Physik-Institut der Universität Zürich
in conjunction with ETH Zürich

Advanced Topics in General Relativity and Gravitational Waves

Autumn Semester 2017

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Contents

Preface

This course is a continuation of the courses “General Relativity” and “Applications of General Relativity in Astrophysics and Cosmology” taught by Prof. Philippe Jetzer in autumn 2016 and spring 2017. We will use the prefix GRI and GRII when referring to equations in those lecture notes, as for example in eq. (GRII 1.1). These lecture notes are far from being a complete treatment of the subject but should enable the student or interested reader to access specialized literature on the subject. For further reading we included a list of useful textbooks at the end. If you find any mistakes, please report them to boetzel@physik.uzh.ch.

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1 Gravitational lensing

The contents covered in this section follow the lecture notes of the course "Gravitational Lensing" taught by Prof. Philippe Jetzer in autumn 2004. The script covering this course can be found on the webpage `www.physik.uzh.ch/groups/jetzer/notes/Gravitational_Lensing.pdf`.

1.1 Historical introduction

Long before Einstein's theory of General Relativity, it was argued that gravity might influence the behavior of light. Already Newton, in the first edition of his book on optics of 1704, discussed the possibility that celestial bodies could deflect the trajectory of light.

In 1804, the astronomer J. G. Soldner published an article in the *Berliner Astronomisches Jahrbuch* in which he computed the error induced by the light deflection on the determination of the position of stars [1]. He used to that purpose the Newtonian theory of gravity, assuming that the light consists of particles. He also estimated that a light ray which just grazes the surface of the sun would be deflected by a value of only 0.84 arc seconds. Within general relativity, this value is about twice as much: 1.75 arc seconds (see eq. (GRI 24.8)). The first measurement of this effect has been made by British astronomers during the total solar eclipse of May 29 in 1919, in northern Brazil and on an island in the western coast of Africa. Both groups confirmed the value predicted by general relativity.

Apart from solar system tests, gravitational lensing remained only a theoretical possibility, deemed to be beyond experimental verification, until 1979 when the first gravitational lens of two point-like quasars was discovered. The spectra showed that both objects had the same redshift and were therefore at the same distance. Later on, the galaxy acting as lens for these two point-like quasars was found, making it clear that the two objects were actually the images of the same lensed quasar.

In the last 40 years gravitational lensing has grown into a major area of research. It allows us to probe the distribution of matter in galaxies and in galaxy clusters independently of the nature of the matter, in particular independently of whether the gravitational potential is due to luminous matter or not.

1.2 Point-like lens

The geometry of an homogeneous and isotropic expanding universe is described by the Friedmann-Lemaître-Robertson-Walker metric. All the inhomogeneities in the metric can be considered as local perturbations. Thus the trajectory of light coming from a distant source can be divided into three distinct pieces. In the first one the light coming from a distant source propagates in a flat unperturbed spacetime. Near the lens the trajectory gets modified due to the gravitational potential of the lens. In the third piece the light again travels in an unperturbed spacetime until it gets to the observer.

To first order approximation, the region around the lens can be described by a flat Minkowskian spacetime with small perturbations induced by the gravitational potential of the lens according to the post-Newtonian approximation (see eq. (GRII 7.31)). This approximation is valid as long as the Newtonian potential Φ is small, which means $|\Phi| \ll c^2$, and if the peculiar velocity v of the lens is negligible compared to c. With these assumptions we can describe the light propagation near the lens in a flat spacetime.

The effect of the spacetime curvature on the light trajectory can be described as an effective refraction index, given by

$$n = 1 - \frac{2\Phi}{c^2} = \left(1 + \frac{2|\Phi|}{c^2}\right) . \tag{1.1}$$

The Newtonian potential Φ is negative and vanishes asymptotically. As in geometrical optics a refraction index $n > 1$ means that the light travels with a speed which is less compared with its speed in the vacuum. Thus the effective speed of light in a gravitational field is given by

$$v = \frac{c}{n} \simeq c - \frac{2|\Phi|}{c} . \tag{1.2}$$

Since the effective speed of light is less in a gravitational field, the travel time gets longer compared to the propagation in empty space. The total time delay Δt, known as Shapiro delay, is obtained by integrating the second term (the one representing the delay) in

$$dt = \frac{dx}{v} = dx\frac{1}{c}\left(1 + \frac{2|\Phi|}{c^2}\right) \tag{1.3}$$

along the light trajectory from the source x_s to the observer position x_{obs}. This gives us

$$\Delta t = \int_{x_s}^{x_{obs}} \frac{2|\Phi|}{c^3} dx . \tag{1.4}$$

The deflection angle for the light rays which go through a gravitational field is given by the integration of the gradient component of n (eq. (1.1)) perpendicular to the trajectory itself

$$\vec{\alpha} = -\int_{-\infty}^{+\infty} \nabla_\perp log(n) dz = \frac{2}{c^2}\int_{-\infty}^{+\infty} \nabla_\perp \Phi dz . \tag{1.5}$$

For all astrophysical applications of interest the deflection angle is always extremely small, so that the computation can be substantially simplified by integrating $\nabla_\perp$ along an unperturbed path. This result can also be derived using Fermat's principle which states that "The actual path between two points taken by a beam of light is the one which is traversed in the least time".

The light trajectory is proportional to $\int n(\vec{x})d\vec{x}$ and the actual light path is the one which minimizes this functional. In other words the light path is the solution of the following variational problem

$$\delta \int n(\vec{x})d\vec{x} = 0 \;\Leftrightarrow\; \delta \int n(\vec{x}(\lambda))\left|\frac{d\vec{x}}{d\lambda}\right| d\lambda = 0 . \tag{1.6}$$

Defining $L(\vec{x}, \dot{\vec{x}}) \equiv n(\vec{x})\sqrt{\dot{\vec{x}}^2}$ and $\dot{\vec{x}} \equiv \frac{d\vec{x}}{d\lambda}$ this problem leads to the Euler-Lagrange equation

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0 , \tag{1.7}$$

with $\frac{\partial L}{\partial \vec{x}} = \frac{\partial n}{\partial \vec{x}} = \nabla n$, $\frac{\partial L}{\partial \dot{\vec{x}}} = n\dot{\vec{x}}$. By defining the tangent vector $\vec{e} := \dot{\vec{x}}$, assumed to be normalized by a suitable choice of the parameter λ, we get from the Euler-Lagrange equation

$$\frac{d}{d\lambda}(n\dot{\vec{x}}) - \nabla n = 0 \Leftrightarrow \frac{d}{d\lambda}(n\vec{e}) - \nabla n = n\dot{\vec{e}} + \vec{e}(\nabla n\vec{e}) - \nabla n = 0 \tag{1.8}$$

$$\Leftrightarrow n\dot{\vec{e}} = \nabla n - \vec{e}(\vec{e}\cdot\nabla n) = \nabla_\perp n\,, \tag{1.9}$$

where we used that $\nabla_\| = \vec{e}(\vec{e}\cdot\nabla)$ is the gradient component along the light path. Integrating this equation, using $\vec{e} \equiv \dot{\vec{x}}$ and $\log(n) \simeq -\frac{2|\Phi|}{c^2}$ we get

$$\vec{e}_{source} - \vec{e}_{observer} = -\int_{\lambda_{source}}^{\lambda_{obs}} \frac{d\vec{e}}{d\lambda} d\lambda = \frac{2}{c^2}\int_{z_{source}}^{z_{obs}} \nabla_\perp \Phi dz\,. \tag{1.10}$$

This holds for a light ray coming from the $-\vec{e}_z$ direction and passing by a point-like lens at $z=0$. As $\vec{\alpha} = \vec{e}_{source} - \vec{e}_{observer}$, we have recovered eq. (1.5).

As an example we consider the deflection angle of a point-like lens of mass M. Its Newtonian potential is given by

$$\Phi(b,z) = -\frac{GM}{(b^2+z^2)^{\frac{1}{2}}}\,, \tag{1.11}$$

where b is the impact parameter of the unperturbed light ray and z denotes the position along the unperturbed path as measured from the point of minimal distance from the lens. This way we get

$$\nabla_\perp \Phi(b,z) = \frac{GM\vec{b}}{(b^2+z^2)^{\frac{3}{2}}}\,, \tag{1.12}$$

where $\vec{b}$ is orthogonal to the unperturbed light trajectory and is directed towards the point-like lens. Inserting this result in eq. (1.5) we find

$$\vec{\alpha} = \frac{2}{c^2}\int_{-\infty}^{\infty} \nabla_\perp \Phi dz = \frac{4GM}{c^2 b}\frac{\vec{b}}{|\vec{b}|} = \frac{2R_s}{b}\frac{\vec{b}}{|\vec{b}|}\,, \tag{1.13}$$

where $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius for a body of mass M. Thus the absolute value of the deflection angle is $\alpha = 2R_S/b$. For the Sun the Schwarzschild radius is 2.95 km, whereas its physical radius is $6.96 \cdot 10^5$ km. A light ray which just grazes the solar surface is deflected by an angle of 1.75".

1.3 Thin lens approximation

From the above considerations one sees that the main contribution to the light deflection comes from the region $\Delta z \simeq \pm b$ around the lens. This distance is typically much smaller than the distance between observer and lens D_d and the distance between lens and source D_{ds}. The lens can thus be assumed to be thin compared to the full length of the light trajectory.

We consider the mass of the lens, for instance a galaxy cluster, projected on a plane perpendicular to the line of sight (between observer and lens) and going through the center of the lens. This plane is usually referred to as the *lens plane*, and similarly one can define the *source plane*. The projection of the lens mass on the lens plane is obtained by integrating the mass density ρ along the direction perpendicular to the lens plane

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z) dz \,, \tag{1.14}$$

where $\vec{\xi}$ is a two dimensional vector in the lens plane and z is the distance from the plane.

The deflection angle at the point $\vec{\xi}$ is then given by summing over the deflection due to all mass elements in the plane as follows

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2} d^2\vec{\xi'} \,. \tag{1.15}$$

Note that for one mass element (1 particle), $\Sigma(\vec{\xi'}) = M\delta^2(\vec{\xi'} - \vec{\xi_0})$.

In the general case the deflection angle is described by a two dimensional vector, however in the special case that the lens has circular symmetry one can reduce the problem to a one-dimensional situation. Then the deflection angle $\vec{\alpha}$ is a vector directed towards the center of the symmetry with absolute value given by

$$\alpha = \frac{4GM(\xi)}{c^2\xi} \,, \tag{1.16}$$

where ξ is the distance from the center of the lens and $M(\xi)$ is the total mass inside a radius ξ from the center, defined as

$$M(\xi) = 2\pi \int_0^{\xi} \Sigma(\xi')\xi' d\xi' \,. \tag{1.17}$$

1.4 Lens equation

The geometry for a typical gravitational lens is given in Figure 1.1. A light ray coming from a source S, laying in the source plane, is deflected by the lens by an angle α, with ξ as the impact parameter, and reaches the observer located in O.

We arbitrarily define β to be the angle between the optical axis and the true source position, whereas θ is the angle between the optical axis and the image position. As mentioned before, the distances along the line of sight between observer and lens, lens and source and observer and source are D_d, D_{ds} and D_s, respectively.

Assuming small angles, we can easily derive $\vec{\theta} D_s = \vec{\beta} D_s + \vec{\alpha} D_{ds}$ from the geometry of the problem. Rearranging terms we get

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \frac{D_{ds}}{D_s} \,, \tag{1.18}$$

which is known as the *lens equation*. This is a non-linear equation and it is possible to have several images $\vec{\theta}$ corresponding to a single source position $\vec{\beta}$.

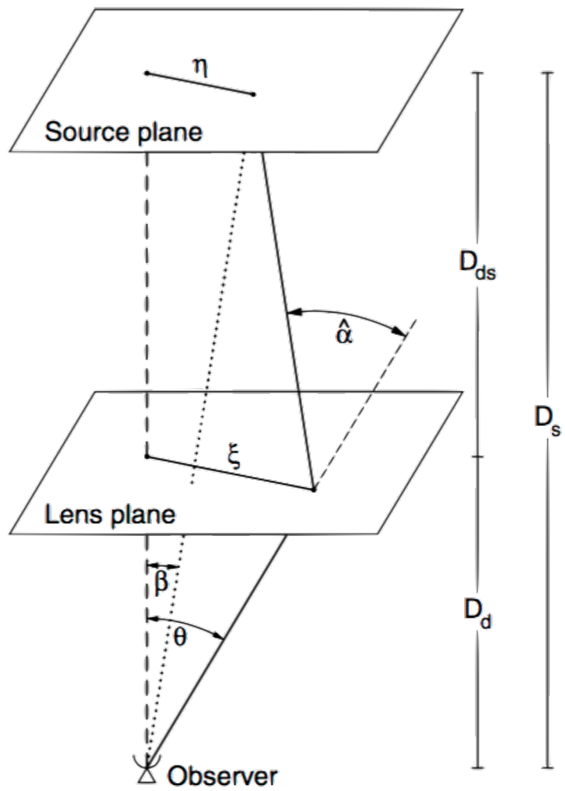

Figure 1.1: Geometry of a gravitational lens. [2]

The lens equation (1.18) can also be derived using Fermat's principle, which is identical to the classical one in geometrical optics but with the refraction index as defined in eq. (1.1). The light trajectory is given by the variational principle

$$\delta \int n(\vec{x}) d\vec{x} = 0 \tag{1.19}$$

as we did in eq. (1.6). It expresses the fact that the light trajectory will be such that the traveling time will be extremal.

Let us consider a light ray emitted from the source S at time $t = 0$. The light ray will proceed straight till it reaches the lens, located at L in the lens plane, where it will be deflected and then proceed again straight to the observer O. We can thus compute

$$t = \int_{x_S}^{x_O} \frac{n}{c} dx = \frac{1}{c} \int_{x_S}^{x_O} \left(1 - \frac{2\Phi}{c^2}\right) dx = \frac{l}{c} - \frac{2}{c^3} \int_{x_S}^{x_O} \Phi dx \,, \tag{1.20}$$

where l is the Euclidean distance SLO. From the geometry of the problem we obtain

$$l = \sqrt{(\vec{\xi} - \vec{\eta})^2 + D_{ds}^2} + \sqrt{\vec{\xi}^2 + D_d^2} \simeq D_{ds} + D_d + \frac{1}{2D_{ds}} (\vec{\xi} - \vec{\eta})^2 + \frac{1}{2D_d} \vec{\xi}^2 \,, \tag{1.21}$$

where $\vec{\eta}$ is a two dimensional vector on the source plane, and where we did a simple Taylor expansion.

The second term of eq. (1.20) containing Φ has to be integrated along the light trajectory. If we assume a point like lens then the Newtonian potential takes the form $\Phi = -GM/|\vec{x}|$, we get

$$\int_{x_S}^{x_L} \frac{2\Phi}{c^3} dx = \frac{2GM}{c^3} \left(\ln \frac{|\vec{\xi}|}{2D_{ds}} + \frac{\vec{\xi}(\vec{\eta}-\vec{\xi})}{|\vec{\xi}|D_{ds}} + \mathcal{O}\left(\frac{(\vec{\eta}-\vec{\xi})^2}{D_{ds}} \right) \right) , \tag{1.22}$$

and a similar expression for $\int_{x_L}^{x_O} 2\Phi/c^3 dx$. While for the thin lens approximation with a surface density $\Sigma(\xi)$ given by eq. (1.14) we obtain, neglecting higher order contributions,

$$\int_{x_S}^{x_O} \frac{2\Phi}{c^3} dx = \frac{4G}{c^3} \int d^2\vec{\xi}' \Sigma(\vec{\xi}') \ln \left(\frac{|\vec{\xi}-\vec{\xi}'|}{\xi_0} \right) , \tag{1.23}$$

where ξ_0 is a characteristic length in the lens plane and the right hand side term is defined up to a constant.

The difference in arrival time between a photon deflected by the lens and one not deflected at all is obtained by subtracting the travel time without deflection from S to O to the deflected travel time given by eq. (1.20). Thus the two first terms of the expansion in eq. (1.21) disappear. This way one gets

$$c\Delta t = \hat{\phi}(\vec{\xi}, \vec{\eta}) + const. , \tag{1.24}$$

where $\hat{\phi}(\vec{\xi}, \vec{\eta})$ is the *Fermat potential* and it is

$$\hat{\phi}(\vec{\xi}, \vec{\eta}) = \frac{D_d D_s}{2D_{ds}} \left(\frac{\vec{\xi}}{D_d} - \frac{\vec{\eta}}{D_s} \right)^2 - \hat{\psi}(\vec{\xi}) , \tag{1.25}$$

where

$$\hat{\psi}(\vec{\xi}) = \frac{4G}{c^2} \int d^2\vec{\xi}' \Sigma(\vec{\xi}') \ln \left(\frac{|\vec{\xi}-\vec{\xi}'|}{\xi_0} \right) , \tag{1.26}$$

is the *deflection potential* and it doesn't depend on $\vec{\eta}$.

The Fermat principle can thus be written as $d\Delta t/d\vec{\xi} = 0$, inserting eq. (1.24) we get again the lens equation

$$\vec{\eta} = \frac{D_s}{D_d}\vec{\xi} - D_{ds}\vec{\alpha}(\vec{\xi}) , \tag{1.27}$$

where $\vec{\alpha}$ is defined in eq. (1.15). Defining $\vec{\beta} = \vec{\eta}/D_s$ and $\vec{\theta} = \vec{\xi}/D_d$ we recover eq. (1.18). We can also rewrite eq. (1.27) as

$$\nabla_{\vec{\xi}} \hat{\phi}(\vec{\xi}, \vec{\eta}) = 0 , \tag{1.28}$$

which is an equivalent form of the Fermat principle. The arrival time delay of light rays coming from two different images located at $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$ from the same source $\vec{\eta}$ is given by

$$c(t_1 - t_2) = \hat{\phi}(\vec{\xi}^{(1)}, \vec{\eta}) - \hat{\phi}(\vec{\xi}^{(2)}, \vec{\eta}) . \tag{1.29}$$

It can be shown[1] that the *magnification* factor μ for a given image θ is given by

$$\mu = \frac{1}{\det A(\theta)} , \tag{1.30}$$

where $A(\theta) = \frac{d\vec{\beta}}{d\vec{\theta}}$ and $A_{ij} = \frac{d\beta_i}{d\theta_j}$.

[1]See Sec. 2.4 of the "Gravitational Lensing" script.

1.5 Schwarzschild lens

For a lens with axial symmetry, using the deflection angle obtained in eq. (1.16), we get the following lens equation

$$\beta(\theta) = \theta - \frac{D_{ds}}{D_s D_d} \frac{4GM(\theta)}{c^2\theta} \,. \tag{1.31}$$

From this equation we see that the image of a perfectly aligned source ($\beta = 0$) is a ring if the lens is supercritical[2]. By setting $\beta = 0$ in eq. (1.31) we get the radius of the ring

$$\theta_E = \left(\frac{4GM(\theta_E)}{c^2} \frac{D_{ds}}{D_d D_s} \right)^{\frac{1}{2}} , \tag{1.32}$$

which is called *Einstein radius*[3]. The Einstein radius depends not only on the characteristics of the lens but also on the various distances.

