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PROPAGATION OF COVARIANCE

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ABSTRACT

We propagate covariance through the coadd creation process.

1. NOTATION

Individual visits written as $V^{(n)}$ where n is the visit number. So $V^{(4)}$ would be the 4$^{\rm th}$ visit used to create a coadd. The $^{(i)}$ will usually be supressed when referring to the transforms related to a single visit for brevity & clarity. Pixels are indexed with ordered pairs $[i, j]$ where the usual parentheses have been replaced with brackets because actual pixel values in an image array are index using brackets. Hence, $V^{(5)}[3, 4]$ refers to the value of the pixel in row 3, column 4 on visit 5.

Coadds are written as K. Usually, we will use Greek letters to index pixels on the coadd. Consider the pixel $[\alpha, \beta]$ on the coadd K. Many individual visits are used to create the coadd. In general, the gridding of the individual visits will not line up with the gridding of the coadd. In particular, the gridding of any individual visit may be

1. offset
2. rotated
3. sheared

from the gridding of the coadd. Generating the coadd requires that we know the value of our image on an individual visit at a location corresponding to a coadd grid-point. Consider the coadd grid-point $[\alpha, \beta]$. Define the mapping $S^{(n)} : [\alpha, \beta] \rightarrow [a, b]$ to be the mapping from the coadd gridding to the visit gridding for visit n i.e. $S^{(n)}$ allows us to convert pixel co-ordinates in the coadd to the corresponding (non-integral) pixel co-ordinates in visit n. Essentially, S uses the coadd WCS to compute the RA & Dec of the point and then uses the visit WCS to convert the computed RA & Dec to pixel co-ordinates in the visit. E.g., $S^{(5)}$ maps coadd grid-point $[3, 4]$ to visit grid-point $[3.3, 4.2]$. As with visits, when propagating covariance for a single visit across interpolation, we will supress the visit number when writing S.

Under the assumption that the visit has been oversampled, the value of the image at a given location on the visit can be interpolated using an interpolation kernel. All interpolation kernels can be written in the generic form $\mathfrak{K}([i, j] - S(\alpha, \beta))$. The coadd grid-point $V[S(\alpha, \beta)]$ may then be computed using

$$V[S(\alpha, \beta)] = \sum_{i,j} \mathfrak{K}([i, j] - S(\alpha, \beta))V[i, j]. \tag{1}$$

Theoretically, if the (band-limited) signal/image is sampled at its Nyquist rate and is of infinite duration, using *sinc* interpolation allows us to perfectly recover the sigal at any un-sampled grid-point (Marks II 1990). The 2-d seperable

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sinc kernel $\mathfrak{S}$ is defined by

$$\mathfrak{S}(x, y) = \operatorname{sinc} x \operatorname{sinc} y = \frac{\sin \pi x}{x} \frac{\sin \pi y}{y}, \tag{2}$$

and has infinite support. In practice, the *seperable Lánczos Interpolation* is preferred since it has finite support. The seperable 2-d Lánczos kernel $\mathfrak{L}_o$ of order-o given by

$$\mathfrak{L}_o(x, y) = \frac{o \sin(\pi x) \sin(\pi x/o)}{\pi^2 x^2} \frac{o \sin(\pi y) \sin(\pi y/o)}{\pi^2 y^2}, \tag{3}$$

2. VISIT MODEL

Let $\mathfrak{g}$ be the gain of the camera and let $\mathfrak{N}$ be the instrument noise level. We assume that the measured value of a pixel $V[i, j]$ is given by

$$V[i, j] = V_{\mathrm{true}}[i, j] + w_{[i,j]}, \tag{4}$$

where $w_{[i,j]} \sim \mathcal{N}(0, V[i, j]/\mathfrak{g} + \mathfrak{N})$. The mean of the noise in a given pixel is

$$\langle w_{[i,j]} \rangle = 0, \tag{5}$$

and the noise covariance between pixels is given by

$$\langle w_{[i,j]} w_{[k,l]} \rangle - \langle w_{[i,j]} \rangle \langle w_{[k,l]} \rangle = \langle w_{[i,j]} w_{[k,l]} \rangle = C_{V^{(n)}V^{(m)}}[i, j, k, l], \tag{6}$$

where $C_{V^{(n)}V^{(m)}}[i, j, k, l]$ is the covariance between the noise in visit pixels $[i, j]$ in visit $V^{(n)}$ & $[k, l]$ in visit $V^{(m)}$. Under the assumption that the pixel noise is uncorrelated across both visits& pixels, $C_{V^{(n)}V^{(m)}}[i, j, k, l]$ takes the form

$$C_{V^{(n)}V^{(m)}}[i, j, k, l] = \left(\frac{V^{(n)}[i, j]}{\mathfrak{g}} + \mathfrak{N} \right) \delta_{i,k} \delta_{j,l} \delta_{n,m}, \tag{7}$$

where $\delta_{i,j}$ is the Kronecker delta.

