 (with respect to the masses) for the 8 planets, keeping all the terms up to degree 5 in the eccentricity-inclination variables. This system gives the secular variations of the variables eccentricity $\bar{z}_i$ and inclination $\bar{\zeta}_i$ $(i = 1, \ldots, 8)$. It includes 153824 monomial terms and can be expressed in the form:

$$\dot{\alpha} = \sqrt{-1}(\Phi_1\alpha + \Phi_3(\alpha, \bar{\alpha}) + \Phi_5(\alpha, \bar{\alpha})) \tag{6}$$

where

$$\alpha = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_8, \bar{\zeta}_1, \ldots \bar{\zeta}_8)$$

Φ_1 is a real matrix with constant coefficients.

Φ_3 gathers all the terms of degree 3.

Φ_5 gathers all the terms of degree 5.

All the 153824 numerical coefficients are real and are computed with a relative precision of 10^{-6} (Laskar, 1984, 1985).

The usual approach in general planetary theory is to integrate analytically such a system in order to obtain a solution expanded as a quasi-periodic function having the form

$$\alpha = \sum_k \alpha_k e^{\sqrt{-1}(\Sigma_i k_i \beta_i)t} \tag{7}$$

(Brumberg and Egorova, 1971; Brumberg, 1980; Bretagnon, 1974; Duriez, 1977, 1979).

We have shown that the presence of small divisors in the computation make this method very difficult if one wants to obtain an accurate solution (Laskar, 1984). On the other hand, the main periods of this system are very large, over 50000 years (the solution of the linear Laplace Lagrange system $\dot{\alpha} = \sqrt{-1}\Phi_1$ α gives a good approximation of these periods).

The secular variations of the elliptic elements are then very smooth, and it is quite easy to perform a numerical integration of the entire system (6), despite the 153824 terms, because the step size can be as large as 500 years. This numerical integration has been made over 1 million years (Laskar, 1984), and the results will be presented in a forthcoming paper. In the present paper, we are interested in the very beginning of this solution, over about 10000 years which will provide us enough points to compute numerically the Taylor expansion of the elliptic elements at the origin up to degree 10. This expansion will give us an accurate representation of the solution for historical use, and will allow us to make direct comparisons with the classical theories. For a longer span of time (up to several million years), there is no necessity for such high precision and a quasi-periodical representation which gives the main long periods of the solution should be preferred.

2.3. *Perturbations due to relativity and the Moon*

To compute the relativistic perturbations in the elliptical elements, we need consider only the first order terms in the motion of the perihelion, as given in (Brumberg, 1972) or (Lestrade and Bretagnon, 1982).

These terms are limited to the post-Newtonian approximation in $1/c^2$ (c is the speed of light). In the system (6), we just have to add the terms:

$$\left.\frac{d\bar{z}}{dt}\right|_R = -\sqrt{-1}\,\delta_R \bar{z} \tag{8}$$

with

$$\delta_R = 3\frac{n^3a^2}{c^2(1 + m/m_\odot)} \times \frac{1}{1 - e^2} = \kappa_R \frac{1}{1 - e^2} \tag{9}$$

Here, a is the mean value of the semi major axis of the considered planet, and n is connected with a by Kepler's law $n^2a^3 = k^2(1 + m/m_\odot)$. The values of a and n are taken from (Bretagnon, 1982) and are given in Table 1, as well as the computed values of κ_R. We use the value of the astronomical unit (AU) and the speed of light (c) adopted by the IAU of Grenoble (1976):

$$c = 299792458\,\mathrm{m s^{-1}}$$

$$1\,\mathrm{AU} = 1.49597870\,10^{11}\,\mathrm{m}$$

from which

$$c = 63241.0774\,\mathrm{AU/yr}$$

The perturbations due to the moon are limited to a single term:

$$\left.\frac{d\bar{z}}{dt}\right|_L = -\sqrt{-1}\,\delta_L \bar{z}_3 \tag{10}$$

with

$$\delta_L = 3.192472\,10^{-7}$$

Table 1. Initial values for the mean motion n and corresponding values of the semi-major axis a for J2000 (Bretagnon, 1982). Values of the relativistic perturbation. κ_R (Eq. 9)

Planet	n (rd/yr)	a (AU)	$\kappa_R \times 10^{12}$
Mercury	26.0879360339024	0.387098350584818	1995650
Venus	10.2133357162869	0.723329859446194	418116
Earth	6.2830662287852	1.00000105726665	186053
Mars	3.3406528698589	1.5236793816472	64924
Jupiter	0.5297217887326	5.202603230909	3015
Saturn	0.2127618949734	9.554909635329	659
Uranus	0.0745768020043	19.2184461013	114
Neptune	0.0380284307602	30.1103869089	37

3. The NGT solution for the secular terms of the inner planets

3.1. *Polynomial expression of the secular terms*

In classical planetary theories, like VSOP82, the secular variations of the mean variables $a_i, \eta_i, \mathbf{k}_i, \mathbf{h}_i, \mathbf{q}_i, \mathbf{p}_i$, are given in a polynomial form:

$$x = x_0 + x_1t + x_2t^2 + x_3t^3 + \cdots \tag{11}$$

which is supposed to be the Taylor expansion of the considered variable $x(t)$ at the origin ($t = 0$). The coefficients are computed order by order with respect to the masses, and each order requires more and more computations. The VSOP82 solution is of order 3 with respect to the masses; the secular terms are then given to degree 3 for all the planets. This limitation to degree 3

Table 2. Integration constants J2000 (2451545) of NGT and VSOP82 (Bretagnon, 1982)

Planet	$1/m$	N (rd/1000 yr)	$\mathbf{k}_0$	$\mathbf{h}_0$	$\mathbf{q}_0$	$\mathbf{p}_0$
Mercury	6023600	26087.9031415742	0.04466059760	0.20072331368	0.04061563384	0.04563550461
Venus	408523.5	10213.2855462110	−0.00449282133	0.00506684726	0.00682410142	0.02882285775
Earth	328900.5	6283.0758491800	−0.00374081650	0.01628447663	0.	0.
Mars	3098710.	3340.6124314923	0.08536560252	−0.03789973236	0.01047042574	0.01228449307
Jupiter	1047.355	529.6909650946	0.04698572124	0.01200385748	−0.00206561098	0.01118377157
Saturn	3498.5	213.2990954380	−0.00296003595	0.05542964254	−0.00871747436	0.01989147301
Uranus	22869.	74.7815985673	−0.04595132376	0.00563791307	0.00185915075	0.00648617008
Neptune	19314.	38.1330356378	0.00599977571	0.00669242413	−0.01029147819	0.01151683985

with respect to the time makes the precision decrease rapidly after 1000 years, and the solution was extended later on up to degree 6 for the outer planets by an iterative method (Bretagnon, 1982).

Our method is different from the classical theories. We have computed the differential system (6) in a fully analytical way. It gives the secular variations of the mean variables for any time at the order 2 with respect to the masses. The solution of this system is obtained numerically and then expanded up to any power of the time in the form of the classical theories (11). The main limitation is then the precision of the differential system (6).

We performed the numerical integration of the system (6), including the effects due to the relativity (9) and the Moon (10), with a fourth-order Runge-Kutta method and a step size of 250 yr. The integration spanned 10000 yr on each sides of the origin (J2000) and used the initial values given in Table 2. The global accuracy of the numerical integration is estimated by comparing the results with a 125 yr step size integration, and is better than 10^{-10}.