The Einstein radius sets a natural scale for the angles entering the description of the lens. Indeed, for multiple images the typical angular separation between the different images turns out to be of order $2\theta_E$. Moreover sources with angular distance smaller than θ_E from the optical axis of the system get magnified quite substantially whereas sources which are at a distance much greater than θ_E are only weakly magnified.

A particular case of a lens with axial symmetry is the Schwarzschild lens, for which $\Sigma(\vec{\xi}) = M\delta^{(2)}(\vec{\xi})$ and thus $m(\theta) = \theta_E^2$. The source is also considered as point-like, this way we get the following lens equation

$$\beta = \theta - \frac{\theta_E^2}{\theta} \,. \tag{1.33}$$

This equation has two solutions

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right) . \tag{1.34}$$

Therefore, there will be two images of the source located one inside the Einstein radius and the other outside. For a lens with axial symmetry the magnification is given by

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta} \,. \tag{1.35}$$

For the Schwarzschild lens, which is a limiting case of an axial symmetric one, we can use β given by eq. (1.33) and obtain the magnification of the two images

$$\mu_\pm = \left(1 - \left(\frac{\theta_E}{\theta_\pm}\right)^4\right)^{-1} = \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2} \,, \tag{1.36}$$

where $u = r/R_E$ is the ratio between the impact parameter $r = \beta D_d$ and the Einstein radius given by $R_E = \theta_E D_d$. The parameter u can thus also be expressed as $u = \beta/\theta_E$. Since

[2] A lens which has $\Sigma > \Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_s}{D_d D_{ds}}$ somewhere in it is defined as supercritical, and has in general multiple images.

[3] Instead of the angle θ_E one often uses $R_E = \theta_E D_d$.

$\theta_- < \theta_E$ we have that $\mu_- < 0$. The negative sign for the magnifications indicates that the parity of the image is inverted with respect to the source.

The total amplification is given by the sum of the absolute values of the magnifications for each image

$$\mu = |\mu_+| + |\mu_-| = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}\,. \tag{1.37}$$

If $r = R_E$ then we get $u = 1$ and $\mu = 1.34$, which corresponds to an increase of the apparent magnitude of the source of $\Delta m = -2.5 \log \mu = -0.32$. For lenses with a mass of the order of a solar mass and which are located in the halo of our galaxy the angular separation between the two images is far too small to be observable. Instead, one observes a time dependent change in the brightness of the the source star. This situation is also referred to as *microlensing*.

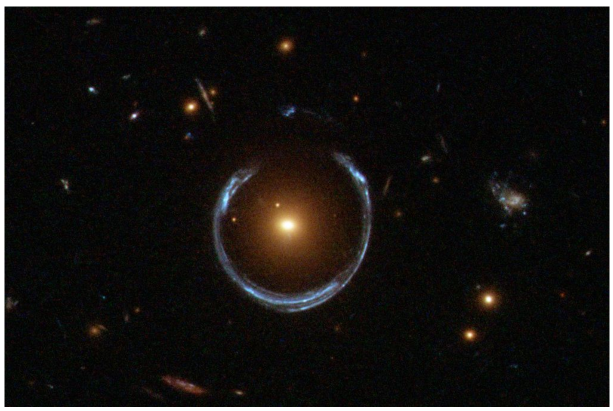

Figure 1.2: Einstein Ring, taken with the NASA/ESA Hubble Space Telescope. [3]

2 General relativistic stellar structure equations

The content covered in this section follows the book of N. Straumann [4].

2.1 Introduction

Although the applications of general relativity to astronomical problems has a long history dating back to Einstein himself, it was not until the discovery of pulsars[4] in 1967 that a great deal of interest was directed toward the impact of the theory on stellar structure.

The first pioneering theoretical works were done 30 years earlier by Landau [5] and later on by Oppenheimer and Volkoff [6]. It is only the sure existence of neutron stars and white dwarfs that lead to the construction of modern models representing our contemporary view of these objects.

Before proceeding, it is useful to remember how a star can evolve into either a *white dwarf*, *neutrons star* or *black hole*.

A low mass star with mass $M < 4\,M_\odot$ cannot start the carbon fusion process and thus evolves into a white dwarf by a collapse, where the relatively gentle mass ejection forms a planetary nebulae. The white dwarf is the stellar core left behind that is supported by electron degeneracy pressure.

Throughout the life of heavier stars, a core of iron is formed. At the end of the star's life, the outer layers collapse. The gravitational energy of the layers make it possible to fuse or fission the iron core. The collapse hence continues past the electron degeneracy pressure until the neutron degeneracy pressure stops it. The layers rebounce on the core of the star, now made mostly of neutrons, and form a supernovae. For stars with masses $10 - 30M_\odot$, a neutron star with mass 1.4 to $2 - 3$ $M_\odot$ is remnant. A neutron star is hence almost entirely composed of neutrons and held by neutron degeneracy pressure.

For sufficiently massive stars, the core will accrete too much mass to be stable and will collapse into a proto-neutron star, continue to gather mass through a stalled shock until it becomes unstable and undergoes a collapse into a black hole.

2.2 General relativistic stellar structure equations

We derive the general relativistic stellar structure equations for non-rotating, static and spherically symmetric compact stars. For such stars the metric is the Schwarzschild metric, and has the following form[5]

$$ds^2 = -e^{2a(r)}dt^2 + e^{2b(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\,, \tag{2.1}$$

where we are using units in which $c = 1$ and signature $(-,+,+,+)$. Note that by changing to the convention with signature $(+,-,-,-)$ and by defining $B(r) \equiv e^{2a(r)}$, $A(r) \equiv e^{2b(r)}$ we recover eq. (GRI 22.2).

[4]A pulsar is a highly magnetized, rotating neutron star or white dwarf.

[5] See section 7 of Sean M. Carroll lecture notes [7] for the derivation.

The Einstein tensor corresponding to the above metric is

$$G^0{}_0 = -\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right), \tag{2.2}$$

$$G^1{}_1 = -\frac{1}{r^2} + e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right), \tag{2.3}$$

$$G^2{}_2 = G^3{}_3 = e^{-2b}\big(a'^2 - a'b' + a'' + \frac{a'-b'}{r}\big), \tag{2.4}$$

$$G^\mu{}_\nu = 0 \text{ for all the other components.} \tag{2.5}$$

Here the $'$ indicate the derivative with respect to r.

We choose to model the matter and energy in the star as a perfect fluid neglecting any anisotropic stresses and heat conduction. The energy-momentum tensor of an isotropic perfect fluid is $T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p)$, where ρ and p are the energy density and pressure as measured in the rest frame. This makes intuitively sense, as you cannot have different pressures in every direction with a spherical symmetry.

The field equations $G^\mu{}_\nu = 8\pi G T^\mu{}_\nu$ where $c = 1$ are

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = 8\pi G\rho\,, \tag{2.6}$$

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = -8\pi G p\,. \tag{2.7}$$

Defining $u/r \equiv e^{-2b(r)}$, the first field equation can be rewritten as

$$u' = -8\pi G\rho r^2 + 1\,, \tag{2.8}$$

and through integration we obtain

$$u = r - 2GM(r)\,, \tag{2.9}$$

where

$$M(r) = 4\pi \int_0^r \rho(r') r'^2 dr'\,, \tag{2.10}$$

and conclude

$$e^{-2b} = 1 - \frac{2GM(r)}{r}\,. \tag{2.11}$$

Now if we subtract the second field equation to the first one, we obtain

$$e^{-2b}(a' + b') = 4\pi G(\rho + p) r\,, \tag{2.12}$$

and after integration

$$a = -b + 4\pi G \int_\infty^r e^{2b(r')} r' (\rho + p)(r') dr'\,. \tag{2.13}$$

If ρ and p are known one can then determine the gravitational field.

An additional useful relation can be derived from the conservation law $\nabla_\nu T^{\mu\nu} = 0$.

$$T^{\mu\nu}{}_{;\nu} = 0 \Leftrightarrow T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\lambda\nu} T^{\lambda\nu} + \Gamma^{\nu}_{\lambda\nu} T^{\mu\lambda} = 0 \tag{2.14}$$

$$\Leftrightarrow \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\nu} (\sqrt{|g|} T^{\mu\nu}) + \Gamma^{\mu}_{\mu\lambda} T^{\lambda\nu} = 0 \,. \tag{2.15}$$

The energy-momentum tensor of a perfect fluid is

$$T_{\mu\nu} = \Big(\rho + \frac{p}{c^2}\Big) u_\mu u_\nu - p g_{\mu\nu} \,, \tag{2.16}$$

where u^μ is the four-velocity of the fluid. Inserting $T_{\mu\nu}$ for $\mu = 1$ in the above expression and noticing that $g^{\mu\nu}{}_{;\nu} = 0$, we obtain

$$a' = \frac{-p'}{(p+\rho)} \,. \tag{2.17}$$

On the other hand, combining eq. (2.11), eq. (2.10) and eq. (2.13) we recover

$$a' = \frac{G}{1 - \frac{2GM(r)}{r}} \left(\frac{M(r)}{r^2} + 4\pi r p \right) . \tag{2.18}$$

Comparing this equation with eq. (2.17) we find the *Tolman-Oppenheimer-Volkoff* (TOV) equation

$$\frac{dp}{dr} = -\frac{GM(r)\rho}{r^2} \left(1 + \frac{p}{\rho c^2}\right) \left(1 + \frac{4\pi r^3 p}{Mc^2}\right) \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1} , \tag{2.19}$$

where the pressure vanishes at the stellar radius R and the gravitational mass is given by

$$M(r) = 4\pi \int_0^r \rho(r') r'^2 dr' \,. \tag{2.20}$$

The TOV equation is the generalization of the *hydrostatic equilibrium* (HE) equation

$$\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2} \,, \tag{2.21}$$

of the Newtonian theory, the non-relativistic case. Within general relativity the radial pressure gradient increases for the following three reasons:

- Since gravity also acts on the pressure p, the density ρ is replaced by $(\rho + p)$ (first term in brackets of TOV equation).
- Since the pressure also acts as a source of the gravitational field, there is a term proportional to p in addition to $M(r)$ (second term in brackets of TOV equation).
- The gravitational force increases faster than $1/r^2$, in GR this quantity is replaced by $r^{-2}(1 - 2GM(r)/c^2 r)^{-1}$ (third term in brackets of TOV equation).

In order to construct a stellar model, one needs an *equation of state* $p = p(\rho)$. If this is known[6] the density profile of the star is determined by the central energy density $\rho_c \equiv \rho(r = 0)$.

The general relativistic stellar structure equations are (in units where $c = 1$)

$$-p' = \frac{G(\rho + p)(M(r) + 4\pi r^3 p)}{r^2(1 - 2GM(r)/r)}\,, \tag{2.22}$$

$$M' = 4\pi r^2 \rho\,, \tag{2.23}$$

$$a' = \frac{G}{1 - 2GM(r)/r}\left(\frac{M(r)}{r^2} + 4\pi r p\right). \tag{2.24}$$

The initial condition $M(r = 0) = 0$ is obtain from eq. (2.20), the other two boundary conditions are $p(r = 0) = p(\rho_c)$ and $e^{2a(R)} = 1 - 2GM(R)/R$ assuming a Schwarzschild metric outside the star.

2.3 Interpretation of M

We want to compare the gravitational mass M of eq. (2.20) with the baryonic mass $M_0 = Nm_N$, where N is the total number of nucleons in the star and m_N the mass of the nucleon. To get this quantity we have to integrate the baryon number density $n(r)$ over the volume as follows

$$N = 4\pi \int_0^R n(r) e^{b(r)} r^2 dr = 4\pi \int_0^R \frac{n(r) r^2 dr}{\sqrt{1 - \frac{2GM(r)}{r}}}\,. \tag{2.25}$$

Note that here the volume element is not Minkowskian[7] but, according to the metric given in eq. (2.1), given by $dV = e^{b(r)} r^2 dr \sin\theta d\theta d\phi$ [8] .

The proper internal material energy density is defined as

$$\varepsilon(r) \equiv \rho(r) - m_N n(r)\,, \tag{2.26}$$

and the total internal energy is correspondingly

$$E = M - M_0 = M - m_N N\,. \tag{2.27}$$

We use eq. (2.20) and (2.25) to decompose $E = T + V$ as follows

$$E = 4\pi \int_0^R r^2 \left(\rho(r) - \frac{n(r) m_N}{\sqrt{1 - \frac{2GM(r)}{r}}}\right) dr \tag{2.28}$$

$$= 4\pi \int_0^R \frac{r^2 \varepsilon(r)}{\sqrt{1 - \frac{2GM(r)}{r}}} dr + 4\pi \int_0^R r^2 \rho(r) \left(1 - \frac{1}{\sqrt{1 - \frac{2GM(r)}{r}}}\right) dr \equiv T + V \tag{2.29}$$

[6] One usually makes some assumptions and choses the equation of state accordingly. In general the equation of state is of the form $p = p(\rho, T, \{X_i\})$, where $\{X_i\}$ is the set of the abundances of all elements i and T their temperature. These dependencies increase the complexity of the model. Furthermore one has to consider the temperature structure of the star, its abundance ratios governed by nuclear physics and the various heat transport mechanisms.

[7] A Minkoweskian volume element is defined by $dV = dxdydz$.

[8] Here $e^{b(r)} r^2 dr$ denotes the radial part, $\sin\theta d\theta d\phi = d\Omega$ the angular one.

In order to find the connection with Newtonian theory, we expand the square roots in T and V, assuming that $GM(r)/r \ll 1$. This leads us to

$$T = 4\pi \int_0^R r^2 \varepsilon(r) \left(1 + \frac{GM(r)}{r} + ...\right) dr\,, \tag{2.30}$$

$$V = -4\pi \int_0^R r^2 \rho(r) \left(\frac{GM(r)}{r} + \frac{3G^2M^2(r)}{2r^2} + ...\right) dr\,. \tag{2.31}$$

The leading terms in T and V are the Newtonian values for the internal and gravitational energy of the star. Note that the extremely small value G^2 appears in the second term of the taylor expansion for the potential.

2.4 The interior of neutron stars

The physics of the interior of a neutron star is extremely complicated. A major problem is to establish a reliable equation of state. We assume that the neutron star is made up by an *ideal* mixture of nucleons and electrons in β-equilibrium.

In a system where quantum degeneracy is prevalent, the states in momentum space occupy a Fermi sphere of radius p_F. The energy density ρ, number density n and pressure p of an ideal Fermi gas at temperature $T = 0$ are given (in units where $c = 1$) by:

$$\rho = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{p_F} \sqrt{p^2 + m_N^2} p^2 dp = 3\rho_0 \int_0^{\frac{p_F}{m_N}} (u^2+1)^{\frac{1}{2}} u du\,, \tag{2.32}$$

$$n = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{p_F} p^2 dp\,, \tag{2.33}$$

$$p = \frac{1}{3}\frac{8\pi}{(2\pi\hbar)^3} \int_0^{p_F} \frac{p^2}{(p^2+m_N^2)^{\frac{1}{2}}} p^2 dp = \rho_0 \int_0^{\frac{p_F}{m_N}} (u^2+1)^{-\frac{1}{2}} u du\,, \tag{2.34}$$

where

$$\rho_0 = \frac{8\pi m_N^4 c^3}{3(2\pi\hbar)^3} = 6.11 \cdot 10^{15} \frac{\text{g}}{\text{cm}^3} \tag{2.35}$$

In 1939, Oppenheimer and Volkoff [6], performed the first neutron star calculations assuming that such objects are entirely made of a gas of non-interacting relativistic neutrons. They used the following equation of state

$$\rho = \rho\left(\frac{p_F}{m_N}\right),\; p = p\left(\frac{p_F}{m_N}\right) \;\Rightarrow\; p = p(\rho)\,. \tag{2.36}$$

This equation of state is extremely "soft" meaning that very little additional pressure is gained with increasing density and it predicts a maximum neutron star mass of just $M_{max} = 0.71\,M_\odot$ with radius $R_{max} = 9.3$ km (Figure 2.1). Since the neutron star's mass is a function of the central density $M = M(\rho_c)$, one can recover $\rho_{c,max} = 4 \cdot 10^{15}$ g/cm^3. For values of $\rho_c > \rho_{c,max}$ the star is unstable.

For more realistic equations of state, when other particles and interactions among the neutrons are considered, the star's maximum mass increases from $0.71\,M_\odot$ to $2 - 3\,M_\odot$. Indeed most observed neutron stars have masses around $\simeq 1.4\,M_\odot$.

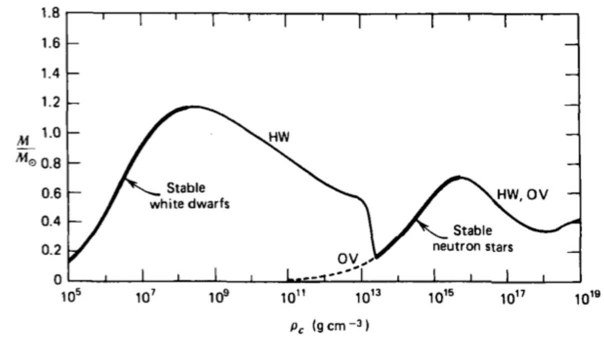

Figure 2.1: Gravitational mass versus central density for the Oppenheimer and Volkoff (OV) and Harrison and Wheeler (HW) equations of state. [8]

We can now use the listed properties of a neutron star to obtain an estimate of the gravitational redshift of a photon emitted at the surface of the neutron star. Using eq. (GRI 10.5), (GRI 10.6) and (GRI 22.4), we find

$$z = \frac{\Delta\lambda}{\lambda} = \left(1 - \frac{2GM_{max}}{R_{max}}\right)^{-\frac{1}{2}} - 1 \simeq 0.13 \tag{2.37}$$

Furthermore, a rapidly rotating neutron star can also emit gravitational waves. Its rapid rotational motion flattens its structure and generate a quadrupole radiation (see GRII Sec. 5.5). The flattening of the neutron star is given by the *ellipticity*

$$\epsilon \equiv \frac{r_{\text{equatorial}} - r_{\text{polar}}}{r_{\text{average}}} = \frac{a-b}{(a+b)/2} \,. \tag{2.38}$$

One can use the flattening ϵ together with the moment of inertia I of the neutron star to calculate the emitted energy per unit time (see eq. (GRII 6.8))

$$P = \dot{E}_{\text{grav}} = -\frac{32}{5}\frac{G}{c^5} I^2 \epsilon^2 \Omega^6 \,. \tag{2.39}$$

A recent paper from the LIGO and Virgo Scientific Collaboration et al. [9] exclude neutron stars with ellipticity $\epsilon > 10^{-5}\sqrt{10^{38}\,\text{kg}\,\text{m}^2/I}$ within 100 pc of Earth for frequencies above 55 Hz.

3 Tetrad formalism

3.1 Introduction

The content presented in this section follows the notation of S. Weinberg [10].
To derive the effects of gravity on physical systems, we have until now followed the covariance principle: write the appropriate special relativistic equations that hold in the absence of gravitation, replace $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$, and replace all derivatives with covariant derivatives. The resulting equations will be generally covariant and true in the presence of gravitational fields. However, this method actually works only for objects that behave like tensors under Lorentz transformation, and not for spinors fields. We will see that the *tetrad formalism* is useful to incorporate spinor fields.