For individual visits, we have

$$\langle V[i, j] \rangle = \langle V_{\mathrm{true}}[i, j] + w_{[i,j]} \rangle = \langle V_{\mathrm{true}}[i, j] \rangle + \langle w_{[i,j]} \rangle = \langle V_{\mathrm{true}}[i, j] \rangle = V_{\mathrm{true}}[i, j],$$

i.e.

$$\langle V[i, j] \rangle = V_{\mathrm{true}}[i, j]. \tag{8}$$

Similarly,

$$\begin{gathered}
\langle V[i, j] V[k, l] \rangle - \langle V[i, j] \rangle \langle V[k, l] \rangle = \langle (V_{\mathrm{true}}[i, j] + w_{[i,j]})(V_{\mathrm{true}}[k, l] + w_{[k,l]}) \rangle - V_{\mathrm{true}}[i, j] V_{\mathrm{true}}[k, l] \\
= \\
\langle V_{\mathrm{true}}[i, j] V_{\mathrm{true}}[k, l] + V_{\mathrm{true}}[i, j] w_{[k,l]} + V_{\mathrm{true}}[k, l] w_{[k,l]} + w_{[i,j]} w_{[k,l]} \rangle - V_{\mathrm{true}}[i, j] V_{\mathrm{true}}[k, l] \\
= \\
\langle V_{\mathrm{true}}[i, j] V_{\mathrm{true}}[k, l] \rangle + \langle V_{\mathrm{true}}[i, j] w_{[k,l]} \rangle + \langle V_{\mathrm{true}}[k, l] w_{[k,l]} \rangle + \langle w_{[i,j]} w_{[k,l]} \rangle - V_{\mathrm{true}}[i, j] V_{\mathrm{true}}[k, l] \\
= \\
V_{\mathrm{true}}[i, j] V_{\mathrm{true}}[k, l] + V_{\mathrm{true}}[i, j] \langle w_{[k,l]} \rangle + V_{\mathrm{true}}[k, l] \langle w_{[k,l]} \rangle + \langle w_{[i,j]} w_{[k,l]} \rangle - V_{\mathrm{true}}[i, j] V_{\mathrm{true}}[k, l] \\
= \\
\langle w_{[i,j]} w_{[k,l]} \rangle = C_{VV}[i, j, k, l] = \left(\frac{V[i, j]}{\mathfrak{g}} + \mathfrak{N} \right) \delta_{i,k} \delta_{j,l},
\end{gathered}$$

i.e.

$$\langle V[i, j] V[k, l] \rangle - \langle V[i, j] \rangle \langle V[k, l] \rangle = C_{VV}[i, j, k, l] = \left(\frac{V[i, j]}{\mathfrak{g}} + \mathfrak{N} \right) \delta_{i,k} \delta_{j,l}. \tag{9}$$

3. WARP MODEL

The value of the warp grid-point $[a,b] = S(\alpha,\beta)$ corresponding to grid-point $[\alpha,\beta]$ on the coadd is given by equation (1). The expectation value of the warp grid-point $[a,b]$ is

$$\langle V[S(\alpha,\beta)]\rangle = \langle\sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))V[i,j]\rangle$$
$$= \sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))\langle V[i,j]\rangle$$
$$= \sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))\langle V_{\rm true}[i,j]+w_{[i,j]}\rangle$$
$$= \sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))\langle V_{\rm true}[i,j]\rangle+\langle w_{[i,j]}\rangle$$
$$= \sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))V_{\rm true}[i,j],$$

i.e.

$$\langle V[S(\alpha,\beta)]\rangle = \sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))V_{\rm true}[i,j]. \tag{10}$$

Similarly,

$$\langle V[S(\alpha,\beta)]V[S(\gamma,\delta)]\rangle - \langle V[S(\alpha,\beta)]\rangle\langle V[S(\gamma,\delta)]\rangle$$
$$= \langle(\sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))V[i,j])(\sum_{k,l}\mathfrak{K}([k,l]-S(\gamma,\delta))V[k,l])\rangle - \sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))V_{\rm true}[i,j]\sum_{k,l}\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\rm true}[k,l]$$
$$= \langle(\sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))V[i,j])(\sum_{k,l}\mathfrak{K}([k,l]-S(\gamma,\delta))V[k,l])\rangle - \sum_{i,j,k,l}\mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\rm true}[i,j]V_{\rm true}[k,l]$$
$$= \langle(\sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))(V_{\rm true}[i,j]+w_{[i,j]}))(\sum_{k,l}\mathfrak{K}([k,l]-S(\gamma,\delta))(V_{\rm true}[k,l]+w_{[k,l]}))\rangle - \sum_{i,j,k,l}\mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\rm true}[i,j]V_{\rm true}[k,l]$$
$$= \langle(\sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))V_{\rm true}[i,j]+\mathfrak{K}([i,j]-S(\alpha,\beta))w_{[i,j]})(\sum_{k,l}\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\rm true}[k,l]+\mathfrak{K}([k,l]-S(\gamma,\delta))w_{[k,l]})\rangle - \sum_{i,j,k,l}\mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\rm true}[i,j]V_{\rm true}[k,l],$$

i.e.

