UQ Astrophysics III - Theoretical Notes - Lecturer: Tamara Davis

1 The Robertson-Walker Metric

The Robertson-Walker (RW) metric comes in several equivalent forms. The main differences between them being the definition of comoving distance, and which variable carries the dimensions. Seven main variations are described below with a description of the links between them. The list is not comprehensive, but includes the most common and most useful forms.

$$ds^2 = -c^2dt^2 + R^2(t)\left[d\chi^2 + S_k^2(\chi)d\psi^2\right] \qquad \begin{array}{l}\chi = \frac{c}{R_0}\int\frac{dz}{H(z)} \text{ - } \textit{dimensionless} \\ R \text{ - dimensions of } \textit{distance}\end{array} \tag{1}$$

Set $r = S_k(\chi)$. Differentiating gives $d\chi^2 = \frac{dr^2}{1-kr^2}$, where $k = -1, 0, 1$ for open, flat, closed respectively.

$$ds^2 = -c^2dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\psi^2\right] \qquad \begin{array}{l}r \text{ - } \textit{dimensionless} \\ R \text{ - dimensions of } \textit{distance}\end{array} \tag{2}$$

Set $R(t) = R_0a(t)$ and $x = R_0\chi$. Substitute into Equation 1.

$$ds^2 = -c^2dt^2 + a^2(t)\left[dx^2 + R_0^2S_k^2\left(\frac{x}{R_0}\right)d\psi^2\right] \qquad \begin{array}{l}x \text{ - dimensions of } \textit{distance} \\ a \text{ - } \textit{dimensionless}\end{array} \tag{3}$$

Set $R(t) = R_0a(t)$ and $\mathsf{r} = R_0r$. Substitute into Equation 2.

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{d\mathsf{r}^2}{1-k\mathsf{r}^2/R_0^2} + \mathsf{r}^2\mathsf{d}\psi^2\right] \qquad \begin{array}{l}\mathsf{r} \text{ - dimensions of } \textit{distance} \\ a \text{ - } \textit{dimensionless}\end{array} \tag{4}$$

Define $\kappa = k/R_0^2$, which is the curvature parameter that can take on any value, so closed, flat, and open universes correspond to $\kappa > 0, = 0, < 0$ respectively.

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{d\mathsf{r}^2}{1-\kappa\mathsf{r}^2} + \mathsf{r}^2\mathsf{d}\psi^2\right] \qquad \begin{array}{l}\mathsf{r} \text{ - dimensions of } \textit{distance} \\ a \text{ - } \textit{dimensionless}\end{array} \tag{5}$$

Set $d\eta = cdt/R(t)$. Substitute into Equation 1.

$$ds^2 = R^2(t)\left[-d\eta^2 + d\chi^2 + S_k^2(\chi)d\psi^2\right] \qquad \begin{array}{l}\chi \text{ and } \eta \text{ - } \textit{dimensionless} \\ R \text{ - dimensions of } \textit{distance}\end{array} \tag{6}$$

Set $dn = cdt/a(t)$. Substitute into Equation 3.

$$ds^2 = a^2(t)\left[-dn^2 + dx^2 + R_0^2S_k^2\left(\frac{x}{R_0}\right)d\psi^2\right] \qquad \begin{array}{l}x \text{ and } n \text{ - dimensions of } \textit{distance} \\ a \text{ - } \textit{dimensionless}\end{array} \tag{7}$$

Different situations are more easily addressed with different metrics. For example, if you are calculating radial distance, then x or χ (1,3) would be simpler to deal with than r or r (2,4) because there are no curvature components in the radial part of the metric. On the other hand, the calculation of angular size distance and luminosity distance which depend on the angular components of the metric would be simplified by using r or r. It is often easier to deal with the present day scalefactor as normalised to 1 (i.e. a), instead attributing dimensions to the comoving distances. Changing the time coordinate to conformal time η or n (6,7) renders spacetime diagrams special-relativistic-like (when you plot the values within the square brackets). Without $a(t)$ or $R(t)$ past light cones have a gradient of 1 and comoving worldlines are straight.