The *tetrad formalism* is a different approach to the problem of determining the effects of gravitation on physical systems. As a formalism rather than a theory, it does not make different predictions but does allow the relevant equations to be expressed differently.

In general relativity, coordinate basis methods provide a straightforward procedure to calculate many quantities such as, for example, ∇_α and $R^\alpha{}_{\mu\beta\nu}$. Sometimes it is of advantage to use an orthonormal basis in tensor calculations. Remember from (GRI) that a coordinate basis $\left\{\frac{\partial}{\partial x^\mu}\right\}$ is orthonormal only for the trivial case of flat spacetime.

Tetrads are basis vectors which transform the metric onto a Minkowski structure. Due to the Principle of Equivalence, at every point X, we can erect a set of coordinates ξ^α_X that are locally inertial at X. Note that you can create a *single* inertial coordinate system that is locally inertial everywhere only if the spacetime is flat. As shown in Sec. 7 & 8.3 of (GRI) the general metric of non-inertial coordinate system is given by:

$$g_{\mu\nu}(x) = \left.\frac{\partial \xi^\alpha_X(x)}{\partial x^\mu}\right|_{x=X} \left.\frac{\partial \xi^\beta_X(x)}{\partial x^\nu}\right|_{x=X} \eta_{\alpha\beta} \equiv V^\alpha{}_\mu(X) V^\beta{}_\nu(X)\eta_{\alpha\beta}\,, \tag{3.1}$$

where $V^\alpha{}_\mu(X) := \frac{\partial \xi^\alpha_X(x)}{\partial x^\mu}|_{x=X}$. If we want to change our non-inertial coordinates from x^μ to x'^μ the partial derivatives $V^\alpha{}_\mu$ transform according to the rule

$$V^\alpha{}_\mu \to V'^\alpha{}_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V^\alpha{}_\nu\,. \tag{3.2}$$

We can think of $V^\alpha{}_\mu$ as forming *four* covariant vector fields, rather than a single tensor. This set of four vectors is known as a *tetrad* or *vierbein*.

Now, given any contravariant vector field $A^\mu(x)$, we can use the tetrad to refer its components at x to the coordinate system ξ^α_X locally inertial at x:

$$^*A^\alpha \equiv V^\alpha{}_\mu A^\mu\,. \tag{3.3}$$

We are contracting a contravariant vector A^μ with four covariant vectors $V^\alpha{}_\mu$, so this has the effect of replacing the single four-vector A^μ with the four scalars $^*A^\alpha$. We can do the same for covariant vector fields and general tensor fields as follows:

$$^*A_\alpha \equiv V_\alpha{}^\mu A_\mu\,, \tag{3.4a}$$
$$^*B^\alpha{}_\beta \equiv V^\alpha{}_\mu V_\beta{}^\nu B^\mu{}_\nu\,, \tag{3.4b}$$

where $V_\beta{}^\nu$ is the tetrad $V^\alpha{}_\mu$ with α-index lowered with the Minkowski tensor $\eta_{\alpha\beta}$ and μ-index are raised with the metric tensor $g^{\mu\nu}$ as follows

$$V_\beta{}^\nu = \eta_{\alpha\beta} g^{\mu\nu} V^\alpha{}_\mu \,. \tag{3.5}$$

According to eq. (3.1) this is just the inverse of the tetrad:

$$\delta^\mu{}_\nu = V_\beta{}^\mu V^\beta{}_\nu \tag{3.6}$$

Proof. Using the definition of $V_\beta{}^\mu$ and eq. (3.1) we easily get

$$V_\beta{}^\mu V^\beta{}_\nu = \eta_{\alpha\beta} g^{\mu\bar\nu} V^\alpha{}_{\bar\nu} V^\beta{}_\nu = g_{\bar\nu\nu} g^{\mu\bar\nu} = \delta^\mu{}_\nu \,.$$

□

and analogously

$$\delta^\alpha{}_\beta = V^\alpha{}_\mu V_\beta{}^\mu \,. \tag{3.7}$$

The scalar components of the metric tensor are then:

$$^*g_{\alpha\beta} = V_\alpha{}^\mu V_\beta{}^\nu g_{\mu\nu} = \eta_{\alpha\beta} \tag{3.8}$$

Proof. Using the definition of $V_\alpha{}^\mu$ and $V^\mu{}_\alpha$ we get

$$\begin{aligned} V_\alpha{}^\mu V_\beta{}^\nu g_{\mu\nu} &= \eta_{\alpha\bar\beta} g^{\mu\bar\nu} V^{\bar\beta}{}_{\bar\nu} \eta_{\beta\bar\alpha} g^{\nu\bar\mu} V^{\bar\alpha}{}_{\bar\mu} g_{\mu\nu} = \eta_{\alpha\bar\beta} \eta_{\beta\bar\alpha} g^{\mu\bar\nu} \delta^{\bar\mu}{}_\mu V^{\bar\beta}{}_{\bar\nu} V^{\bar\alpha}{}_{\bar\mu} \\ &= \eta_{\alpha\bar\beta} V_\beta{}^{\bar\nu} V^{\bar\beta}{}_{\bar\nu} = \eta_{\alpha\bar\beta} \delta^{\bar\beta}{}_\beta = \eta_{\alpha\beta} \end{aligned}$$

□

Note that one could also start defining the tetrad from eq. (3.8). We have now shown how to make any tensor field into a set of scalars. As mentioned at the beginning of this chapter, the tetrad formalism will be useful to incorporate spinor fields, like the Dirac's electron 1/2 spin field.

The equivalence principle requires that special relativity applies to locally inertial frames and that it makes no difference which locally inertial frame we chose. Thus, when going from one local inertial frame at a given point to another at the same point, the fields transform with respect to a Lorentz transformation $\Lambda^\alpha{}_\beta(x)$ as follows

$$^*A^\alpha(x) \to \Lambda^\alpha{}_\beta(x)\, ^*A^\beta(x) \tag{3.9}$$

$$^*T_{\alpha\beta}(x) \to \Lambda^\gamma{}_\alpha(x) \Lambda^\delta{}_\beta(x)\, ^*T_{\gamma\delta}(x) \tag{3.10}$$

where

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\gamma(x) \Lambda^\beta{}_\delta(x) = \eta_{\gamma\delta} \tag{3.11}$$

satisfies the requirement for the Lorentz transformation. Note that the Lorentz transformation depends on the position and therefore is not a constant.

The tetrad changes according to the same rule of ${}^*A^\alpha$:

$$V^\alpha{}_\mu(x) \to \Lambda^\alpha{}_\beta(x) V^\beta{}_\mu(x)\,. \tag{3.12}$$

Next we define the spin connection $\omega^a{}_b$ starting from the Levi-Civita connection

$$(\omega^a{}_b)_\mu := V^a{}_\lambda \nabla_\mu V_b{}^\lambda\,, \tag{3.13}$$

where the covariant derivative $\nabla_\mu V_b{}^\lambda$ is given by, using eq. (GRI 16.2),

$$\begin{aligned}\nabla_\mu V_b{}^\lambda &= \nabla_{\frac{\partial}{\partial x^\mu}}\left(\frac{\partial}{\partial x_\lambda}\xi_b\right) = \Gamma^\lambda_{\mu\sigma}\frac{\partial}{\partial x_\sigma}\xi_b + \frac{\partial}{\partial x_\lambda}\left(\frac{\partial}{\partial x^\mu}\xi_b\right)\\ &= \Gamma^\lambda_{\mu\sigma}V_b{}^\sigma + \frac{\partial}{\partial x^\mu}V_b{}^\lambda = \partial_\mu V_b{}^\lambda + \Gamma^\lambda_{\mu\sigma}V_b{}^\sigma\,,\end{aligned} \tag{3.14}$$

and therefore

$$\begin{aligned}(\omega^a{}_b)_\mu &= V^a{}_\lambda \nabla_\mu V_b{}^\lambda = V^a{}_\lambda\Big(V_b{}^\sigma\Gamma^\lambda_{\mu\sigma} + \partial_\mu V_b{}^\lambda\Big) = V_b{}^\sigma V^a{}_\lambda\Gamma^\lambda_{\mu\sigma} - V_b{}^\lambda\partial_\mu V^a{}_\lambda =\\ &= V_b{}^\sigma\Big(V^a{}_\lambda\Gamma^\lambda_{\mu\sigma} - \partial_\mu V^a{}_\lambda\Big)\,,\end{aligned} \tag{3.15}$$

where we used the identity $\delta^a{}_b = V^a{}_\lambda V_b{}^\lambda$ to compute $V^a{}_\lambda\partial_\mu V_b{}^\lambda = \partial_\mu(V^a{}_\lambda V_b{}^\lambda) - V_b{}^\lambda\partial_\mu V^a{}_\lambda = -V_b{}^\lambda\partial_\mu V^a{}_\lambda$. Note that there are other conventions for tetrads: Straumann [4] uses $V_a{}^\mu \doteq \theta^\mu$ and $V^a{}_\mu \doteq e_\mu$.

Note also that a, b are not tensor indices, they do not transform correctly under Lorentz transformations. It follows also that

$$\nabla_\mu V_b{}^\lambda = (\omega^c{}_b)_\mu V_c{}^\lambda\,. \tag{3.16}$$

This relation can be checked through the definition of the 1-form $(\omega^\mu{}_\nu)_a$ as follows

$$(\omega^a{}_b)_\mu = V^a{}_\lambda\nabla_\mu V_b{}^\lambda = V^a{}_\lambda(\omega^c{}_b)_\mu V_c{}^\lambda = \delta^a{}_c(\omega^c{}_b)_\mu = (\omega^a{}_b)_\mu\,. \tag{3.17}$$

Sometimes the tetrad $V_\alpha{}^\mu$ is also denoted with $e_\alpha{}^\mu$ and $V^\alpha{}_\mu$ with $e^\alpha{}_\mu$ (or $e^a{}_\mu$ when α, β are replaced by a, b). Finally note that μ, ν are raised or lowered with $g_{\mu\nu}$ and α, β (a,b) with $\eta_{\alpha\beta}$.

3.2 Some differential geometry

Here we follow the notation by N. Straumann [4].
Let $\mathcal{M}$ be a differentiable manifold with a Levi-Civita connection ∇. Let $(e_1, \ldots, e_n)$ be a basis of C^∞ vector fields on an open subset U. Let $(\theta^1, \ldots, \theta^n)$ be the corresponding dual basis of 1-forms, i.e. $\theta^i(e_j) = \delta^i_j$. We use latin letters to make clear that we work in the e_i frame.
We define the *connection forms* $\omega^i{}_j \in \Lambda^1(U)$ by

$$\nabla_X e_j = \omega^i{}_j(X) e_i\,. \tag{3.18}$$

We know from the definition of the Christoffel symbols (in the basis e_i) that

$$\nabla_{e_k} e_j = \Gamma^i_{kj} e_i = \omega^i{}_j(e_k) e_i \,, \tag{3.19}$$

thus

$$\omega^i{}_j = \Gamma^i_{kj} \theta^k \,. \tag{3.20}$$

Since ∇_X commutes with contraction, one can show that

$$\nabla_X \theta^i = -\omega^i{}_j(X) \theta^j \,. \tag{3.21}$$

We define the *torsion forms* $\Theta^i \in \Lambda^2(U)$ and *curvature forms* $\Omega^i{}_j \in \Lambda^2(U)$ by

$$T(X,Y) = \Theta^i(X,Y) e_i \,, \tag{3.22a}$$

$$R(X,Y) e_j = \Omega^i{}_j(X,Y) e_i \,. \tag{3.22b}$$

We can expand $\Omega^i{}_j$ as follows

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl} \theta^k \otimes \theta^l \,. \tag{3.23}$$

The torsion forms and curvature forms satisfy the *Cartan structure equations*:

$$\mathrm{d} g_{ij} = \omega_{ij} + \omega_{ji} \,, \tag{3.24a}$$

$$\Theta^i = \mathrm{d}\theta^i + \omega^i{}_j \wedge \theta^j \,, \tag{3.24b}$$

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \,, \tag{3.24c}$$

where $\omega_{ij} = g_{ik}\omega^k_j$ and $\mathrm{d}g_{ij} = e_k(g_{ij})\theta^k = \partial_k(g_{ij}) dx^k$ is the exterior derivative of the components of the metric in the basis e_i.

Let $\mathcal{M}$ now be a 4-dimensional Lorentzian manifold with orthonormal basis $(e_0, \ldots, e_3)$ and dual basis $(\theta^0, \ldots, \theta^3)$, i.e. such that $g(e_i, e_j) = \eta_{ij}$. In these coordinates we can write the metric as $g = \eta_{ij}\theta^i \otimes \theta^j$. In this case the Cartan structure equations simplify to

$$\omega_{ij} = -\omega_{ji} \,, \tag{3.25a}$$

$$\mathrm{d}\theta^i = -\omega^i{}_j \wedge \theta^j \,, \tag{3.25b}$$

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \,. \tag{3.25c}$$

Note also that $\Omega_{ij} = \eta_{ik}\Omega^k_j = -\Omega_{ji}$.

3.3 The Schwarzschild metric revisited

The Cartan structure equations allow us to easily calculate the Riemann tensor and its contractions. We will now rederive the Schwarzschild metric using the tetrad formalism.

The static isotropic metric can be written in the form

$$g = -e^{2a(r)} dt^2 + e^{2b(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \,. \tag{3.26}$$

Note that the factor e^{2a} corresponds to B and e^{2b} to A in eq. (GRI 22.2). The metric should be asymptotically flat, so $a(r \to \infty) = b(r \to \infty) = 0$.

3.3.1 Orthonormal coordinates

The Schwarzschild coordinates (t, r, θ, ϕ) are well adapted to the symmetries of the spacetime, but they are not orthonormal, as $g(\partial_i, \partial_j) = g_{ij} \neq \eta_{ij}$. We thus construct an orthonormal frame (e_0, e_1, e_2, e_3) at every point in the spacetime (i.e. a tetrad field), such that $g(e_i, e_j) = \eta_{ij}$. By observation of eq. (3.26) we find

$$\begin{aligned} e_0 &= e^{-a}\,\partial_t\,, \quad e_2 = r^{-1}\,\partial_\theta\,, \\ e_1 &= e^{-b}\,\partial_r\,, \quad e_3 = (r\sin\theta)^{-1}\,\partial_\phi\,. \end{aligned} \tag{3.27}$$

The corresponding co-frame $(\theta^0, \theta^1, \theta^2, \theta^3)$ is found by the condition $\theta^i(e_j) = \delta^i_j$, so

$$\begin{aligned} \theta^0 &= e^a\,dt\,, \quad \theta^2 = r\,d\theta\,, \\ \theta^1 &= e^b\,dr\,, \quad \theta^3 = r\sin\theta\,d\phi\,. \end{aligned} \tag{3.28}$$

3.3.2 Connection forms

We now want to solve the Cartan structure equations (3.25), meaning we have to calculate the exterior derivatives $\mathrm{d}\theta^i$. We find

$$\begin{aligned} \mathrm{d}\theta^0 &= \mathrm{d}(e^a dt) = \partial_i(e^a)\,dx^i \wedge dt = \partial_r(e^a)\,dr \wedge dt = a' e^a\,dr \wedge dt\,, \\ \mathrm{d}\theta^1 &= 0\,, \\ \mathrm{d}\theta^2 &= dr \wedge d\theta\,, \\ \mathrm{d}\theta^3 &= \sin\theta\,dr \wedge d\phi + r\cos\theta\,d\theta \wedge d\phi\,. \end{aligned} \tag{3.29}$$

We want to work in the basis given by the θ^i, so we rewrite these as follows:

$$\begin{aligned} \mathrm{d}\theta^0 &= -a' e^{-b}\,\theta^0 \wedge \theta^1\,, \\ \mathrm{d}\theta^1 &= 0\,, \\ \mathrm{d}\theta^2 &= r^{-1} e^{-b}\,\theta^1 \wedge \theta^2\,, \\ \mathrm{d}\theta^3 &= r^{-1} e^{-b}\,\theta^1 \wedge \theta^3 + r^{-1}\cot\theta\,\theta^2 \wedge \theta^3\,. \end{aligned} \tag{3.30}$$

From the first and second Cartan equation we now can read of the connection forms. For e.g. we have

$$a' e^{-b}\,\theta^0 \wedge \theta^1 = \omega^0_{\ 1} \wedge \theta^1 + \omega^0_{\ 2} \wedge \theta^2 + \omega^0_{\ 3} \wedge \theta^3\,. \tag{3.31}$$

From this we can read of that

$$\begin{aligned} \omega^0_{\ 1} &= \omega^1_{\ 0} = -a' e^{-b}\theta^0 + c_1\theta^1\,, \\ \omega^0_{\ 2} &= \omega^2_{\ 0} = c_2\theta^2\,, \\ \omega^0_{\ 3} &= \omega^3_{\ 0} = c_3\theta^3\,, \end{aligned} \tag{3.32}$$

where we cannot yet determine the constants c_i due to the fact that $\theta^i \wedge \theta^i = 0$. Solving the other equations we finally find

$$\begin{aligned}
\omega^0_{\ 1} &= \omega^1_{\ 0} = a' e^{-b} \theta^0 \,, \\
\omega^0_{\ 2} &= \omega^2_{\ 0} = 0 \,, \\
\omega^0_{\ 3} &= \omega^3_{\ 0} = 0 \,, \\
\omega^1_{\ 2} &= -\omega^2_{\ 1} = -r^{-1} e^{-b} \theta^2 \,, \\
\omega^1_{\ 3} &= -\omega^3_{\ 1} = -r^{-1} e^{-b} \theta^3 \,, \\
\omega^2_{\ 3} &= -\omega^3_{\ 2} = -r^{-1} \cot\theta \, \theta^3 \,.
\end{aligned} \tag{3.33}$$

3.3.3 Curvature forms

We are now in a place to calculate the curvature forms from the third Cartan structure equation (3.25c). For example, we find

$$\begin{aligned}
\Omega^0_{\ 1} &= \mathrm{d}\omega^0_{\ 1} + \omega^0_{\ 2} \wedge \omega^2_{\ 1} + \omega^0_{\ 3} \wedge \omega^3_{\ 1} = \mathrm{d}\omega^0_{\ 1} = \mathrm{d}(a' e^{-b} \theta^0) = \mathrm{d}(a' e^{-b}) \wedge \theta^0 + a' e^{-b} \mathrm{d}\theta^0 \\
&= \partial_r (a' e^{-b}) dr \wedge \theta^0 - a'^2 e^{-2b} \, \theta^0 \wedge \theta^1 \\
&= -e^{-2b} (a'^2 - a'b' + a'') \, \theta^0 \wedge \theta^1 \,,
\end{aligned} \tag{3.34}$$

where we used eq. (3.28) for the last equality. The other curvature forms follow similarly and the results are

$$\begin{aligned}
\Omega^0_{\ 1} &= \Omega^1_{\ 0} = -e^{-2b} (a'^2 - a'b' + a'') \, \theta^0 \wedge \theta^1 \,, \\
\Omega^0_{\ 2} &= \Omega^2_{\ 0} = -\frac{a' e^{-2b}}{r} \, \theta^0 \wedge \theta^2 \,, \\
\Omega^0_{\ 3} &= \Omega^3_{\ 0} = -\frac{a' e^{-2b}}{r} \, \theta^0 \wedge \theta^3 \,, \\
\Omega^1_{\ 2} &= -\Omega^2_{\ 1} = \frac{b' e^{-2b}}{r} \, \theta^1 \wedge \theta^2 \,, \\
\Omega^1_{\ 3} &= -\Omega^3_{\ 1} = \frac{b' e^{-2b}}{r} \, \theta^1 \wedge \theta^3 \,, \\
\Omega^2_{\ 3} &= -\Omega^3_{\ 2} = \frac{1 - e^{-2b}}{r^2} \, \theta^2 \wedge \theta^3 \,.
\end{aligned} \tag{3.35}$$