We have then computed numerically the derivatives at the origin with the method of symmetric differences and a step of 500 yr up to order 10. The relative precision of these derivatives decreases with the order, but the higher powers of t require less precision than the lower powers. Our solution will be denoted by NGT (Numerical General Theory) in the remainder of this paper.

We obtain a Taylor expansion which we can directly compare with the secular terms of the classical theory VSOP82 for the different planets (Table 3). In this table, we have limited the

Table 3. Comparison of the secular polynomials of NGT (above), and VSOP82 (below). The time t is measured in units of 10000 julian years from J2000 (JD 2451545.0)

		t	t^2	t^3	t^4
Mercury	$\mathbf{k} \times 10^{10}$	−552206151	−18628892	7904951	589540
		−552114624	−18603970	6336200	
	$\mathbf{h} \times 10^{10}$	143780476	−79764913	−3043725	811285
		143750118	−79746890	−2630900	
	$\mathbf{q} \times 10^{10}$	65445517	−10713235	2245279	−376780
		65433117	−10712150	2114900	
	$\mathbf{p} \times 10^{10}$	−127599238	−9132313	1898818	−640089
		−127633657	−9133500	1800400	
Venus	$\mathbf{k} \times 10^{10}$	31262529	6045841	−6834889	493964
		31259019	6059130	−6923900	
	$\mathbf{h} \times 10^{10}$	−36123807	18469749	328049	−613650
		−36121239	18396270	97100	
	$\mathbf{q} \times 10^{10}$	138139141	−10911318	−18641793	601726
		138133826	−10909420	−18592000	
	$\mathbf{p} \times 10^{10}$	−40387970	−62329244	2473042	4228784
		−40384791	−62328910	2513700	
Earth	$\mathbf{k} \times 10^{10}$	−82273540	27632106	11695572	−2695722
		−82266699	27489390	10421700	
	$\mathbf{h} \times 10^{10}$	−62033371	−33841635	8510121	2770542
		−62030259	−33538880	7118500	
	$\mathbf{q} \times 10^{10}$	−113462152	12373396	12654170	−1371808
		−113469002	12373140	12705000	
	$\mathbf{p} \times 10^{10}$	10183600	47019367	−5417367	−2507948
		10180391	47019980	−5382900	

(Continued)

Table 3 (continued)

		t	t^2	t^3	t^4
Mars	$\mathbf{k} \times 10^{10}$	376295028	−246525938	−36760524	11112422
		376330152	−246574160	−39524100	
	$\mathbf{h} \times 10^{10}$	624615290	155237412	−63487894	−6592895
		624657465	155272320	−67194000	
	$\mathbf{q} \times 10^{10}$	17131135	−40767021	−13883445	916176
		17138526	−40775910	−13860000	
	$\mathbf{p} \times 10^{10}$	−107996526	−19223063	8718504	3090121
		−108020083	−19221950	8837300	
Jupiter	$\mathbf{k} \times 10^{10}$	111977082	−107308403	−42835068	18629325
		113010377	−109301260	−42874800	20539000
	$\mathbf{h} \times 10^{10}$	216186108	97412918	−49954664	−10191892
		217149360	98585390	−51310900	−9007000
	$\mathbf{q} \times 10^{10}$	−31351179	−16648979	7994237	3567383
		−31340156	−16673920	7692600	
	$\mathbf{p} \times 10^{10}$	−23437577	20825055	5342201	−3397595
		−23427562	20867600	5072100	
Saturn	$\mathbf{k} \times 10^{10}$	−524323398	301658976	127946660	−59122530
		−529602626	309284050	129621500	−59959000
	$\mathbf{h} \times 10^{10}$	−371623742	−314855651	153439338	30789323
		−375593887	−319902360	159863300	32451000
	$\mathbf{q} \times 10^{10}$	80165285	41304350	−20016004	−8728168
		80171499	41422820	−19604900	−9439000
	$\mathbf{p} \times 10^{10}$	59460176	−52230829	−13164931	8546654
		59439766	−52351170	−12721900	8295000
Uranus	$\mathbf{k} \times 10^{10}$	18145856	−672379	−4377283	1576471
		18344050	−808490	−4539600	2185000
	$\mathbf{h} \times 10^{10}$	−74663156	11837002	−4049094	−1177159
		−74964350	12102000	−4208800	−1714000
	$\mathbf{q} \times 10^{10}$	−12375589	−2050962	778117	159537
		−12449382	−2073730	762100	
	$\mathbf{p} \times 10^{10}$	−11687971	3134975	706005	−488530
		−11744733	3177990	731700	
Neptune	$\mathbf{k} \times 10^{10}$	815372	−1138645	−290625	112243
		871279	−1199020	−403400	
	$\mathbf{h} \times 10^{10}$	7559591	767685	−325657	−83896
		7824336	808010	−395500	
	$\mathbf{q} \times 10^{10}$	−78472	−64004	89989	24765
		−72727	−65680	166800	
		2570836	194213	74755	42727
	$\mathbf{p} \times 10^{10}$	2575536	193770	133100	

degree of our solution to 4 to make the comparisons easier (for higher degrees, see Table 5). In Table 3, the time t is expressed in units of 10000 yr.

3.2. *Comparison of VSOP82 and NGT*

VSOP82 is an order 3 classical theory. In such a theory, the powers of the time are computed order by order: the coefficients of t appear at order 1, the coefficients of t^2 appear only at order 2, and so on. The coefficient x_1 of t in (11) is then computed at order 3 with respect to the masses, the coefficient x_2 of t^2 at the order 2, and the coefficient x_3 of t^3 at the order 1 with respect to the masses. As with NGT, the coefficient x_1 of VSOP82 does include the perturbations due to relativity and the effects of the Moon on the Earth-Moon center of mass. However, in VSOP82, the eccentricity and inclination variables are included numerically, so their contributions exist at all degrees.

In NGT each coefficient x_i of the Taylor expansion (11) is given with the same precision (order 2 with respect to the masses and degree 5 in the eccentricity-inclination.) We can then consider the coefficients x_1 of t in VSOP82 as a solution of reference which should differ very little from the exact value (this difference is of order 4 with respect to the masses). This coefficient x_1 is also the value of the derivative of $x(t)$ at the origin. An estimate of the precision of our differential system (6), and hence of our

Table 4. Relative precision of the General Theory. This precision is estimated by comparing the values of the derivatives at the origin (Eq. 6) with the corresponding values of VSOP82 (Table 3). For each variable x, $\Delta x = |x_{NGT} - x_{VSOP82}|$

Mercury	$\Delta\mathbf{k}_1/\sqrt{\mathbf{k}_1^2+\mathbf{h}_1^2}$	$\Delta\mathbf{h}_1/\sqrt{\mathbf{k}_1^2+\mathbf{h}_1^2}$	$\Delta\mathbf{q}_1/\sqrt{\mathbf{q}_1^2+\mathbf{p}_1^2}$	$\Delta\mathbf{p}_1/\sqrt{\mathbf{p}_1^2+\mathbf{p}_1^2}$
Mercury	0.000160	0.000053	0.000086	0.000240
Venus	0.000073	0.000054	0.000037	0.000022
Earth	0.000066	0.000030	0.000060	0.000028
Mars	0.000048	0.000058	0.000068	0.000215
Jupiter	0.004244	0.003956	0.000282	0.000256
Saturn	0.008215	0.006178	0.000062	0.000204
Uranus	0.002579	0.003920	0.004335	0.003335
Neptune	0.007353	0.034819	0.002234	0.001827

solution NGT, is thus given by the differences between the coefficients of t in VSOP82 and in NGT (Table 4). In this table, we have put the relative precision of the second hand member of (6) for each of the variables $\mathbf{k}, \mathbf{h}, \mathbf{q}, \mathbf{p}$. We can see that the precision is high, especially in the case of the inner planets. For the inclination of the Earth, for example, it gives a difference of about 0″015 after 1000 yr. This table shows that the differential system (6) obtained by the methods of the general planetary theory is very accurate. This precision still exists in the numerical solution NGT, and it allows us to use these results in order to extend VSOP82.