2 The Friedmann Equation

Friedmann's equation is often written with the scalefactor R. Equally often you will see it expressed in terms of the normalised scalefactor $a = R/R_0$. Present day quantities are given the subscript zero. Note that you cannot arbitrarily set $R_0 = 1$ *and* $k = -1, 0, 1$. Generally we use the curvature parameter $\kappa = k/R_0^2$, which is negative, zero, or positive in open, flat and closed universes respectively. By converting Friedmann's equation to a form with $(\Omega_\mathrm{M}, \Omega_\Lambda)$ instead of κ, the R_0's cancel and then R and a can be used interchangeably.

The Friedmann Equation is given by,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{\kappa}{R^2}. \tag{8}$$

Substitute $R = R_0 a$, and $\rho = \rho_o a^{-3}$, then use the critical denstiy $\rho_\mathrm{c} = 3H^2/(8\pi G)$, and define a normalised density $\Omega = \rho/\rho_\mathrm{c}$. The definition of Λ is such that $\rho_\Lambda = \Lambda/(8\pi G)$.

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho_o}{3}a^{-3} - \frac{\Lambda}{3} = \frac{-\kappa c^2}{R_0^2 a^2}, \tag{9}$$

$$R_0^2\, a^2 H_0^2 \left[\frac{H^2}{H_0^2} - \Omega_\mathrm{M} a^{-3} - \Omega_\Lambda\right] = -\kappa c^2. \tag{10}$$

Note there is still a pesky unnormalised scalefactor R_0 in this version of the equation. Since κ is a constant it is convenient to evaluate it at the present epoch,

$$R_0^2\, H_0^2\,[1 - \Omega_\mathrm{M} - \Omega_\Lambda] = -\kappa c^2. \tag{11}$$

Substituting this present day value for κ (11) back into (10),

$$R_0^2\, a^2 H_0^2 \left[\frac{H^2}{H_0^2} - \Omega_\mathrm{M} a^{-3} - \Omega_\Lambda\right] = R_0^2 H_0^2\,[\Omega_M + \Omega_\Lambda - 1]\,. \tag{12}$$

Conveniently, the R_0's cancel, leaving an equation that uses neither the unnormalised scalefactor R *nor* the curvature parameter κ,

$$\frac{da}{dt} = H_0 \left[1 + \Omega_M\left(\frac{1}{a} - 1\right) + \Omega_\Lambda(a^2 - 1)\right]^{1/2}. \tag{13}$$

Since a flat universe is one in which the sum of all the Ω_x components equals 1, the curvature is sometimes quantified as another density parameter Ω_k such that $\Omega_M + \Omega_\Lambda + (\Omega_\text{anything else}) + \Omega_k = 1$. So, in the absence of anything else,

$$\Omega_k = 1 - \Omega_M - \Omega_\Lambda, \tag{14}$$

which lets us write Friedmann's equation as,

$$\frac{da}{dt} = H_0 \left[\Omega_k + \Omega_M \frac{1}{a} + \Omega_\Lambda a^2\right]^{1/2}. \tag{15}$$

Using the equation of state $w \equiv p/\rho$ means that for general fluids you can write $\Omega_x = \Omega_{x0} a^{-3(1+w)}$, which allows us to write this in a more condensed form,

$$H^2 = H_0^2 \sum \Omega_x a^{-3(1+w)}. \tag{16}$$

The Friedmann equation for a varying w is given by Eq. 16 with the following replacement,

$$a^{-3(1+w)} \to \exp\left(3\int_a^1 \frac{1 + w(a)}{a} da\right) \equiv \exp\left(3\int_0^z [1 + w(z)]\, d\ln(1+z)\right). \tag{17}$$

Using the popular parameterization $w(a) = w_0 + w_a(1 - a)$, Eq. 17 simplifies to (Linder, 2003)

$$a^{-3(1+w_0)} \to a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}. \tag{18}$$

3 Other quantities of interest

3.1 What is the current value of the scalefactor, R_0?

By rearranging Friedmann's equation, you find,

$$R_0 = \frac{c}{H_0}\left(\frac{-k}{1-\Omega_{\rm M}-\Omega_\Lambda}\right)^{1/2} = \frac{c}{H_0\sqrt{|\Omega_{\rm k}|}}. \tag{19}$$

3.2 Which Universes collapse?

The condition on $\Omega_{\rm M}$ and Ω_Λ for the universe to expand forever is:

$$\Omega_\Lambda \geq \begin{cases} 0 & 0 \leq \Omega_{\rm M} \leq 1 \\ 4\Omega_{\rm M}\left\{\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{1-\Omega_{\rm M}}{\Omega_{\rm M}}\right)+\frac{4\pi}{3}\right]\right\}^3 & \Omega_{\rm M} > 1 \end{cases} \tag{20}$$

(Carroll, Press and Turner 1992).

