Study Note of VAR and Local Projections

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Abstract

This study note provides a review of some previous researches on Vector Autoregressions (VARs) and Local Projections (LPs). I will give detail review of the equivalence of these two methods when the data generating process (DGP) follows the VAR process Jordà (2005), and their approximate equivalence in the scheme under Plagborg-Møller and Wolf (2021) when the DGP is unknown.

Keywords: local projection, VAR, impulse response function, Choleskey decomposition

1. Introduction

Vector Autoregressions (VARs) and Local Projections (LPs) are both multivariate time series models. The VAR introduced by Sims (1980) is considered the most commonly used multivariate time series model by economists, while the LP by Jordà (2005) has become an increasingly popular alternative econometric approach. In modern dynamic macroeconomics researches, VAR has embraced some transformations which variate from its traditional form, such as the Structural Vector Autogressions (SVAR), which deal with the dynamic response with respect to a structural shock; Threshold Vector Autogressions (TVAR), which set thresholds to certain variable in the equation and examine the IRF in different schemes; BVAR, which treats its model parameters as random variables with a prior probability rather than a fixed value.

In conventional analysis, it holds that VARs are more efficient and contain more economic explanations, while LPs are more robust to model misspecification and provide simpler joint or point-wise analytic inference. However, in empirical results, it shows that VAR and LP based approaches might give different results (Ramey, 2016), and such implementation differences might cause studies to different conclusions (Nakamura and Steinsson, 2018). By Jordà (2005), when the true model is a VAR(p) process, the impulse response functions by VAR(p) and linear LP(p)

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This note is for junior students who are studying Econometrics and Time Series

are exactly the same; when the data generating process (DGP) does not follow a VAR(p) process, then flexible LP(p) can be more robust to model misspecification by assuming a nonlinear time series to be expressed as a generic projection function of its past values.

In recent paper from Plagborg-Møller and Wolf (2021), they argued the equivalence of impulse response functions between VARs and linear LPs under the condition of unknown data generating process. In a short prediction window, the ratio of impulse response functions between VAR(p) and linear LP(p) is close to a constant, while in the distant prediction, this ratio will diverge eventually. To make studies above more straight-forward for learners, this study note first gives a brief overview of VAR and Local projection so it would be clear to see the detail structure of the impulse response functions. To visualize the comparisons in Plagborg-Møller and Wolf (2021), this study note further uses an empirical example with different data set to re-illustrate some of the key comparisons in that article.

2. Review of VAR

2.1 VAR with white noise error term

Both vector autoregressions (VARs) and local projections (LPs) lies in scheme of multivariate time series. To begin with the summary, we first introduce them distinctively. The traditional form of the p^{th} order vector autogressive (VAR) model is:

$$\boldsymbol{y}_t = \boldsymbol{\alpha}_0 + \boldsymbol{A}_1\boldsymbol{y}_{t-1} + \boldsymbol{A}_2\boldsymbol{y}_{t-2} + \cdots + \boldsymbol{A}_p\boldsymbol{y}_{t-p} + \boldsymbol{e}_t$$

suppose $\boldsymbol{y}_t = (y_{1t}, y_{2t}, \cdots, y_{mt})'$ is a $m \times 1$ vector which contains m variables at time t, in which case $\boldsymbol{A}_l$ will be a $m \times m$ coefficient matrix and $\boldsymbol{e}_t = (e_{1t}, e_{2t}, \cdots, e_{mt})'$ will be a $m \times 1$ error vector. Therefore, the contemporaneous covariance matrix is set to be $\boldsymbol{\Sigma} = \mathbb{E}\left[\boldsymbol{e}_t\boldsymbol{e'}_t\right]$.

$$\boldsymbol{A}_l = \begin{bmatrix} a_{11,l} & a_{12,1} & \cdots & a_{1m,l} \\ a_{21,l} & a_{22,1} & \cdots & a_{2m,l} \\ \vdots & \vdots & & \vdots \\ a_{m1,l} & a_{m2,1} & \cdots & a_{mm,l} \end{bmatrix}$$

In the most simple scheme, we assume $\boldsymbol{y}_t$ is covariance stationary and has the projection equation $\boldsymbol{y}_t = \boldsymbol{\alpha}_0 + \Sigma_{l=1}^{\infty}\boldsymbol{A}_l\boldsymbol{y}_{t-l} + \boldsymbol{e}_t = \boldsymbol{y}_t$. All those innovations $\boldsymbol{e}_t$ satisfy:

$$
\begin{aligned}
\mathbb{E}[\boldsymbol{e}_t] &= 0 \\
\mathbb{E}[\boldsymbol{y}_{t-l}\boldsymbol{e}_t'] &= 0, \qquad l \geq 1 \\
\mathbb{E}[\boldsymbol{e}_{t-l}\boldsymbol{e}_t'] &= 0, \qquad l \geq 1
\end{aligned}
$$

and

$$
\boldsymbol{\Sigma} = \mathbb{E}[\boldsymbol{e}_t\boldsymbol{e}_t'] < \infty
$$

Under this scheme, the multivariate innovations in $\boldsymbol{e}_t$ are mean zero and serially uncorrelated. This is what we call a multivariate white noise process. Under this scheme, the innovations $\boldsymbol{e}_t$ happens in period t, which was unpredictable in the past, represents us with the unique information in the period t. Therefore, we obtain the linear projection of the vector $\boldsymbol{y}_t$ in its past history as:

$$
\mathscr{P}_{t-1}\left[\boldsymbol{y}_t\right] = \mathscr{P}_{t-1}\left[\boldsymbol{y}_t \middle| \left(\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \cdots\right)\right] = \boldsymbol{\alpha}_0 + \Sigma_{l=1}^{\infty}\boldsymbol{A}_l\boldsymbol{y}_{t-l} = \boldsymbol{y}_t - \boldsymbol{e}_t
$$

By Wold decomposition $\boldsymbol{y}_t = \boldsymbol{\mu} + \Sigma_{l=0}^{\infty}\Theta_l\boldsymbol{e}_{t+h-l}$, we can calculate the projection of $\boldsymbol{y}_t$ onto its past history as:

$$
\mathscr{P}_t\left[\boldsymbol{y}_{t+h}\right] = \boldsymbol{\mu} + \Sigma_{l=h}^{\infty}\boldsymbol{\Theta}_l\boldsymbol{e}_{t+h-l}
$$

Since we define our impulse response function of variable i with respect to innovation j as: $IRF_{ij}\left(h\right) = \frac{\partial\mathscr{P}_t\left[y_{it+h}\right]}{\partial e_{jt}}$, therefore our impulse response matrix would be:

$$
\mathrm{IRF}\left(h\right) = \frac{\partial}{\partial\boldsymbol{e}_t'}\mathscr{P}_t\left[\boldsymbol{y}_{t+h}\right] = \boldsymbol{\Theta_h} = \begin{bmatrix} IRF_{11}(h) & \cdots & IRF_{1j}(h) \\ \vdots & \ddots & \vdots \\ IRF_{i1}(h) & \cdots & IRF_{ij}(h) \end{bmatrix}
$$

What's more, there're some recursive relationships between $\boldsymbol{\Theta}_i$ and $\boldsymbol{\Theta}_j$. First let's assume $\boldsymbol{\alpha}_0 = \mathbf{0}$ and $\boldsymbol{\Theta}_0 = \boldsymbol{I}_m$:

$$
\begin{aligned}
&\boldsymbol{y}_0 = \boldsymbol{e}_0 = \boldsymbol{\Theta}_0\boldsymbol{e}_0 \\
&\boldsymbol{y}_1 = \boldsymbol{A}_1\boldsymbol{y}_0 = \boldsymbol{A}_1\boldsymbol{\Theta}_0\boldsymbol{e}_0 = \boldsymbol{\Theta}_1\boldsymbol{e}_0 \\
&\boldsymbol{y}_2 = \boldsymbol{A}_1\boldsymbol{y}_1 + \boldsymbol{A}_2\boldsymbol{y}_0 = \boldsymbol{A}_1\boldsymbol{\Theta}_1\boldsymbol{e}_0 + \boldsymbol{A}_2\boldsymbol{\Theta}_0\boldsymbol{e}_0 = \boldsymbol{\Theta}_2\boldsymbol{e}_0 \\
&\boldsymbol{y}_3 = \boldsymbol{A}_1\boldsymbol{y}_2 + \boldsymbol{A}_2\boldsymbol{y}_1 + \boldsymbol{A}_3\boldsymbol{y}_0 = \boldsymbol{A}_1\boldsymbol{\Theta}_2\boldsymbol{e}_0 + \boldsymbol{A}_2\boldsymbol{\Theta}_1\boldsymbol{e}_0 + \boldsymbol{A}_3\boldsymbol{\Theta}_0\boldsymbol{e}_0 = \boldsymbol{\Theta}_3\boldsymbol{e}_0
\end{aligned}
$$