3.3.4 Ricci tensor

The Ricci tensor is given by the contraction of the Riemann tensor. Using the curvature forms we find

$$R_{ij} \stackrel{\text{(GRI 17.4)}}{=} \theta^l \left(R(e_l, e_j) e_i \right) \stackrel{\text{(3.22b)}}{=} \theta^l \left(\Omega^k_{\ i}(e_l, e_j) e_k \right) = \Omega^l_{\ i}(e_l, e_j) \,. \tag{3.36}$$

We can thus easily calculate the components of the Ricci tensor. For e.g. the 00-component is given by

$$R_{00} = \Omega^1_{\ 0}(e_1, e_0) + \Omega^2_{\ 0}(e_2, e_0) + \Omega^3_{\ 0}(e_3, e_0) \,. \tag{3.37}$$

Note that $\theta^i \wedge \theta^j = \theta^i \otimes \theta^j - \theta^j \otimes \theta^i$ and thus

$$\theta^i \wedge \theta^j(e_k, e_l) = \theta^i \otimes \theta^j(e_k, e_l) - \theta^j \otimes \theta^i(e_k, e_l) = \delta^i_k \delta^j_l - \delta^j_k \delta^i_l \,. \tag{3.38}$$

The 00-component of the Ricci tensor then evaluates to

$$R_{00} = e^{-2b}(a'^2 - a'b' + a'') + \frac{a'e^{-2b}}{r} + \frac{a'e^{-2b}}{r} = e^{-2b}\left(a'^2 - a'b' + a'' + \frac{2a'}{r}\right) \,. \tag{3.39}$$

The other components evaluate similarly to

$$R_{11} = -e^{-2b}\left(a'^2 - a'b' + a'' - \frac{2b'}{r}\right) \,, \tag{3.40a}$$

$$R_{22} = e^{-2b}\left(\frac{b' - a'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \,, \tag{3.40b}$$

$$R_{33} = R_{22} \,. \tag{3.40c}$$

The Ricci scalar then is

$$\mathcal{R} = \eta^{ij} R_{ij} = -2e^{-2b}\left(a'^2 - a'b' + a'' + \frac{2(a' - b')}{r} + \frac{1}{r^2}\right) + \frac{2}{r^2} \,. \tag{3.41}$$

The non-diagonal terms are trivially zero. The components of the Einstein tensor are computed using $G_{ij} = R_{ij} - \frac{1}{2}\eta_{ij}\mathcal{R}$

$$G_{00} = \frac{1}{r^2} + e^{-2b}\left(\frac{2b'}{r} - \frac{1}{r^2}\right) \,, \tag{3.42a}$$

$$G_{11} = -\frac{1}{r^2} + e^{-2b}\left(\frac{2a'}{r} + \frac{1}{r^2}\right) \,, \tag{3.42b}$$

$$G_{22} = e^{-2b}\left(a'^2 - a'b' + a'' + \frac{1}{r}(a' - b')\right) \,, \tag{3.42c}$$

$$G_{33} = G_{22} \,, \tag{3.42d}$$

$$G_{\mu\nu} = 0 \quad \text{for all other components.} \tag{3.42e}$$

3.3.5 Solving the Einstein Field Equations

We now want to solve the vacuum field equations $G_{ij} = 0$. We combine the first two of these equations to find

$$G_{00} + G_{11} = \frac{2e^{-2b}}{r}(a' + b') = 0 \,. \tag{3.43}$$

This is equal to $a' + b' = 0$ or $a + b = \text{const} = 0$. The 00-equation reads

$$G_{00} = \frac{1}{r^2} + e^{-2b}\left(\frac{2b'}{r} - \frac{1}{r^2}\right) = 0 \,, \tag{3.44}$$

which is equivalent to

$$\frac{d}{dr}\left(re^{-2b}\right) = 1 \quad \Rightarrow \quad e^{-2b} = 1 - \frac{2m}{r}\,. \tag{3.45}$$

m is an integration constant and can be determined by the Newton limit $m = GM/c^2$. The static isotropic metric thus reads

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\,. \tag{3.46}$$

We see here the advantage of the tetrad formalism: Instead of computing 32 Christoffel symbols, we only had to compute 6 curvature forms to get the same result.

4 Spinors in curved spacetime

In this section we assume the reader is familiar with group theory, spinors and the Dirac equation. The content presented here follows the books of S. Weinberg [10] and S. Chandrasekhar [11].

4.1 Representations of the Lorentz group

Under a general Lorentz transformation a set of quantities ψ_n transform into the new quantities

$$\psi'_n = \sum_m \big[D(\Lambda)\big]_{nm}\psi_m\,, \tag{4.1}$$

where Λ is a Lorentz transformation (LT) and $D(\Lambda)$ is a *representation* of the Lorentz group. A representation has to fulfill the following property

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)\,, \tag{4.2}$$

which means that the Lorentz transformation Λ_1 followed by the Lorentz transformation Λ_2 gives the same result as the Lorentz transformation $\Lambda_1\Lambda_2$. The eq. (4.2) is understood as a matrix multiplication. For instance, if ψ is a contravariant vector V^β, then $D(\Lambda)$ is

$$\big[D(\Lambda)\big]^\alpha{}_\beta = \Lambda^\alpha{}_\beta\,, \tag{4.3}$$

while if ψ is a covariant tensor $S_{\alpha\beta}$, the corresponding D-matrix is

$$\big[D(\Lambda)\big]_{\gamma\delta}{}^{\alpha\beta} = \Lambda_\gamma{}^\alpha\Lambda_\delta{}^\beta\,. \tag{4.4}$$

One can verify that eq. (4.3) and eq. (4.4) satisfy the group multiplication rule (4.2).

The most general true reprensentations of the homogeneous Lorentz group are provided by tensor representations such as eq. (4.3) and eq. (4.4). Hence, we might expect that all quantities of physical interest should be tensors. However, there are additional representations of the *infinitesimal* Lorentz group, the *spinor representations*, that play an important role in quantum field theory. The infinitesimal Lorentz group consists of Lorentz transformations infinitesimally close to the identity, described by

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \alpha^\mu{}_\nu\,, \tag{4.5}$$

where $|\alpha^\mu{}_\nu| \ll 1$. Furthermore, to first order in α, the condition $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ (eq. (GRI 3.10)) implies that the matrix is antisymmetric $\alpha_{\mu\nu} = -\alpha_{\nu\mu}$.

We now define the *generators* $T_{\mu\nu}$ of the infinitesimal Lorentz group using the representation of the infinitesimal Lorentz transformation

$$D(\Lambda) = 1 + \frac{1}{2}\alpha_{\mu\nu}T^{\mu\nu}\,, \tag{4.6}$$

where $T_{\mu\nu}$ are a fixed set of antisymmetric matrices, i.e. $T^{\mu\nu} = -T^{\nu\mu}$.

It turns out that the generators $T^{\alpha\beta}$ of the infinitesimal Lorentz group for a contravariant vector have the components

$$(T^{\alpha\beta})^{\mu}{}_{\nu} = \eta^{\alpha\mu}\delta^{\beta}{}_{\nu} - \eta^{\alpha\beta}\delta^{\mu}{}_{\nu}\,. \tag{4.7}$$

Fore example, one can use this result to compute the transformation of the vector x^μ and recover $x'^\mu = \Lambda^\mu{}_\nu x^\nu = x^\mu + \frac{1}{2}\alpha_{\rho\sigma}(T^{\rho\sigma})^\mu{}_\nu x^\nu = x^\mu + \alpha^\mu{}_\nu x^\nu$. On the other hand the generators of a covariant tensor have components

$$\big[T_{\alpha\beta}\big]_{\gamma\delta}{}^{\varepsilon\xi} = \eta_{\alpha\gamma}\delta^{\varepsilon}{}_{\beta}\delta^{\xi}{}_{\delta} - \eta_{\beta\gamma}\delta^{\varepsilon}{}_{\alpha}\delta^{\xi}{}_{\delta} + \eta_{\alpha\delta}\delta^{\xi}{}_{\beta}\delta^{\varepsilon}{}_{\gamma} - \eta_{\alpha\beta}\delta^{\xi}{}_{\alpha}\delta^{\varepsilon}{}_{\gamma}\,. \tag{4.8}$$

Furthermore it can be shown that $T_{\alpha\beta}$ satisfies the commutation relations

$$\big[T_{\alpha\beta}, T_{\gamma\delta}\big] \equiv T_{\alpha\beta}T_{\gamma\delta} - T_{\gamma\delta}T_{\alpha\beta} = \eta_{\gamma\beta}T_{\alpha\delta} - \eta_{\gamma\alpha}T_{\beta\delta} + \eta_{\delta\beta}T_{\gamma\alpha} - \eta_{\delta\alpha}T_{\gamma\beta}\,. \tag{4.9}$$

One can easily check this inserting the matrices (4.7) and (4.8) in the above relation. Thus the problem of finding the general representation of the infinitesimal homogeneous Lorentz group is reduced to finding all matrices which satisfy eq. (4.9) (see next section).

Next we have to consider how a quantity ψ_n transforms under a position dependent Lorentz transformation or a more general coordinate transformation. Under the transformation $x^\mu \to x'^\mu$ the derivative changes according to

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\,. \tag{4.10}$$

Remember that ψ_n transforms according to eq. (4.1), but now Λ depends on the position and thus we need to extend the definition of covariant derivative to a general representation D. Note that in the case of tensor fields, this is just the covariant derivative. Here we give directly the final result[9].

The covariant derivative of a field ψ transforming in a representation of the Lorentz group with generators $T^{\mu\nu}$ is, in an orthonormal basis,

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{2}(\omega_{\nu\rho})_\mu T^{\nu\rho}\psi\,, \tag{4.11}$$

where $\partial_\mu\psi = V^b{}_\mu\partial_b\psi$ and $(\omega_{\nu\rho})_\mu$ is given in eq. (3.13). Weinberg uses a different notation than the one we are using: our $T^{\mu\nu}$ corresponds to his $\sigma^{\mu\nu}$ and he uses

$$\widetilde{\Gamma}_\mu \equiv \frac{1}{2}\sigma^{\nu\rho}V_\nu{}^\delta V_{\rho\delta;\mu} = \frac{1}{2}T^{\nu\rho}(\omega_{\nu\rho})_\mu\,, \tag{4.12}$$

where $(\omega_\nu{}^\rho)_\mu = V_\nu{}^\kappa\nabla_\mu V^\rho{}_\kappa$ which is equivalent to our notation. Note that $\widetilde{\Gamma}$ is neither a Christoffel symbol nor a Dirac matrix. One can introduce

$$\begin{aligned}\mathscr{D}_\alpha \equiv V_\alpha{}^\mu\mathcal{D}_\mu = V_\alpha{}^\mu\Big[\frac{\partial}{\partial x^\mu} + \widetilde{\Gamma}_\mu\Big] &= V_\alpha{}^\mu\frac{\partial}{\partial x^\mu} + \frac{1}{2}\sigma^{\nu\rho}V_\nu{}^\delta V_\alpha{}^\mu V_{\rho\delta;\mu} \\ &= V_\alpha{}^\mu\frac{\partial}{\partial x^\mu} + \frac{1}{2}T^{\nu\rho}V_\alpha{}^\mu(\omega_{\nu\rho})_\mu\end{aligned} \tag{4.13}$$

The effect of gravity is such that the equations which hold in special relativity are modified by replacing $\frac{\partial}{\partial x^\mu}$ with $\mathscr{D}_\alpha$.

[9]See S. Weinberg's book [10] pp. 368-370 for the explicit derivation.

4.2 Dirac gamma matrices

The Dirac gamma matrices are a set of square matrices $\{\Gamma^\mu\}$ which obey the anticommutation relation

$$\{\Gamma^\mu, \Gamma^\nu\} = \Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu} I_4 \,, \tag{4.14}$$

where $\{\,,\,\}$ denotes the anticommutator and I_4 the 4×4 identity matrix. Covariant gamma matrices are defined by

$$\Gamma_\mu = \eta_{\mu\nu}\Gamma^\nu = (\Gamma^0, -\Gamma^1, -\Gamma^2, -\Gamma^3)\,. \tag{4.15}$$

In 4-dimentional spacetime, the smallest representation of the gamma matrices is given by the Dirac representation in which Γ^μ are the 4×4 matrices

$$\Gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}, \Gamma^1 = \begin{bmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{bmatrix}, \Gamma^2 = \begin{bmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{bmatrix}, \Gamma^3 = \begin{bmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{bmatrix}, \tag{4.16}$$

where $\sigma_x\,, \sigma_y\,, \sigma_z$ are the 2×2 Pauli matrices and I_2 is the 2×2 identity matrix. The Dirac representation is used to describe spin $1/2$ particles.

Furthermore, the matrices

$$T^{\mu\nu} = -\frac{1}{4}\left[\Gamma^\mu, \Gamma^\nu\right] \tag{4.17}$$

form a representation of the Lorentz algebra, as given by eq. (4.9).

4.3 Dirac equation in flat space

The Dirac equation for a free particle of mass m reads

$$\big(i\hbar\Gamma^\mu\partial_\mu - mc\big)\psi = 0 \;\Leftrightarrow\; (i\hbar\not{\partial} - mc)\psi = 0\,, \tag{4.18}$$

where $\psi = (\psi_1, \psi_2)$ is a bispinor consisting of two spinors, ψ_1 and ψ_2, and $\not{\partial} = \Gamma^\mu\partial_\mu$ is the Feynman slash notation.

The Dirac equation for a particle in an external electromagnetic field, described by the four potential $A_\mu(x)$, is obtained by replacing ∂_μ with $\partial_\mu + i\frac{e}{c\hbar}A_\mu$. In natural units we obtain

$$(i\not{\partial} - e\not{A} - m)\psi = 0\,. \tag{4.19}$$

For example an electron in the field of a proton, i.e. an hydrogen atom, is described by this equation.

4.4 Dirac equation in curved spacetime

In curved spacetime we replace ∂_μ by $\mathscr{D}_\alpha$ and we thus get

$$(i\hbar\Gamma^a V_a{}^\mu \mathcal{D}_\mu - mc)\psi = 0\,, \tag{4.20}$$

with $V_a{}^\mu\mathcal{D}_\mu = \mathscr{D}_a$. If there is an external electromagnetic field we have the extra term $-\frac{e}{c}\Gamma^\mu A_\mu$ as we had in eq. (4.19)

$$(i\hbar\Gamma^a V_a{}^\mu \mathcal{D}_\mu - \frac{e}{c}\Gamma^\mu A_\mu - mc)\psi = 0\,, \tag{4.21}$$

where A_μ is the four potential.

4.5 Dirac equation in Kerr-Newman geometry

Here we study the Dirac equation around a charged, rotating black hole (see GRII Sec. 8). We follow the papers of S. Chandrasekhar [12] and the paper of D. Page [13]. We will not give the full derivation, we just list the main results. Let P^B and Q^B be two-component spinors representing the wave functions particles and antiparticles, respectively, then

$$\psi = \begin{bmatrix} P^B \\ Q^B \end{bmatrix} . \tag{4.22}$$

For the Kerr-Newman black hole we consider the Bayer-Lindquist coordinates given in eq. (GRII 8.12)

$$\Delta = r^2 - 2Mr + a^2 + Q^2 , \tag{4.23}$$

$$\rho^2 = r^2 + a^2 cos^2\theta , \tag{4.24}$$

where M is the total mass, a is the intrinsic angular momentum and Q the total charge. Page is using a different notation in his paper: $\rho \leftrightarrow \Sigma$.

The Dirac equation in natural units takes the following form

$$\sqrt{2}\big(\nabla_{BB'} + ieA_{BB'}\big)P^B + i\mu_e {Q^*}_{B'} = 0 , \tag{4.25}$$

$$\sqrt{2}\big(\nabla_{BB'} - ieA_{BB'}\big)Q^B + i\mu_e {P^*}_{B'} = 0 , \tag{4.26}$$

where μ_e is the electron mass and $A_{BB'}$ is the electromagnetic potential, which in the second equation enter with a minus sign to preserve gauge invariance. Here the asterisk denote the complex conjugate, while in Chandrasekhar's papers he uses the over-bar. The indices A, B, A', $B' = \{0, 1\}$.

To lower and raise indices of a spinor[10] we use two-dimensional Levi-Civita symbol ε_{AB} and ε^{AB}, respectively (where $\varepsilon_{00} = \varepsilon_{11} = 0$ and $\varepsilon_{01} = -\varepsilon_{10} = 1$), thus

$$P^A = \varepsilon^{AB} P_B \;,\; P_A = \varepsilon_{AB} P^B . \tag{4.27}$$

We shall assume that the four components of the wave function have a dependence on time t and on the azimuthal angle ϕ given by

$$e^{i(\sigma t + m\phi)} . \tag{4.28}$$

This dependence is that of a wave function with frequency σ and wavenumber m. Then one can make the following ansatz to solve the eq. (4.25) and (4.26)

$$P^0 = (r - ia\cos\theta)^{-1} e^{i(\sigma t + m\phi)} f_1(r, \theta) , \tag{4.29}$$

$$P^1 = e^{i(\sigma t + m\phi)} f_2(r, \theta) , \tag{4.30}$$

$$Q^{*1} = e^{i(\sigma t + m\phi)} g_1(r, \theta) , \tag{4.31}$$

$$Q^{*0} = -(r + ia\cos\theta)^{-1} e^{i(\sigma t + m\phi)} g_2(r, \theta) . \tag{4.32}$$

[10] See Chandrasekhar's book [11] p. 534.