3.3 Which Universes have no big bang?

The condition for no big bang (universe has a turning point in its past, it collapsed from infinite size to a finite radius and is now reexpanding) is,

$$\Omega_\Lambda \geq 4\Omega_{\rm M}\left\{coss\left[\frac{1}{3}coss^{-1}\left(\frac{1-\Omega_{\rm M}}{\Omega_{\rm M}}\right)\right]\right\}^3, \tag{21}$$

where "*coss*" is defined as being cosh when $\Omega_{\rm M} < 1/2$ and cos when $\Omega_{\rm M} > 1/2$ (Carroll, Press and Turner 1992).

3.4 What is the scalefactor at maximum expansion in a recollapsing universe?

Turnaround occurs at $\dot{a} = 0$,

$$\dot{a} = H_0\left[1+\Omega_M\left(\frac{1}{a}-1\right)+\Omega_\Lambda(a^2-1)\right]^{1/2}, \tag{22}$$

$$0 = 1+\Omega_M\left(\frac{1}{a_{\rm max}}-1\right)+\Omega_\Lambda(a_{\rm max}^2-1), \tag{23}$$

$$0 = \Omega_\Lambda a_{\rm max}^3 + a_{\rm max}(1-\Omega_M-\Omega_\Lambda)+\Omega_M. \tag{24}$$

Solve this to find $a_{\rm max}$.

3.5 How do densities evolve?

An interesting thing to plot is how the relative contribution from each of the components of the universe changes as the universe expands (see Fig. 1). For example, the relative contribution from matter is,

$$\frac{\Omega_{\rm M}(t)}{\Omega_{\rm tot}(t)} = \frac{\Omega_{\rm M}(t)}{\Omega_{\rm M}(t)+\Omega_{\rm R}(t)+\Omega_{\Lambda}(t)} = \frac{\Omega_{\rm M}a^{-3}}{\Omega_{\rm M}a^{-3}+\Omega_{\rm R}a^{-4}+\Omega_{\Lambda}} \tag{25}$$

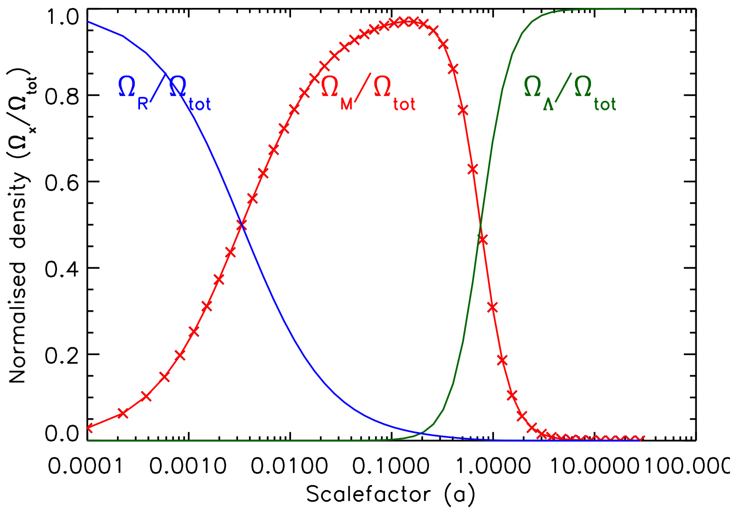


Figure 1: The relative contributions to the total denstiy from radiation (blue), matter (red), and cosmological constant (green). I've plotted these against scalefactor. I plotted the points on the $\Omega_{\rm M}/\Omega_{\rm tot}$ curve so you can see I did not use evenly spaced points. (To get back to where radiation was important I had to plot scalefactor logarithmically.) I actually used an array of 50 scale factors starting from 0.0001, where each was 1.25 *times* the previous one.

3.6 How does inflation drive the universe towards flatness?

Friedmann's equation can be rewritten:

$$\Omega_{\rm tot}(t) - 1 = \frac{k}{H^2a^2} \tag{26}$$

$$= \frac{k}{\dot{a}^2}. \tag{27}$$

This shows that if the universe decelerates ($\dot{a}$ decreases) then $\Omega_{\rm tot}(t)-1$ increases. That is, $\Omega_{\rm tot}(t)$ is driven away from one. However, if the universe accelerates $\Omega_{\rm tot}$ is driven towards one as $\dot{a}$ increases. The universe is driven towards *spatial* flatness. This is how inflation removes the flatness problem.