To summarize briefly, we can get the impulse response function of VAR at different stages as:

$$
\begin{aligned}
&\boldsymbol{\Theta}_0 = \boldsymbol{I}_m \\
&\boldsymbol{\Theta}_1 = \boldsymbol{A}_1\boldsymbol{\Theta}_0 \\
&\boldsymbol{\Theta}_2 = \boldsymbol{A}_1\boldsymbol{\Theta}_1 + \boldsymbol{A}_2\boldsymbol{\Theta}_0 \\
&\boldsymbol{\Theta}_3 = \boldsymbol{A}_1\boldsymbol{\Theta}_2 + \boldsymbol{A}_2\boldsymbol{\Theta}_1 + \boldsymbol{A}_3\boldsymbol{\Theta}_0 \\
&\cdots = \cdots \\
&\boldsymbol{\Theta}_h = \Sigma_{i=1}^{h}\boldsymbol{A}_i\boldsymbol{\Theta}_{h-i} = \Sigma_{i=1}^{h}\boldsymbol{A}_i\left(\frac{\partial}{\partial \boldsymbol{e}_t'}\mathscr{P}_t\left[\boldsymbol{y}_{t+h-i}\right]\right)
\end{aligned}
\tag{1}
$$

Here we write the h-step impulse response function as a linear function of the previous impulse response functions. This is a typical form to compute impulse response fucntion of in VAR in programming. We can also view the IRF of VAR from another equivalent perspective. From this perspective, we will introduce the explicit form of the h-step IRF of VAR.

Suppose $\boldsymbol{y}_t$ is truly a VAR(p) process, then $\boldsymbol{y}_{t+h}$ can be written as:

$$
\begin{aligned}
\boldsymbol{y}_{t+h} &= \boldsymbol{\alpha}_0 + \boldsymbol{A}_1\boldsymbol{y}_{t+h-1} + \boldsymbol{A}_2\boldsymbol{y}_{t+h-2} + \cdots + \boldsymbol{A}_p\boldsymbol{y}_{t+h-p} + \boldsymbol{e}_{t+h} \\
&= \boldsymbol{\alpha}_0 + \boldsymbol{A}_1\left(\boldsymbol{\alpha}_0 + \boldsymbol{A}_1\boldsymbol{y}_{t+h-2} + \boldsymbol{A}_2\boldsymbol{y}_{t+h-3} + \cdots + \boldsymbol{A}_p\boldsymbol{y}_{t+h-p-1} + \boldsymbol{e}_{t+h-1}\right) \\
&+ \boldsymbol{A}_2\boldsymbol{y}_{t+h-2} + \cdots + \boldsymbol{A}_p\boldsymbol{y}_{t+h-p} + \boldsymbol{e}_{t+h} \\
&= \underbrace{\boldsymbol{\alpha}_0 + \boldsymbol{A}_1\boldsymbol{\alpha}_0}_{\text{constant}} + \left(\boldsymbol{A}_1\boldsymbol{A}_1 + \boldsymbol{A}_2\right)\boldsymbol{y}_{t+h-2} + \cdots + \boldsymbol{A}_1\boldsymbol{A}_p\boldsymbol{y}_{t+h-p-1} + \underbrace{\boldsymbol{A}_1\boldsymbol{e}_{t+h-1} + \boldsymbol{e}_{t+h}}_{\boldsymbol{\mu}_{t+h-2}} \\
&= \cdots\cdots \\
&= \boldsymbol{b}_0 + \Sigma_{j=1}^{p}\boldsymbol{B}_j\boldsymbol{y}_{t+1-j} + \underbrace{\Sigma_{i=1}^{h}\boldsymbol{C}_i\boldsymbol{e}_{t+i}}_{\boldsymbol{\mu}_t} \\
&= \boldsymbol{b}_0 + \boldsymbol{B}_1\boldsymbol{y}_t + \Sigma_{j=2}^{p+1}\boldsymbol{B}_j\boldsymbol{y}_{t+1-j} + \boldsymbol{\mu}_t \\
&= \boldsymbol{b}_0 + \boldsymbol{B}_1\left(\boldsymbol{\alpha}_0 + \boldsymbol{A}_1\boldsymbol{y}_{t-1} + \boldsymbol{A}_2\boldsymbol{y}_{t-2} + \cdots + \boldsymbol{A}_p\boldsymbol{y}_{t-p} + \boldsymbol{e}_t\right) + \Sigma_{j=2}^{p+1}\boldsymbol{B}_j\boldsymbol{y}_{t+1-j} + \boldsymbol{\mu}_t \\
&= \tilde{\boldsymbol{b}}_0 + \boldsymbol{B}_1\boldsymbol{e}_t + \Sigma_{j=2}^{p+1}\tilde{\boldsymbol{B}}_j\boldsymbol{y}_{t+1-j} + \boldsymbol{\mu}_t
\end{aligned}
\tag{2}
$$

where $\boldsymbol{B} = \boldsymbol{f}\left(\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_p\right)$

By the assumption we argued before ($\mathbb{E}(\boldsymbol{e}_t) = 0$, $\mathbb{E}(\boldsymbol{y}_{t-l}\boldsymbol{e}_t') = 0$, $\mathbb{E}(\boldsymbol{e}_{t-l}\boldsymbol{e}_t') = 0$), we have:

$$
\begin{aligned}
Cov\left(\boldsymbol{e}_t, \boldsymbol{y}_{t+1-j}\right) &= 0, \ \ \forall j \in [2, p+1] \\
Cov\left(\boldsymbol{e}_t, \Sigma_{i=1}^{h}\boldsymbol{e}_{t+i}\right) &= Cov\left(\boldsymbol{e}_t, \boldsymbol{\mu}_t\right) = 0
\end{aligned}
$$

Thus:

$$\text{IRF}_h = \frac{\partial}{\partial \boldsymbol{e}_t'} \mathscr{P}_t \left[\boldsymbol{y}_{t+h}\right] = \boldsymbol{\Theta}_h = \boldsymbol{B}_1 \tag{3}$$

Notice (3) is the same impulse response function as we deducted recursively in (1). Here we can see that deriving correct impulse responses functions from VARs can be extremely complicated (Hansen, 2000) .

In computer software, we estimate the h-step IRF through estimator of coefficient matrix, as $\hat{\boldsymbol{\Theta}}_h = \hat{\boldsymbol{B}}_1 = \boldsymbol{f}\left(\hat{\boldsymbol{A}}_1, \cdots, \hat{\boldsymbol{A}}_p\right)$. We can achieve this by minimize the GLS (or LS) MSE of VAR(p).

For a stationary, stable VAR(p): $\boldsymbol{y}_t = \boldsymbol{\alpha}_0 + \boldsymbol{A}_1 \boldsymbol{y}_{t-1} + \cdots + \boldsymbol{A}_p \boldsymbol{y}_{t-p} + \boldsymbol{e}_t$ with $\boldsymbol{y}_t = (y_{t,1}, y_{t,2}, \cdots, y_{t,k})'$, we write it as:

$$\boldsymbol{Y}_t = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{U}$$

where,

$$\boldsymbol{Y}_t = \begin{bmatrix} y_{t,1} & y_{t,2} & \cdots & y_{t,k} \end{bmatrix}^T, \ \underbrace{\boldsymbol{A} = (\boldsymbol{\alpha}_0, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_p)}_{k\times(kp+k)}$$

$$\boldsymbol{X} = \underbrace{\begin{bmatrix} \boldsymbol{1}_{(1\times k)} & \boldsymbol{Y}_{t-1}' & \cdots & \boldsymbol{Y}_{t-p}' \end{bmatrix}^T}_{k(p+1)\times 1}, \ \boldsymbol{U} = \begin{bmatrix} \mu_{t,1} & \mu_{t,2} & \cdots & \mu_{t,k} \end{bmatrix}^T$$

