Project 3: Production of thermal relics

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Background

In this project you will learn how to compute the cosmological abundance ('relic density') Ωh^2 of stable particles produced in the thermal bath of the early Universe. Many popular dark matter candidates are examples of such so-called 'thermal relics', which start off in thermal equilibrium with all the Standard Model (SM) particles, but eventually 'freeze out' as the Universe expands and cools.

Here we will go over the general form of the standard thermal production calculation. This follows the treatment in Refs. [1–4] and consists of solving the Boltzmann Equation

$$\frac{dn}{dt} = -3Hn + \langle\sigma_{\rm eff}v\rangle \left(n_{\rm eq}^2 - n^2\right) , \tag{1}$$

for the comoving number density n of a particle X with mass m, as a function of time t. Here $H = \dot{a}/a$ is the Hubble rate and $n_{\rm eq}$ is the number density that the particle would have if its production and annihilation were in equilibrium with the thermal bath. The thermal average over all particle velocities is denoted by the brackets $\langle ...\rangle$, such that $\langle\sigma_{\rm eff}v\rangle$ is the average effective annihilation cross-section weighted by the relative velocity v between annihilating particles.

In order to solve Eq. (1) numerically, it is far more stable and accurate to work with dimensionless quantities. We thus swap n for $Y \equiv n/s'$, the abundance of X per unit entropy, with s' the comoving entropy density of the Universe.[1] The present-day value of Y is related to the actual cosmological abundance via

$$Y_0 \approx 3.63 \times 10^{-9}\,\Omega h^2 \left(\frac{\rm GeV}{m}\right). \tag{2}$$

We also swap t for the dimensionless time co-ordinate $x \equiv m/T$, with T the temperature of the thermal bath. By dividing both sides of Eq. (1) by the total entropy per comoving volume $S \equiv a^3 s'$ (which we can do because S is constant in time, assuming no new source of entropy production), we can rewrite the Boltzmann equation as

$$\frac{{\rm d}Y(x)}{{\rm d}x} = \sqrt{\frac{\pi}{45}}\frac{mM_{\rm Pl}}{x^2}\sqrt{g_*(x)}\langle\sigma_{\rm eff}v\rangle(x)\left[Y_{\rm eq}^2(x) - Y^2(x)\right] , \tag{3}$$

where $M_{\rm Pl} = G^{-1/2} \approx 1.2 \times 10^{19}\,{\rm GeV}$ is the Planck mass, and

$$\sqrt{g_*(x)} \equiv \frac{h_{\rm eff}(x)}{\sqrt{g_{\rm eff}(x)}}\left(1 - \frac{x}{3h_{\rm eff}(x)}\frac{{\rm d}h_{\rm eff}(x)}{{\rm d}x}\right) , \tag{4}$$

with $h_{\rm eff}$ and $g_{\rm eff}$ the effective entropic and energetic degrees of freedom. To a rough approximation, for computing freeze out of weak-scale particles these can be estimated from the SM field content to be constant values $g_{\rm eff} \sim h_{\rm eff} \sim 86.25$. The equilibrium abundance in the Maxwell-Boltzmann approximation is

$$Y_{\rm eq}(x) = \frac{45}{4\pi^4}\frac{x^2}{h_{\rm eff}(x)}\sum_i g_i\left(\frac{m_i}{m_1}\right)^2 K_2\left(x\frac{m_i}{m_1}\right), \tag{5}$$

with K_j the jth order modified Bessel function of the second kind. The sum runs over all particles that can eventually decay to the stable relic species (denoted by $i = 1$). The factor g_i is the number of internal degrees of freedom for the ith particle: $g_i = 2J_i + 1$, where J_i is the ith particle's spin.

[1] The prime here is just to distinguish s' from another s that appears later.