With a cosmological constant the situation is no different. It is the total density (including Ω_Λ) which is driven towards one. The universe is driven towards spatial flatness by any acceleration. The fact that our Universe is accelerating at the moment does not remove the need for inflation because the amount of acceleration required is far greater than the amount we have had in the universe so far.

References: See Liddle (1998), Sect. 6.2, 12.1 and 12.3.

4 Glossary

Here's some useful definitions:

Redshift, z, vs scalefactor, a:

$$z = \frac{1}{a} - 1, \qquad a = \frac{1}{1+z}$$

$D(t,z) = R(t)\chi(z)$ = Proper distance of object at redshift z at time, t.
$D_L(t,z) = R(t)S_k(\chi)(1+z)$ = Luminosity distance.
$D_A(t,z) = R(t)S_k(\chi)(1+z)^{-1}$ = Angular diameter distance. Where,

$$S_k(x) = \begin{cases} \sin(x) & k = 1 \\ x & k = 0 \\ \sinh(x) & k = -1 \end{cases}$$

k is the curvature of the universe, where $k = 1, 0$ or -1 corresponds to a closed, flat or open universe respectively.

Cosmological Parameters

Ω_M represents the nomalised, present day matter density.
Ω_Λ represents the normalised, present day cosmological constant.
They are given by:

$$\Omega_M \equiv \frac{8\pi G \rho_o}{3H_0^2} \qquad \Omega_\Lambda \equiv \frac{\Lambda_o}{3H_0^2} = \frac{8\pi G \rho_\Lambda}{3H_0^2}$$

They have been normalised to the present day critical density,

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G}$$

so that the universe is flat when:

$$\Omega_M + \Omega_\Lambda = 1$$

Matter Density

Density is inversely proportional to volume.

$$\frac{\rho(t)}{\rho_o} = \frac{V_o}{V(t)}$$

Volume is proportional to scale factor cubed:

$$\frac{V(t)}{V_o} = \left(\frac{R(t)}{R_0}\right)^3 = a^3(t) = (z+1)^{-3}$$

Therefore, if the mass of matter stays constant as the universe expands:

$$\rho(t) = \rho_o \left(\frac{R_0}{R(t)}\right)^3 = \rho_o\, a^{-3}(t) = \rho_o(z+1)^3$$