For the outer planets, our solution is less accurate, due to our limitation in degree and order. Indeed, for the Jupiter-Saturn couple, the contribution of degree 7 and of order 3 resulting from the great inequality $2N_5 - 5N_6$ is probably important and would explain the differences between NGT and VSOP82.

On the other hand, with the classical theories, it is possible to obtain higher degree terms, x_4, x_5, x_6 for the outer planets with an iterative method (Bretagnon, 1982), and even higher for the Jupiter-Saturn couple by harmonic analysis (Simon and Francou, 1982). For these two reasons, we shall now restrict ourselves to the inner planets only (Mercury, Venus, the Earth, and Mars).

3.3. *New secular terms for the variables* $\mathbf{k}, \mathbf{h}, \mathbf{q}, \mathbf{p}$ *of the inner planets*

We shall derive new secular terms for the inner planets up to degree 10 in the time t, using the results of NGT and VSOP82.

The initial values x_0 are taken from VSOP82 (Table 2).

We have already said that the x_1 term in VSOP82 is always the best, and we shall keep it.

The x_2 terms of both solutions are computed at the same order, but there are some slight differences between them:

- in NGT, the degree in the eccentricity-inclination variables is limited to 5.
- in VSOP82, all the contributions from relativity and of the Moon are not included in the x_2 terms.

So we have kept the Keplerian part of x_2 from VSOP82 and added to it the Moon and relativity contributions from NGT.

The x_3 terms are taken from NGT. Indeed, they are computed up to order 2 while the VSOP82 x_3 terms are computed up to order 1.

The terms $x_4, x_5, \ldots, x_{10}$ appear only in NGT.

The resulting secular polynomials of degree 10 for the inner planets are given in Table 5 for each of the eccentricity-inclination variables $\mathbf{k}, \mathbf{h}, \mathbf{q}, \mathbf{p}$.

3.4. *Semi-major axis a_i and mean longitude λ_i*

Poisson's Theorem (Duriez, 1978) shows that there are no secular terms in the semi-major axis of the planets at the order 2 with respect to the masses. However, some secular terms appear in the classical theories at the order 3 with respect to the masses (Simon and Bretagnon, 1978).

NGT is a second order theory, but the identification order by order is not strictly made at the order 2 with respect to the masses, and some secular terms may appear in the variable p_i related to the semi-major axis a_i (5). These secular terms are just a little part of the order 3 contribution and are then meaningless until we compute the complete order 3 (which is a difficult task): for the present, NGT cannot give reliable information about the secular variations of the semi-major axis.

The study of the mean longitude λ is more difficult because it involve the study of the variable p connected with the semi-major axis a and of the mean longitude of the initial epoch, ε(4). Indeed, $n = N + Np$ and

$$\frac{d\lambda}{dt} = N + Np + \frac{d\varepsilon}{dt} \tag{12}$$

The secular variation of ε is given by an autonomous system similar to (6), even in the eccentricity-inclination variables, and limited to degree 4 in our computations (Laskar, 1985).

$$\frac{d\varepsilon}{dt} = K_0 + K_2(\alpha, \bar{\alpha}) + K_4(\alpha, \bar{\alpha}) \tag{13}$$

with $\alpha = (\bar{z}_1, \ldots, \bar{z}_8, \ldots, \bar{\zeta}_1, \ldots, \bar{\zeta}_8)$.

We must add to these terms the relativistic perturbation, given in (Lestrade and Bretagnon, 1982):

$$\left.\frac{d\lambda}{dt}\right|_R = \frac{n^3a^2}{c^2(1+m/m_\odot)}\left\{-\frac{1-e^2-\sqrt{1-e^2}}{e^2}\left[-10\left\langle\frac{a^4}{r^4}\right\rangle(1-e^2)+\left\langle\frac{a^3}{r^3}\right\rangle(17+e^2)-7\left\langle\frac{a^2}{r^2}\right\rangle\right]+8\left\langle\frac{a^3}{r^3}\right\rangle(1-e^2)-20\left\langle\frac{a^2}{r^2}\right\rangle+6\right\} \tag{14}$$

where r is the radius vector, and $\langle x\rangle$ denotes the averaged value of x.

Table 5. Secular terms for the inner planets. The time t is measured in units of 10000 julian years from J2000 (JD 2451545.0)

	$\lambda \times 10^{10}$	$\mathbf{k} \times 10^{10}$	$\mathbf{h} \times 10^{10}$	$\mathbf{q} \times 10^{10}$	$\mathbf{p} \times 10^{10}$
Mercury					
	44026088424	446605976	2007233137	406156338	456355046
t	2608790314157420	−552114624	143750118	65433117	−127633657
t^2	−9084250	−18607467	−79744997	−10713296	−9134193
t^3	1796404	7904951	−3043725	2245279	1898818
t^4	805422	589540	811285	−376780	−640089
t^5	−59877	−156482	−78243	−30978	−25951
t^6	−101925	−52991	27580	10508	47156
t^7	16604	18290	8853	−8728	−5477
t^8	13915	4274	−4219	3403	−2271
t^9	−1795	−1597	−383	509	1944
t^{10}	−290	−49	10	−99	−14
Venus					
	31761466969	−44928213	50668473	68241014	288228577
t	1021328554621100	31259019	−36121239	138133826	−40384791
t^2	2846522	6041681	18468752	−10909716	−62328916
t^3	690608	−6834889	328049	−18641793	2473042
t^4	−10986	493964	−613650	601726	4228784
t^5	−148464	597550	−168598	746057	−57042
t^6	12166	−109138	−123616	−40592	−116943
t^7	21487	−68614	49912	−17319	−12239
t^8	2956	16888	20100	9132	2584
t^9	−944	5132	−6528	254	3467
t^{10}	−454	−170	−330	−254	−166
Earth					
	17534703144	−37408165	162844766	0	0
t	628307584918000	−82266699	−62030259	−113469002	10180391
t^2	−9793168	27626329	−33829810	12372674	47020439
t^3	429738	11695572	8510121	12654170	−5417367
t^4	734935	−2695722	2770542	−1371808	−2507948
t^5	83525	−715070	−467407	−320334	463486
t^6	−59447	218146	−62395	5072	56431
t^7	−52555	22635	247	−6941	−50813
t^8	13798	−19921	403	15095	−2799
t^9	14426	−2032	686	−72	8609
t^{10}	−564	475	−423	−352	−67
Mars					
	62034809134	853656025	−378997324	104704257	122844931
t	334061243149230	376330152	624657465	17138526	−108020083
t^2	4416007	−246579527	155295878	−40776201	−19221776
t^3	−317199	−36760524	−63487894	−13883445	8718504
t^4	718030	11112422	−6592895	916176	3090121
t^5	−765245	259071	729862	1759071	37687
t^6	−68083	7855	113707	112984	8722
t^7	197772	27211	184557	−128937	−21863
t^8	26923	−16901	12638	−1438	−18876
t^9	−30823	−5266	−31421	4131	7764
t^{10}	−1561	1094	−1668	−296	−266

We have integrated numerically the variables α_i over 10000 yr. We can use these computed values to compute the values of $\frac{d\varepsilon}{dt}$ every 250 yr. The numerical derivation gives then the Taylor expansion of $\frac{d\varepsilon}{dt}$ up to degree 9, and by integration, the Taylor expansion of ε up to degree 10 for all the inner planets.