Let us define the following quantitiy

$$K := (r^2 + a^2)\sigma + am - eQr\,, \tag{4.33}$$

and make the separation ansatz for f_1, f_2, g_1 and g_2:

$$f_1(r,\theta) = R_{-\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta) \equiv R(r)S(\theta)\,, \tag{4.34}$$
$$f_2(r,\theta) = R_{+\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)\,, \tag{4.35}$$
$$g_1(r,\theta) = R_{+\frac{1}{2}}(r)S_{-\frac{1}{2}}(\theta)\,, \tag{4.36}$$
$$g_2(r,\theta) = R_{-\frac{1}{2}}(r)S_{+\frac{1}{2}}(\theta)\,, \tag{4.37}$$

where $R_{\pm\frac{1}{2}}(r)$ and $S_{\pm\frac{1}{2}}(\theta)$ are functions of r and θ respectively, with $R_{+\frac{1}{2}}(r)$ the radial wave function for spin up particle. Inserting the ansatz into the eq. (4.25) and (4.26), gets us four constants of separation $\lambda_1\,, \lambda_2\,, \lambda_3\,, \lambda_4$ which satisfy for consistency $\lambda_1 = \lambda_3 = \frac{1}{2}\lambda_2 = \frac{1}{2}\lambda_4 = \lambda$. To write these equations, we use the following definitions

$$\mathscr{D}_n := \partial_r + \frac{iK}{\Delta} + 2n\frac{r-M}{\Delta} \equiv \mathscr{D}_0 + 2n\frac{r-M}{\Delta}\,, \tag{4.38}$$
$$\mathscr{D}_n^\dagger := \partial_r - \frac{iK}{\Delta} + 2n\frac{r-M}{\Delta} \equiv \mathscr{D}_0^\dagger + 2n\frac{r-M}{\Delta}\,, \tag{4.39}$$
$$\mathscr{L}_n := \partial_\theta + \widetilde{Q} + n\cot\theta\,, \tag{4.40}$$
$$\mathscr{L}_n^\dagger := \partial_\theta - \widetilde{Q} + n\,\mathrm{cosec}\,\theta\,, \tag{4.41}$$

where $\widetilde{Q} = a\sigma\sin\theta + m\,\mathrm{cosec}\,\theta$. Note that $\mathscr{D}_n$, $\mathscr{D}_n^\dagger$ are purely radial operators, while $\mathscr{L}_n$, $\mathscr{L}_n^\dagger$ are purely angular operators. Note also that r and M have the same dimension in natural units. Furthermore when these operators are applied to "background" quantities, i.e. quantities independent of t and ϕ, they reduce to ∂_r and ∂_θ respectively[11]. One then gets the following equations

$$\mathscr{D}_0 R_{-\frac{1}{2}} = (\lambda + i\mu_e r)R_{+\frac{1}{2}}(r)\,, \tag{4.42}$$
$$\Delta\mathscr{D}_{\frac{1}{2}}^\dagger R_{+\frac{1}{2}} = 2(\lambda - i\mu_e r)R_{-\frac{1}{2}}(r)\,, \tag{4.43}$$
$$\mathscr{L}_{\frac{1}{2}} S_{+\frac{1}{2}} = -\sqrt{2}(\lambda - a\mu_e\cos\theta)S_{-\frac{1}{2}}(\theta)\,, \tag{4.44}$$
$$\mathscr{L}_{\frac{1}{2}}^\dagger S_{-\frac{1}{2}} = \sqrt{2}(\lambda + a\mu_e\cos\theta)S_{+\frac{1}{2}}(\theta)\,. \tag{4.45}$$

We can for instance eliminate $R_{+\frac{1}{2}}(r)$ from the first pair of equations to get an equation for $R_{-\frac{1}{2}}(r)$, one gets

$$\left[\Delta\mathscr{D}_{\frac{1}{2}}^\dagger\mathscr{D}_0 - \frac{i\mu_e}{\lambda + i\mu_e r}\mathscr{D}_0 - 2(\lambda^2 + \mu_e^2 r^2)\right]R_{-\frac{1}{2}}(r) = 0\,, \tag{4.46}$$

where λ^2 is a characteristic eigenvalue determined by the condition that $S_{+\frac{1}{2}}(\theta)$ and $S_{-\frac{1}{2}}(\theta)$ are regular at $\theta = 0$ and $\theta = \pi$. We can do the same for the second pair of equation and

[11] See Chandrasekhar's book [11] p. 383.

eliminate $S_{+\frac{1}{2}}(\theta)$ to get an equation for $S_{+\frac{1}{2}}(\theta)$

$$\left[\mathscr{L}_{\frac{1}{2}}\mathscr{L}_{\frac{1}{2}}^{\dagger} + \frac{a\mu_e \sin\theta}{\lambda + a\mu_e \cos\theta}\mathscr{L}_{\frac{1}{2}}^{\dagger} + 2(\lambda^2 + a^2\mu_e^2\cos^2\theta)\right]S_{-\frac{1}{2}}(\theta) = 0\,. \tag{4.47}$$

We have thus reduced the solution of the Dirac equation outside a charged, rotating black hole to the solution of a pair of decoupled eq. (4.46) and (4.47). The general solution can be expressed as a linear superposition of the various solutions belonging to the different characteristic eigenvalues of λ^2.

4.6 Dirac equation in Schwarzschild geometry

Here we study the Dirac equation around a Schwarzschild black hole. We follow the derivation of the paper of B. Mukhopadhyay and S. K. Chakrabarti [14], for more details see S. Chandrasekar's book [11].

In Schwarzschild geometry we consider only the radial eq. (4.46). Furthermore here the black hole does not rotate and is not charged, therefore the Bayer-Lindquist coordinates reduce to $\Delta = r^2 - 2Mr$ and $\rho^2 = r^2$, where M is the mass of the black hole. The quantity defined in eq. (4.33) becomes $K = r^2\sigma$, where σ is the frequency of the incoming Dirac wave. Also in this section we set $G = c = \hbar = 1$. Let us define

$$r_* = r + 2M\log|r - 2M|\,, \tag{4.48}$$

where $r > r_+ = 2M$, with r_+ the Schwarzshild radius. We see that $r_* \in (-\infty, +\infty)$. We then have

$$\frac{d}{dr_*} = \frac{\Delta}{r^2}\frac{d}{dr}\,. \tag{4.49}$$

The operators $\mathscr{D}_0$ and $\mathscr{D}_0^{\dagger}$ take the following forms in terms of r_*

$$\mathscr{D}_0 = \frac{r^2}{\Delta}\Big(\frac{d}{dr_*} + i\sigma\Big)\,, \tag{4.50}$$

$$\mathscr{D}_0^{\dagger} = \frac{r^2}{\Delta}\Big(\frac{d}{dr_*} - i\sigma\Big)\,. \tag{4.51}$$

Furthermore define $\theta := \arctan(\mu_e r)$, which gives

$$\cos\theta = \frac{1}{\sqrt{1 + \mu_e^2 r^2}}\,, \tag{4.52}$$

$$\sin\theta = \frac{\mu_e r}{\sqrt{1 + \mu_e^2 r^2}}\,, \tag{4.53}$$

$$1 \pm i\mu_e r = e^{\pm i\theta}\sqrt{1 + \mu_e^2 r^2}\,. \tag{4.54}$$

We choose $R_{-\frac{1}{2}} = P_{-\frac{1}{2}}$ and $\Delta^{\frac{1}{2}}R_{+\frac{1}{2}} = P_{+\frac{1}{2}}$. Following Chandrasekhar approach, we write

$$P_{+\frac{1}{2}} = \psi_{+\frac{1}{2}}\exp\Big(-\frac{1}{2}i\arctan(\mu_e r)\Big)\,, \tag{4.55}$$

$$P_{-\frac{1}{2}} = \psi_{-\frac{1}{2}}\exp\Big(+\frac{1}{2}i\arctan(\mu_e r)\Big)\,. \tag{4.56}$$

Finally, we define $\hat{r}_* := r_* + \frac{1}{2\sigma}\arctan(\mu_e r)$, which yields

$$d\hat{r}_* = \Big(1 + \frac{\Delta}{r^2}\frac{\mu_e}{2\sigma}\frac{1}{1+\mu_e^2 r^2}\Big)dr_* \,. \tag{4.57}$$

Defining $z_\pm := \psi_{+\frac{1}{2}} \pm \psi_{-\frac{1}{2}}$ the equations (4.42) and (4.43) become

$$\Big(\frac{d}{d\hat{r}_*} - W\Big)z_+ = i\sigma z_- \,, \tag{4.58}$$

$$\Big(\frac{d}{d\hat{r}_*} + W\Big)z_- = i\sigma z_+ \,, \tag{4.59}$$

where

$$W = \frac{\Delta^{\frac{1}{2}}(1+\mu_e^2 r^2)^{\frac{3}{2}}}{r^2(1+\mu_e^2 r^2) + \mu_e\frac{\Delta}{2\sigma}} \,. \tag{4.60}$$

One important point to note is that the transformation of the spacial coordinate r to $\hat{r}_*$ shifts the horizon from $r = r_+$ to $\hat{r}_* = -\infty$. From the eqs. (4.58) and (4.59) we get a pair of independent one-dimensional wave-equations

$$\Big(\frac{d^2}{d\hat{r}_*^2} + \sigma^2\Big)z_\pm = V_\pm z_\pm \,, \tag{4.61}$$

where

$$\begin{aligned} V_\pm &= W^2 \pm \frac{dW}{d\hat{r}_*} \\ &= \frac{\Delta^{\frac{1}{2}}(1+\mu_e^2 r^2)^{\frac{3}{2}}}{[r^2(1+\mu_e^2 r^2)+\mu_e\frac{\Delta}{2\sigma}]^2}[\Delta^{\frac{1}{2}}(1+\mu_e^2 r^2)^{\frac{3}{2}} \pm ((r-M)(1+\mu_e^2 r^2) + 3\mu_e^2 r\Delta)] \\ &\mp \frac{\Delta^{\frac{3}{2}}(1+\mu_e^2 r^2)^{\frac{5}{2}}}{[r^2(1+\mu_e^2 r^2)+\mu_e\frac{\Delta}{2\sigma}]^3}[2r(1+\mu_e^2 r^2 + 2\mu_e^2 r^3 + \mu_e\frac{r-M}{\sigma})] \,. \end{aligned} \tag{4.62}$$

The parameter space is spanned by the frequency σ and the rest mass of the incoming particle μ_e. The parameter space is unphysical[12] when $\sigma < \mu_e$. The rest of the parameter space, i.e. when $\sigma > \mu_e$, is divided into two regions: I-region $E > V_m$ and II-region $E < V_m$, where $V_m = \max(V_+, V_-)$ and $E = \sigma^2$. This is shown on Figure 4.1.

In I-region, the wave is locally sinusoidal since the wave number k is real for the entire region of $\hat{r}_*$ (eq. (4.63)). In II-region the wave is decaying when $E < V$, i.e. where the wave "hits" the potential barrier, while in the rest of the region the wave is propagating.

In I-region eq. (4.61) is

$$\frac{d^2 z_+}{d\hat{r}_*^2} + (\sigma^2 - V_+)z_+ = 0 \,, \tag{4.63}$$

where $\sigma^2 - V_+ > 0$. This is like a Schrödinger equation, with σ^2 being the total energy of the wave. The solutions are locally sinusoidal. Let

$$k(\hat{r}_*) = (\sigma^2 - V_+)^{\frac{1}{2}} \,, \tag{4.64}$$

$$u(\hat{r}_*) = \int k(\hat{r}_*)d\hat{r}_* + \text{const.} \,, \tag{4.65}$$

[12] The energy $E \sim \sigma^2$ should be bigger than the rest energy $E = (m^2 + p^2)^{\frac{1}{2}} \geq m = \mu_e$.

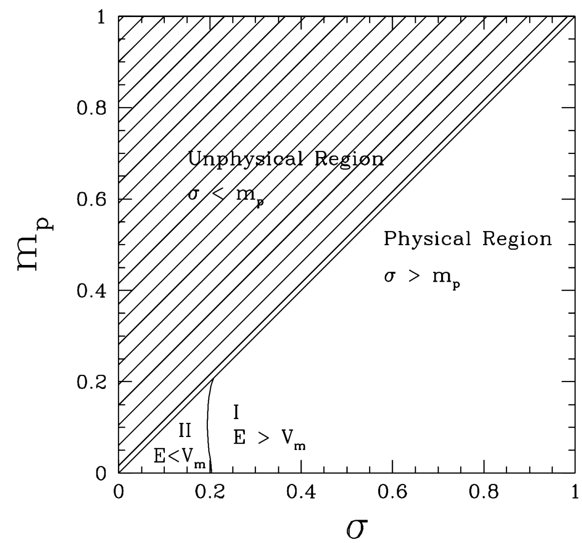

Figure 4.1: Classification of the parameter space in terms of the energy and rest mass of the particles. The physical region $\sigma > m_p$ is further classified in terms of whether the particle actually "hits" the barrier or not. [14]

where k is the wavenumber of the incoming wave and u the *Eiconal*. Then the solution of eq. (4.63) using the WKB methods is

$$z_+ \sim \frac{A_+}{\sqrt{k}} \exp(iu) + \frac{A_-}{\sqrt{k}} \exp(-iu)\,, \tag{4.66}$$

where $A_+^2 + A_-^2 = k$.

In II-region σ^2 is no longer greater than $V_\pm$ at all radii. As a result $k^2 = \sigma^2 - V_\pm$ may be negative in some regions, therefore k and u are imaginary and the solution is proportional to

$$z_+ \sim \frac{\exp(-u)}{\sqrt{k}} + \frac{\exp(+u)}{\sqrt{k}}\,. \tag{4.67}$$

The reflection and transmission coefficients are given by

$$R = \frac{A_-^2}{k}\ ,\ T = \frac{A_+^2}{k}\,. \tag{4.68}$$

With these results (radial function, transmission and reflection coefficients) we can study the scattering of 1/2 spin particles from a Schwarzschild black hole. Sometimes transmission and reflection coefficients are defined at a single point. This is meaningful only if the potential sharply changes in a small region (around the considered point). Recently it has been argued that no bound solutions exist for the Schwarzschild case, see [15]. This is connected to the fact that there is a finite probability to find the particle inside the horizon.

5 The four laws of black hole dynamics

The content presented here follows the book of N. Straumann [4].

There are close analogies between the laws of thermodynamics and the physics of black holes. The reasons behind this is still not fully understood. The first steps in this direction were made by J. Bekenstein and S. Hawking. In 1972, Bekenstein was the first to suggest that black holes should have a well-defined entropy and that it is proportional to its event horizon. Two years later Hawking proposed the existence of blackbody radiation released by black holes due to quantum effects near the event horizon. His work was inspired by a visit to the Soviet scientists Y. Zeldovich and A. Starobinsky who showed him that, according to the Heisenberg uncertainty principle, rotating black holes should create and emit a thermal flux of particles. Hawking radiation is also known as *black hole evaporation* because it reduces the mass and energy of black holes. Because of this, black holes that do not gain mass through other means are expected to shrink and ultimately vanish.

5.1 The zeroth law

The zeroth law of thermodynamics states that the temperature T throughout a body in thermal equilibrium is uniform. For black holes the analogue of the temperature is the surface gravity κ. The zeroth law of black hole dynamics states that *the surface gravity of a stationary black hole is uniform over the entire event horizon.*

It can be shown[13] that for a Kerr-Newmann black hole, the value of the surface gravity κ at the horizon radius r_H is

$$\kappa = \frac{r_H - M}{2Mr_H - Q^2} \, . \tag{5.1}$$

From this relation we see that the surface gravity is constant, therefore Kerr-Newman black holes satisfy the zeroth law of black hole dynamics.

5.2 The first law

The first law of thermodynamics states that the change in the internal energy $U \equiv U(S, V, N, ...)$ is equal to the amount of heat $\mathcal{Q}$ supplied to the system, minus the amount of work $\mathcal{W}$ done by the system on its surroundings

$$dU = \delta\mathcal{Q} - \delta\mathcal{W} = TdS - pdV + \mu dN + \dots \, . \tag{5.2}$$

In black hole physics the analogue of the thermodynamical potential U is the total mass M which is a function of the surface area A of the horizon, the angular momentum J and the electric charge Q.

In order to derive the surface area of the Kerr-Newman horizon we consider its induced metric in Boyer-Lindquist coordinates[14] for fixed t

$$ds^2 = \rho^2 d\theta^2 + \frac{(r^2 + a^2)^2}{\rho^2} \sin^2\theta d\phi^2 \, , \tag{5.3}$$

[13] See Straumann's book [4] pp. 492-494 for the explicit derivation.
[14] See Straumann's book [4] pp. 467-468.

where

$$r = r_H = M + \sqrt{M^2 - a^2 - Q^2}\,. \tag{5.4}$$

The corresponding volume form is $\bar{\eta} = (r_H^2 + a^2)d\theta \wedge \sin\theta d\phi$ and the surface area of the horizon is

$$A = 4\pi(r_H^2 + a^2) = 4\pi(2Mr_H - Q^2)\,. \tag{5.5}$$

This equation can be solved for the total mass M of the black hole. Using the abbreviation $M_0^2 = \frac{A}{16\pi}$, we find

$$M(A, J, Q) = \left(\left(M_0 + \frac{Q^2}{4M_0}\right)^2 + \frac{J^2}{4M_0^2}\right)^{\frac{1}{2}}\,. \tag{5.6}$$

From this equation we can recover

$$\frac{\partial M}{\partial A} = \frac{1}{8\pi}\kappa\,, \tag{5.7}$$

$$\frac{\partial M}{\partial J} = \Omega_H = \frac{a(2Mr_H - Q^2)}{(r_H^2 + a^2)^2} = \frac{a}{r_H^2 + a^2}\,. \tag{5.8}$$

Before writing down the expression for $\frac{\partial M}{\partial Q}$ we give the electric potential of the black hole

$$\phi = \frac{Q}{\rho^2}r(1 - \Omega_H a \sin^2\theta)\,, \tag{5.9}$$

see Straumann's book for an explicit derivation. At the horizon we have

$$\phi_H = \phi|_H = \frac{Qr_H}{r_H^2 + a^2} = \frac{Qr_H}{2Mr_H - Q^2}\,, \tag{5.10}$$

which is constant, thus

$$\frac{\partial M}{\partial Q} = \phi_H\,. \tag{5.11}$$

Putting all these results together we get a relation for the differential of M, this is known as the first law of black holes dynamics

$$dM = \frac{1}{8\pi}\kappa dA + \Omega_H dJ + \phi_H dQ\,. \tag{5.12}$$

The analogy with eq. (5.2) is striking, which suggest that the temperature T corresponds to the surface gravity κ and the entropy S correspond to the surface area A of the horizon of the black hole.

Hawking radiation suggests that one should associate a temperature and an entropy to a black hole as follows

$$T_H = \frac{\hbar\kappa}{2\pi k_B}\;,\; S = \frac{k_B A}{4\hbar}\,, \tag{5.13}$$

where k_B is the Boltzmann constant and $\hbar$ is the Planck constant. The quantity S is known as *Bekenstein-Hawking* entropy and T_H as Hawking temperature. For a complete derivation of these quantities we refer to pp. 497-502 of Straumann's book [4].

5.3 The second law

The second law of black holes dynamics is due to S. Hawking and states that *in any (classical) interaction of matter and radiation with black holes, the total surface area of the boundaries of these holes (as formed by their horizons) can never decrease.*

The limitation to classical interactions means that we do not consider changes in the quantum theory of matter due to the presence of the strong external gravitational fields of black holes. This is justified for macroscopic black holes, while for “mini-holes” quantum effects, such as spontaneous Hawking radiation, would become important. The physics of the latter case requires the use of Dirac equations in curved spacetime.

To provide an example, consider two black holes colliding and coalescing into a single one. The second law tells us that the resulting surface area is larger than the sum of the surface area of the event horizons of both original black holes.