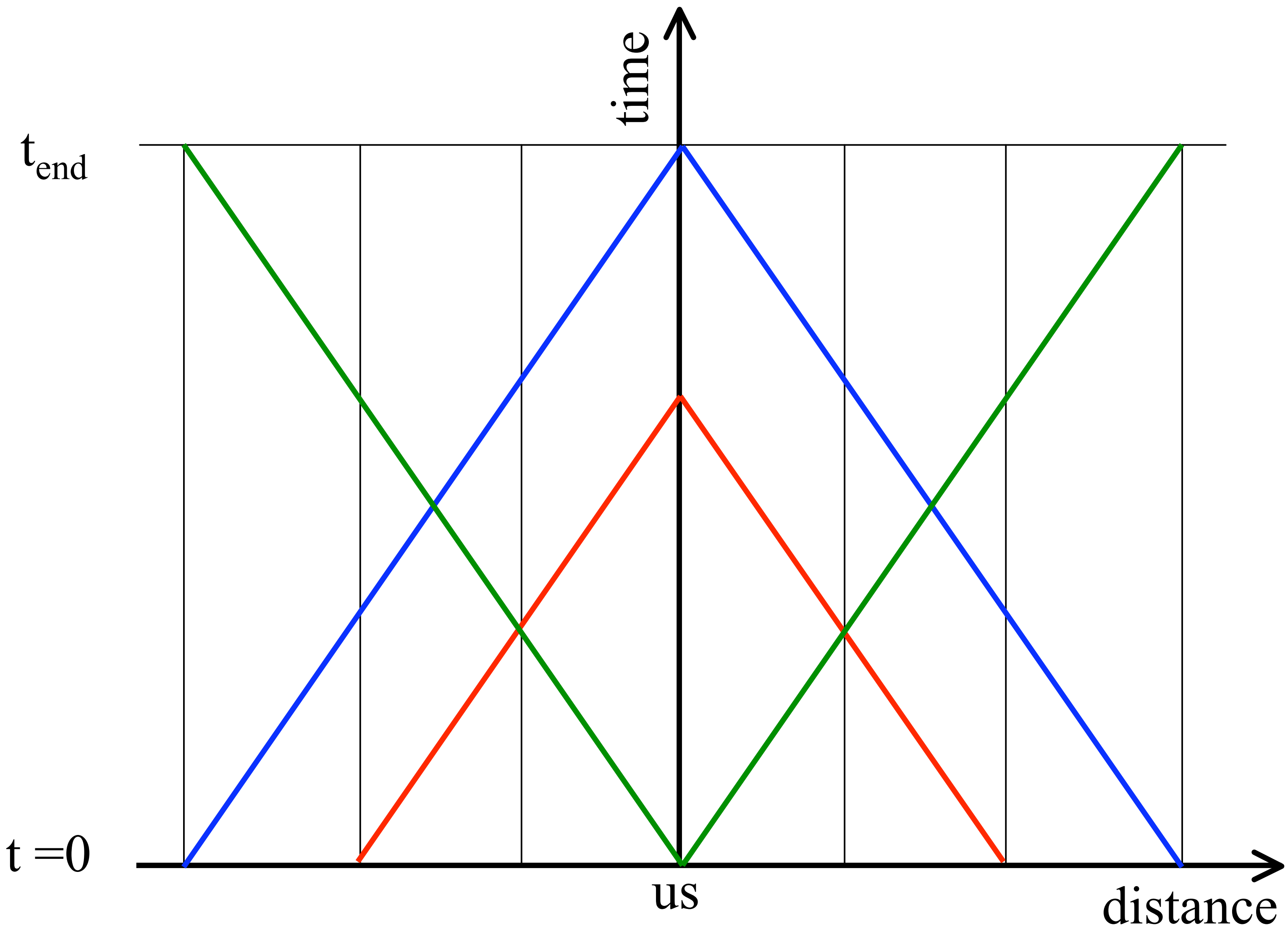
For a radiation dominated universe, there is an extra factor
since the energy of photons is reduced as their wavelengths stretch:

$$\rho(t) = \rho_o \left(\frac{R_0}{R(t)}\right)^4 = \rho_o\, a^{-4}(t) = \rho_o(z+1)^4$$

Energy density $\varepsilon = \rho c^2$.

Equation of state $w = \frac{p}{\rho c^2}$, which is usually just written as $w = p/\rho$, setting $c = 1$.