Therefore,

$$\boldsymbol{Y}_t = \text{vec}\left(\boldsymbol{Y}_t\right) = \left(\boldsymbol{X}' \otimes \boldsymbol{I}_k\right) \bullet \text{vec}\left(\boldsymbol{A}\right) + \boldsymbol{U}$$

We have to choose $\boldsymbol{\alpha} = \text{vec}\left(\boldsymbol{A}\right)$ which minimizes:

$$\begin{aligned}
\boldsymbol{S}\left(\boldsymbol{\alpha}\right) &= \boldsymbol{U}'\boldsymbol{U} \\
&= \left(\boldsymbol{Y}_t - \left(\boldsymbol{X}' \otimes \boldsymbol{I}_k\right)\boldsymbol{\alpha}\right)' \left(\boldsymbol{Y}_t - \left(\boldsymbol{X}' \otimes \boldsymbol{I}_k\right)\boldsymbol{\alpha}\right) \\
&= \boldsymbol{Y}_t'\boldsymbol{Y}_t - 2\boldsymbol{\alpha}'\left(\boldsymbol{X} \otimes \boldsymbol{I}_k\right)\boldsymbol{Y}_t + \boldsymbol{\alpha}'\left(\boldsymbol{X} \otimes \boldsymbol{I}_k\right)\left(\boldsymbol{X}' \otimes \boldsymbol{I}_k\right)\boldsymbol{\alpha} \\
&= \boldsymbol{Y}_t'\boldsymbol{Y}_t - 2\boldsymbol{\alpha}'\left(\boldsymbol{X} \otimes \boldsymbol{I}_k\right)\boldsymbol{Y}_t + \boldsymbol{\alpha}'\left(\boldsymbol{X}\boldsymbol{X}' \otimes \boldsymbol{I}_k\right)\boldsymbol{\alpha}
\end{aligned}$$

Hence,

$$\frac{\partial \boldsymbol{S}(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} = 2\left(\boldsymbol{X}\boldsymbol{X}' \otimes \boldsymbol{I}_k\right)\boldsymbol{\alpha} - 2\left(\boldsymbol{X} \otimes \boldsymbol{I}_k\right)\boldsymbol{Y}_t$$

Equating this term to zero using *First Order Condition* provides us with:

$$\begin{aligned}
\hat{\boldsymbol{\alpha}} &= \left(\left(\boldsymbol{X}\boldsymbol{X}'\right)^{-1} \otimes \boldsymbol{I}_k\right) \bullet \left(\boldsymbol{X} \otimes \boldsymbol{I}_k\right)\boldsymbol{Y}_t \\
&= \left(\left(\boldsymbol{X}\boldsymbol{X}'\right)^{-1}\boldsymbol{X} \otimes \boldsymbol{I}_k\right) \bullet \text{vec}\left(\boldsymbol{Y}_t\right) \\
&= \text{vec}\left(\boldsymbol{Y}_t\left(\left(\boldsymbol{X}\boldsymbol{X}'\right)^{-1}\boldsymbol{X}\right)'\right) \\
&= \text{vec}\left(\left(\boldsymbol{Y}_t\boldsymbol{X}'\right)\left(\boldsymbol{X}\boldsymbol{X}'\right)^{-1}\right) = \text{vec}\left(\hat{\boldsymbol{A}}\right)
\end{aligned}$$

Above, we conclude that our coefficient matrix estimator would be $\hat{\boldsymbol{A}} = \left(\boldsymbol{Y}_t\boldsymbol{X}'\right)\left(\boldsymbol{X}\boldsymbol{X}'\right)^{-1}$. We put this estimator into explicit IRF form and achieve the impulse response function.

2.2 VAR with contemporaneously correlated error term

In the previous discussions, we only assumes $\boldsymbol{e}_t$ to be mean zero, uncorrelated with lagged $\boldsymbol{y}_{t-1}$, and are serially uncorrelated. Under this condition, we define our impulse response function as $IRF_{ij}(h) = \frac{\partial \mathscr{P}_t[y_{it+h}]}{\partial e_{jt}}$. However, this definition may sounds problematic if e_{jt} and e_{it} are (in general) not independent, and consequently it would be improper to treat e_{jt} and e_{it} as simply fundamental ”shocks”.

Therefore, it would be a natural solution to orthogonalized the innovations so that they’re uncorrelated. There are various ways to decompose the variance matrix $\boldsymbol{\Sigma}$, a common way is to use a Cholesky decomposition. I can write the innovations as a function of the orthogonalized errors as:

$$\boldsymbol{e}_t = \boldsymbol{B}\boldsymbol{\epsilon}_t$$

Therefore, we can decompose $\boldsymbol{\Sigma}$ into the product of an $m \times m$ matrix $\boldsymbol{B}$ with its transpose. This set $\boldsymbol{B}$ to be a **lower triangular**.

$$\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{B}', \quad \boldsymbol{B} = chol\left(\boldsymbol{\Sigma}\right) = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}$$

Equivalently, all innovations related with the orthonalized error term can be writtern as:

$$
\begin{aligned}
e_{1t} &= b_{11}\epsilon_{1t} \\
e_{2t} &= b_{21}\epsilon_{1t} + b_{22}\epsilon_{2t} \\
e_{3t} &= b_{31}\epsilon_{1t} + b_{32}\epsilon_{2t} + b_{33}\epsilon_{3t} \\
\cdots &= \cdots \\
e_{mt} &= b_{m1}\epsilon_{1t} + b_{m2}\epsilon_{2t} + \cdots + b_{mm}\epsilon_{mt}
\end{aligned}
$$

In this way, the orthonalized impulse response function would be:

$$
\mathrm{IRF}(h) = \frac{\partial}{\partial \boldsymbol{\epsilon}_t'} \mathscr{F}_t \left[\boldsymbol{y}_{t+h} \right]
$$

By Choleskey decomposition, we decompose the error terms into several uncorrelated shocks. However, as we can see from $\boldsymbol{B}$, there's an obvious drawback that we put too much restrictions on relationships of $(e_{it}, e_{jt})_{\forall i \neq j}$. In practice, the decomposition of $\boldsymbol{\Sigma}$ doesn't limit to Cholesky decomposition. A broader class of models can be written as:

$$
\boldsymbol{H}\boldsymbol{e}_t = \boldsymbol{B}\boldsymbol{\epsilon}_t
$$

where the diagonal of H are set to 1 due to normalizetion. Explicitly we can view this equation as:

$$
\begin{aligned}
e_{1t} &= -h_{12}e_{2t} - h_{13}e_{3t} - \cdots - h_{1p}e_{pt} + b_{11}\epsilon_{1t} + b_{12}\epsilon_{2t} + \cdots + b_{1p}\epsilon_{pt} \\
e_{2t} &= -h_{21}e_{1t} - h_{23}e_{3t} - \cdots - h_{2p}e_{pt} + b_{21}\epsilon_{1t} + b_{22}\epsilon_{2t} + \cdots + b_{2p}\epsilon_{pt} \\
\cdots &= \cdots \\
e_{pt} &= -h_{p1}e_{1t} - h_{p2}e_{2t} - \cdots - h_{pt}e_{p-1,t} + b_{p1}\epsilon_{1t} + b_{p2}\epsilon_{2t} + \cdots + b_{pp}\epsilon_{pt}
\end{aligned}
\tag{4}
$$

This is the essence of Structural Vector Autoregressive models (SVARs). From (4), we can see that the key to decomposition is to identify the coefficients in $\boldsymbol{H}$ and $\boldsymbol{B}$, which solution is not unique. The identification can be combined with specific economic conditions, and economists can articulate it on their own.

When defining the IRF of orthogonalized shocks, we can first write $\boldsymbol{y}_t$ into its Wold decomposition forms:

$$
\boldsymbol{y}_t = \boldsymbol{\mu} + \Sigma_{l=0}^{\infty} \boldsymbol{\Theta}_l \boldsymbol{e}_{t-l} = \Sigma_{l=0}^{\infty} \boldsymbol{\Theta}_l \boldsymbol{H}^{\mathbf{-1}} \boldsymbol{B} \boldsymbol{\epsilon}_{t-l}
$$

In this way, our true IRF of orthogonalized shocks would be:

$$\text{IRF}\,(h) = \frac{\partial}{\partial \boldsymbol{\epsilon}_t'} \mathscr{F}_t \left[\boldsymbol{y}_{t+h}\right] = \boldsymbol{\Theta}_h \boldsymbol{H}^{-1} \boldsymbol{B}$$

3. Review of Local Projection

3.1 When the true model is $\mathbf{VAR(p)}$

From discussion from above, it's obvious to conclude that the IRF form of VARs is a complicated iterated form, which associated with the specification and estimation of the unknown matrix $\boldsymbol{A}$, $\boldsymbol{H}$, and $\boldsymbol{B}$. Excitingly, there is an alternative solution to achieve the same impulse response function, such as running a direct regression of $\boldsymbol{y}_{t+h}$ with respect to $\left(\boldsymbol{y}_t, \cdots, \boldsymbol{y}_{t-p}\right)$, a method called *Local Projection* which introduced by Jordà in 2005. I will show the equivalence of IRF between VAR and Local Projection when the true data is generated by a VAR process.

