Elliptical Core Grain Geometry

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1 Introduction

Most individuals who encounter solid rocket motor grain geometries, typically at a hobby level, deal solely with BATES grain geometries in which one takes a cylindrical grain and bores a perfectly circular hole through the center of the grain. This has the ability, assuming other parameters are met, to provide a fairly neutral pressure and/or thrust curve throughout the burn. Other grain geometries can be implemented but are typically very complicated to work with and construct; finocyl, conocyl, star, anchor, dendrite - such geometries require special equipment and careful procedures to get a grain that has a specific pressure and/or thrust curve. However, one such geometry which appears to be glossed over quite a bit (at least, in literature) is utilizing an elliptical port through a grain's center rather than a circular port. Such a grain could be made for a mid-size motor (or maybe smaller), utilizing CNC milling machines or 3D-printing technologies to construct a base and mandrel to pack propellant with. Furthermore, the production of such a grain need not worry about sharp angles on the mandrel and thus makes it easier to remove the grain, as well as provide a means for film-like materials like Saran wrap or electrical tape to easily conform to the mandrel in the case that a spray mold release is unavailable.

2 General Grain Structure

The structure of the Elliptical core grain is in the name itself. It is a solid rocket motor propellant grain, but instead of having a circular bore through the center of the grain, the port is an ellipse defined by a semi-major axis, a, and a semi-minor axis b. The length of the propellant grain is defined as being L_0, and the radius of the grain is R_g. This is shown in Figure 1. As we will soon find out, the elliptical core brings up some really nasty-looking terms that, while intimidating to look at from a Calculus II student's perspective, aren't really too bad in the end and help structure approaches to other potentially more challenging grain geometries.

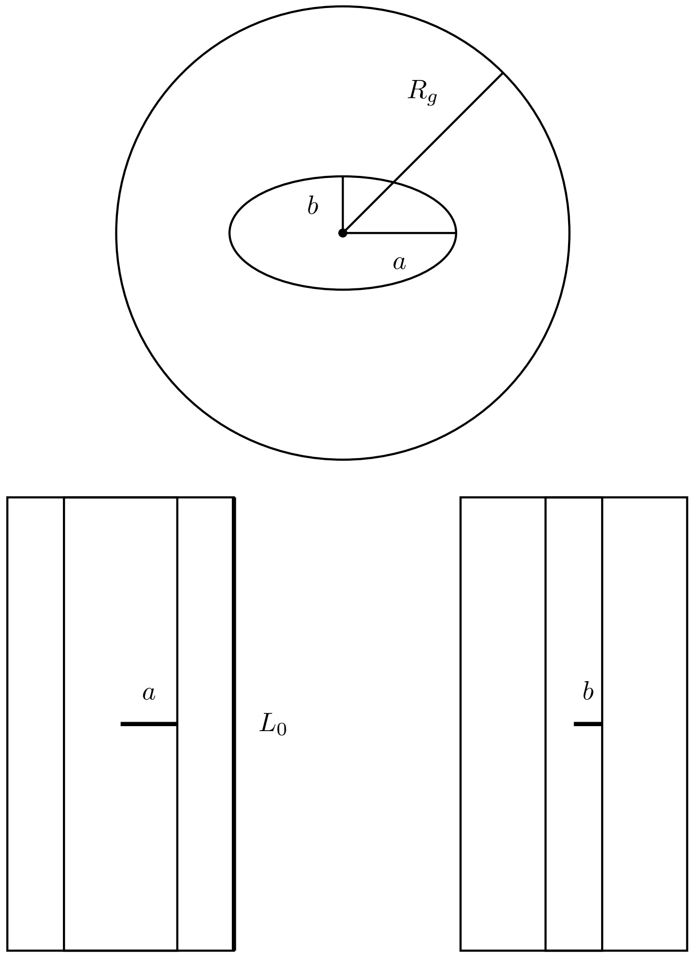

Figure 1: Cross Sections of Elliptical Core Grain

3 Surface Area as a Function of Web Consumed

Every solid rocket motor has a prescribed web thickness associated with it, or rather in most cases, multiple web thicknesses that help define how long certain phases of burning occur. In an elliptical core, we have three distinct web thicknesses to keep track of: $w_a = R_g - a$, $w_b = R_g - b$, and $w_L = \frac{L_0}{2}$. With this information in mind, we may continue on to determining the surface area of the port and ends of the grain during a burn.