We further investigate this situation assuming that these two black holes are Kerr black holes. Furthermore, let us assume that the final black hole, resulting from the coalescence of the two, is stationary. Due to the second law we have

$$M\left(M+\sqrt{M^2-a^2}\right) \geq M_1\left(M_1+\sqrt{M_1^2-a_1^2}\right)+M_2\left(M_2+\sqrt{M_2^2-a_2^2}\right), \tag{5.14}$$

where M_1, M_2 are the mass of the two colliding black holes and M is the mass of the final black hole. From this relation we find the *efficiency*:

$$\varepsilon := \frac{M_1+M_2-M}{M_1+M_2} < \frac{1}{2}\,. \tag{5.15}$$

As a special case let us take $M_1 = M_2 = \frac{1}{2}\mathcal{M}$ and $a_1 = a_2 = a = 0$, then $M > \frac{1}{\sqrt{2}}\mathcal{M}$ and hence

$$\varepsilon \leq 1 - \frac{1}{\sqrt{2}} = 0.293\,. \tag{5.16}$$

In principle, a lot of energy can be released as gravitational radiation. It should be stressed that within classical theory the correspondence between temperature and surface gravity is just formal since the physical temperature of a black hole is zero. This in not the case in quantum theory where the physical temperature is given the Hawking temperature $T_H = \frac{\hbar\kappa}{2\pi k_B}$.

5.4 The third law

Nernst’s third law of thermodynamics states that it is impossible for any process to reduce the entropy of a system to its absolute-zero value in a finite number of operations. This is equivalent to the unattainability formulation that states it is impossible to bring a system to absolute zero temperature a finite number operations. The equivalent law for black holes dynamics states that *the surface gravity of a black hole cannot be reduced to zero within a finite advanced time.*

6 Hawking radiation

The contents covered in this section are taken from pp. 84-89 of the lecture script of Prof. Gian Michele Graf [16], the book of R. M. Wald [17] and the page "Hawking Radiation" on Wikipedia.

6.1 Introduction

Energy emission is possible from a static black hole, provided quantum effects are taken into account. Let us assume a pair of particle and antiparticle is created through vacuum quantum fluctuations, where one is created inside and the other one outside the event horizon of the black hole. In this scenario the latter one can reach a distant observer with energy $E_2 > 0$, this type of particles constitute *Hawking radiation*.

A detailed treatment of the phenomenon requires quantum field theory in curved spacetime. In order to make the analysis easier we consider a scalar field describing a scalar particle and thus the Klein-Gordon equation, rather than the Dirac equation. The Klein-Gordon equation reads

$$(\Box_g + \mu_p^2)\phi = 0\,, \tag{6.1}$$

where μ_p is the mass of the scalar field of the particle p and $\Box_g$ is the Laplacian for the metric g

$$\Box_g = \frac{1}{|g|}\partial_\nu\left(\sqrt{|g|}g^{\mu\nu}\partial_\mu\right)\,. \tag{6.2}$$

We consider a Schwarzschild black hole for which we introduce Regge-Wheeler coordinates $(t, r_*, \theta, \varphi)$ with $r > 2M$, for particles outside the black hole. Using natural units, i.e. $G = c = \hbar = 1$, the transition function is given by

$$r_* = r + 2M\log\left(\frac{r}{2M} - 1\right)\,, \tag{6.3}$$

with t, θ, φ fixed. This function maps $r \in\,]2M, \infty[\ \mapsto r_* \in\,]-\infty, \infty[$. Since

$$\frac{dr_*}{dr} = 1 + \left(\frac{r}{2M} - 1\right)^{-1} = \left(1 - \frac{2M}{r}\right)^{-1}\,, \tag{6.4}$$

the metric, with $r = r(r_*)$, reads

$$ds^2 = \left(1 - \frac{2M}{r}\right)(dt^2 - dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{6.5}$$

We write the Klein-Gordon equation in Regge-Wheeler coordinates, after separating the angular part according to

$$f(t, r_*, \theta, \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{f_{lm}(t, r_*)}{r}Y_{lm}(\theta, \varphi)\,, \tag{6.6}$$

as, without proof,

$$(\partial_t^2 - \partial_{r_*}^2 + V_l)f_{lm} = 0\,, \tag{6.7}$$

where V_l is the effective potential

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + \mu_p^2\right), \tag{6.8}$$

and has limits

$$V_l(r) \to \begin{cases} 0\,, & r_* \to -\infty\,, \text{i.e. } r \to 2M \\ \mu_p^2\,, & r_* \to \infty\,, \text{i.e. } r \to \infty \end{cases} . \tag{6.9}$$

Eq. (6.7) has precisely the form of the wave equation for a massless scalar field in a two-dimensional flat spacetime (with cartesian coordinates t, r_*) with scalar potential (6.8).

Thus for $r_* \to -\infty$, i.e. $r \to 2M$, the solutions are of the form

$$f_{lm}(t, r_*) = f_{out}(t - r_*) + f_{in}(t + r_*)\,, \tag{6.10}$$

where f_{in} describe the incoming wave from the white hole[15] and f_{out} the outgoing wave from black hole.

6.2 Expected number of outgoing particles

One can continue the analysis of the particle creation effect using the theory of scattering by potentials in flat spacetime, as it is done for e.g. particle physics. A complete proof of Hawking radiation using scattering theory does not yet exist.

Let us assume $\mu_p \neq 0$. Then every wave packet should behave, in the asymptotic past, like a free (i.e. V=0) massless solution in (t, r_*)-space, propagating in from $r_* \to -\infty$ (i.e. the white hole horizon), together with a massive solution (distorted by a $\frac{1}{r_*}$ potential[16]) propagating in from $r_* \to \infty$. Similarly, in the asymptotic future, every wave packet should behave as a free massless wave propagating to $r_* \to -\infty$ together with a (distorted) massive wave propagating to $r_* \to \infty$.

Near the horizon, for $r_* \to -\infty$ (i.e. $r \to 2M$), the "free wave" looks like

$$\sim a \exp(-i\omega u) + b \exp(-i\omega v)\,, \tag{6.11}$$

where $\omega > \mu_p$, $u = t - r_*$ and $v = t + r_*$. The case of purely outgoing solution is given by $b = 0$. It is found (no proof) that each outgoing mode of frequency ω passes an "infinite oscillation" singularity on the black hole horizon. The behavior of these solution is

$$\sim a \exp(i\omega k^{-1} \ln(-U))\,, \tag{6.12}$$

where $k = \frac{1}{4M}$ and $U = e^{-ku}$. Next, one finds that ϕ, the solution of Klein-Gordon equation, is $\phi \sim \frac{f(r,t)}{r} Y_{lm}(\theta, \varphi)$. For outgoing particle one has

$$\phi(v) = \begin{cases} 0\,, & v > 0 \\ \phi_0 \exp\left(\frac{i\omega}{k}\ln(-\alpha v)\right)\,, & v < 0 \end{cases} , \tag{6.13}$$

[15] In general relativity, a white hole is a hypothetical region of spacetime which cannot be entered from the outside, although matter and light can escape from it. In this sense, it is the reverse of a black hole, which can only be entered from the outside and from which matter and light cannot escape.

[16] Eq. (6.8) behaves as $\sim (\mu_p^2 - 2M\mu_p^2/r_*)$ for $r_* \to \infty$.

where $\alpha \sim \frac{dU}{d\lambda}\big|_{\lambda=0}$ (λ is the affine parameter describing a geodesic entering the black hole, with $\lambda = 0$ chosen to correspond to the intersection between the geodesic and the horizon) and $U = 0$ on the event horizon. Furthermore since λ depends smoothly on U and satisfy $\frac{dU}{d\lambda} \neq 0$ on the horizon, it follows that near $\lambda = 0$, each outgoing mode oscillate as a function of λ as $\alpha \sim \exp(i\omega k^{-1}\ln(-\alpha\lambda))$.

It has to be noted that even when starting from purely positive frequencies for ω, ϕ also contains negative frequencies antiparticles.

Next, one can show that the number of particles spontaneously created is

$$\langle N \rangle = |t|^2 \frac{e^{-2\pi\omega/k}}{1 - e^{-2\pi\omega/k}} , \tag{6.14}$$

where $|t|^2$ is the square of the Klein-Gordon norm of the wave packet which would enter the white hole in the extended Schwarzschild solution. Eq. (6.14) is precisely the formula which holds for a perfect black body emitter at a temperature given by[17]

$$E = k_B T = \frac{\hbar k}{2\pi c} = \frac{\hbar c^3}{8\pi G M} , \tag{6.15}$$

where $k = \frac{c^4}{4\pi MG}$ and

$$T \simeq 6 \cdot 10^{-8} \frac{M_\odot}{M}\ \mathrm{K} . \tag{6.16}$$

Eq. (6.15) gives us the *Hawking temperature* as

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} . \tag{6.17}$$

6.3 Black hole evaporation

We now want to perform a crude analytic estimate of the radiated power. We assume that only photons are radiated away and use the Stefan-Boltzmann law of blackbody radiation. The Stefan-Boltzmann constant is given by $\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$, the Schwarzschild radius is $r_s = \frac{2GM}{c^2}$. The Schwarzschild sphere surface of the black hole is

$$A_s = 4\pi r_s^2 = 4\pi \left(\frac{2GM}{c^2}\right)^2 = \frac{16\pi G^2 M^2}{c^4} . \tag{6.18}$$

To obtain our rough estimate of the emitted power we use the Stefan-Boltzmann power law $P = A\sigma\varepsilon T^4$ assuming the black hole is a perfect blackbody $\varepsilon = 1$

$$P = A_s \sigma T_H^4 = \left(\frac{16\pi G^2 M^2}{c^4}\right)\left(\frac{\pi^2 k_b^4}{60\hbar^3 c^2}\right)\left(\frac{\hbar c^3}{8\pi G M k_B}\right)^4 = \frac{\hbar c^6}{15360\pi G^2 M^2} = \frac{K_{ev}}{M^2} , \tag{6.19}$$

where $K_{ev} = 3.5 \cdot 10^{32}\ \mathrm{W\,kg^2}$. Under the assumption of an otherwise empty universe, it is possible to calculate how long it would take for the black hole to evaporate. We relate the rate of evaporation energy loss of the black hole

$$P = -\frac{dE}{dt} = \frac{K_{ev}}{M^2} , \tag{6.20}$$

[17] Note that $\frac{h\nu}{k_B T} \overset{!}{=} \frac{2\pi\omega c}{k}$

to its mass loss using the Einstein's mass-energy relation $E = Mc^2$

$$P = -\frac{dE}{dt} = -c^2 \frac{dM}{dt}\,, \tag{6.21}$$

and get through integration

$$\int_{M_0}^{0} M^2 dM = -\frac{K_{ev}}{c^2} \int_0^{t_{ev}} dt\,, \tag{6.22}$$

where M_0 is the initial black hole mass and t_{ev} the *evaporation time*. Solving this equation with respect to t_{ev} give us

$$t_{ev} = \frac{c^2 M_0^3}{3K_{ev}} = \left(\frac{c^2 M_0^3}{3}\right)\left(\frac{15360\pi G^2}{\hbar c^6}\right) = \frac{5120\pi G^2 M_0^3}{\hbar c^4} = 8.4 \cdot 10^{-17} \left(\frac{M_0}{\mathrm{kg}}\right)^3 \mathrm{s}\,. \tag{6.23}$$

For $M_0 = M_\odot$ we get

$$t_{ev} = \frac{5120\pi G^2 M_\odot^3}{\hbar c^4} = 6.6 \cdot 10^{74} \simeq 2.1 \cdot 10^{67}\,\mathrm{yr}\,, \tag{6.24}$$

this is a huge number compared to the age of the universe $t_{universe} \simeq 1.4 \cdot 10^{10}\,\mathrm{yr}$.

7 Test of general relativity

The contents presented in this section are taken form the review of C. Will [18].

7.1 The Einstein equivalence principle

Einstein used the principle of equivalence as a basic element in the development of general relativity. Between the 1960 and 1965, R. Dicke contributed with crucial ideas about the foundations of gravitation theory and generalized the principle to what today is called the *Einstein equivalence principle* (EEP). The EEP is a powerful and far-reaching concept stating that the following three principles are valid:

1. The *weak equivalence principle* (WEP) stating that the property of a body called inertial mass is equal to the gravitational mass.

2. The *local Lorentz invariance* (LLI) stating that the outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed.

3. The *local position invariance* (LPI) stating that the outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed.

It is possible to argue convincingly that if EEP is valid, then the effects of gravity must be equivalent to the effects of living in a curved spacetime. As a consequence, the only theories of gravity that can fully embody EEP are those that satisfy the postulates of "metric theories of gravity" which are:

1. Spacetime is endowed with a symmetric metric.

2. The trajectories of freely falling test bodies are geodesics of that metric.

3. In local freely falling reference frames, the non-gravitational laws of physics are those written in the language of special relativity.

Indeed, if EEP holds, then in local freely falling frames the laws governing experiments must be independent of the velocity of the frame (LLI), with constant values for the various atomic constants (in order to satisfy LPI, the independence of location). The only laws we know of that fulfill this are those that are compatible with special relativity, such as Maxwell's equations of electromagnetism.

7.2 Test of the weak equivalence principle

A direct test of WEP is the comparison of the acceleration of two bodies of different composition in an external gravitational field. The violation of this principle would result in a different acceleration of the two bodies. Let us suppose that the inertial mass m_I is different from the gravitational mass m_P. Then, in a gravitational field g the acceleration is given by $m_I a = m_P g$. The inertial mass m_I is made up of several types of mass-energy: rest energy, electromagnetic energy, weak-interaction energy, and so on. If one of these energies contributes to m_P differently than it does to m_I a violation of WEP would result. We could then write

$$m_P = m_I + \sum_A \frac{\eta^A E^A}{c^2}\,, \tag{7.1}$$

where E^A is the inertial energy of the body due to the interaction A, η^A dimensionless parameter that measures the strength of the violation of WEP. A measurement on the fractional difference in acceleration between two bodies, called the *"Eötvös ratio"*, is then given by

$$\eta := 2\left|\frac{a_1 - a_2}{a_1 + a_2}\right| = \sum_A \eta^A \left(\frac{E_1^A}{m_1 c^2} - \frac{E_2^A}{m_2 c^2}\right)\,, \tag{7.2}$$

where we dropped the index "I" for the inertial masses m_1, m_2. Experimental limits on η place limits on the WEP-violation parameters η^A. Many high-precision experiments have been performed to measure the Eötvös, see Figure 7.1 for a selection of these tests. The last experiment to date is being made by the MICROSCOPE satellite launched in April 2016 and is designed to give an upper limit of $\eta \leq 10^{-15}$. As of December 2017, the first paper of the MICROSCOPE experiment has been published [19] and gives an upper limit of $\eta \leq 10^{-14}$. The hope is to reach the $\eta \leq 10^{-15}$ limit by 2020, when more orbits and more data analysis provide a gain of one order of magnitude. We also expect future measurements from STE-QUEST with atom-interferometry which will further improve the accuracy.

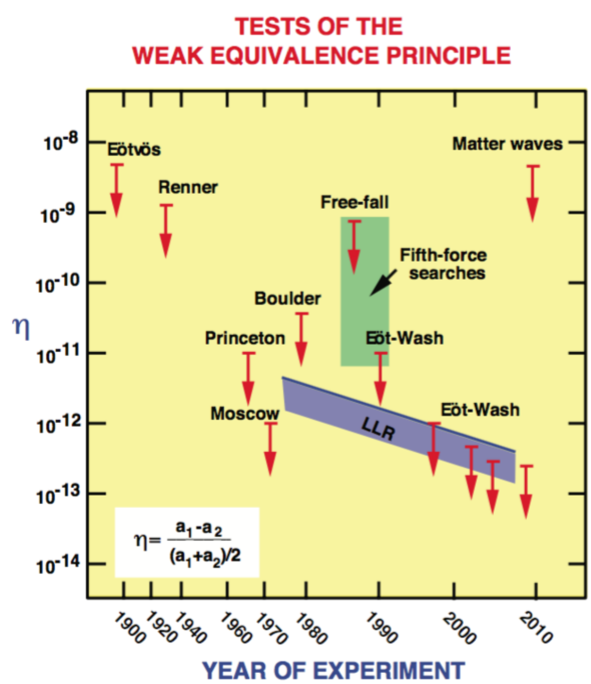

Figure 7.1: Selected tests of the weak equivalence principle, showing bounds on the Eötvös ratio η. The green band shows the results of many experiments performed to search for a fifth force. The blue band shows evolving bounds on η for gravitating bodies from lunar laser ranging. [18]

7.3 Test of local Lorentz invariance

Special relativity has been tested in depth, see for example particle physics experiments. In the past several years, a vigorous theoretical and experimental effort has been launched to find violations of special relativity to look for evidence of new physics "beyond" Einstein, such as violations of Lorentz invariance that might result from certain models of quantum gravity. Quantum gravity asserts that there is a fundamental length scale given by Planck length $l_p = (\frac{\hbar G}{c^3})^{\frac{1}{2}} = 1.6 \cdot 10^{-33}\,\mathrm{cm}$, but since length is not an invariant quantity, there could be a violation of Lorentz invariance at some level in quantum gravity.

A useful way to interpret some experiments is to suppose that the electromagnetic interactions suffer a slight violation of Lorentz invariance, through a change in the speed of electromagnetic radiation c relative to the limiting speed of material test particles, in other words $c \neq 1$. Such a violation necessarily selects a preferred universal rest frame, presumably that of the cosmic background radiation, through which we are moving at 370 km s^{-1}. Such a Lorentz-non-invariant electromagnetic interaction would cause shifts in the energy levels of atoms and nuclei that depend on the orientation of the quantization axis of the state relative to our universal velocity vector, and on the quantum numbers of the state.

The results of these experiments are shown in Figure 7.2, introducing the quantity $\delta := |\frac{1}{c^2} - 1|$. Astrophysical observations have also been used to put bound on Lorentz violations. For example, if photons satisfy the Lorentz violating dispersion relation

$$E^2 = p^2c^2 + E_{Pl} f^{(1)}|p|c + f^{(2)}p^2c^2 + \frac{f^{(3)}}{E_{Pl}}|p|^3c^3 + \ldots\,, \tag{7.3}$$

where $E_{Pl} = (\frac{\hbar c^5}{G})^{\frac{1}{2}}$ is the Planck energy, then the speed of light $v_\gamma = \frac{\partial E}{\partial p}$ would be given to linear order in the $f^{(n)}$ by

$$\frac{v_\gamma}{c} \simeq 1 + \sum_{n\geq 1} \frac{(n-1) f_\gamma^{(n)} E^{n-2}}{2E_{Pl}^{n-2}}\,. \tag{7.4}$$

Such a Lorentz-violating dispersion relation could be a relic of quantum gravity.