Furthermore, in real Macroeconomics analysis, it is unreasonable to assume the true DGP always follows the VAR process, therefore some linear or nonlinear misspecification may come into our sight. Under this consideration, traditional Local Projection can also be extended to a nonlinear form, thus showing more robustness in model misspecification.

First we discuss the condition when $\boldsymbol{y}_t$ follows a VAR(p) form of $\boldsymbol{y}_t = \boldsymbol{\alpha}_0 + \boldsymbol{A}_1 \boldsymbol{y}_{t-1} + \boldsymbol{A}_2 \boldsymbol{y}_{t-2} + \cdots + \boldsymbol{A}_p \boldsymbol{y}_{t-p} + \boldsymbol{e}_t$. By definition of *Local Projection*, it can be viewed as a projection of $\boldsymbol{y}_{t+h}$ onto its linear space generated by $\left(\boldsymbol{y}_t, \boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \cdots, \boldsymbol{y}_{t-p}\right)$ with $\boldsymbol{y}_t = (y_{t,1}, y_{t,2}, \cdots, y_{t,k})'$. Therefore:

$$\begin{aligned}
\boldsymbol{y}_{t+h} &= \gamma_0 + \boldsymbol{\Gamma}_1 \boldsymbol{y}_t + \cdots + \boldsymbol{\Gamma}_{p+1} \boldsymbol{y}_{t-p} + \boldsymbol{u}_t \\
&= \gamma_0 + \boldsymbol{\Gamma}_1 \left(\boldsymbol{\alpha}_0 + \boldsymbol{A}_1 \boldsymbol{y}_{t-1} + \boldsymbol{A}_2 \boldsymbol{y}_{t-2} + \cdots + \boldsymbol{A}_p \boldsymbol{y}_{t-p} + \boldsymbol{e}_t\right) + \cdots + \boldsymbol{\Gamma}_{p+1} \boldsymbol{y}_{t-p} + \boldsymbol{u}_t \\
&= \tilde{\gamma}_0 + \boldsymbol{\Gamma}_1 \boldsymbol{e}_t + \Sigma_{j=2}^{p+1} \tilde{\boldsymbol{\Gamma}}_j \boldsymbol{y}_{t+1-j} + \boldsymbol{u}_t
\end{aligned} \tag{5}$$

where

$$\boldsymbol{u}_t = \boldsymbol{y}_{t+h} - \mathbb{E}\left(\boldsymbol{y}_{t+h} | \boldsymbol{y}_t, \boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \cdots, \boldsymbol{y}_{t-p}\right), \;\; \mathbb{E}\left(\boldsymbol{u}_t\right) = 0$$

We can match it with equation (2). For convenience I write equation (2) below again:

$$\begin{aligned}\boldsymbol{y}_{t+h} &= \tilde{\boldsymbol{b}}_0 + \boldsymbol{B}_1\boldsymbol{e}_t + \Sigma_{j=2}^{p+1}\tilde{\boldsymbol{B}}_j\boldsymbol{y}_{t+1-j} + \boldsymbol{\mu}_t \\ &= \tilde{\boldsymbol{b}}_0 + \boldsymbol{B}_1\boldsymbol{e}_t + \Sigma_{j=2}^{p+1}\tilde{\boldsymbol{B}}_j\boldsymbol{y}_{t+1-j} + \underbrace{\Sigma_{i=1}^{h}\boldsymbol{C}_i\boldsymbol{e}_{t+i}}_{\boldsymbol{\mu}_t}\end{aligned} \tag{6}$$

where

$$\boldsymbol{\mu}_t = \boldsymbol{C}_1\boldsymbol{e}_{t+1} + \cdots + \boldsymbol{C}_h\boldsymbol{e}_{t+h} = \boldsymbol{y}_{t+h} - \mathbb{E}\left(\boldsymbol{y}_{t+h}|\boldsymbol{y}_t, \boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \cdots, \boldsymbol{y}_{t-p}\right), \;\; \mathbb{E}\left(\boldsymbol{\mu}_t\right) = 0$$

From both equation (5) and eqaution (6), we can conclude that:

$$\begin{aligned}\mathbb{E}\left(\boldsymbol{e}_t'\boldsymbol{\mu}_t\right) &= 0 \\ \mathbb{E}\left(\boldsymbol{e}_t'\boldsymbol{u}_t\right) &= 0 \\ \mathbb{E}\left(\boldsymbol{e}_t'\boldsymbol{y}_{t+1-j}\right) &= 0, \;\; \forall j \in [2, p+1]\end{aligned}$$

therefore using Frisch–Waugh–Lovell (FWL) theorem,

$$\boldsymbol{B}_1 = \boldsymbol{\Gamma}_1 = \frac{\partial}{\partial \boldsymbol{e}_t'}\mathscr{F}\left(\boldsymbol{y}_{t+h}\right) = \mathbb{E}\left(\left(\boldsymbol{y}_{t+h}\boldsymbol{e}_t'\right)\left(\boldsymbol{e}_t\boldsymbol{e}_t'\right)^{-1}\right) = \text{IRF}\left(h\right) \tag{7}$$

The direct intuition from equation (7) shows that the IRF (h) between VAR and LP (when the true model is VAR(p)) are exactly the same. Since $\boldsymbol{e}_t$ is unobservable, we cannot directly use (7) to calculate the IRF. Recall the explicit form of $\hat{\boldsymbol{B}}_1$ is complicated, we can calculate the $\widehat{\text{IRF}}\,(h)$ using $\hat{\boldsymbol{\Gamma}}_1$ by running a simple regression of $\boldsymbol{y}_{t+h}$ with respect to $\left(\boldsymbol{y}_t, \cdots, \boldsymbol{y}_{t-p}\right)$ instead.

3.2 Extension of LP – when the true model is not VAR$(\mathbf{p})$

Assume $\boldsymbol{y}_t$ can be expressed as a function of its past white noise innovation, $\Phi(\cdot)$, and assume this function form is sufficiently well behaved, we can write *local projection* in a new form of $\Phi\left(\boldsymbol{e}_t, \boldsymbol{e}_{t-1}, \boldsymbol{e}_{t-2}, \cdots\cdots\right)$. An example of this is as Jordà showed in 2005:

$$\boldsymbol{y}_{t+h} = \boldsymbol{\gamma}_0 + \boldsymbol{\Gamma}_1\boldsymbol{y}_t + \boldsymbol{D}_1\boldsymbol{y}_t^2 + \boldsymbol{C}_1\boldsymbol{y}_t^3 + \boldsymbol{\Gamma}_2\boldsymbol{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{p+1}\boldsymbol{y}_{t-p} + \boldsymbol{u}_t$$

As Jordà and Salyer (2003) argued, such flexible local projections actually perform more reasonable than VAR when approximating non-linear models. This conclusion can be understood by a direct intuition. VAR (if not considered Threshold VAR) is a form of linear model, therefore the misspecification error will compound when the calculation of standard error from a non-linear model is more complicated than linear ones.

In conclusion, *Local Projection* provides direct moethod when estimating impulse responses, and is more robust to model misspecification than conventional methods.

4. Review of Equivalence of VAR and linear LP

In previous sections, this note gives review to the explicit form of IRF in VAR and local projections. When the true data generating process (DGP) follows a true VAR process, the IRF of VAR and local projections are exactly the same. By convention, we say that VARs are more efficient and the coefficients are easier to interpret, while LPs are more direct and robust to model misspecification. However, Stock and Watson (2018) pointed out that this opinion is in lack of formal analysis. By Plagborg-Møller and Wolf (2021), they concluded that by not putting too much restrictions on the linearity or dimensionality of DGP, linear LPs and VARs are not conceptually differently models.