Under the assumption that the the annihilating particles are in kinetic equilibrium with the heat bath, their velocities follow a Maxwell-Boltzmann distribution, allowing us to write

$$\langle\sigma_{\rm eff}v\rangle \equiv \sum_{ij}\langle\sigma_{ij}v_{ij}\rangle\frac{n_i^{\rm eq}}{n^{\rm eq}}\frac{n_j^{\rm eq}}{n^{\rm eq}} = \frac{\int_0^\infty dp_{11}p_{11}^2 W_{\rm eff}K_1\left(\frac{\sqrt{s}}{T}\right)}{m_1^4 T\left[\sum_i \frac{g_i}{g_1}\frac{m_i^2}{m_1^2}K_2\left(\frac{m_i}{T}\right)\right]^2}. \tag{6}$$

The sums are over all initial states in the heat bath that can eventually decay to X and therefore contribute to its final cosmological abundance, and include only processess that actually deplete the total number of such particles. The temperature of the heat bath is given by T, and s is the Mandelstam variable corresponding to the square of the total energy of the colliding particles in the centre of mass frame. $W_{\rm eff}$ is the effective invariant annihilation rate, which is independent of the temperature and defined as

$$W_{\rm eff} \equiv 2\sum_{ij}\frac{p_{ij}}{p_{11}}\frac{g_i g_j}{g_1^2}(s-m_i^2-m_j^2)\sigma_{ij}v_{ij,{\rm lab}}\,, \tag{7}$$

with the relative momentum between particles i and j given by

$$p_{ij} = \frac{1}{2\sqrt{s}}\left[s-(m_i+m_j)^2\right]^{1/2}\left[s-(m_i-m_j)^2\right]^{1/2}. \tag{8}$$

The lab-frame relative velocity is related to the relative velocity in the centre-of-mass frame as

$$v_{ij,{\rm cms}} = 2\left(1-2m^2/s\right)v_{ij,{\rm lab}}. \tag{9}$$

Problems

Q.1 Warm-up (2 marks)

Consider the case of a stable relic particle to which no other particle can decay.

(a) Show that if it annihilates with itself (or its antiparticle) to SM particles with velocity-weighted cross-section $\sigma v_{\rm cms}(s)$, the thermally-averaged annihilation cross-section, Eq. (6), can be written as [4]

$$\langle\sigma_{\rm eff}v\rangle(x) = \int_{4m^2}^{\infty}\frac{xs\sqrt{s-4m^2}\,K_1(x\sqrt{s}/m)\,\sigma v_{\rm cms}(s)}{16m^5\,K_2^2(x)}\,{\rm d}s\,. \tag{10}$$

Q.2 Code to solve Boltzman equation (6 marks)

Write a computer program to calculate the present-day cosmological density $\Omega_{\rm DM}h^2$ of a thermal relic dark matter candidate of mass m, by solving Eq. (3). Your solver should make use of another function that returns the thermally-averaged effective cross-section times relative velocity $\langle\sigma_{\rm eff}v\rangle(x)$, as a function of $x = m/T$. The idea is that you should be able to swap in alternative versions of this function in order to compute the relic density of different dark matter candidates with different cross-sections (you'll need this for Q4).

- You can write your code in whatever language you like, but make sure to comment it appropriately. Your goal should be to explain to the reader what basically every line of code does. If you're not sure which to choose, use python.

- You are also welcome to use a public DE solver, e.g., from GSL, scipy or similar. Make sure that you get sensible results out of whatever external package you use! (If you really want, you may code this yourself Ref. [5]).

- Note that getting the solver to converge can be tricky. In order to maximise the numerical accuracy, you will probably want to scale your Y variable, and experiment with different initial values of x from which to begin your integration. [Why would choosing an initial x too high or too low make solving (3) difficult?]

Q.3 Thermal abundance of Majorana Fermion WIMP (6 marks)

Use your program to compute the thermal abundance of a generic Majorana fermion (self-conjugate, spin 1/2) WIMP dark matter candidate χ, parameterised purely in terms of its mass m_χ and a constant (i.e. temperature-independent) thermally-averaged cross-section times relative velocity $\langle\sigma_{\rm eff}v\rangle(x) = \langle\sigma v\rangle$ = a number (most typically stated in units of $\mathrm{cm}^3\,\mathrm{s}^{-1}$). Make plots of the following, including captions explaining the main features of your plots:

(a) The dimensionless comoving abundance Y as a function of x (i.e. plot the dark matter abundance as the Universe cools), for a few indicative values of m_χ and $\langle\sigma v\rangle$

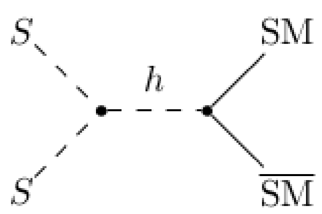
- Typical values you may wish to start from: $m_\chi \sim 100\,\mathrm{GeV}$, $\langle\sigma v\rangle \sim 10^{-26}\,\mathrm{cm}^2/\mathrm{s}$

(b) The present-day abundance $\Omega_\chi h^2$ as a function of m_χ (in GeV) for a few values of $\langle\sigma v\rangle$.