In VSOP82, some secular terms appear in the semi-major axis, and thus in the variable p, but the contribution of the term Np in (12) is negligible for Mercury, Venus, and the Earth; it remains small in the case of Mars unlike the outer planets where it represents the main part of $\frac{d\lambda}{dt}$ (Simon and Francou, 1981). As we deal only with the inner planets, we shall only compute the contribution of Np for Mars.

In VSOP82, we have:

$$a_4 = a_{40} + a_{41}t \tag{15}$$

The Kepler relation $n^2a^3 = k^2(1 + m/m_{\odot})$ gives then:

$$\frac{dn}{dt} = -\frac{3}{2}\frac{da}{dt}\frac{n}{a} \tag{16}$$

We thus shall add to the coefficient λ_{42} of t^2 in the longitude of Mars, the term:

$$\lambda'_{42} = -\frac{3}{4}a_{41}\frac{n_4}{a_4} \tag{17}$$

That is, with $a_{41} = 31\,10^{-10}$ AU/10000 yr, and the values of n_4 and a_4 given in Table 1:

$$\lambda'_{42} = -509750\,10^{-10}/(10000\,\text{yr})^2$$

which is not negligible beyond the total value of λ_{42} given in Table 5.

The full results of the secular terms of the mean longitude for the inner planets, λ_i are given in Table 5. In the case of the variables, $\mathbf{k}_i$, $\mathbf{h}_i$, $\mathbf{q}_i$, $\mathbf{p}_i$, the initial values are the values of the considered variable for $t = 0$ as given in Table 2 (Bretagnon, 1982). In the case of the mean longitude, the initial value is given by the relation:

$$\left.\frac{d\lambda_i}{dt}\right|_{t=0} = N_i \tag{18}$$

where N_i is the mean mean motion (Bretagnon, 1982, Table 2). The coefficients of t in the secular variation of the mean longitude is then equal to N_i. Except for the correction to the coefficient of t^2 of Mars which we have just quoted, all the coefficients of $t^2, t^3, \ldots, t^{10}$ in Table 5 are taken from NGT for the mean longitudes.

3.5. *Mixed terms in the inequality $4N_3 - 8N_4 + 3N_5$*

We have partially extended the results of Laskar (1985) in order to compute the mixed terms of the form $t \sin t$, $t^2 \sin t$, etc... for an inequality different from the secular inequality. These results have been applied to compute the contribution of the inequality $4N_3 - 8N_4 + 3N_5$ in the longitudes of the Earth and Mars, coming from the Np part of (12). (In this case, the contribution of $\frac{d\varepsilon}{dt}$ is negligible in regard to the accuracy of our computations).

The variation of the part of p involving the inequality $4N_3 - 8N_4 + 3N_5$ is then given by a differential system similar to (6):

$$\frac{dp}{dt} = \Phi_{4N_3-8N_4+3N_5}(\alpha,\bar{\alpha})\exp\sqrt{-1}(4\lambda_3 - 8\lambda_4 + 3\lambda_5) \tag{19}$$

where $\Phi_{4N_3-8N_4+3N_5}(\alpha,\bar{\alpha})$ is similar to $\Phi_1\alpha + \Phi_3(\alpha,\bar{\alpha}) + \Phi_5(\alpha,\bar{\alpha})$ in (6), and gathers monomials of degree 1, 3, and 5.

The integration of the system (19) is processed in the same way as the integration of (13): we compute the values of $\Phi_{4N_3-8N_4+3N_5}(\alpha,\bar{\alpha})$ every 250 yr and then compute its Taylor expansion up to degree k with respect to the time. That gives:

$$\left.\frac{dp}{dt}\right|_{4N_3-8N_4+3N_5} = (\phi_0 + \phi_1 t + \cdots + \phi_k t^k + o(t^k)) \times \exp\sqrt{-1}(4\lambda_3 - 8\lambda_4 + 3\lambda_5) \tag{20}$$

where ϕ_i are complex numbers and $o(t^k)$ is the classical notation for the remainder of the Taylor expansion. We just keep the linear part of the longitudes in the exponential, and the other part is expanded in power series of the time. This system is then integrated, and by (12), gives the mixed terms in the longitude on the form:

$$\begin{aligned}\lambda_{4N_3-8N_4+3N_5} = {} & (x_0 + x_1 t + \cdots + x_k t^k + o(t^k)) \\ & \times \exp\sqrt{-1}((4\lambda_{30} - 8\lambda_{40} + 3\lambda_{50}) \\ & + (4N_3 - 8N_4 + 3N_5)t)\end{aligned} \tag{21}$$

The x_i are complex numbers, but $\lambda_{4N_3-8N_4+3N_5}$ is real, and (21) can also be expanded in the form:

$$\begin{aligned}\lambda_{4N_3-8N_4+3N_5} = {} & (y_0^s + y_1^s t + \cdots + y_k^s t^k + o(t^k)) \\ & \times \sin(4\lambda_{30} - 8\lambda_{40} + 3\lambda_{50} + (4N_3 - 8N_4 + 3N_5)t) \\ & + (y_0^c + y_1^c t + \ldots + y_k^c t^k + o(t^k)) \\ & \times \cos(4\lambda_{30} - 8\lambda_{40} + 3\lambda_{50} + (4N_3 - 8N_4 + 3N_5)t)\end{aligned} \tag{22}$$

The values of the coefficients y_i^s, y_i^c are given in Table 6. Practically, we have limited our computation to the degree $k = 4$.

In Table 6, we compare our values to the similar values obtained by Bretagnon in VSOP82 for the coefficients of degree 0 and 1.

