ASTRONOMY SOUND OF THE MONTH

ACCELERATED EXPANSION OF THE UNIVERSE SONIFICATION

Greg Salvesen — gregsalvesen@gmail.com

I first heard the illusion of an endlessly rising/falling tone from a promotional video for a race put on by the Endurance Society. In addition to inciting dread into the hearts and minds of race participants, this auditory illusion also nicely conveys the accelerated expansion of the Universe!

1 Data

The temporal evolution of the Hubble parameter is given by,

$$H(z) = \frac{\dot{a}}{a} = H_0 \sqrt{\Omega_{\rm m}(1+z)^3 + \Omega_{\rm r}(1+z)^4 + \Omega_\Lambda}, \tag{1}$$

where $a = 1/(1+z)$ is the scale factor, $\dot{a} = da/dt$ is its time derivative, and z is redshift. You can think of redshift and time as being interchangeable. By construction, $a(t_0) = 1$ for the present day ($z = 0$) age of the Universe $t_0 \simeq 13.8$ Gyr. You can interpret $a(t)$ as describing the physical distance between two objects in space at time t relative to their current separation today. We will be rather colloquial and say that the scale factor describes the "size" of the Universe at some time in the past or future compared to its "size" today. The size of the Universe was smaller in the past ($a < 1$) and will be bigger in the future ($a > 1$). For example, when $a = 2$ the Universe will have expanded in all three spatial dimensions by a factor of two, so we say it is twice as big in size and $2 \times 2 \times 2 = 8$ times bigger in volume. The parameters $\Omega_{\rm m}$, $\Omega_{\rm r}$, Ω_Λ in Equation 1 are, respectively, the matter, radiation, and dark energy densities relative to the critical density $\rho_{\rm crit}$ for a "flat" Universe, which ours appears to be.

In Equation 1, H_0 is the famous Hubble constant, which tells us how fast the Universe is expanding and whose exact value is a long-standing debate. The *Planck* collaboration measured $H_0 \simeq 67.74 \pm 0.46$ km s^{-1} Mpc^{-1}, which means that the expansion of the Universe is causing a hypothetical galaxy at a distance of 1 Mpc (3.3 million light-years) away from us to be moving even further away at a speed of 67.74 km/s (42 miles every second). To make things interesting, Riess et al. (2016) used the *Hubble Space Telescope* to measure the Hubble constant to be $H_0 = 73.24 \pm 1.74$ km s^{-1} Mpc^{-1}, implying an even faster expansion rate than that measured by *Planck*. Any way you cut it, the Universe is expanding crazy fast and getting faster as time goes on. We're all about compromise here at Astronomy Sound of the Month, so we'll adopt the average of these two values in our analysis: $H_0 = 70.49$ km s^{-1} Mpc^{-1}.

Around six billion years ago and for reasons unknown ("dark energy"), the expansion rate of the Universe started increasing. This "accelerated expansion" continues today ($z = 0$) and it looks like the expansion rate will continue to increase forever. We can describe this accelerated expansion mathematically as follows. Today, the density content of the Universe as measured by *Planck* is ($\Omega_{\rm m,0} = 0.3089 \pm 0.0062$, $\Omega_{\rm r,0} \sim 10^{-4}$, $\Omega_{\Lambda,0} = 0.6911 \pm 0.0062$). Dark energy dominates the mass-energy content of the Universe today ($\Omega_{\Lambda,0} > \Omega_{\rm m,0} \gg \Omega_{\rm r,0}$), though not by terribly much, and astronomers think this trend of dark energy dominance will continue into the ultimate future ($\Omega_{\rm m} \to 0$, $\Omega_{\rm r} \to 0$, $\Omega_\Lambda \to 1$) for a flat Universe. Therefore, we make the approximation that dark energy is the *only* thing contributing to the future dynamical evolution of our Universe. This approximation is called a *de Sitter universe* and means we are completely ignoring the effects of both ordinary matter and dark matter on the future expansion of the Universe. Today, this assumption is "just okay" because $\Omega_{\Lambda,0} \sim \Omega_{\rm m,0}$, but a de Sitter universe is a very good approximation far into the future when $\Omega_{\Lambda,0} \gg \Omega_{\rm m,0}$. Assuming a de Sitter universe now and forever ($\Omega_{\rm m} = 0$, $\Omega_{\rm r} = 0$, $\Omega_\Lambda = 1$), Equation 1 becomes,