Speed of sound $c_s = \sqrt{\frac{dp}{d\rho}}$.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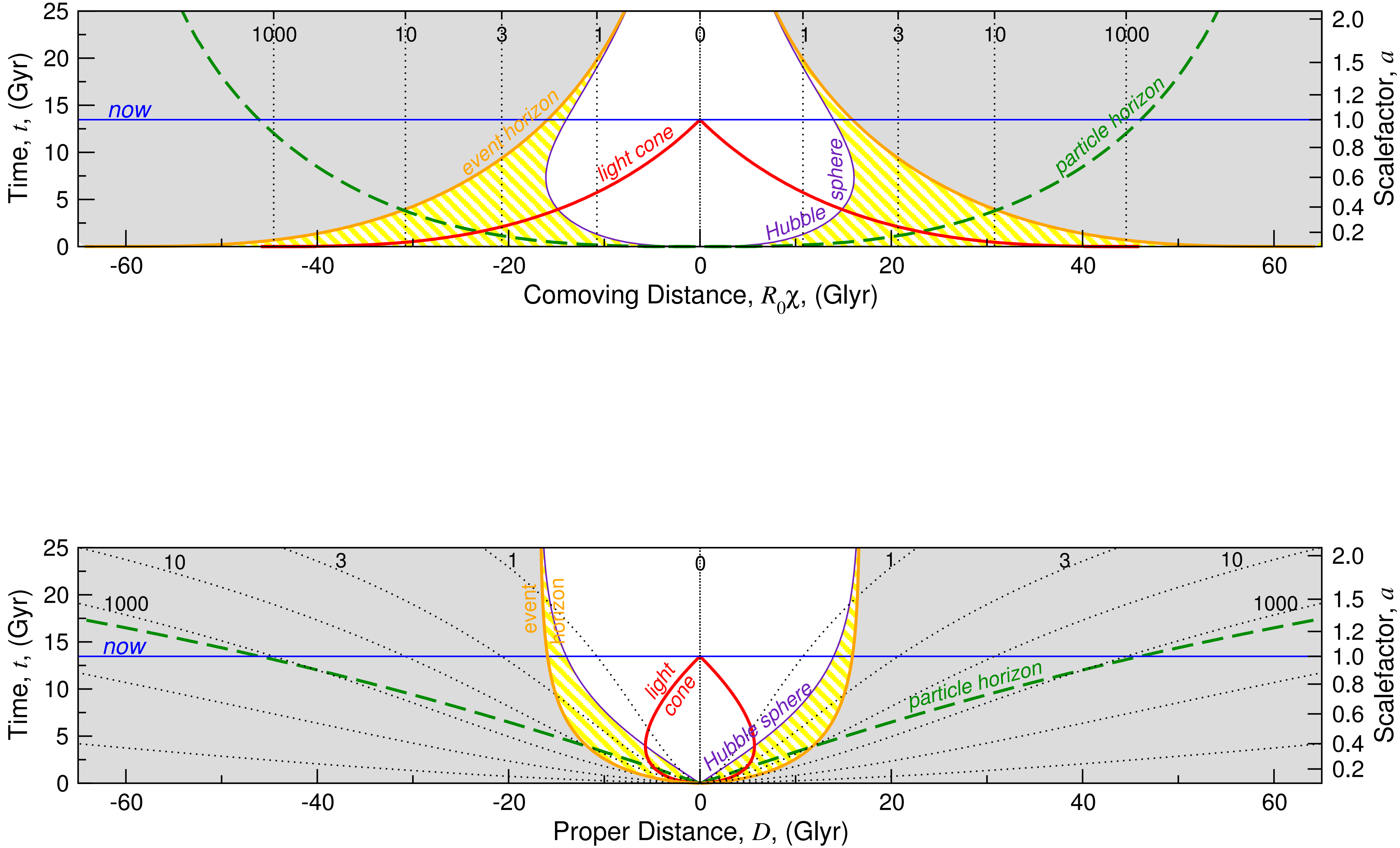

now
event horizon
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Time, t, (Gyr)
Scalefactor, a
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Hubble sphere
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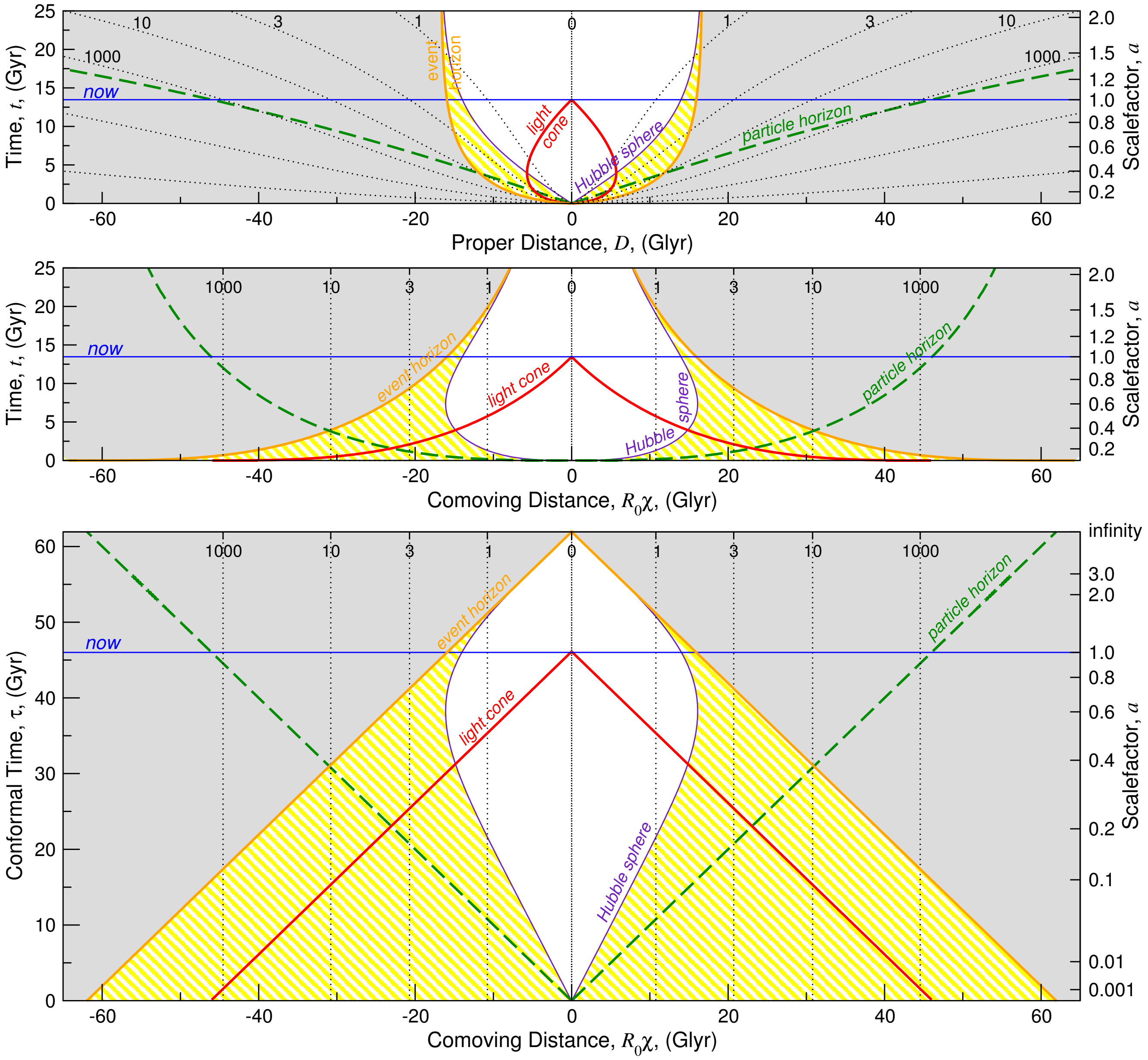