$$
\begin{aligned}
&\langle V[S(\alpha,\beta)]V[S(\gamma,\delta)]\rangle - \langle V[S(\alpha,\beta)]\rangle\langle V[S(\gamma,\delta)]\rangle \\
&= \\
&\langle \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]V_{\text{true}}[k,l] \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]w_{[k,l]} \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[k,l]w_{[i,j]} \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))w_{[i,j]}w_{[k,l]}\rangle \\
&- \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]V_{\text{true}}[k,l] \\
&= \\
&\langle \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]V_{\text{true}}[k,l]\rangle \\
&\quad + \langle \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]w_{[k,l]}\rangle \\
&\quad + \langle \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[k,l]w_{[i,j]}\rangle \\
&\quad + \langle \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))w_{[i,j]}w_{[k,l]}\rangle \\
&- \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]V_{\text{true}}[k,l] \\
&= \\
&\sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\langle V_{\text{true}}[i,j]V_{\text{true}}[k,l]\rangle \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\langle V_{\text{true}}[i,j]w_{[k,l]}\rangle \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\langle V_{\text{true}}[k,l]w_{[i,j]}\rangle \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\langle w_{[i,j]}w_{[k,l]}\rangle \\
&- \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]V_{\text{true}}[k,l] \\
&= \\
&\sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]V_{\text{true}}[k,l] \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]\langle w_{[k,l]}\rangle \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[k,l]\langle w_{[i,j]}\rangle \\
&\quad + \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\langle w_{[i,j]}w_{[k,l]}\rangle \\
&\qquad - \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))V_{\text{true}}[i,j]V_{\text{true}}[k,l],
\end{aligned}
$$

i.e.

$$\langle V[S(\alpha,\beta)]V[S(\gamma,\delta)]\rangle - \langle V[S(\alpha,\beta)]\rangle\langle V[S(\gamma,\delta)]\rangle = \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\langle w_{[i,j]}w_{[k,l]}\rangle = \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))C_{VV}[i,j,k,l].$$

We see that generically, the covariance between pixels on the warp is given by

$$\langle V[S(\alpha,\beta)]V[S(\gamma,\delta)]\rangle - \langle V[S(\alpha,\beta)]\rangle\langle V[S(\gamma,\delta)]\rangle = \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))C_{VV}[i,j,k,l]. \tag{11}$$

Under the stricter assumption that the original visit pixel noise is un-correlated, the covariance between the noise $C_{VV}[i,j,k,l]$ can be simplified giving

$$\begin{aligned}\langle V[S(\alpha,\beta)]V[S(\gamma,\delta)]\rangle - \langle V[S(\alpha,\beta)]\rangle\langle V[S(\gamma,\delta)]\rangle &= \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))C_{VV}[i,j,k,l] \\ &= \sum_{i,j,k,l} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\left(\frac{V[i,j]}{\mathfrak{g}}+\mathfrak{N}\right)\delta_{i,k}\delta_{j,l} \\ &= \sum_{i,j} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([i,j]-S(\gamma,\delta))\left(\frac{V[i,j]}{\mathfrak{g}}+\mathfrak{N}\right),\end{aligned}$$

i.e.

$$\langle V[S(\alpha,\beta)]V[S(\gamma,\delta)]\rangle - \langle V[S(\alpha,\beta)]\rangle\langle V[S(\gamma,\delta)]\rangle = \sum_{i,j} \mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([i,j]-S(\gamma,\delta))\left(\frac{V[i,j]}{\mathfrak{g}}+\mathfrak{N}\right). \tag{12}$$

4. COADD MODEL

The coadd is constructed by averaging over N warps, i.e.

$$K[\alpha,\beta] = \frac{1}{N}\sum_n V^{(n)}[S^{(n)}(\alpha,\beta)]. \tag{13}$$

The expectation value of the coadd pixels is

$$\langle K[\alpha,\beta]\rangle = \langle \frac{1}{N}\sum_n V^{(n)}[S^{(n)}(\alpha,\beta)]\rangle = \frac{1}{N}\sum_n \langle V^{(n)}[S^{(n)}(\alpha,\beta)]\rangle = \frac{1}{N}\sum_n\sum_{i,j} \mathfrak{K}([i,j]-S^{(n)}(\alpha,\beta))V^{(n)}_{\mathrm{true}}[i,j],$$

i.e.