First, let's start with the initial surface area.

$$A_b(0) = 2\pi(R_g^2 - ab) + L_0 \int_0^{2\pi} \sqrt{a^2 \sin^2(\theta) + b^2 \cos^2(\theta)} d\theta$$

Already, we're off to an amazing start. Since one cannot precisely determine the arc length (or circumference for that matter) of an ellipse using an easy-to-use formula, an integral is required to compute the circumference of the port. If we were to set $a = b = r_0$, we would find that the initial surface area matches that of a BATES geometry as it should: $A_b(0) = 2\pi(R_g^2 + r_0L_0 - r_0^2)$. Moving onto the more interesting matter, what happens when we start chipping away at our web thicknesses? Well, we can inject a web thickness consumed term, δ. Then, for phase I of the burn, we obtain the following surface area:

$$A_{bI}(\delta) = 2\pi(R_g^2 - (a+\delta)(b+\delta)) + (L_0 - 2\delta) \int_0^{2\pi} \sqrt{(a+\delta)^2 \sin^2(\theta) + (b+\delta)^2 \cos^2(\theta)} d\theta$$

The above equation is under the condition that $0 \leq \delta \leq \min(w_a, w_b, w_L)$.

Assuming that the length of the grain is sufficient, we move on to the next phase of the burn. This one is definitely quite something to express.

$$\mathrm{A}_{bII}(\delta) = 4(L_0 - 2\delta) \int_\phi^{\frac{\pi}{2}} \sqrt{(a+\delta)^2 \sin^2(\theta) + (b+\delta)^2 \cos^2(\theta)} d\theta$$
$$+ 4 \int_{-x_0}^{x_0} \left(\sqrt{R_g^2 - l^2} - (b+\delta)\sqrt{1 - (\frac{l}{a+\delta})^2} \right) dl$$

This equation holds until the end of the burn, or in other words, $\min(w_a, w_b, w_L) < \delta \leq \min(w_L, \max(w_a, w_b))$. In the equation, $\phi = \arctan\left(\frac{b+\delta}{a+\delta}\sqrt{\frac{(a+\delta)^2 - R_g^2}{(a+\delta)^2 - (b+\delta)^2}}\right)$, and $x_0 = (a+\delta)\sqrt{\frac{R_g^2 - (a+\delta)^2}{(a+\delta)^2 - (b+\delta)^2}}$.

Now that we've displayed the final result, let's explain where we got to this point. First, we must recognize that for the grain to be burning on the second phase, A_{bII}, we must have already found that $\delta = \min(w_a, w_b)$. This means that at that instant, if we were to see the face of the grain it would appear as if the ellipse contacts the boundary of the grain defined by R_g, and therefore we transition into having to deal with the fact that there are intersections between an ellipse and a circle. Fortunately, Wolfram Mathworld kindly gives us the intersection coordinates of a circle of any radius and an ellipse of any semi-major and semi-minor axes without us having to derive those from scratch unless we absolutely wanted to. With that out of the way, we define ϕ as being the angle formed from the arctangent of the y and x coordinates of the intersection of the circle and ellipse in the first quadrant. Once this is established, we can find the arc length between $-x_0$ and x_0 on the ellipse, where x_0 is just the x-component of the intersection's coordinates. This is straightforward to determine, and is shown as follows:

$$S = \int_0^{\frac{\pi}{2}} \sqrt{(a+\delta)^2 \sin^2(\theta) + (b+\delta)^2 \cos^2(\theta)} d\theta$$
$$- \int_0^{\arctan\left(\frac{b+\delta}{a+\delta}\sqrt{\frac{(a+\delta)^2 - R_g^2}{(a+\delta)^2-(b+\delta)^2}}\right)} \sqrt{(a+\delta)^2 \sin^2(\theta) + (b+\delta)^2 \cos^2(\theta)} d\theta$$
$$= \int_{\arctan\left(\frac{b+\delta}{a+\delta}\sqrt{\frac{(a+\delta)^2 - R_g^2}{(a+\delta)^2-(b+\delta)^2}}\right)}^{\frac{\pi}{2}} \sqrt{(a+\delta)^2 \sin^2(\theta) + (b+\delta)^2 \cos^2(\theta)} d\theta$$