Let us notice that $4N_3 - 8N_4 + 3N_5$ is an inequality which appears only at the second order. In VSOP82, the coefficients y_1^s and y_1^c are then computed only at the first order with respect

Table 6. Mixed terms for the inequality $4N_3 - 8N_4 + 3N_5$ in NGT and VSPO82 (Eq. 22). The time t is measured in units of 10000 julian years from J2000 (JD 2451545.0)

	Earth		Mars	
	$y_i^c \times 10^{10}$	$y_i^s \times 10^{10}$	$y_i^c \times 10^{10}$	$y_i^s \times 10^{10}$
NGT				
	327749	−96842	−2660804	786122
t	19178	−424825	−155762	3448928
t^2	−272641	−21359	2213451	173426
t^3	−4479	128681	36371	−1044706
t^4	41445	971	−336478	7870
VSOP82				
	321853	−97066	−2613137	788039
t	−13470	−399400	109970	3244110

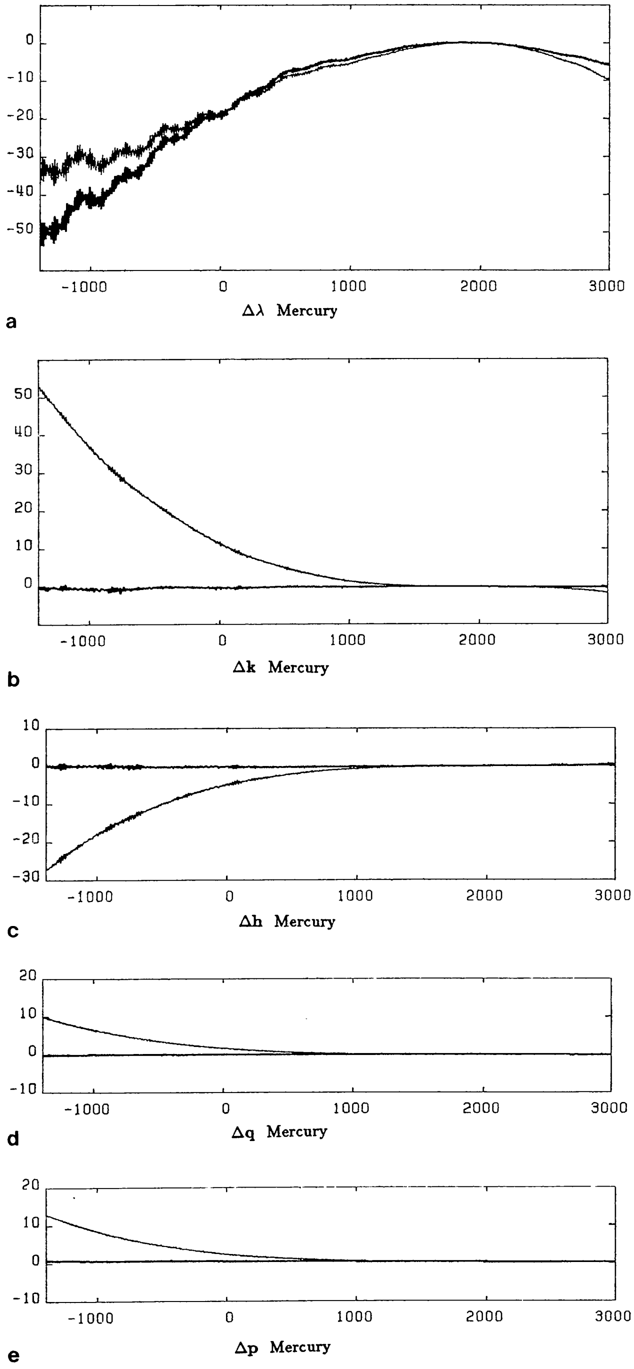

Fig. 1a–e. *Mercury.* Differences VSOP82 minus DE102 (thin curve) and (NGT + VSOP82) minus DE102 (bold curve) over the period -1400, $+3000$. Unit is 10^{-7} rad ($50\,10^{-7}$ rad $\approx 1''$)

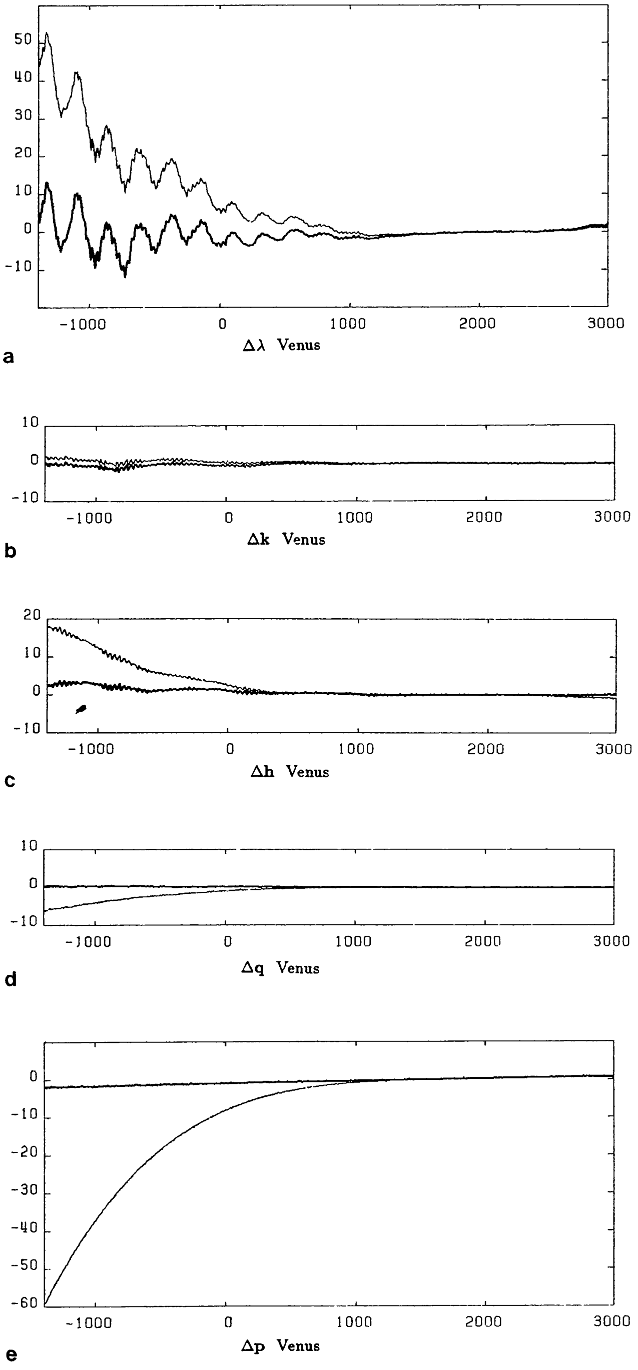

Fig. 2a–e. *Venus.* Differences VSOP82 minus DE102 (thin curve) and (NGT + VSOP82) minus DE102 (bold curve) over the period -1400, $+3000$. Unit is 10^{-7} rad ($50\,10^{-7}$ rad $\approx 1''$)

in (Kinoshita, 1975, 1977):

$$\frac{dp_A}{dt} = R(\epsilon_A) - \cot\varepsilon_A\left(\sin p_A + \Omega\,\frac{di}{dt} + \sin i\cos p_A + \Omega\,\frac{d\Omega}{dt}\right) - (1-\cos i)\,\frac{d\Omega}{dt} \tag{23}$$

$$\frac{d\epsilon_A}{dt} = \cos p_A + \Omega\,\frac{di}{dt} - \sin i\sin p_A + \Omega\,\frac{d\Omega}{dt}$$

with:

$$R(\epsilon_A) = \frac{3k^2 m_M}{a_L^3\omega}\,\frac{2C-A-B}{2C}\left[(\mathrm{M}_0 - \mathrm{M}_2/2)\cos\epsilon_A + \mathrm{M}_1\,\frac{\cos 2\epsilon_A}{\sin\epsilon_A} - \mathrm{M}_3\,\frac{m_M}{m_E+m_M}\,\frac{n_M^2}{\omega n_\Omega}\,\frac{2C-A-B}{2C}\,(6\cos^2\epsilon_A - 1)\right] + \frac{3k^2 m_\odot}{a_\odot^3\omega}\,\frac{2C-A-B}{2C}\,[(\mathrm{S}_0 - \mathrm{S}_2/2)\cos\epsilon_A] \tag{24}$$