In this section, the study note will sort out the core opinions in Plagborg-Møller and Wolf (2021) below. Their basic assumption is that the data $\{\boldsymbol{w}_t\}$ are covariance stationary and purely non-deterministic, with absolutely summable Wold decomposition coefficients. What's more, $\{\boldsymbol{w}_t\}$ is assumed to be a jointly Gaussian vector time series for notation simplicity. Here, this study note shows their lemma proofs in a more detailed way for learners.

Lemma 4.1. *The ratio of $LP(\infty)$ and $VAR(\infty)$ impulse response functions are up to a constant, which implies the convertibility of those two methods.*

Proof. For $LP(\infty)$, the linear projection can be written as:

$$y_{t+h} = \mu_h + \boldsymbol{\delta}'_h \boldsymbol{w}_t + \sum_{l=1}^{\infty} \boldsymbol{\delta}'_{h,l} \boldsymbol{w}_{t-l} + v_{h,t}$$

where $\boldsymbol{w}_t = (\boldsymbol{r}'_t, x_t, y_t, \boldsymbol{q}'_t)'$, from which $\boldsymbol{r}_t$ and $\boldsymbol{q}_t$ are controlled variables while x_t and y_t are scaler time series, no predeterminedness assumptions are required. Here we mainly focus on the impulse response of y_t to x_t.

The reduced form of $LP(\infty)$ can be written as:

$$
y_{t+h}=\mu_h+\beta_h x_t+\boldsymbol{\gamma}_h'\boldsymbol{r}_t+\sum_{l=1}^{\infty}\boldsymbol{\delta}_{h,l}'\boldsymbol{w}_{t-l}+\epsilon_{h,t}
$$

where the projection residual $\epsilon_{h,t}=y_{t+h}-\mathbb{E}\left(y_{t+h}|x_t,\boldsymbol{r}_t,\{\boldsymbol{w}_\tau\}_{\tau<t}\right)$, $Cov\left(\epsilon_{h,t},x_t\right)=0$ is under the assumption that w_t is a jointly Gaussian vector time series.

Using Frisch-Waugh-Lovell (FWL) theorem, the impulse response function of $LP(\infty)$ is defined as follows:

$$
\begin{aligned}
\text{IRF}\left(h\right)_{LP(\infty)}&=\mathbb{E}\left(y_{t+h}|x_t=1,\boldsymbol{r}_t,\{\boldsymbol{w}_\tau\}_{\tau<t}\right)-\mathbb{E}\left(y_{t+h}|x_t=0,\boldsymbol{r}_t,\{\boldsymbol{w}_\tau\}_{\tau<t}\right)\\
&=\frac{\mathbb{E}\left(y_{t+h}\widetilde{x}_t\right)}{\mathbb{E}\left(\widetilde{x}_t^2\right)}=\frac{\mathbb{E}\left(y_{t+h}\widetilde{x}_t\right)-\mathbb{E}\left(y_{t+h}\right)\mathbb{E}\left(\widetilde{x}_t\right)}{\mathbb{E}\left(\widetilde{x}_t^2\right)}\\
&=\frac{Cov\left(y_{t+h},\widetilde{x}_t\right)}{\mathbb{E}\left(\widetilde{x}_t^2\right)}=\beta_h
\end{aligned}
\tag{8}
$$

where the projection residual of $\tilde{x}_t$ to $\left(\boldsymbol{r}_t,\{\boldsymbol{w}_\tau\}_{\tau<t}\right)$ is defined as $\tilde{x}_t=x_t-\mathbb{E}\left(x_t|\boldsymbol{r}_t,\{\boldsymbol{w}_\tau\}_{\tau<t}\right)$, therefore $E(\tilde{x}_t)=0$. Notice that the $\hat{\beta}_h$ would converge to the unbiased coefficient β_h of x_t to y_t **only if** x_t is uncorrelated with $\epsilon_{h,t}$, while this condition does not always hold in the real world since there might be omitted variable bias.

For a multivariate linear $VAR(\infty)$, the structural form of the projection can be written as:

$$
\boldsymbol{w}_t=\boldsymbol{c}+\sum_{l=1}^{\infty}\boldsymbol{A}_l\boldsymbol{w}_{t-l}+\boldsymbol{u}_t
$$

where the projection residual is defined as $\boldsymbol{u}_t=\boldsymbol{w}_t-\mathbb{E}\left(\boldsymbol{w}_t|\,\{\boldsymbol{w}_\tau\}_{\tau<t}\right)=\left(\boldsymbol{u}_{rt}',u_{xt},u_{yt},\boldsymbol{u}_{qt}'\right)'$.

Let $\Sigma_u\equiv\mathbb{E}(u_tu_t{}')$ and define the Cholesky decomposition $\boldsymbol{\Sigma}_u=\boldsymbol{B}\boldsymbol{B}'$, we can get the structural form of VAR:

$$
\left(\boldsymbol{I}-\sum_{l=1}^{\infty}\boldsymbol{A}_lL^l\right)\boldsymbol{w}_t=\boldsymbol{A}\left(L\right)\boldsymbol{w}_t=\boldsymbol{c}+\boldsymbol{B}\boldsymbol{\eta}_t
$$

Set $\boldsymbol{C}\left(L\right)=\Sigma_{l=0}^{\infty}\boldsymbol{C}_lL^l=\boldsymbol{A}\left(L\right)^{-1}$, and set $\tilde{\boldsymbol{c}}=\boldsymbol{c}\cdot\boldsymbol{A}(L)^{-1}$. Then:

$$\boldsymbol{w}_t = \widetilde{\boldsymbol{c}} + \boldsymbol{C}(L)\boldsymbol{B}\boldsymbol{\eta}_t, \quad \begin{bmatrix} \boldsymbol{r}_t \\ x_t \\ y_t \\ \boldsymbol{q}_t \end{bmatrix} = \widetilde{\boldsymbol{c}} + \boldsymbol{C}(L)\boldsymbol{B} \begin{bmatrix} \boldsymbol{\eta}_{rt} \\ \eta_{xt} \\ \eta_{yt} \\ \boldsymbol{\eta}_{qt} \end{bmatrix}$$

As a result, since $\boldsymbol{\eta}_t$ is a vector of uncorrelated fundamental shocks and η_{xt} follows a normal distribution by assumption, the impulse response function of VAR is:

$$\text{IRF}(h)_{VAR(\infty)} = \boldsymbol{C}(h)_{(n_{r+2},\bullet)}\boldsymbol{B}_{(\bullet,n_r+1)} = \frac{\mathbb{E}(y_{t+h}\eta_{xt})}{\mathbb{E}(\eta_{xt}^2)} = Cov(y_{t+h}, \eta_{xt})$$

Since $\boldsymbol{B}\boldsymbol{\eta}_t = \boldsymbol{u}_t$ where $\boldsymbol{B}$ is a lower triangle, η_{xt} and $\boldsymbol{\eta}_{rt}$ are uncorrelated. We can have:

$$u_{rt}(i) = \boldsymbol{B}_{(i,1:i)}\boldsymbol{\eta}_{rt}(1:i) + \underbrace{\boldsymbol{B}_{(i,i+1:r)}\boldsymbol{\eta}_{rt}(i+1:r) + \boldsymbol{B}_{(n_r+1,n_r+1)}\eta_{xt} + \boldsymbol{B}_{(n_r+1,n_r+2)}\eta_{yt} + \cdots}_{0}$$

$$u_{xt} = \boldsymbol{B}_{(n_r+1,1:n_r)}\boldsymbol{\eta}_{rt} + \boldsymbol{B}_{(n_r+1,n_r+1)}\eta_{xt} + \underbrace{\boldsymbol{B}_{(n_r+1,n_r+2)}\eta_{yt} + \boldsymbol{B}_{(n_r+1,n_r+3:n)}\boldsymbol{\eta}_{qt}}_{0}$$

Therefore, for $\forall i \in [1, r]$, $Cov(u_{rt}(i), \eta_{xt}) = 0$. Actually, as we can find from the deduction above, this outcome is based on a strict assumption, that we have to use the Cholesky decomposition, and would fail otherwise if $\boldsymbol{\mu}_{rt}$ and η_{xt} are correlated.