7.4 Test of local position invariance

The principle of local position invariance can be tested by the gravitational redshift experiment. The gravitational redshift experiment measures the frequency or wavelength shift $z = \frac{\Delta\nu}{\nu} = -\frac{\Delta\lambda}{\lambda}$ between two identical frequency standards (clocks) placed at rest at different heights in a static gravitational field. If the frequency of a given type of atomic clock is the same when measured in a local, momentarily comoving freely falling frame, independent of the location or velocity of that frame, then the comparison of frequencies of two clocks at rest at different locations boils down to a comparison of the velocities of two local Lorentz frames. The frequency shift is then a consequence of the first-order Doppler shift between the frames. The structure of the clock plays no role whatsoever. The result is a shift

$$z = \frac{\Delta U}{c^2}\,, \tag{7.5}$$

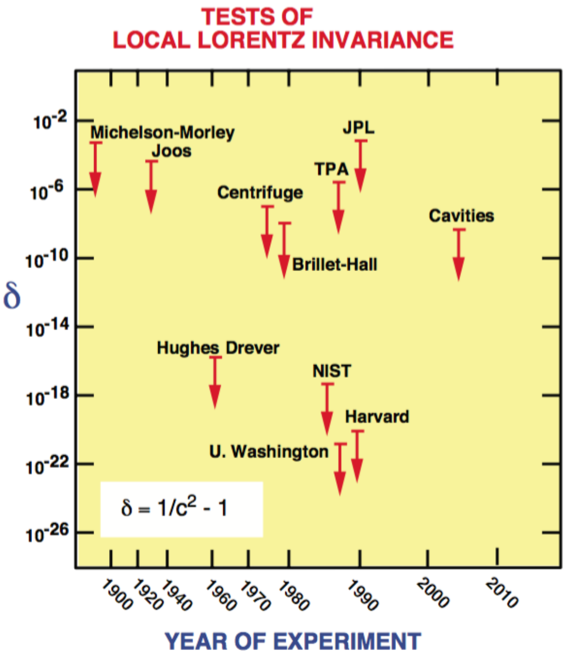

Figure 7.2: Selected tests of local Lorentz invariance showing the bounds on the parameter δ, which measures the degree of violation of Lorentz invariance in electromagnetism. The limits assume a speed of Earth of 370 km/s relative to the mean rest frame of the universe. [18]

where ΔU is the difference in the Newtonian gravitational potential between the receiver and the emitter. If the LPI is not valid, then the shift can be written as

$$z = (1 + \alpha)\frac{\Delta U}{c^2}, \tag{7.6}$$

where the parameter α depends upon the nature of the clock whose shift is being measured.

First high precision measurements were done by the Pound-Rebka-Snider experiments between the 1960 and 1965 using γ-ray photons from ^{53}Fe as they ascended and descended the Jefferson Physical Laboratory tower at Harvard.

The most precise standard redshift test to date was the Vessot-Levine rocket experiment known as Gravity Probe-A (GPA) that took place in June 1976. A hydrogen-maser was flown on a rocket to an altitude of 10'000 km and its frequency compared to a hydrogen-maser on the ground. The analysis of the data yielded an upper limit of $|\alpha| < 2 \cdot 10^{-4}$. The results of these experiments are shown in Figure 7.2.

The *Atomic Clock Ensemble in Space* (ACES), scheduled to be launched at the end 2018 or beginning 2019, is going to be put on the International Space Station (ISS) and consist of an advanced hydrogen-maser designed and built in Switzerland and a cold atom clock based on cesium (PHARAO). An improvement of the measurements of two order of magnitude is expected, which will bring the upper limit to $\alpha < 10^{-6}$.

If the LPI is satisfied, then the fundamental constants of non-gravitational physics should be constants in time. More recent experiments have used Strontium-87 or Rubidium-87 atoms trapped in optical lattices and compared with Cesium-133 to obtain

$$\frac{\dot{\alpha}_{EM}}{\alpha_{EM}} < 10^{-16}\,\mathrm{yr}^{-1}\,, \tag{7.7}$$

where $\alpha_{EM} = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ is the fine structure constant.

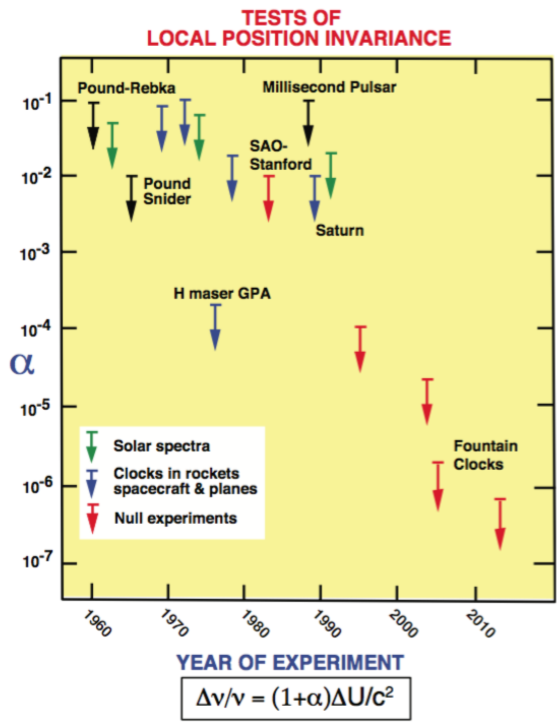

Figure 7.3: Selected tests of local position invariance via gravitational redshift experiments, showing bounds on α, which measures degree of deviation of redshift. [18]

7.5 Schiff's conjecture

The three parts of the Einstein equivalence principle that seem at first glance to be independent theoretical principles are however related in a "complete" theory. Around 1960, L. Schiff conjectured that *"any complete, self-consistent theory of gravity that embodies WEP necessary embodies EEP"*. In other words, the validity of WEP alone guarantees the validity of local Lorentz and position invariance, and thereby EEP.

A rigorous proof of the conjecture is impossible (some counter-examples are known), yet a number of powerful "plausibility" arguments can be formulated. The most general ones are based on the assumption of energy conservation. For example consider a system in quantum state A that decays to a state B, emitting a quantum of frequency ν. The quantum falls a height H in an external gravitational field and is shifted to frequency ν' while the system in state B falls with acceleration g_B. At the bottom, state A is rebuilt out of state B, the quantum of frequency ν', and the kinetic energy $m_B g_B H$ that state B gained during its fall. The energy left over must be exactly enough, $m_A g_A H$, to raise state A to its original location. If g_A and g_B depend on that portion of the inertial energy of the states that was involved in the quantum transition from A to B according to

$$g_A = g\left(1+\frac{\alpha E_A}{m_A c^2}\right) \ , \ g_B = g\left(1+\frac{\alpha E_B}{m_B c^2}\right) \ , \ E_A - E_B = h\nu\,, \tag{7.8}$$

(violation of WEP), then by conservation of energy, there must be a corresponding violation of LPI in the frequency shift of the form to lowest order in $\frac{h\nu}{mc^2}$ (see eq. (7.6))

$$z = \frac{\nu' - \nu}{\nu'} = (1+\alpha)\frac{gH}{c^2} = (1+\alpha)\frac{\Delta U}{c^2}\,. \tag{7.9}$$

One may check this as follows: the ground energy conservation requires

$$m_A g_A H + m_A c^2 = m_B g_B H + m_B c^2 + h\nu' \tag{7.10}$$

in order to lift the atom again, where ν' is given by the following Taylor expanded relation

$$\nu' = \frac{\nu}{1-(1+\alpha)\frac{\Delta U}{c^2}} \simeq \nu\left(1+(1+\alpha)\frac{\Delta U}{c^2}\right)\,. \tag{7.11}$$

Inserting eq. (7.8) in eq. (7.10) and rearranging the terms, we recover eq. (7.11)

$$h\nu' = m_A c^2 - m_B c^2 + m_A g H - m_B g H + \alpha g H\frac{E_A}{c^2} - \alpha g H\frac{E_B}{c^2}\,, \tag{7.12}$$

where we use $E_A - E_B = m_A c^2 - m_B c^2 = h\nu$ and $gH = \Delta U$ to get $h\nu' = h\nu(1+(1+\alpha)\Delta U/c^2)$. Furthermore one can compute the Eötvös ratio

$$\eta = 2\left|\frac{g_A - g_B}{g_A + g_B}\right| = 2\left|\frac{g(1+\frac{\alpha E_A}{m_A c^2}) - g(1+\frac{\alpha E_B}{m_B c^2})}{g(1+\frac{\alpha E_A}{m_A c^2}) + g(1+\frac{\alpha E_B}{m_B c^2})}\right| = \frac{2\frac{\alpha}{c^2}\left(\frac{E_A}{m_A} - \frac{E_B}{m_B}\right)}{2+\frac{\alpha}{c^2}\left(\frac{E_A}{m_A} + \frac{E_B}{m_B}\right)}\,, \tag{7.13}$$

assuming α very small, $\eta \simeq \frac{\alpha}{c^2}\left(\frac{E_A}{m_A} + \frac{E_B}{m_B}\right)$ can be neglected. Approximating $m_A \sim m_B \sim m$ with

$$\frac{E_A - E_B}{mc^2} \simeq \frac{h\nu}{mc^2} \simeq \frac{10\text{eV}}{10GeV} \simeq 10^{-9}\,, \tag{7.14}$$

and therefore $\eta \simeq \alpha(10^{-9} - 10^{-10})$. Note that if $\alpha \simeq 10^{-5} - 10^{-6}$ then $\eta \simeq 10^{-14} - 10^{-16}$ or less. Remember that η is a dimensionless parameter that measures the strength of the violation of WEP. What was derived here is just an example, for generalization including LLI violations see the review of C. M. Will [18].

7.6 The strong equivalence principle

The empirical evidence supporting the Einstein equivalence principle supports the conclusion that the only theories of gravity that have a hope of being viable are metric theories. In any metric theory of gravity, matter and non-gravitational fields respond only to the spacetime metric $g_{\mu\nu}$. In principle, however, there could exist other gravitational fields besides the metric, such as scalar fields, vector fields, and so on.

General relativity is a purely dynamical theory since it contains only one gravitational field, the metric itself, and its structure and evolution are governed by the Einstein's field equations.

By discussing metric theories of gravity from this broad point of view, it is possible to draw some general conclusions about the nature of gravity in different metric theories, conclusions that are reminiscent of the Einstein equivalence principle, but that are subsumed under the name "strong equivalence principle". The strong equivalence principle (SEP) states that:

1. WEP is valid for self-gravitating bodies as well as for test bodies.

2. The outcome of any local test experiment is independent of the velocity of the freely falling apparatus.

3. The outcome at any local test experiment is independent of where and when in the universe it is performed.

The distinction between the SEP and EEP is the inclusion of bodies with self-gravitational interactions, planets, stars and of experiments involving gravitational forces, see for example Cavendish experiments and gravimeter measurements. Empirically it has been found that almost every metric theory other than GR introduces auxiliary gravitational fields, either dynamical or prior geometric, and thus predicts violations of SEP at some level. General relativity seems to be the only viable metric theory that embodies SEP completely.

8 Binaries on elliptic orbits

The contents presented in this section follow the book of M. Maggiore [20]. In the previous lecture (GRII) we have considered the emission of gravitational waves assuming circular orbits for two coalescing black holes. Here we will consider elliptical orbit. Let m_1, m_2 be the masses of the compact bodies, such as black holes or neutron stars, at positions $\vec{r}_1$ and $\vec{r}_2$. In the center of mass frame (CM), the problem reduces to a one-body problem for a particle of mass equal to the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ and acceleration $\ddot{\vec{r}} = -(G\frac{m}{r^2})\hat{\vec{r}}$, where $m = m_1 + m_2$ and $\vec{r} = \vec{r}_2 - \vec{r}_1$.

8.1 Elliptic Keplerian orbits

There are two integrals of motions, the angular momentum $\vec{L}$ and the energy E. The conservation of the angular momentum implies that the orbit lies on a plane. We introduce polar coordinates (r, ϕ) in the orbital plane with origin at the CM to describe the orbits. Then, the angular momentum and the energy are given by

$$L = |\vec{L}| = \mu r^2 \dot{\phi} \tag{8.1}$$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{G\mu m}{r} = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{G\mu m}{r}\,. \tag{8.2}$$

Form eq. (8.2) we get $\dot{r}$ as a function of r and from eq. (8.1) we get $\dot{\phi}$, combining the two expressions we find $\frac{dr}{d\phi}$ as a function of r. Integrating the obtained expression, we get the equation of the orbit

$$\frac{1}{r} = \frac{1}{R}(1 + e\cos\phi)\,. \tag{8.3}$$

The *length-scale* R and the *eccentricity* e are constants of motion and are related to the energy E of the system[18] and to the orbital angular momentum L by

$$R = \frac{L^2}{Gm\mu^2}\,, \tag{8.4}$$

and

$$e^2 = 1 + \frac{2EL^2}{G^2m^2\mu^3}\,, \tag{8.5}$$

where the eccentricity e for an ellipse satisfies $0 \leq e < 1$. For $e = 0$ the ellipse becomes a circle while for $e = 1$ the orbit corresponds to a parabola. The two semi-axes of the ellipse are given by

$$a = \frac{R}{1-e^2}\,, \tag{8.6}$$

$$b = \frac{R}{(1-e^2)^{\frac{1}{2}}}\,. \tag{8.7}$$

[18] For bound system, $E < 0$.

Inserting the expression for e, given in eq. (8.5), we obtain

$$a = \frac{Gm\mu}{2|E|}\,. \tag{8.8}$$

Note that a is independent of L, so orbits with the same energy have the same value of the semi-major axis a. We can rewrite eq. (8.3) as

$$r = \frac{a(1-e^2)}{1+e\cos\phi}\,. \tag{8.9}$$

Combining eq. (8.1) and eq. (8.4), we get

$$\dot{\phi} = \frac{(GmR)^{\frac{1}{2}}}{r^2}\,. \tag{8.10}$$

Integrating eqs. (8.1) and (8.2) get us $r(t)$ and $\phi(t)$. The parametric solutions are

$$r = a(1-e\cos u)\,, \tag{8.11}$$

$$\cos\phi = \frac{\cos u - e}{1-e\cos u}\,, \tag{8.12}$$

where u is the *eccentric anomaly*, and is related to t by the *Kepler equation*

$$\beta \equiv u - e\sin u = \omega_0 t\,, \tag{8.13}$$

where $\omega_0^2 = \frac{Gm}{a^3}$. We have chosen the origin of time such that at $t=0$ we have $\phi = 0$, i.e. the periastron. Using trigonometric identities, eq. (8.12) can also be rewritten as

$$\tan\frac{\phi}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{u}{2}\,, \tag{8.14}$$

or

$$\phi = A_e(u) = 2\arctan\left(\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{u}{2}\right)\,, \tag{8.15}$$

where $A_e(u)$ is called *true anomaly*. Note that in eq. (8.13) if $t \to t + \frac{2\pi}{\omega_0}$, then we have $\beta \to 2\pi$ and $u \to u + 2\pi$, so the coordinates r, ϕ are periodic functions of t, with period

$$T = \frac{2\pi}{\omega_0}\,. \tag{8.16}$$

Since u runs between $-\pi$ and π, ϕ also runs between $-\pi$ and π. Observe that for $e = 0$, $\phi = u$. In Cartesian coordinates, centered on the focus of the ellipse, the orbits is described by

$$x = r\cos\phi = a(\cos u(t) - e)\,, \tag{8.17}$$

$$y = r\sin\phi = b\sin u(t)\,, \tag{8.18}$$

note that the focus of the ellipse is the point where $\vec{r}_{cm} = 0$.

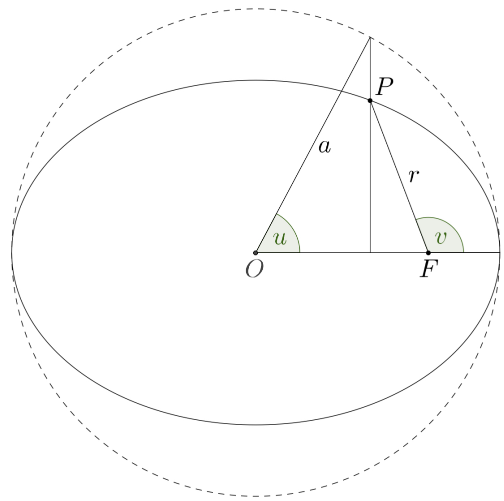

Figure 8.1: Geometry of the elliptical Kepler orbit. The eccentric anomaly u takes the role of the angle as measured from the center of the ellipse. Note that ϕ is here represented by v.

8.2 Radiated power

Here we use the same notation as in Sec. 6.3 and 6.4 of (GRII), remembering that a rotating rigid body can be interpreted as a mass distribution whose rotational frequency Ω leads to gravitational waves of frequency $\omega = 2\Omega$. We will here compute the total radiated power in gravitational waves, integrating over all frequencies and the solid angle. We choose a reference frame such that the orbit is in the (x, y)-plane. In this frame the *second mass moment* is given by the 2×2 matrix

$$M_{ab} = \mu r^2 \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}_{ab}, \tag{8.19}$$

where $a, b = 1, 2$ are indices in the (x, y)-plane. We need the third time derivative of M_{ab} to compute the total power emitted in the quadrupole approximation (GRII)

$$P = \frac{G}{5c^5}\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}(\dddot{M}_{kk})^2\rangle\,. \tag{8.20}$$

In eq. (8.19) the time dependence is in $r(t)$ and $\phi(t)$. To compute these derivatives, the simplest way is to write M_{ab} as a function of ϕ only, eliminating r with eq. (8.9), one gets for e.g.

$$M_{11} = \mu r^2\cos\phi = \mu a^2(1-e^2)^2\frac{\cos\phi}{(1+e\cos\phi)^2}\,, \tag{8.21}$$

and

$$\dot{\phi} = \frac{(GmR)^{\frac{1}{2}}}{r^2} = \left(\frac{Gm}{a^3}\right)^{\frac{1}{2}}(1-e^2)^{-\frac{3}{2}}(1+e\cos\phi)^2\,. \tag{8.22}$$

Then we can compute

$$\dddot{M}_{11} = \widetilde{\beta}(1+e\cos\phi)^2(2\sin(2\phi)+3e\sin\phi\cos^2\phi)\,, \tag{8.23}$$
$$\dddot{M}_{22} = \widetilde{\beta}(1+e\cos\phi)^2(-2\sin(2\phi)-e\sin\phi(1+3\cos^2\phi))\,, \tag{8.24}$$
$$\dddot{M}_{12} = \widetilde{\beta}(1+e\cos\phi)^2(-2\cos(2\phi)+e\cos\phi(1-3\cos^2\phi))\,, \tag{8.25}$$

where

$$\widetilde{\beta}^2 = \frac{4G^3\mu^2m^3}{a^5(1-e^2)^5}\,. \tag{8.26}$$

Inserting this into the formula for the quadrupole radiation, we get P as a function of the position angle ϕ along the orbit,

$$P(\phi) = \frac{G}{5c^2}\left(\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 2\dddot{M}_{12}^2 - \frac{1}{3}(\dddot{M}_{11}^2+\dddot{M}_{22}^2)^2\right) \tag{8.27}$$
$$= \frac{8G^4}{15c^5}\frac{\mu^2m^3}{a^5(1-e^2)^5}(1+e\cos\phi)^4(12(1+\cos\phi)^2+e^2\sin^2\phi)\,. \tag{8.28}$$

A particle in a Keplerian elliptic orbit emits gravitational waves at frequencies which are integer multiple of the frequency ω_0, defined in eq. (8.13). Therefore the period of the gravitational waves is a fraction of the orbital period T as given in eq. (8.16). Thus a well defined quantity is the average of $P(\phi(t))$ over one orbital period T:

$$\begin{aligned} P &= \frac{1}{T}\int_0^T dtP(\phi) = \frac{\omega_0}{2\pi}\int_0^{2\pi}\frac{d\phi}{\dot{\phi}}P(\phi(t)) = (1-e^2)^{\frac{3}{2}}\int_0^{2\pi}\frac{d\phi}{2\phi}\frac{P(\phi)}{(1+e\cos\phi)^2} = \\ &= \frac{8G^4\mu^2m^3}{15c^5a^5}(1-e^2)^{-\frac{7}{2}}\int_0^{2\pi}\frac{d\phi}{2\pi}(12(1+e\cos\phi)^4+e^2(1+e\cos\phi)^2\sin^2\phi)\,. \end{aligned} \tag{8.29}$$

Evaluating the integral gets us to

$$P = \frac{32G^4\mu^2m^3}{5c^5a^5}f(e)\,, \tag{8.30}$$

where

$$f(e) = \frac{1}{(1-e^2)^{\frac{7}{2}}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\,, \tag{8.31}$$

this formula is due to Peters and Mathews (1963). Using eq. (8.13), we can eliminate m in favor of ω_0 and rewrite the above result as

$$P = \frac{32}{5}\frac{G\mu^2}{c^5}a^4\omega_0^6f(e)\,. \tag{8.32}$$

Note that for $e = 0$ we have $f(e) = 1$, a becomes the radius of the circular orbit and ω_0 becomes the same as Ω in eq. (GRII 6.8). The Hulse-Taylor binary pulsar (see GRII) has $e = 0.617$ and $f(e) \simeq 11.8$. As a result, the radiated power is an order of magnitude larger than the power emitted in a circular orbit with radius a (i.e. with the same energy).