$$\frac{da/dt}{a} \simeq H_0, \tag{2}$$

which is separable,

$$\int_1^a \frac{da}{a} = H_0 \int_{t_0}^t dt, \tag{3}$$

and has the solution,

$$a(t) = a_0 \cdot e^{H_0 t}, \tag{4}$$

where $a_0 = e^{-H_0 t_0}$. Let's replace t in Equation 4 with $t' = t - t_0$, which conveniently sets the present day as our reference time rather than using the Big Bang as our reference time. Now Equation 4 becomes,

$$a\left(t'\right) = e^{t'/t_H}, \tag{5}$$

where $t_H = 1/H_0 \simeq 13.9$ Gyr is the Hubble time. Equation 5 describes how the Universe grows in size with time from today ($t' = 0$) into the future ($t' > 0$). We can rearrange Equation 5 to calculate the time it takes $t' = t'_N$ for today's Universe to grow in size by a factor of $a = N$,

$$t'_N = \ln(N)\, t_H. \tag{6}$$

Using this equation, we can see that the Universe will become 15% bigger ($N = 1.15$) in 1 billion years from now, double in size ($N = 2$) in 10 billion years, grow 1000-fold ($N = 1000$) in 100 billion years, and grow one nonillion-fold ($N = 10^{30}$) in one trillion years, which is just a silly big number that means nothing to me, as is the size of the Universe for that matter.

2 Sonification

Let's express the endlessly accelerating expansion of the Universe described by Equation 5 using the auditory illusion of an endlessly rising tone known as a Risset glissando. This is a pretty neat trick that can be constructed as follows[1]. Consider a sine wave of the form,

$$y(t) = A(t) \cdot \sin\left[2\pi f_0 \cdot e^{\alpha t}/\alpha\right], \tag{7}$$

where t is time, f_0 is a reference frequency corresponding to $t = 0$, and α is a constant. We will not worry about the amplitude envelope function $A(t)$ for the moment, but we will come back to this later because it is key to the illusion. Keeping in mind our application of sonifying the expansion of the Universe, comparing Equations 5 and 7 reveals that $H_0 = 1/t_H$ is a nice choice for α,

$$\begin{aligned} y(t) &= A(t) \cdot \sin\left[2\pi f_0 \cdot a(t) \cdot t_H\right] \\ &= A(t) \cdot \sin\left[2\pi f_0 \cdot e^{t/t_H} \cdot t_H\right], \end{aligned} \tag{8}$$

where we point out that we've dropped the prime symbol from t' in Equation 5 from here onward for convenience. Notice that the local frequency of the sine wave changes with time according to,

$$f(t) = f_0 \cdot e^{\alpha t}. \tag{9}$$

Suppose we wanted to experience the exponential expansion of the Universe as the rising tone described by Equation 8. Since the scale factor $a(t) = e^{\alpha t}$ increases forever, we would encounter the problem that the local frequency $f(t)$ will inevitably rise out of the frequency range of human hearing [20 Hz, 20 kHz].

How can we get around this problem? Suppose we write $\alpha = \ln(3/2)/\Delta t_{5\text{th}}$. Then for any time interval of duration $\Delta t_{5\text{th}}$, the local frequency will rise by a factor of 3:2, or one musical fifth: $f(t + \Delta t_{5\text{th}}) = 1.5 f(t)$. The reason to use intervals of musical fifths is just because they sound nice when played together, which

[1] I am borrowing heavily from the book *A First Course in Fourier Analysis* (D. W. Kammler 2000; Chapter 11, pp. 727–727).

will make more sense soon. Equating $\alpha = 1/t_H$ and $\alpha = \ln(3/2)/\Delta t_{\rm 5th}$, we see that the local frequency will rise by a musical fifth in a time interval $\Delta t_{\rm 5th} = \ln(3/2)\, t_H \simeq 5.63$ Gyr. Because we don't have all day, let's choose to let one second in our sonification correspond to one billion years of actual time so that $\Delta t_{\rm 5th} \simeq 5.63$ s.