Now that we have that, we just need to multiply that by 4 (since there are four quadrants) and by $(L_0 - 2\delta)$ to get the surface area of what remains in the core. As for the slivers left on the faces, that's simply a matter of areas between curves and that part of the derivation is left as an exercise to the reader. Once established in the correct quantities, we add it to the above monster of an integral and we get how phase II should behave.

Now, we have our surface area of the grain as a function of consumed web:

$$A_b(\delta) = \begin{cases} 2\pi(R_g^2 - (a+\delta)(b+\delta)) + (L_0 - 2\delta)\displaystyle\int_0^{2\pi} \sqrt{\Big((a+\delta)^2 \sin^2(\theta) + (b+\delta)^2 \cos^2(\theta)\Big)} d\theta \\ : 0 \leq \delta \leq \min(w_a, w_b, w_L) \\ \\ 4(L_0 - 2\delta)\displaystyle\int_\phi^{\frac{\pi}{2}} \sqrt{\Big((a+\delta)^2 \sin^2(\theta) + (b+\delta)^2 \cos^2(\theta)\Big)} d\theta \\ +4\displaystyle\int_{-(a+\delta)\sqrt{\frac{R_g^2-(a+\delta)^2}{(a+\delta)^2-(b+\delta)^2}}}^{(a+\delta)\sqrt{\frac{R_g^2-(a+\delta)^2}{(a+\delta)^2-(b+\delta)^2}}} \Big(\sqrt{R_g^2 - l^2} - (b+\delta)\sqrt{1 - (\frac{l}{a+\delta})^2}\Big) dl \\ : \min(w_a, w_b, w_L) < \delta \leq \min(w_L, \max(w_a, w_b)) \end{cases}$$

Quite the mess we have here, but hey, a theoretical surface area for an elliptical core grain as a function of consumed web.

4 Blurb on Max Surface Area

It is nothing surprising that the grain follows a similar profile to a BATES grain during phase I. So, it should be expected that the grain has at least a local maximum on this phase of the burn. They best way to go about this is to simply plot the area of the grain and visually or numerically obtain an estimate on the maximum surface area. Because of this, I won't go into the gory details of trying to determine maximum surface area by hand, despite how masochistic I have been with the previous derivation. However, feel free to give it a shot.

5 Grain Volume

I am going to keep this section as brief as possible. During phase I, the volume of the propellant grain is as follows:

$$V_{gI}(\delta) = \pi(R_g^2 - (a+\delta)(b+\delta))(L_0 - 2\delta)$$

and during phase II,

$$V_{gII}(\delta) = 2(L_0 - 2\delta)\int_{-x_0}^{x_0} \left(\sqrt{R_g^2 - l^2} - (b+\delta)\sqrt{1 - (\frac{l}{a+x})^2}\right) dl$$

Deriving phase I's grain volume is simply taking the cylinder that is the grain and removing the core's volume. As we get to phase II, however, things get a bit trickier but not too bad to deal with. We just need to compute the area of one of the endcaps and multiply it by $(L_0 - 2\delta)$ to obtain the volume those slivers of propellant have.

Now that we have that, we may now condense the grain volume into one large expression.

$$V_g(\delta) = \begin{cases} \pi(R_g^2 - (a+\delta)(b+\delta))(L_0 - 2\delta) : 0 \le \delta \le \min(w_a, w_b, w_L) \\ 2(L_0 - 2\delta)\int_{-x_0}^{x_0} \left(\sqrt{R_g^2 - l^2} - (b+\delta)\sqrt{1 - (\frac{l}{a+x})^2}\right) dl \\ : \min(w_a, w_b, w_L) < \delta \le \min(w_L, \max(w_a, w_b)) \end{cases}$$