$$\langle K[\alpha,\beta]\rangle = \frac{1}{N}\sum_n\sum_{i,j} \mathfrak{K}([i,j]-S^{(n)}(\alpha,\beta))V^{(n)}_{\mathrm{true}}[i,j]. \tag{14}$$

Similarly, the covariance between coadd pixels is given by

$$
\begin{aligned}
\langle K[\alpha,\beta]K[\gamma,\delta]\rangle - \langle K[\alpha,\beta]\rangle\langle K[\gamma,\delta]\rangle &= \langle \frac{1}{N}\sum_n V^{(n)}[S^{(n)}(\alpha,\beta)]\frac{1}{M}\sum_m V^{(m)}[S^{(m)}(\gamma,\delta)]\rangle \\
&\quad - \langle \frac{1}{N}\sum_n V^{(n)}[S^{(n)}(\alpha,\beta)]\rangle\langle \frac{1}{M}\sum_m V^{(m)}[S^{(m)}(\gamma,\delta)]\rangle \\
&= \frac{1}{NM}\sum_{n,m}\langle V^{(n)}[S^{(n)}(\alpha,\beta)]V^{(m)}[S^{(m)}(\gamma,\delta)]\rangle \\
&\quad - \frac{1}{NM}\sum_{n,m}\langle V^{(n)}[S^{(n)}(\alpha,\beta)]\rangle\langle V^{(m)}[S^{(m)}(\gamma,\delta)]\rangle \\
&= \frac{1}{NM}\sum_{n,m}\langle V^{(n)}[S^{(n)}(\alpha,\beta)]V^{(m)}[S^{(m)}(\gamma,\delta)]\rangle - \langle V^{(n)}[S^{(n)}(\alpha,\beta)]\rangle\langle V^{(m)}[S^{(m)}(\gamma,\delta)]\rangle \\
&= \frac{1}{NM}\sum_{n,m}\sum_{i,j,k,l}\mathfrak{K}([i,j]-S^{(n)}(\alpha,\beta))\mathfrak{K}([k,l]-S^{(m)}(\gamma,\delta))C_{V^{(n)}V^{(m)}}[i,j,k,l],
\end{aligned}
$$

i.e. generally covaraiance between coadd pixels $K[\alpha,\beta]$ and $K[\gamma,\delta]$ is given by

$$
\langle K[\alpha,\beta]K[\gamma,\delta]\rangle - \langle K[\alpha,\beta]\rangle\langle K[\gamma,\delta]\rangle = \frac{1}{NM}\sum_{n,m}\sum_{i,j,k,l}\mathfrak{K}([i,j]-S^{(n)}(\alpha,\beta))\mathfrak{K}([k,l]-S^{(m)}(\gamma,\delta))C_{V^{(n)}V^{(m)}}[i,j,k,l]. \tag{15}
$$

Under the stricter assumption that the pixel noise is un-correlated across both visits and pixels within a visit, the noise covariance $C_{V^{(n)}V^{(m)}}[i,j,k,l]$ can be simplified giving

$$
\begin{aligned}
\langle K[\alpha,\beta]K[\gamma,\delta]\rangle - \langle K[\alpha,\beta]\rangle\langle K[\gamma,\delta]\rangle &= \frac{1}{NM}\sum_{n,m}\sum_{i,j,k,l}\mathfrak{K}([i,j]-S^{(n)}(\alpha,\beta))\mathfrak{K}([k,l]-S^{(m)}(\gamma,\delta))C_{V^{(n)}V^{(m)}}[i,j,k,l] \\
&= \frac{1}{NM}\sum_{n,m}\sum_{i,j,k,l}\mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([k,l]-S(\gamma,\delta))\left(\frac{V[i,j]}{\mathfrak{g}}+\mathfrak{N}\right)\delta_{i,k}\delta_{j,l}\delta_{n,m} \\
&= \frac{1}{\min(N,M)^2}\sum_{n}^{\min(n,m)}\sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([i,j]-S(\gamma,\delta))\left(\frac{V[i,j]}{\mathfrak{g}}+\mathfrak{N}\right),
\end{aligned}
$$

i.e.

$$
\langle K[\alpha,\beta]K[\gamma,\delta]\rangle - \langle K[\alpha,\beta]\rangle\langle K[\gamma,\delta]\rangle = \frac{1}{\min(N,M)^2}\sum_{n,m}^{\min(n,m)}\sum_{i,j}\mathfrak{K}([i,j]-S(\alpha,\beta))\mathfrak{K}([i,j]-S(\gamma,\delta))\left(\frac{V[i,j]}{\mathfrak{g}}+\mathfrak{N}\right). \tag{16}
$$

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