Now let's construct a chord that is composed of infinitely many tones spaced apart by musical fifths, which will sound "good" to our ears. At a given time t, this chord is given by,

$$\begin{aligned} w(t) &= \sum_{m=-\infty}^{+\infty} y\left(t + m\cdot\Delta t_{\rm 5th}\right) \\ &= \sum_{m=-\infty}^{+\infty} A\left(t + m\cdot\Delta t_{\rm 5th}\right)\cdot \sin\left[2\pi f_0 \cdot e^{(t+m\cdot\Delta t_{\rm 5th})/t_H}\cdot t_H\right] \\ &= \sum_{m=-\infty}^{+\infty} A\left(t + m\cdot\Delta t_{\rm 5th}\right)\cdot \sin\left[2\pi\,(3/2)^m\,\tilde{f}_0 \cdot e^{t/t_H}\right], \end{aligned} \tag{10}$$

where m is an integer. To arrive at the bottom line of Equation 10, we remember that $\Delta t_{\rm 5th} = \ln(3/2)\, t_H$ and we absorbed t_H into $\tilde{f}_0$ by defining the dimensionless frequency $\tilde{f} \equiv f\cdot t_H$. As our chord $w(t)$ evolves forward in time, the infinitely many tones rise in pitch individually and any two adjacent tones always remain separated by a musical fifth. Consider an individual tone in the chord at a particular time with frequency $f_m(t)$. At the end of the time interval $\Delta t_{\rm 5th}$ that it takes this tone to rise in frequency by one musical fifth, it will have reached the starting frequency of its leading tone $[f_m(t+\Delta t_{\rm 5th}) \to f_{m+1}(t)]$, while its trailing tone will have reached its starting frequency $[f_{m-1}(t+\Delta t_{\rm 5th}) \to f_m(t)]$. In this way, we have constructed a "sonic barber pole," whereby a local frequency wraps back around on itself with a period of $\Delta t_{\rm 5th}$.

Now, as one tone in the chord described by Equation 10 exits the audible band, it is replaced by a tone that enters the audible band. This means we will always hear rising tones within the range of human hearing, which solves our previous problem; however, we have a new problem. Every time a new tone in this never ending train of tones separated by musical fifths enters our audible band, we will hear it distinctly. In effect, we will hear a scrambled mess of rising tones. What we want is a chord that sounds like it is rising forever, not one that sounds like a mess.

Here's the clever trick that solves this problem. Choose an amplitude envelope $A(t)$ that causes the incoming/outgoing tones at the low-/high-frequency limits of human hearing to be barely audible (i.e., at a low decibel level), while amplifying the tones in the central portion of the audible band. We can accomplish this with the Gaussian amplitude envelope,

$$A(t) = L_{\rm max}\cdot \exp\left[\frac{-\left(t-t_{\rm mid}\right)^2}{2\sigma^2}\right]. \tag{11}$$

The idea here is to localize the tone in the time interval $[t_{\rm min}, t_{\rm max}]$ that spans the full frequency sweep. Peak amplitude $L_{\rm max} = A(t_{\rm mid})$ occurs at the midpoint of the time interval $t_{\rm mid} = (t_{\rm max}+t_{\rm min})/2$. We design the width of the Gaussian to be,

$$\sigma^2 = \frac{-\left|\Delta t/2\right|^2}{2\ln\left(L_{\rm min}/L_{\rm max}\right)}, \tag{12}$$

where $\Delta t/2 = (t_{\rm max} - t_{\rm mid}) = -(t_{\rm min} - t_{\rm mid})$. This form for σ^2 allows us to specify the minimum amplitude $L_{\rm min} = A(t_{\rm min}) = A(t_{\rm max})$ at the extremes of the time interval. Inserting Equation 12 into Equation 11 gives,

$$A(t) = L_{\rm max}\left\{\exp\left[\left(\frac{t-t_{\rm mid}}{\Delta t/2}\right)^2\right]\right\}^{\ln(L_{\rm min}/L_{\rm max})}. \tag{13}$$

Now if we listened to Equation 7 — setting $L_{\text{min}} = 22$ dB, which is as loud as a whisper, and $L_{\text{max}} = 56$ dB, which is as loud as an ordinary conversation — we would just barely hear a low tone start to rise at time t_{min}, then getting louder until reaching peak at a frequency $\tilde{f}_0$ and time t_{mid}, and then continuing to rise but getting softer until becoming barely audible again at time t_{max}.

Now here is where the magic happens. Applying the Gaussian amplitude envelope of Equation 13 to Equation 10 gives the scary looking expression,

$$w(t) = \sum_{m=-\infty}^{+\infty} L_{\text{max}} \left\{ \exp\left[\left(\frac{t + m \cdot \Delta t_{5\text{th}} - t_{\text{mid}}}{\Delta t/2} \right)^2 \right] \right\}^{\ln(L_{\text{min}}/L_{\text{max}})} \cdot \sin\left[2\pi (3/2)^m \tilde{f}_0 \cdot e^{t/t_H} \right]. \tag{14}$$

Don't be too scared, though, as we've built this up in a very intentional way. Again, the important aspect of this sonification is that the local frequency is rising as an exponential in time (Equation 9, just like the scale factor of the Universe $a(t)$.