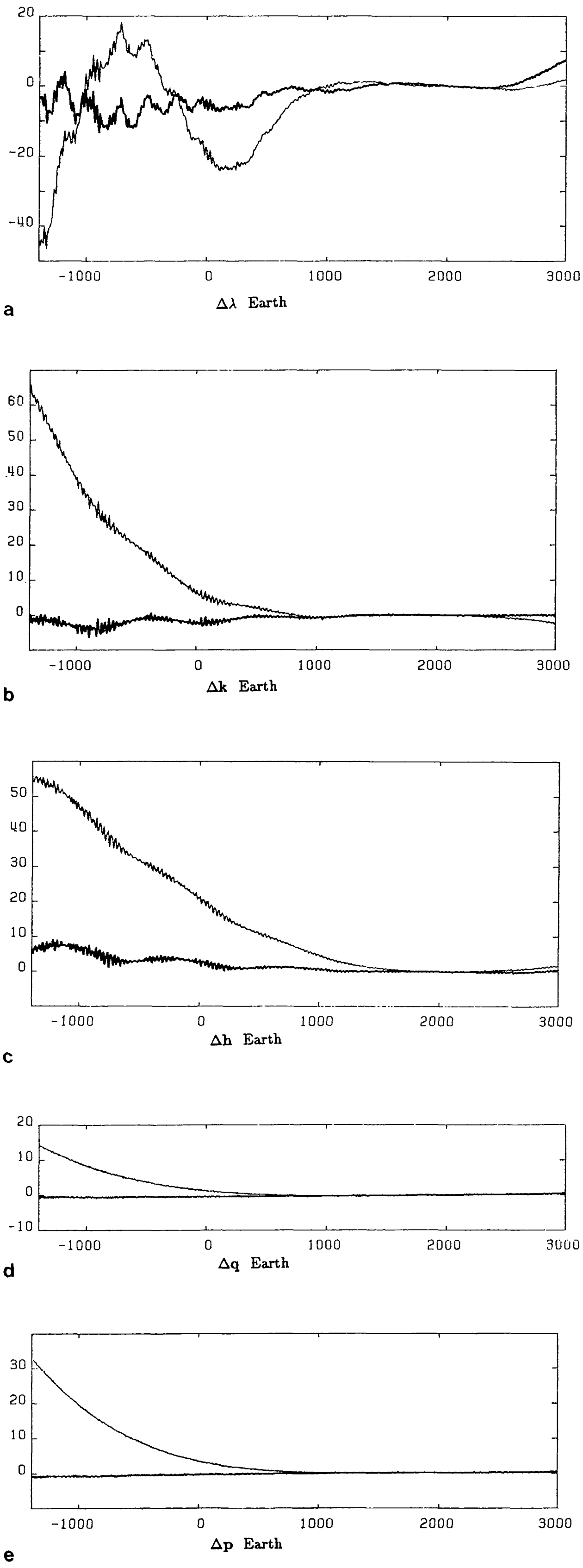

Fig. 3a–e. *Earth.* Differences VSOP82 minus DE102 (thin curve) and (NGT + VSOP82) minus DE102 (bold curve) over the period −1400, +3000. Unit is 10^{-7} rad ($50\,10^{-7}$ rad $\approx 1''$)

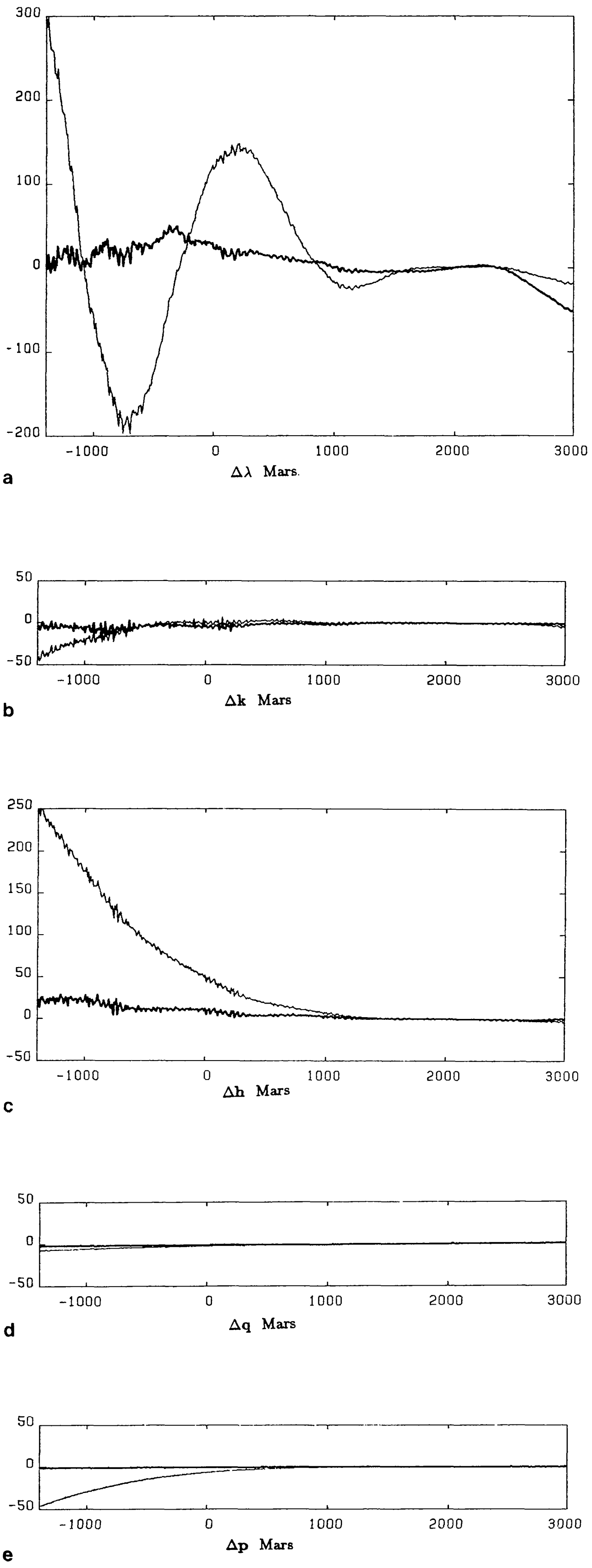

Fig. 4a–e. *Mars.* Differences VSOP82 minus DE102 (thin curve) and (NGT + VSOP82) minus DE102 (bold curve) over the period −1400, +3000. Unit is 10^{-7} rad ($50\,10^{-7}$ rad $\approx 1''$)

to the masses. We have then retained the values of NGT given in Table 6 for $y_1^s, y_2^s, \ldots, y_6^s$, and $y_1^c, y_2^c, \ldots, y_6^c$.

3.6. Comparison with DE102

We have obtained new values for the secular terms of the inner planets up to degree 10 with respect to the time for the mean longitude λ, and the eccentricity-inclination variables **k**, **h**, **q**, **p**, (Table 5) for the 4 inner planets. We have also computed the mixed terms up to degree 4 for the contribution of the inequality $4N_3 - 8N_4 + 3N_5$ in the longitudes of the Earth and Mars. To check the accuracy of these new terms, we have compared our solution NGT with the results of the numerically integrated JPL ephemeris, DE102 (Newhall et al., 1982), over the whole range of DE102, that is -1400, 3000 yr.