$$\begin{aligned} \boldsymbol{B}_{(n_r+1,n_r+1)}\eta_{xt} &= u_{xt} - \mathbb{E}(u_{xt}|\boldsymbol{\eta}_{rt}) \\ &= u_{xt} - \mathbb{E}(u_{xt}|\boldsymbol{u}_{rt}) \\ &= \widetilde{u}_{xt} \\ \eta_{xt} &= \frac{\widetilde{u}_{xt}}{\boldsymbol{B}_{(n_r+1,n_r+1)}} \end{aligned}$$

Since $\tilde{x}_t = x_t - \mathbb{E}\left(x_t \middle| \boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{\tau<t}\right)$, $u_{xt} = x_t - \mathbb{E}\left(x_t \middle| \{\boldsymbol{w}_\tau\}_{\tau<t}\right)$, therefore the relationship between $\widetilde{u}_{xt}$ and $\tilde{x}_t$ is:

$$\begin{aligned}
u_{xt} - \widetilde{x}_t &= \mathbb{E}\left(x_t \middle| \boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{\tau<t}\right) - \mathbb{E}\left(x_t \middle| \{\boldsymbol{w}_\tau\}_{\tau<t}\right) \\
&= \mathbb{E}\left[u_{xt} + \mathbb{E}\left(x_t \middle| \{\boldsymbol{w}_\tau\}_{\tau<t}\right) \middle| \boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{\tau<t}\right] - \mathbb{E}\left(x_t \middle| \{\boldsymbol{w}_\tau\}_{\tau<t}\right) \\
&= \mathbb{E}\left(u_{xt} \middle| \boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{\tau<t}\right) + \mathbb{E}\left(x_t \middle| \{\boldsymbol{w}_\tau\}_{\tau<t}\right) - \mathbb{E}\left(x_t \middle| \{\boldsymbol{w}_\tau\}_{\tau<t}\right) \\
&= \mathbb{E}\left(u_{xt} \middle| \boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{\tau<t}\right) \\
&= \mathbb{E}\left(u_{xt} \middle| \boldsymbol{\mu}_{rt}, \{\boldsymbol{w}_\tau\}_{\tau<t}\right) = \mathbb{E}\left(\mu_{xt} | \boldsymbol{\mu}_{rt}\right) \\
&= u_{xt} - \widetilde{u}_{xt}
\end{aligned}$$

Therefore,

$$\widetilde{x}_t = \widetilde{u}_{xt}, \;\; \boldsymbol{B}^2_{n_r+1,n_r+1} = E\left(\widetilde{u}^2_{xt}\right) = E(\widetilde{x}^2_t)$$

Therefore the impulse response function for VAR(∞) is:

$$\mathrm{IRF}_{VAR(\infty)}\left(h\right) = Cov\left(y_{t+h}, \eta_{x,t}\right) = \frac{Cov(y_{t+h}, \widetilde{x}_t)}{\sqrt{E(\widetilde{x}^2_t)}} \tag{9}$$

Compare (8) and (9), we draw the conclusion that:

$$\mathrm{IRF}_{LP(\infty)}\left(h\right) = \mathrm{IRF}_{VAR(\infty)}\left(h\right) \times \sqrt{\mathbb{E}\left(\tilde{x}^2_t\right)} \tag{10}$$

□

Recall that in secton 3, we mentioned that if the true DGP follows a VAR(p) process, then the impulse response functions by linear local projection and VAR(p) are same. This is a special case of what we conclude from equation (10). If the true model is VAR(∞), then $\mathbb{E}\left(\widetilde{u}^2_{xt}\right) = \mathbb{E}\left(\widetilde{x}^2_t\right) = 1$ and the constant term in equation (10) would become 1.

In practice, we can scale $\widetilde{x}_t$ to have variance 1, thus the linear LP and VAR impulse response function coincide. Be aware that we make no assumption about the true DGP model for the equality in (10). The real model can as well be a non-parametric model, and the validity about the equivalence of linear LPs and VARs still holds.

Lemma 4.2. *The non-negative integers h, if p satisfies $h \leq p$, the IRF ratio of LP(p) and VAR(p) is approximately up to a constant, which implies the convertibility of those two methods.*

Proof. For LP(p), the linear projection can be written as:

$$y_{t+h} = \mu_h + \beta_h x_t + \boldsymbol{\gamma}_h' \boldsymbol{r}_t + \sum_{l=1}^{p} \boldsymbol{\delta}_{h,l}' \boldsymbol{w}_{t-l} + \epsilon_{h,t}, \quad w_t = (r_t', x_t, y_t, q_t\prime)$$

And the impulse response function is:

$$\begin{aligned}
\mathrm{IRF}_{LP(p)}(h) &= \mathbb{E}\left(y_{t+h}|x_t = 1, \boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{p\leq\tau<t}\right) - \mathbb{E}\left(y_{t+h}|x_t = 0, \boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{p\leq\tau<t}\right) \\
&= \frac{\mathbb{E}\left(y_{t+h}\widetilde{x}_t(p)\right)}{\mathbb{E}\left(\widetilde{x}_t^2(p)\right)} = \frac{\mathbb{E}\left(y_{t+h}\widetilde{x}_t\right) - \mathbb{E}\left(y_{t+h}\right)\mathbb{E}\left(\widetilde{x}_t(p)\right)}{\mathbb{E}\left(\widetilde{x}_t^2(p)\right)} \\
&= \frac{Cov\left(y_{t+h}, \widetilde{x}_t(p)\right)}{\mathbb{E}\left(\widetilde{x}_t^2(p)\right)} = \beta_h
\end{aligned}$$

where the projection residual of $\tilde{x}_t(p)$ to $\left(\boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{p\leq\tau<t}\right)$ is defined as $\tilde{x}_t(p) = x_t(p) - \mathbb{E}\left(x_t(p)|\boldsymbol{r}_t, \{\boldsymbol{w}_\tau\}_{p\leq\tau<t}\right)$, therefore $E\left(\tilde{x}_t(p)\right) = 0$.

For VAR(p), the linear projection can be written as:

$$\boldsymbol{w}_t = \boldsymbol{c} + \sum_{l=1}^{p} \boldsymbol{A}_l \boldsymbol{w}_{t-l} + \boldsymbol{u}_t$$

where the projection residual is defined as $\boldsymbol{u}_t = \boldsymbol{w}_t - \mathbb{E}\left(\boldsymbol{w}_t|\{\boldsymbol{w}_\tau\}_{p\leq\tau<t}\right) = \left(\boldsymbol{\mu}_{rt}', \mu_{xt}, \mu_{yt}, \boldsymbol{\mu}_{qt}'\right)'$.

Decompose $\boldsymbol{\mu}_t$ using Cholesky decomposition, therefore,

$$\left(\boldsymbol{I} - \sum_{l=1}^{p} \boldsymbol{A}_l L^l\right) \boldsymbol{w}_t = \boldsymbol{A}^{(\boldsymbol{p})} \boldsymbol{w}_t = \boldsymbol{c} + \boldsymbol{B}^{(\boldsymbol{p})} \bar{\boldsymbol{\eta}}_t, \quad \boldsymbol{w}_t = \widetilde{\boldsymbol{c}} + \boldsymbol{C}^{(\boldsymbol{p})}(L) \boldsymbol{B}^{(\boldsymbol{p})} \bar{\boldsymbol{\eta}}_t$$

Define $Cov^p\left(y_{t+h}, x_t\right)$ as the covariance calculated from VAR(p) model. In this way, the impulse response function for VAP(p) has a similar form as VAR(∞), which is:

$$\mathrm{IRF}_{VAR(p)}(h) = \boldsymbol{C}^{(\boldsymbol{p})}(h)_{(n_{r+2},\bullet)} \boldsymbol{B}^{(\boldsymbol{p})}{}_{(\bullet,n_r+1)} = \frac{\mathbb{E}(y_{t+h}\bar{\eta}_{xt})}{\mathbb{E}(\bar{\eta}_{xt}^2)} = Cov^p\left(y_{t+h}, \bar{\eta}_{xt}\right)$$