(c) The band in the m_χ, $\langle\sigma v\rangle$ plane for which the relic density of your WIMP matches the dark matter abundance observed in the latest *Planck* analysis of the CMB, to within the 3σ confidence level. Also indicate on your plot the region of parameter space where your WIMP produces only a fraction of the observed dark matter, and is therefore still viable, but somewhat less appealing as a theory than where it explains all of dark matter.

You might find a root-finding algorithm, such as bisection or Brent's method, useful for making the last plot. Bisection is easy to code yourself; public implementations of root-finding methods can be found online in all major languages. Again, more info on these algorithms can be found in Ref. [5].

Q.4 Thermal abundance of scalar singlet dark matter (6 marks)



Use your program to compute the thermal abundance of scalar singlet dark matter, S. Use the thermally-averaged effective annihilation cross-section given by Eq. (10), and a momentum-dependent cross-section (times relative velocity) for so-called 's-channel' annihilation via Higgs bosons into all other SM particles, given by

$$\sigma v_{\rm cms}(s) = \frac{2\lambda_{hS}^2 v_0^2}{\sqrt{s}}|D_h(s)|^2\Gamma_h(\sqrt{s})\,, \tag{11}$$

where λ_{hS} is a dimensionless coupling that parameterises the strength of the interaction between dark matter and the Higgs boson, and

$$|D_h(s)|^2 \equiv \frac{1}{(s-m_h^2)^2 + m_h^2\Gamma_h^2(m_h)} \tag{12}$$

is the internal Higgs boson propagator of the corresponding Feynman diagram, $m_h = 125\,\mathrm{GeV}$ is the Higgs mass, $v_0 = 246\,\mathrm{GeV}$ is its vacuum expectation value, and $\Gamma_h(Q)$ is the total decay width that the SM Higgs boson would theoretically have, were its mass equal to Q. For the latter, you can use the data in Table 1.

Make plots of the following, include captions explaining the main features of your plots:

(a) The thermally-averaged cross-section (Eq. (10)) as a function of m_S, for $x = 10, 20, 50$ and 100. Plot the curves for all four x values on the same axes, and assume $\lambda_{hS} = 10^{-3}$.

(b) The dimensionless comoving abundance Y as a function of x, for a few indicative values of m_S and λ_{hS}.

(c) The present-day abundance $\Omega_S h^2$ as a function of m_S, for $\lambda_{hS} = 10^{-3}$ and $50\,\mathrm{GeV} < m_S < 200\,\mathrm{GeV}$. Indicate the central observed value of $\Omega_\chi h^2$ with a dashed line.

- In order to compute the thermal average, you will need to perform the numerical integration in Eq. (10). You may write your own code for computing definite integrals, or call one from an external package such as GSL, scipy, etc. Be careful to ensure your results for the integration are converging reasonably.

- In order to interpolate the Higgs widths in Table 1, you will need to use an interpolation routine; cubic splines work nicely, and are widely available from GSL, scipy, etc. Linear interpolation is fine if you want to quickly do it yourself without external routines. In both cases, make sure to think about what numerical artefacts it might introduce to your results, and identify them in your plot captions if they do appear. You might find that it works better to interpolate in the log of the width rather than the width itself, but this isn't essential.

Table 1: Predicted decay width Γ_h of the Higgs boson, as a function of its mass Q [6]. For $Q >$ 300 GeV, it is good enough for your purposes to just assume that the width continues to rise linearly with increasing Q, i.e. Γ_h = a + bQ, with $b = 0.1$.

Q (GeV)	Γ_h (MeV)
80	1.99
90	2.22
100	2.48
110	2.85
120	3.51
130	4.91
140	8.17
150	17.3
160	83.1
170	380
180	631
190	1040
200	1430
220	2310
240	3400
260	4760
280	6430
300	8430

References

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