A mere difference NGT − DE102 will not be very convenient because NGT does not include the short period terms which can reach $8\,10^{-5}$ in the variables **k**, **h** of Mars (for example). To avoid these problems, we used the short-period terms of VSOP82, and we have plotted the differences (NGT + VSOP82) − DE102 for the 5 variables λ, **k**, **h**, **q**, **p** of the inner planets (Fig. 1-4). For comparison, we have also plotted VSOP82 − DE102 for the same variables in Fig. 1-4.

Except for the longitude of Mercury, where some uncertainty remains in the interpretation of the results, we can see that the utilisation of the secular terms given by the general theory NGT leads to a real improvement of VSOP82 beyond 1000 yr.

For the longitude of Venus (Fig. 2a), this improvement is due to the secular terms of Table 5, while for the longitude of the Earth and Mars, it is mainly due to the mixed terms coming from the inequality $4N_3 - 8N_4 + 3N_5$ (Table 6).

In Table 7, we have put the maximum values of |(NGT + VSOP82) − DE102| over the whole range of DE102. If we just consider the secular terms, these values should even be reduced, owing to the fact that some short periods appear in Fig. 1-4, due to the lack of knowledge of the high-degree mixed terms for some of the inequalities.

4. New formulas for the precession, valid over 10000 year

The precessional quantities are completely determined by the two motions of the equatorial and ecliptic pole. The actual formulas given in *Connaissance des Temps* (1984) have been computed by Lieske et al. (1977), and are based upon the secular variations of the ecliptic pole from Newcomb's Theory of the Sun. The improvements of the VSOP82 theory lead Bretagnon and Chapront (1981) to compute new formulas; but they did not take into account the secular variations of the eccentricity of the Sun in their computations of the precessional quantities. On the other hand, Kinoshita (1975, 1977) has improved the theory of the rotation of the Earth, but the precessional quantities he computed were based on Newcomb's theory of the Sun and on initial values which slightly differ from the IAU (Grenoble, 1976) values.

Our computation of the precessional quantities is based on Kinoshita's theory of the rotation of the rigid Earth and on the secular motion of the ecliptic, given in Table 5. They are developed up to degree 10 with respect to the time, and are valid over 10000 yr with a precision estimated at 0.″01 after 1000 yr and a few seconds of arc after 10000 yr (Table 8).

Table 7. Maximum value of |(NGT + VSOP82) − DE102| over the whole range of DE102, −1400/3000 yr. This value is given in units of 10^{-10} rad (above) and in seconds of arc (below)

	$\Delta\lambda$	Δ**k**	Δ**h**	Δ**q**	Δ**p**
Mercury	53236	1657	889	171	800
	1″098	0″035	0″018	0″004	0″017
Venus	13152	2251	3756	548	1992
	0″271	0″046	0″077	0″011	0″041
Earth	13283	6227	9090	816	1192
	0″274	0″128	0″187	0″017	0″025
Mars	51767	16634	29433	2037	1219
	1″068	0″343	0″607	0″042	0″025

Table 8. Formulas for the precession. The general accumulated precession p_A and the obliquity ε_A are given in arcseconds and the time t is measured in units of 10000 julian years from J2000 (JD 2451545.0). NGT denotes our solution (Numerical General Theory). L denotes the solution of Lieske et al. (1977). BC denotes the solution of Bretagnon and Chapront (1981)

	NGT	L	BC
		p_A (″)	
t	502909.66	502909.66	502909.66
t^2	11119.71	11111.3	11137.0
t^3	77.32	−6.	76.
t^4	−2353.16		
t^5	−180.55		
t^6	174.51		
t^7	130.95		
t^8	24.24		
t^9	−47.59		
t^{10}	−8.66		
		ε_A (″)	
t	−4680.93	−4681.50	−4680.93
t^2	−1.55	−5.9	−1.5
t^3	1999.25	1813.	2001.
t^4	−51.38		
t^5	−249.67		
t^6	−39.05		
t^7	7.12		
t^8	27.87		
t^9	5.79		
t^{10}	2.45		

4.1. Equations for the precession

We shall use the notations of (Lieske et al., 1977) for the precessional quantities: p_A denotes the general precession, and ε_A the obliquity of the equatorial plane on the ecliptic plane. For the other variables, we keep the notations already given in Part I. The angle between the axis of figure and the angular momentum axis is of order 10^{-6} and we neglect this effect. We use the equations for the precessional motion of the mean equator given

$R(\varepsilon_A)$ is the secular term due to the direct lunisolar perturbations. The factors M_0, M_1, and M_2 in the Eq. (24) come from the secular terms of $\frac{1}{2(a/r)^3(1-3\sin^2\beta)}$, $(a/r)^3 \sin\beta\cos\beta\sin\lambda$, and $(a/r)^3\cos^2\beta\cos^2\lambda$, respectively, where λ and β are the longitude and latitude of the Moon referred to the ecliptic of date and the mean equinox of date. The factors S_0 and S_2 are the same quantities for the Sun. The term with M_3 comes from the second-order secular perturbations. The quantities M_0, M_1, M_2, M_3, S_0, and S_2 depend only on the orbital elements of the Moon and the Sun, and their numerical values are obtained from Brown's theory of the Moon as improved by Eckert, et al. (1966), and Newcomb's theory of the Sun. The terms having M_1, M_2, M_3, as factors are not included in Newcomb's precessional theory; M_1, M_2, and S_2 come from the long-periodic terms in the motions of the Moon and the Sun. The principal moments of inertia of the Earth are denoted by A, B, and C, and the angular velocity of the Earth is ω. The masses of the Sun, the Earth, and the Moon are denoted by $m_\odot$, m_E, and m_M; the sidereal mean motion of the Sun and of the Moon by $n_\odot$ and n_M; and the mean motion of the node of the Moon by n_Ω. The other terms present in Eq. (23) represent the effects of the secular variation of the ecliptic, caused by the secular planetary perturbations. The numerical values of M_0, M_1, M_2, M_3 are given in (Kinoshita, 1977):

$$\begin{aligned} M_0 &= 496303.3\,10^{-6} \\ M_1 &= -20.7\,10^{-6} \\ M_2 &= -0.1\,10^{-6} \\ M_3 &= 3020.2\,10^{-6} \end{aligned} \tag{25}$$

The constant S_2 of Kinoshita is smaller than 10^{-7}; it is very small in comparison to S_0 and we shall neglect it in (24). If the orbit of the Sun is assumed to be Keplerian, we have:

$$S_0 = \tfrac{1}{2}(1-e^2)^{-3/2} \tag{26}$$

The actual value of S_0 differs from this value, due to the secular terms δS_0 coming from the short-period terms. We have then:

$$S_0 = \tfrac{1}{2}(1-e^2)^{-3/2} + \delta S_0 \tag{27}$$

Kinoshita gives:

$$S_{0,1900} = 500210.1 \times 10^{-6}$$

that is, as $e_{1900} = 0.01675104$, $\delta S_{0,1900} = -0.422 \times 10^{-6}$. This value is very small and we shall consider that δS_0 is a constant. The value of S_0 is then given by:

$$S_0 = \tfrac{1}{2}(1-e^2)^{-3/2} - 0.422\,10^{-6} \tag{28}$$

We shall keep this analytical expression for S_0 in (23) and (24) so we do not have to consider the $\Delta S_0 t$ term which appears in (Kinoshita, 1975a, 1975b). Moreover, for a very long time like 10000 yr, a linear estimation $S_0(t) = S_0 + \Delta S_0 t$ may not be enough to represent the variations of S_0 with good accuracy.