Since $\boldsymbol{B}\bar{\boldsymbol{\eta}}_t = \boldsymbol{u}_t$ and $\bar{\eta}_{x,t}$ and $\bar{\boldsymbol{\eta}}_{rt}$ are uncorrelated, we can have:

$$u_{xt} = \boldsymbol{B}_{n_r+1,1:n_r}(p)\bar{\boldsymbol{\eta}}_{rt} + \boldsymbol{B}_{n_r+1,n_r+1}(p)\bar{\eta}_{xt}$$

and similarily,

$$
\begin{aligned}
\boldsymbol{B}_{n_r+1,n_r+1}(p)\,\bar{\eta}_{x,t} &= u_{x,t} - \mathbb{E}\left(u_{x,t}|\bar{\boldsymbol{\eta}}_{rt}\right) \\
&= u_{x,t} - \mathbb{E}\left(u_{x,t}|\boldsymbol{u}_{rt}\right) \\
&= \widetilde{u}_{x,t} = \widetilde{x}_t(p) \\
&= x_t - \sum_{l=0}^{p} \boldsymbol{\rho}_l(p)' \boldsymbol{w}_{t-l}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\boldsymbol{B}_{n_r+1,n_r+1}(p)^2 &= \mathbb{E}\left(\widetilde{x}_t(p)^2\right) \\
\bar{\eta}_{x,t} &= \frac{\widetilde{x}_t(p)}{\boldsymbol{B}_{n_r+1,n_r+1}(p)} = \frac{\widetilde{x}_t(p)}{\sqrt{\mathbb{E}\left(\widetilde{x}_t(p)^2\right)}}
\end{aligned}
$$

And finally get the impulse response function of VAR(p):

$$
\begin{aligned}
\mathrm{IRF}_{VAR(p)}(h) = Cov^p\left(y_{t+h}, \bar{\eta}_{x,t}\right) &= \frac{Cov^p\left(y_{t+h}, \widetilde{x}_t(p)\right)}{\sqrt{E\left(\widetilde{x}_t(p)^2\right)}} \\
&= \mathrm{IRF}_{LP(p)}(h) \times \sqrt{E\left(\widetilde{x}_t(p)^2\right)} + \frac{Cov^p\left(y_{t+h,}\widetilde{x}_t(p)\right) - Cov\left(y_{t+h}, \widetilde{x}_t(p)\right)}{\sqrt{E\left(\widetilde{x}_t(p)^2\right)}}
\end{aligned}
$$

Notice that the first term equals to 0 if $h \leq p$, follows the argument in Brockwell and Davis (2009). Therefore:

$$
\begin{aligned}
Cov^p\left(y_{t+h,}\widetilde{x}_t(p)\right) - Cov\left(y_{t+h}, \widetilde{x}_t(p)\right) &= \Sigma_{l=0}^{p} \underbrace{\left\{Cov\left(y_{t+h}, w_{t-l}\right) - Cov^p\left(y_{t+h}, w_{t-l}\right)\right\}}_{=0 \text{ when } (t+h)-(t-l)\leq p} \rho_l(p) \\
&= \Sigma_{l=p-h+1}^{p} \left\{Cov\left(y_{t+h}, w_{t-l}\right) - Cov^p\left(y_{t+h}, w_{t-l}\right)\right\} \rho_l(p)
\end{aligned}
$$

However, as Plagborg-Møller and Wolf (2021) argues in their paper, empirically relevant DGP often have $\rho_l(p) \approx 0$ for long lag l. Therefore, for $h \ll p$, we can typically observe the equivalence IRF between LP(p) and VAR(p) in the short horizon:

$$
\mathrm{IRF}_{VAR(p)}(h) \approx \sqrt{E\left(\widetilde{x}_t(p)^2\right)} \times \mathrm{IRF}_{LP(p)}(h), \quad h \ll p
$$

□

In the research of Macroeconomics, the use of external instruments is growing at a rapid pace (Stock and Watson, 2018). Sometimes we are interested in an exogenous shock, such as the shock of monetary policy measured by change of federal fund futures, which cannot be affected by other structural shocks or by lags of itself. An extension of VAR and local projections lies in the interest to instrumental variable, which is an observable variable z_t that is correlated with ϵ_{zt} but not with other shocks. Notice that under the scheme of SVAR with first-ordered external instrument, and if we decompose the structural shocks using Choleskey decomposition, then the external shock is uncorrelated with other shocks by definition. Follow such consideration, we may want to discover the equivalence of VAR and local projection with external instrumental variables (LP-IV).

Lemma 4.3. *The impulse response function of Local Projection Instrumental Variable (LP-IV) estimation us equivalent to estimating a SVAR with the instrument ordered first.*

Proof. Suppose z_t is the instrumental variable of x_t that follows the previous definition of $\boldsymbol{w}_t = (\boldsymbol{r}_t, x_t, y_t, \boldsymbol{q}_t)'$, then the new data set including external instrumental variable can be defined as $\boldsymbol{W}_t = (z_t, \boldsymbol{w}_t')'$.

The first-stage linear LP-IV projection would be:

$$x_t = \mu_F + \beta_F z_t + \Sigma_{l=1}^{\infty} \boldsymbol{\delta}_{F,l}' \boldsymbol{W}_{t-1} + \epsilon_{F,t}$$

The second-stage "reduced-form" LP-IV projection can be written as:

$$y_{t+h} = \mu_{S,h} + \beta_{S,h} z_t + \Sigma_{l=1}^{\infty} \boldsymbol{\delta}_{S,h,l}' \boldsymbol{W}_{t-1} + \epsilon_{S,h,t}$$

The corresponding impulse response function (IRF) for the two stage local projection would be:

$$\begin{aligned} \text{IRF}_{\text{SVAR}}\left(x_t, z_t\right) &= \text{IRF}_{\text{f-LPIV}}\left(x_t, z_t\right) \times \sqrt{E\left(\tilde{z}_t^2\right)} \\ \text{IRF}_{\text{SVAR}}\left(y_{t+h}, z_t\right) &= \text{IRF}_{\text{s-LPIV}}\left(y_{t+h}, z_t\right) \times \sqrt{E\left(\tilde{z}_t^2\right)} \end{aligned} \tag{11}$$

where

$$\begin{aligned} \tilde{z}_t &= z_t - \mathbb{E}\left(z_t \middle| \left\{\boldsymbol{W}_\tau\right\}_{\tau<t}\right) \\ &= \alpha \epsilon_{1,t} + v_t \end{aligned}$$

therefore $\sqrt{E\left(\tilde{z}_t^2\right)} = \sqrt{\alpha^2 + \sigma_v^2}$.

Let the first equation in (11) to be divided by the second term:

$$\frac{\text{IRF}_{\text{SVAR}}\left(y_{t+h}, z_t\right)}{\text{IRF}_{\text{SVAR}}\left(x_t, z_t\right)} = \frac{\text{IRF}_{\text{s-LPIV}}\left(y_{t+h}, z_t\right)}{\text{IRF}_{\text{f-LPIV}}\left(x_t, z_t\right)} = \text{IRF}_{\text{LPIV}}\left(y_{t+h}, x_t\right) \tag{12}$$

The second equation in (12) follows the conclusion in Stock and Watson (2018). Similarly, we expect same equivalence to be hold when the infinite lag is truncated by lag p.

□

5. Empirical Examples

In this section, I conduct a similar empirical test with different data from Plagborg-Møller and Wolf (2021). In the recent paper from Jarociński and Karadi (2020), they use the co-movement of two sets of high frequency data (federal fund rate and S&P 500 stock market index) to decompose announcement surprise into monetary policy shock and central bank information effect. The high frequency surprise variables they used are three-month fed funds futures, and the S&P 500 stock market index; and five controlled variables include one year treasury yield as low-frequency monetary policy indicator, monthly average log S&P 500 prices as stock price index, log growth of industrial production as real economic activities (use real GDP for robustness test), log growth of CPI as price level (use GDP deflator in log levels for robustness test), and finally excess bond premium as indicator of financial conditions. The data spans from December 1990 to December 2016.