Figure 2: Spacetime diagrams for the $(\Omega_{\rm M}, \Omega_\Lambda) = (0.3, 0.7)$ universe with $H_0 = 70 {\rm kms^{-1}Mpc^{-1}}$. Dotted lines show the worldlines of comoving objects. The current redshifts of the comoving galaxies shown appear labeled on each comoving worldline. The normalized scalefactor, $a = R/R_0$, is drawn as an alternate vertical axis. Our comoving coordinate is the central vertical worldline. All events that we currently observe are on our past light cone (the cone or "teardrop" with apex at $t =$ now). All comoving objects beyond the Hubble sphere (thin solid line) are receding faster than the speed of light. The speed of photons on our past light cone relative to us (the slope of the light cone) is not constant, but is rather $v_{\rm rec} - c$. Photons we receive that were emitted by objects beyond the Hubble sphere were initially receding from us (outward sloping lightcone at $t \lesssim 5$ Gyr, upper panel). Only when they passed from the region of superluminal recession $v_{\rm rec} > c$ (yellow crosshatching and beyond) to the region of subluminal recession (no shading) could the photons approach us. More detail about early times and the horizons is visible in comoving coordinates (middle panel) and conformal coordinates (lower panel). Our past light cone in comoving coordinates appears to approach the horizontal $(t = 0)$ axis asymptotically, however it is clear in the lower panel that the past light cone reaches only a finite distance at $t = 0$ (about 46 Glyr, the current distance to the particle horizon). Light that has been travelling since the beginning of the Universe was emitted from comoving positions which are now 46 Glyr from us. The distance to the particle horizon as a function of time is represented by the dashed green line. This is the distance to the most distant object we are able to observe at any particular time. Our event horizon is our past light cone at the end of time, $t = \infty$ in this case. It asymptotically approaches $\chi = 0$ as $t \rightarrow \infty$. Many of the events beyond our event horizon (shaded solid gray) occur on galaxies we can see (the galaxies are within our particle horizon). We see them by light they emitted billions of years ago but we will never see those galaxies as they are today. Galaxies with redshift $z \sim 1.8$ are just now passing over our event horizon. Galaxies with redshift $z \sim 1.45$ are just now receding at the speed of light. The vertical axis of the lower panel shows conformal time (proper time divided by the scalefactor). An infinite proper time is transformed into a finite conformal time so this diagram is complete on the vertical axis. The aspect ratio of $\sim 3/1$ in the top two panels represents the ratio between the size of the observable Universe and the age of the Universe, 46 Glyr / 13.5 Gyr.