There are only a few more choices to make to finalize the illusion. Let's have the full frequency sweep last for $N = 192$ musical fifth intervals[2], each of duration $\Delta t_{5\text{th}}$, and let's start at time $t_{\text{min}} = 0$ for convenience. These choices set the ranges in time and frequency for the full sweep. Time: $[t_{\text{min}}, t_{\text{max}}] = [0, \ln(3/2 \cdot N) \cdot t_{\text{H}}] = [0\text{ s}, 78.6\text{ s}]$. Frequency: $\left[\tilde{f}_{\text{min}}, \tilde{f}_{\text{max}}\right] = \left[\tilde{f}_0, \tilde{f}_0 \cdot e^{t_{\text{max}}/t_{\text{H}}}\right] = [16\text{ Hz}, 4608\text{ Hz}]$, where I picked $\tilde{f}_0 = 16$ corresponding to the lowest note of a pipe organ. I chose this combination of N and $\tilde{f}_0$ just because it gives a wide range of frequencies and the resulting $\tilde{f}_{\text{max}}$ is not too irritating to the ear.

Equation 14 describes the chord being played as the frequency sweep $\tilde{f}_{\text{min}} \to \tilde{f}_{\text{max}}$ is happening. Tones are being subtly introduced/dropped at the extremes of the range of human hearing. Without this trick, your brain could easily identify when one tone exits and the next one abruptly enters. Now, your brain cannot pick out these quiet notes that are sneakily entering and leaving the audible band and you are tricked into hearing a tone that is continuously rising!

The `Python` script `risset.py` produces the sonification for the exponential expansion of the Universe.

3 Visualization

There is a neat website (thecmb.org) that let's you play around with the Cosmic Microwave Background (CMB) projected onto a sphere. Inspired by this, I downloaded the "Commander" high-resolution dataset of CMB brightness temperature fluctuations from the *Planck* Legacy Archive (https://pla.esac.esa.int/pla/#maps). After a lot of frustration (surprise, surprise), I managed to project the CMB onto a sphere using the `Python` package `Mayavi` for rendering 3D data. The script `sphere.py` produces the visualization that accompanies the sonification.

4 Lessons Learned in the Process

- I used `PyAudio` for the first time, which lets you stream audio and have complete control over the waveform to be played. This was handy for listening to waveforms I created in real time without having to write to a file.

- Discontinuous waveforms going from one sine wave to the next will cause popping sounds. To prevent these, the phase at the end of the first sine wave needs to be matched to the phase at the start of the sine wave that follows.

- The `Python wave` module, which I use to write a WAV file, is poorly documented. What you choose for the sampling width will set the min/max allowable volume (or amplitude), which may be obvious

[2]Twelve intervals closes the circle of fifths.

to those in the know but that's not me. I choose a sampling width of 2 bytes, which gives me a min/max amplitude of $-32768 = -2^{(8\cdot2-1)}$ and $32767 = 2^{(8\cdot2-1)} - 1$. You also need to use the `pack` attribute of the `struct` module to get your data in proper binary format, which is a black box to me.

- Visualizing the CMB in 3D turned out to be a real pain. I first used `plot_surface()` and `matplotlib`'s 3D projection, which worked but rendered insanely slow for even modest grid resolution. I eventually stumbled upon `Mayavi`, which was a huge headache to install. I managed to get it working in a hack-y way by specifying the backend, which is a concept that is beyond my computer illiterate brain. I'd also like to warn my future self to ***never*** install `PyQT` again because it did something awful to my `PYTHONPATH` that I still don't understand (`wxPython` seems to work just fine as the backend instead). All of that said, I was pleased that `Mayavi` can render surfaces quickly and can see myself using it again for future projects. I was not able to get the "offscreen rendering" to work (I couldn't install `Mesa 3D`), so I am stuck with these annoying windows popping up every time I render something.

- I used `ffmpeg` for the first time to stitch together image frames into a movie and to combine separate audio/video files. I was pleasantly shocked with how well `ffmpeg` worked and how little Google-ing it required on my part to get the job done.