The variables in this empirical test are:

Table 1: Summary Statistics

	Endogenous Variable
x	one year government bond rate
y	excess bond premium
q_1	stock price index: log S&P 500
q_2	output growth (real GDP, Robustness Test: log growth of Industrial Production)
q_3	price level (GDP deflator, Robustness Test: log growth of CPI)
	Exogenous Shock (*but treated as endogenous instrument in this case*)
z	changes of three-month fed funds futures around policy announcements

In this section I will follow the notation from Plagborg-Møller and Wolf (2021) to make clear of the example. Be aware that the illustration listed below is the essence of coding. I define:

$$\boldsymbol{W}_t = \left(z_t, x_t, y_t, \underbrace{q_{1,t}, q_{2,t}, q_{3,t}}_{\boldsymbol{q}_t'} \right)', \quad \boldsymbol{\eta}_t = \left(\eta_{zt}, \eta_{xt}, \eta_{yt}, \underbrace{\eta_{q_1,t}, \eta_{q_2,t}, \eta_{q_3,t}}_{\boldsymbol{\eta}_{qt}'} \right)'$$

Therefore, the multivariate linear SVAR(p) form would be:

$$\boldsymbol{W}_t = \boldsymbol{c} + \Sigma_{l=1}^{p} \boldsymbol{A}_l \boldsymbol{W}_{t-1} + \boldsymbol{B}\boldsymbol{\eta}_t, \quad \boldsymbol{B} = \begin{pmatrix} b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ b_{61} & \cdots & b_{66} \end{pmatrix}$$

Suppose there's an initial shock and $\eta_{zt} = 1$, therefore the initial influence to all variables in $\boldsymbol{W}_t$ equals the first column of $\boldsymbol{B}$. Follow the SVAR's recursive form of achieving IRF, the corresponding impulse response function matrix to this shock would be:

$$\begin{aligned} \text{IRF} &= \begin{pmatrix} b_{11} & \boldsymbol{A}_1(1,:)\text{IRF}(:,1) & \boldsymbol{A}_1(1,:)\text{IRF}(:,2) + \boldsymbol{A}_2(1,:)\text{IRF}(:,1) & \cdots \\ b_{21} & \boldsymbol{A}_1(2,:)\text{IRF}(:,1) & \boldsymbol{A}_1(2,:)\text{IRF}(:,2) + \boldsymbol{A}_2(2,:)\text{IRF}(:,1) & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ b_{61} & \boldsymbol{A}_1(6,:)\text{IRF}(:,1) & \boldsymbol{A}_1(6,:)\text{IRF}(:,2) + \boldsymbol{A}_2(6,:)\text{IRF}(:,1) & \cdots \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \boldsymbol{z}(t,\cdots,t+h)}{\partial \eta_{zt}} & \frac{\partial \boldsymbol{x}(t,\cdots,t+h)}{\partial \eta_{zt}} & \frac{\partial \boldsymbol{y}(t,\cdots,t+h)}{\partial \eta_{zt}} & \cdots & \frac{\partial \boldsymbol{q}_3(t,\cdots,t+h)}{\partial \eta_{zt}} \end{pmatrix}' \end{aligned}$$

therefore

$$\frac{\text{IRF}_{\text{SVAR}}\left(y_t \cdots y_{t+h}, z_t\right)}{\text{IRF}_{\text{SVAR}}\left(x_t, z_t\right)} = \frac{\partial \boldsymbol{y}\left(t, \cdots, t+h\right)}{\partial \eta_{zt}} / \frac{\partial x\left(t\right)}{\partial \eta_{zt}} \tag{13}$$

By the same way, the two-step reduced form of linear Local Projection with instrumental variable can defined as:

$$\begin{cases} x_t = \mu_F + \beta_F z_t + \Sigma_{l=1}^{p} \boldsymbol{\delta}_{F,l}' \boldsymbol{W}_{t-1} + \epsilon_{F,t} \\ y_{t+h} = \mu_{S,h} + \beta_{S,h} z_t + \Sigma_{l=1}^{p} \boldsymbol{\delta}_{S,h,l}' \boldsymbol{W}_{t-1} + \epsilon_{S,h,t} \end{cases}$$

and the impulse response function of linear LPIV(p) is:

$$\frac{\text{IRF}_{\text{s-LPIV}}\left(y_{t+h}, z_t\right)}{\text{IRF}_{\text{f-LPIV}}\left(x_t, z_t\right)} = \text{IRF}_{\text{LPIV}}\left(y_{t+h}, x_t\right) \tag{14}$$

In the following graphs, we expect (13) and (14) to be equal.

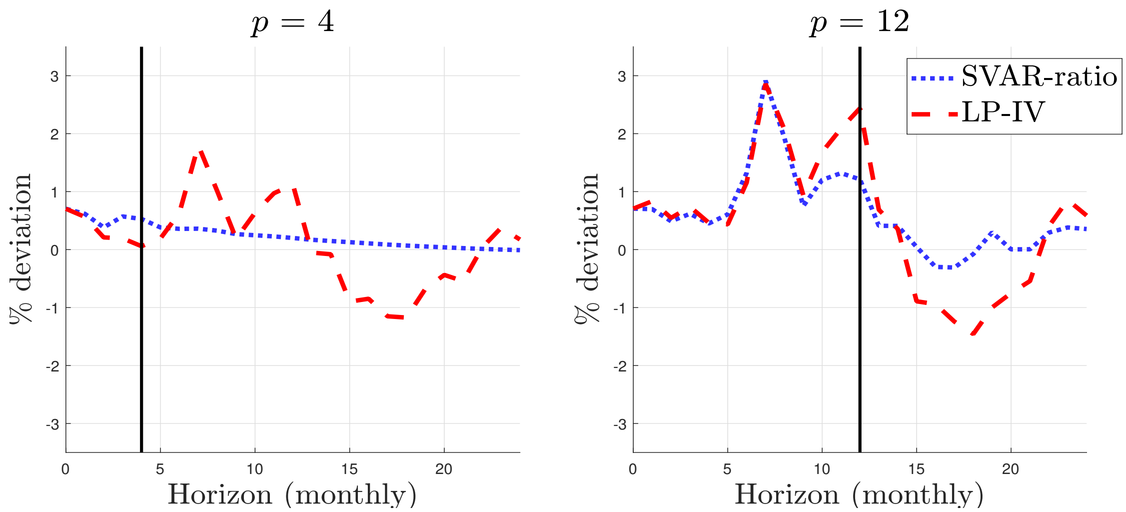

Figure 1. Estimated IRF of the Excess Bond Premium to a monetary policy shock

In figure (1), the lags are set to 4 and 12 respectively, same as the empirical test in Plagborg-Møller and Wolf (2021). We can see the impulse response functions of SVAR and LP-IV are close to each other when horizon is smaller than lag, and deviate from each other when the horizon extends.

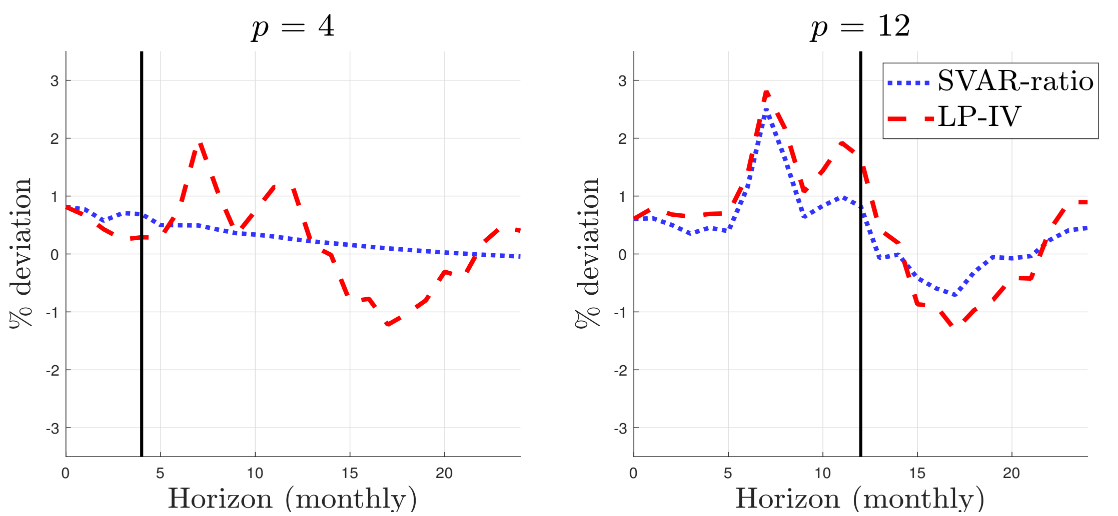

Figure 2. Robustness test: Estimated IRF of the Excess Bond Premium to a monetary policy shock

Figure (2) is a robustness test. I replace the log growth of industrial production and of CPI with monthly real GDP and GDP deflator. Specifically, I use a Kalman filter to distribute quarterly real GDP and GDP deflator series across months using method from Stock and Watson (2010).

More related articles can be viewed in Li, Plagborg-Møller, and Wolf (Working Papers), Plagborg-Møller (2019), Montiel Olea and Plagborg-Møller (Forthcoming), Plagborg-Møller and Wolf (2021).

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