In the case of the Moon, the secular variations of the eccentricity are very small, and we shall keep the values of M_0, M_1, M_2, M_3 constant, as they were computed by Kinoshita.

We use also the following numerical values, taken from the *Connaissance des Temps*, Aoki et al. (1982), Bretagnon (1982), and Chapront-Touzé and Chapront (1983):

$$\begin{aligned} \omega &= 474659981.59757\ \text{arsec yr}^{-1} \\ n_M &= 17325593.4318\ \text{arsec yr}^{-1} \\ n_\Omega &= -69679.1936222\ \text{arsec yr}^{-1} \\ a_\odot &= 1.00000101778\ \text{AU} \\ a_M &= 384747980.645\ \text{m} \\ k &= 0.01720209895 \\ m_\odot/(m_E+m_M) &= 328900.5 \\ m_\odot/m_E &= 332946.0 \end{aligned} \tag{29}$$

In order to use the results of Sect. 1, we shall express the differential system (23) in the variables $\mathbf{q}$, $\mathbf{p}$ (5) we obtain:

$$\begin{aligned} \frac{dp_A}{dt} &= R(\varepsilon_A) - \cot\varepsilon_A[\mathbf{A}(\mathbf{p},\mathbf{q})\sin p_A + \mathbf{B}(\mathbf{p},\mathbf{q})\cos p_A] - 2\mathbf{C}(\mathbf{p},\mathbf{q}) \\ \frac{d\varepsilon_A}{dt} &= -\mathbf{B}(\mathbf{p},\mathbf{q})\sin p_A + \mathbf{A}(\mathbf{p},\mathbf{q})\cos p_A \end{aligned} \tag{30}$$

with:

$$\begin{aligned} \mathbf{A}(\mathbf{p},\mathbf{q}) &= \frac{2}{\sqrt{1-p^2-q^2}}(\dot{\mathbf{q}} + \mathbf{p}(\mathbf{q}\dot{\mathbf{p}} - \mathbf{p}\mathbf{q})) \\ \mathbf{B}(\mathbf{p},\mathbf{q}) &= \frac{2}{\sqrt{1-\mathbf{p}^2-\mathbf{q}^2}}(\dot{\mathbf{p}} - \mathbf{q}(\mathbf{q}\dot{\mathbf{p}} - \mathbf{p}\dot{\mathbf{q}})) \\ \mathbf{C}(\mathbf{p},\mathbf{q}) &= (\mathbf{q}\dot{\mathbf{p}} - \mathbf{p}\dot{\mathbf{q}}) \end{aligned} \tag{31}$$

For $t = 0$, we have $p_A = 0$, $\varepsilon_A = \varepsilon_0$, $\mathbf{p} = \mathbf{q} = 0$, and thus:

$$\left.\frac{dp_A}{dt}\right|_{t=0} = R(\varepsilon_0) - 2\dot{\mathbf{p}}_{t=0}\cot\varepsilon_0 \tag{32}$$

In fact, we must add to $\frac{dp_A}{dt}$ in (30) the geodesic precession due to the general relativity (de Sitter and Brouwer, 1938). As shown in (Barker and O'Connell, 1970), this geodesic precession p_g is one half of the Earth relativistic motion of its perihelion, that is (9), (Table 1):

$$p_g = \delta_R/2 = 191''.88/10000\ \text{yr} \tag{33}$$

The initial values for the resolution of (30) are given by the value of the general precession p and obliquity ε_0 at the origin (J2000). We have:

$$\left.\frac{dp_A}{dt}\right|_{t=0} = p + p_g \tag{34}$$

and the initial values adopted by the IAU (Grenoble, 1976):

$$\begin{aligned} p &= 502909''.66/10000\ \text{yr} \\ \varepsilon_0 &= 23°26'21''.448 \end{aligned} \tag{35}$$

The resolution of (32) with the initial values (35) gives the value of the dynamical ellipticity of the Earth, $\frac{2C-A-B}{2C}$. We can then integrate numerically the differential system (30) and obtain the general precession p_A for all times. The general precession is then corrected by the geodesic precession p_g. The secular variation of the geodesic precession itself is below $0''.03/10000$ yr, so we neglect it and consider the geodesic precession as a constant.

4.2. Formulas for precession

The variations of the variables $\mathbf{p}, \mathbf{q}, \dot{\mathbf{p}}, \dot{\mathbf{q}}$, of the Earth are given in Table 5. We have integrated numerically the differential system (30) with a simple Runge-Kutta method of order 4 and step of 250 yr, which ensure us a global internal precision of 10^{-4} arcseconds.

By numerical derivation, we compute then the Taylor expansion of p_A and ε_A up to degree 10, valid over 10000 yr with a probable accuracy of 0″.02 after 1000 yr, and a few arcseconds after 10000 yr (Table 8).

For comparison, we give also the values for the same quantities computed by Lieske et al. (1977), (L) and by Bretagnon and Chapront (1981), (BC). The solution BC does not include the secular variation of the eccentricity of the Sun. It explains the difference with NGT on the coefficient of t^2 of the precession, while the coefficients of t^3 are quite identical. We can also see in Table 8 that the differences between NGT and the solution L which is generally adopted are far from negligible, and reached 320″ over 6000 yr for the precession, principally due to the presence of a quite large term in t^4 which does not exist in Lieske's formulas. This shows that precessional formulas for the historians must take these terms into account (Bretagnon et al., 1985).

Note: The limitation in the precision of our theory over 1000 yr is principally due to the precision of the variables $\mathbf{q}, \mathbf{p}$; but we did not consider the lack of accuracy of the initial values p and ε_0 (35). In fact, the uncertainty in the value of p is about 1″.5/1000 yr and 0″.1 in ε_0, which is far more important than the uncertainty of our theory.

5. Conclusions

The numerical integration of the autonomous system of our general theory gives accurate results for the secular variations of the inner planets. Using the terms in t^0, t^1 and t^2 from VSOP82 (Bretagnon, 1982), and the terms computed by our theory, we have derived new accurate secular terms for the inner planets, up to t^{10} (Table 5). The comparison with DE102 (Newhall et al. 1983) shows that these secular terms behave very well, and they can be used in the construction of ephemerides over about 10000 yr.

We have also used the general theory to compute the mixed terms of the inequality $4N_3 - 8N_4 + 3N_5$ in the mean longitude of the Earth and Mars, and in this case also, the successive utilisation of analytical and numerical methods is successful.

The accuracy on the secular variations of the ecliptic given by Table 5 is about 0″.02 after 1000 years.

We have then used the theory of (Kinoshita, 1977) for the rotation of the rigid Earth, and we have derived new formulas for the precessional quantities, up to the power 10 of the time (Table 8). The probable accuracy is of about 0″.02 after 1000 years, and a few arcseconds after 10000 years when supposing that the initial constants p and ε_0 (35) are exact.

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