The orbital period T is related to the orbital energy E by $T = const. \times (-E)^{-\frac{3}{2}}$ and therefore

$$\frac{\dot{T}}{T} = -\frac{3}{2}\frac{\dot{E}}{E}\,. \tag{8.33}$$

Using eq. (8.30), together with eq. (8.8), to express the energy loss $\dot{E} = P$ in terms of a, we can rewrite eq. (8.33) as

$$\frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^3\mu m^2}{c^5 a^4} f(e)\,, \tag{8.34}$$

where averaging over one orbital period is understood. Finally expressing a in terms of T, we find

$$\frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^{\frac{5}{3}}\mu m^{\frac{2}{3}}}{c^5}\left(\frac{T}{2\pi}\right)^{-\frac{8}{3}} f(e)\,, \tag{8.35}$$

this equation is of great importance, since it is at the basis of the first experimental evidence for gravitational radiation.

8.3 Parabolic case

The limit $e \to 1^-$ of the radiated power given in eq. (8.30) diverges. This is due to the fact that for $e \to 1^-$, keeping a fixed, we get $R \to 0$ and $b = a(1-e^2)^{\frac{1}{2}} \to 0$. Clearly, at this moment, the approximation of the point-like masses ceases to be valid and we must take into account the finite size of the bodies. Consider eq. (8.3) for the parabolic motion $e = 1$, then

$$r = \frac{R}{1+\cos\phi}\,. \tag{8.36}$$

For $\phi \to -\pi$ the particle is at $r \to \infty$, by increasing ϕ, the value of r decreases down to $r = \frac{R}{2}$ (reached for $\phi = 0$), and then increases again until $r \to +\infty$ as $\phi \to \pi$. In this limit, eliminating a in favor of R using eq. (8.6), we obtain from eq. (8.29) the power radiated along the trajectory,

$$\begin{aligned} P(\phi) &= \frac{8G^4\mu^2 m^3}{15c^5R^5}(1+\cos\phi)^4(12(1+\cos\phi)^2 + \sin^2\phi) \\ &= \frac{16G^4\mu^2 m^3}{15c^5}\frac{1}{r^5}\left(1 + \frac{11R}{2r}\right)\,. \end{aligned} \tag{8.37}$$

Then, $P(\phi)$ goes to zero quite fast as $r \to \infty$, equivalently as $\phi \to \pm\pi$, in the order r^{-5} or r^{-6}. This is expected, since the acceleration vanishes as $\frac{1}{r^2}$. As a result, the total radiated energy in gravitational waves is finite

$$E_{rad} = \int_{-\infty}^{+\infty} dt P(\phi(t)) = \int_{-\pi}^{+\pi}\frac{d\phi}{\dot{\phi}}P(\phi) = \frac{85\pi}{48}\frac{G\mu^2}{R}\left(\frac{v_0}{c}\right)^5\,, \tag{8.38}$$

where $v_0 = (Gm/R)^{\frac{1}{2}}$ is the velocity at $\phi = 0$, which correspond to $r = \frac{R}{2}$, i.e. the maximum velocity attained along the trajectory. Most of the radiation is emitted when $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, when the object is near the periastron.

8.4 Frequency spectrum

We are interested in computing the frequency spectrum of the radiated power, namely $\frac{dP}{d\omega}$ for a Keplerian elliptic orbit. The trajectory is described by eq. (8.17) and eq. (8.18) as a function of time. It is not a simple harmonic motion, thus the first step is to perform the Fourier decomposition of the trajectory. This can be done knowing that $x(t)$ and $y(t)$ are periodic functions of the variable $\beta = \omega_0 t$, defined in eq. (8.13) with period 2π. We restrict therefore β to be in the range $-\pi \le \beta \le \pi$ and perform a discrete Fourier transformation

$$x(\beta) = \sum_{n=-\infty}^{+\infty} \widetilde{x}_n e^{-in\beta}\,, \tag{8.39}$$

$$y(\beta) = \sum_{n=-\infty}^{+\infty} \widetilde{y}_n e^{-in\beta}\,, \tag{8.40}$$

with $\widetilde{x}_n = \widetilde{x}^*_{-n}$ and $\widetilde{y}_n = \widetilde{y}^*_{-n}$, where * indicate the complex conjugate, since $x(\beta)$ and $y(\beta)$ are real functions. We choose the origin of time such that at $t = 0$, i.e. $\beta = 0$, we are at the point $x = a(1-e)$, $y = 0$. Note that the following even and odd properties of the functions $x(-\beta) = x(\beta)$, $y(-\beta) = -y(\beta)$. In the $x(\beta)$ expansion only $\cos(n\beta)$ contribute, whereas in $y(\beta)$ only $\sin(n\beta)$. We rewrite eq. (8.39) and eq. (8.40) as

$$x(\beta) = \sum_{n=0}^{\infty} a_n \cos(n\beta)\,, \tag{8.41}$$

$$y(\beta) = \sum_{n=1}^{\infty} b_n \sin(n\beta)\,, \tag{8.42}$$

where, for $n \le 1$, $a_n = 2\widetilde{x}_n$ and $b_n = -2i\widetilde{y}_n$, while $a_0 = \widetilde{x}_0$ with $\beta = \omega_0 t$ and $\omega_n = n\omega_0$ we have

$$x(t) = \sum_{n=0}^{\infty} a_n \cos(\omega_n t)\,, \tag{8.43}$$

$$y(t) = \sum_{n=1}^{\infty} b_n \sin(\omega_n t)\,, \tag{8.44}$$

where $\omega_0 = (Gm/a^3)^{\frac{1}{2}}$ is the *fundamental frequency* and ω_n are its higher harmonics. The Fourier coefficients, for $n \neq 0$, are obtained from

$$a_n = \frac{2}{\pi}\int_0^{\pi} d\beta\, x(\beta)\cos(n\beta)\,, \tag{8.45}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} d\beta\, y(\beta)\sin(n\beta)\,, \tag{8.46}$$

while for $n = 0$, we have

$$a_0 = \frac{1}{\pi}\int_0^{\infty} d\beta\, x(\beta)\,, \tag{8.47}$$

where $x(\beta)$, $y(\beta)$ are given by eqs. (8.17), (8.18) and β by eq. (8.13) all in terms of $u(t)$. As a result one finds:

$$a_n = \frac{a}{n}(J_{n-1}(ne) - J_{n+1}(ne)) \,, \tag{8.48}$$
$$b_n = \frac{b}{n}(J_{n-1}(ne) + J_{n+1}(ne)) \,, \tag{8.49}$$

where J_n is the *Bessel function*, $n \neq 0$ and $a_0 = -\frac{3}{2}ae$. To compute the spectrum of gravitational waves we actually need the Fourier decomposition of the second mass moment and therefore of $x^2(t)$, $y^2(t)$ and $x(t)y(t)$. One can show that

$$x^2(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t) \,, \tag{8.50}$$
$$y^2(t) = \sum_{n=0}^{\infty} B_n \cos(\omega_n t) \,, \tag{8.51}$$
$$x(t)y(t) = \sum_{n=1}^{\infty} C_n \sin(\omega_n t) \,. \tag{8.52}$$

The first two expressions are symmetric, while the third one is antisymmetric. The coefficients are

$$A_n = \frac{a^2}{n}\left(J_{n-2}(ne) - J_{n+2}(ne) - 2e(J_{n-1}(ne) - J_{n+1}(ne))\right) \,, \tag{8.53}$$
$$B_n = \frac{b^2}{n}\left(J_{n+2}(ne) - J_{n-2}(ne)\right) \,, \tag{8.54}$$
$$C_n = \frac{ab}{n}\left(J_{n+2}(ne) + J_{n-2}(ne) - e(J_{n+1}(ne) + J_{n-1}(ne))\right) \,. \tag{8.55}$$

The second mass moment has the Fourier decomposition

$$\begin{aligned} M_{ab}(t) &= \mu \sum_{n=0}^{\infty} \begin{bmatrix} A_n \cos(\omega_n t) & C_n \sin(\omega_n t) \\ C_n \sin(\omega_n t) & B_n \cos(\omega_n t) \end{bmatrix}_{ab} \,, \\ &\equiv \sum_{n=0}^{\infty} M_{ab}^{(n)}(t) \,. \end{aligned} \tag{8.56}$$

When we compute the radiated power, the temporal averages such as $\langle \sin(\omega_n t)\cos(\omega_m t)\rangle$ are non-vanishing only if $n = m$. So there is no interference term between the different harmonics, and

$$P = \sum_{n=1}^{\infty} P_n \,, \tag{8.57}$$

where P_n is the power radiated in the n-th harmonics. To compute P_n we use the quadrupole formula, written in the form

$$P = \frac{2G}{15c^5}\langle \dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 3\dddot{M}_{12}^2 - \dddot{M}_{11}\dddot{M}_{22}\rangle \,, \tag{8.58}$$

for a orbit in (x, y)-plane, and we replace M_{ab} by $M_{ab}^{(n)}$. Using

$$\dddot{M}_{ab}^{(n)} = \mu\omega_n^3 \begin{bmatrix} A\sin(\omega_n t) & -C_n\cos(\omega_n t) \\ -C_n\cos(\omega_n t) & B_n\sin(\omega_n t) \end{bmatrix}_{ab}, \tag{8.59}$$

we finally get

$$P_n = \frac{G\mu^2\omega_0^6}{15c^5} n^6 (A_n^2 + B_n^2 + 3C_n^2 - A_n B_n)\,, \tag{8.60}$$

where we used $\langle \sin^2(\omega_n t)\rangle = \langle \cos^2(\omega_n t)\rangle = \frac{1}{2}$. Recalling that $\omega_0^2 = \frac{Gm}{a^3}$, we rewrite the previous result as

$$P_n = \frac{32}{5}\frac{G\mu^2}{c^5} a^4 \omega_0^6 g(n, e)\,, \tag{8.61}$$

where

$$g(n, e) = \frac{n^6}{96a^4}(A_n^2(e) + B_n^2(e) + 3C_n^2(e) - A_n(e)B_n(e))\,, \tag{8.62}$$

were the eccentricity is given by $\frac{b^2}{a^2} = (1 - e^2)$. For the limit $e \to 0$, we get back the result for circular motion. For a generic value of e, i.e. $0 < e < 1$, all harmonics contribute and we have radiation at all frequencies $\omega_n = n\omega_0$, for $n \geq 1$.

8.5 Evolution of the orbit under back-reaction

A binary system in a Keplerian orbit radiates both energy and angular momentum. We take the bodies as being point-like, without intrinsic spin. In this picture, the radiation is necessarily drained from the energy and angular momentum of the orbital motion, which therefore undergoes secular changes, both in its semimajor axis and in its ellipticity. We now discuss how the shape and size of the orbit changes, due to gravitational wave radiation. We have already computed the energy radiated in the quadrupole approximation see eq. (8.30) and eq. (8.32). As next, we compute the angular momentum radiated, again in the quadrupole approximation. It can be shown that (summation over same indices)

$$\frac{dL^i}{dt} = -\frac{2G}{5c^5}\varepsilon^{ikl}\langle \ddot{M}_{ka}\dddot{M}_{la}\rangle \tag{8.63}$$

where L^i is the orbital angular momentum of the binary system. We choose our coordinate system such that the orbit is in the (x, y)-plane. M_{ab} is given by eq. (8.19) and we write $L_z = L$ which is orthogonal to the orbital plane. Remember that $\langle ...\rangle = \frac{1}{T}\int_0^T dt...$ means averaging over the time interval. One can thus integrate eq. (8.63) by parts and get the following relation

$$\frac{dL}{dt} = -\frac{2G}{5c^5}\langle \ddot{M}_{1a}\dddot{M}_{2a} - \ddot{M}_{2a}\dddot{M}_{1a}\rangle = \frac{4G}{5c^5}\langle \ddot{M}_{12}(\dddot{M}_{11} - \dddot{M}_{22})\rangle\,, \tag{8.64}$$

The third derivatives have been computed in eqs. (8.23)-(8.25). Similarly we get for

$$\ddot{M}_{12} = \frac{G\mu m}{a(1 - e^2)} \sin\phi(-4(1 + e\cos\phi)^2\cos\phi + 2e(3\cos^2\phi - 1 + 2e\cos^3\phi))\,. \tag{8.65}$$

For periodic motion, the average over several periods of the wave is the same as the average over the one orbital period T, we therefore transform the temporal average over one period into an integration over ϕ

$$\int_0^T \frac{dt}{T} \; ... = (1-e^2)^{\frac{3}{2}} \int_0^{2\pi} \frac{d\phi}{2\pi}(1+e\cos\phi)^{-2} \; ... \,. \tag{8.66}$$

Finally, we get for the angular momentum loss average over one period

$$\begin{aligned} \frac{dL}{dt} &= \frac{8}{5}\frac{G^{\frac{7}{2}}\mu^2 m^{\frac{5}{2}}}{c^5 a^{\frac{7}{2}}}\frac{1}{(1-e^2)^2}\int_0^{2\pi}\frac{d\phi}{2\pi}\sin^2\phi \\ &\quad (-4(1+e\cos\phi)^2\cos\phi + 2e(3\cos^2\phi - 1 + 2e\cos^3\phi))(8\cos\phi + 6e\cos^2\phi + e) = \\ &= -\frac{32}{5}\frac{G^{\frac{7}{2}}\mu^2 m^{\frac{5}{2}}}{c^5 a^{\frac{7}{2}}}\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right). \end{aligned} \tag{8.67}$$

Using eq. (8.5) and eq. (8.8), we can rewrite eq. (8.67) and eq. (8.30) in terms of the evolution of the semimajor axis a and of the eccentricity e in the following form

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^3\mu m^2}{c^5 a^3}\frac{1}{(1-e^2)^{\frac{7}{2}}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), \tag{8.68}$$

$$\frac{de}{dt} = -\frac{304}{15}\frac{G^3\mu m^2}{c^5 a^4}\frac{e}{(1-e^2)^{\frac{5}{2}}}\left(1+\frac{121}{304}e^2\right). \tag{8.69}$$

If $e=0$, i.e. a circular orbit, then from eq. (8.69) we get $\frac{de}{dt}=0$: a circular orbit remains circular. Eq. (8.69) tells us that for $e>0$, $\frac{de}{dt}$ is negative and thus an elliptic orbit becomes more and more circular due to gravitational waves emission. Note that eq. (8.68) and eq. (8.69) have to be integrated numerically. Combining them, one gets

$$\frac{da}{de} = \frac{12}{19}a\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{e(1-e^2)(1+\frac{121}{304}e^2)}. \tag{8.70}$$

This equation can be integrated analytically and gives

$$a(e) = c_0\frac{e^{\frac{12}{19}}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{\frac{870}{2299}} \equiv c_0\tilde{g}(e), \tag{8.71}$$

c_0 is determined by the initial condition $a=a_0$ when $e=e_0$, i.e. $a_0=c_0 g(e_0)$, then $a(e)=\frac{a_0}{g(e_0)}g(e)$.

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References

[1] S. L. Jald, "Johann georg von soldner and the gravitational bending of light," *Foundations of Physics*, vol. 8, no. 11/12, pp. 927–950, 1987.

[2] P. Schneider, C. Kochanek, and J. Wambsganss, *Gravitational Lensing: Strong, Weak and Micro*. Springer Berlin Heidelberg, 2006.

[3] NASA, "Lensshoe hubble," 2011. [Online; accessed December 21, 2011].

[4] N. Straumann, *General Relativity*. Springer Netherlands, 2013.

[5] D. G. Yakovlev, P. Haensel, G. Baym, and C. J. Pethick, "Lev landau and the conception of neutron stars," 2012.

[6] J. R. Oppenheimer and G. M. Volkoff, "On massive neutron cores," *Physical Review*, vol. 55, pp. 374–381, 1939.

[7] S. M. Carroll, "Lecture notes on general relativity." Institute for Theoretical Physics, University of California, 12 1997.

[8] G. Srinivasan, "The maximum mass of neutron stars," *Astronomical Society of India*, pp. 523–547, Sept. 2002.

[9] T. L. S. Collaboration and the Virgo Collaboration et al., "First low-frequency einstein@home all-sky search for continuous gravitational waves in advanced ligo data," 2017.

[10] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. 1972.

[11] S. Chandrasekhar, *The mathematical theory of black holes*. 1983.

[12] S. Chandrasekhar, "The solution of Dirac's equation in Kerr geometry," *Proceedings of the Royal Society of London Series A*, vol. 349, pp. 571–575, June 1976.

[13] D. N. Page, "Dirac equation around a charged, rotating black hole," *Phys. Rev. D*, vol. 14, pp. 1509–1510, Sep 1976.

[14] B. Mukhopadhyay and S. K. Chakrabarti, "Semi-analytical solution of Dirac equation in Schwarzschild geometry," *Classical and Quantum Gravity*, vol. 16, pp. 3165–3181, Oct. 1999.

[15] D. Batic, M. Nowakowski, and K. Morgan, "The Problem of Embedded Eigenvalues for the Dirac Equation in the Schwarzschild Black Hole Metric," *Universe*, vol. 2, p. 31, Dec. 2016.

[16] G. M. Graf, *General Relativity HS14*. ETH Zürich, 2014.

[17] R. M. Wald, *General Relativity*. University of Chicago Press, 1984.

[18] C. M. Will, "The confrontation between general relativity and experiment," *Living Reviews in Relativity*, vol. 17, p. 4, Jun 2014.

[19] P. Touboul, G. Métris, M. Rodrigues, Y. André, Q. Baghi, J. Bergé, D. Boulanger, S. Bremer, P. Carle, R. Chhun, *et al.*, "Microscope mission: First results of a space test of the equivalence principle," *Physical Review Letters*, vol. 119, no. 23, p. 231101, 2017.

[20] M. Maggiore, *Gravitational Waves: Volume 1: Theory and Experiments*. Oxford University Press, 2008.