

# Note of VARs and Local Projections

Xinyi Zhang

## Abstract

This note provides a review of previous researches on Vector Autoregressions (VARs) and Local Projections (LPs). I will give detail review of the equivalence of these two methods when the data generating process (DGP) follows the VAR process Jordà (2005), and their approximate equivalence in the scheme under Plagborg-Møller and Wolf (2021) when the DGP is unknown. In addition, I show that the latter equivalence relies on a strict decomposition method, the Choleskey decomposition. Furthermore, I use an empirical example in Macroeconomics to visualize some of the key comparisons in this article.

**Keywords:** local projection, VAR, impulse response function, Choleskey decomposition

## 1. Introduction

Vector Autoregressions (VARs) and Local Projections (LPs) are both multivariate time series models. The VAR introduced by Sims (1980) is considered the most commonly used multivariate time series model by economists, while the LP by Jordà (2005) has become an increasingly popular alternative econometric approach. In modern dynamic macroeconomics researches, VAR has embraced some transformations which variate from its traditional form, such as the Structural Vector Autogressions (SVAR), which deal with the dynamic response with respect to a structural shock; Threshold Vector Autogressions (TVAR), which set thresholds to certain variable in the equation and examine the IRF in different schemes; BVAR, which treats its model parameters as random variables with a prior probability rather than a fixed value.

In conventional analysis, it holds that VARs are more efficient and contain more economic explanations, while LPs are more robust to model misspecification and provide simpler joint or point-wise analytic inference. However, in empirical results, it shows that VARs and LP based approaches might give different results (Ramey, 2016), and such implementation differences might

---

Xinyi Zhang: astronchad@gmail.com

I am deeply indebted toward Peilin Yang (peiliny@stanford.edu), Xin Geng and Tian Xia for their useful comments. I am also grateful for Marek Jarocinski and Peter Karadi for their high-frequency data.

cause studies to different conclusions (Nakamura and Steinsson, 2018). In this paper, I'll show that when the true model is a VAR(p) process, then impulse response functions by VAR(p) and linear LP(p) are exactly the same; when the data generating process (DGP) does not follow a VAR(p) process, then flexible LP(p) can be more robust to model misspecification by assuming a nonlinear time series to be expressed as a generic projection function of its past values. In this paper, I will first give a brief overview of VAR and *local projection* so it would be clear to see the detail structure of the impulse response functions.

In recent paper from Plagborg-Møller and Wolf (2021), they argued the equivalence of impulse response functions between VARs and linear LPs under the condition of unknown data generating process. In a short prediction window, the ratio of impulse response functions between VAR(p) and linear LP(p) is close to a constant, while in the distant prediction, this ratio will diverge eventually. To visualize such comparison, I will further use examples in fields of rare disaster and equity premium to testify their theory.

## 2. Review of VAR

### 2.1 VAR with white noise error term

Both vector autoregressions (VARs) and local projections (LPs) lies in scheme of multivariate time series. To begin with the summary, we first introduce them distinctively. The traditional form of the  $p^{\text{th}}$  order vector autoregressive (VAR) model is:

$$\mathbf{y}_t = \alpha_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \cdots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{e}_t$$

suppose  $\mathbf{y}_t = (y_{1t}, y_{2t}, \cdots, y_{mt})'$  is a  $m \times 1$  vector which contains  $m$  variables at time  $t$ , in which case  $\mathbf{A}_l$  will be a  $m \times m$  coefficient matrix and  $\mathbf{e}_t = (e_{1t}, e_{2t}, \cdots, e_{mt})'$  will be a  $m \times 1$  error vector. Therefore, the contemporaneous covariance matrix is set to be  $\Sigma = \mathbb{E}[\mathbf{e}_t \mathbf{e}_t']$ .

$$\mathbf{A}_l = \begin{bmatrix} a_{11,l} & a_{12,l} & \cdots & a_{1m,l} \\ a_{21,l} & a_{22,l} & \cdots & a_{2m,l} \\ \vdots & \vdots & & \vdots \\ a_{m1,l} & a_{m2,l} & \cdots & a_{mm,l} \end{bmatrix}$$

In the most simple scheme, we assume  $\mathbf{y}_t$  is covariance stationary and has the projection equation  $\mathbf{y}_t = \alpha_0 + \sum_{l=1}^{\infty} \mathbf{A}_l \mathbf{y}_{t-l} + \mathbf{e}_t = \mathbf{y}_t$ . All those innovations  $\mathbf{e}_t$  satisfy:

$$\begin{aligned}\mathbb{E}[\mathbf{e}_t] &= 0 \\ \mathbb{E}[\mathbf{y}_{t-l}\mathbf{e}'_t] &= 0, \quad l \geq 1 \\ \mathbb{E}[\mathbf{e}_{t-l}\mathbf{e}'_t] &= 0, \quad l \geq 1\end{aligned}$$

and

$$\Sigma = \mathbb{E}[\mathbf{e}_t\mathbf{e}'_t] < \infty$$

Under this scheme, the multivariate innovations in  $\mathbf{e}_t$  are mean zero and serially uncorrelated. This is what we call a multivariate white noise process. Under this scheme, the innovations  $\mathbf{e}_t$  happens in period  $t$ , which was unpredictable in the past, represents us with the unique information in the period  $t$ . Therefore, we obtain the linear projection of the vector  $\mathbf{y}_t$  in its past history as:

$$\mathcal{P}_{t-1}[\mathbf{y}_t] = \mathcal{P}_{t-1}[\mathbf{y}_t | (\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots)] = \alpha_0 + \sum_{l=1}^{\infty} \mathbf{A}_l \mathbf{y}_{t-l} = \mathbf{y}_t - \mathbf{e}_t$$

By Wold decomposition  $\mathbf{y}_t = \boldsymbol{\mu} + \sum_{l=0}^{\infty} \boldsymbol{\Theta}_l \mathbf{e}_{t+h-l}$ , we can calculate the projection of  $\mathbf{y}_t$  onto its past history as:

$$\mathcal{P}_t[\mathbf{y}_{t+h}] = \boldsymbol{\mu} + \sum_{l=h}^{\infty} \boldsymbol{\Theta}_l \mathbf{e}_{t+h-l}$$

Since we define our impulse response function of variable  $i$  with respect to innovation  $j$  as:  $IRF_{ij}(h) = \frac{\partial \mathcal{P}_t[\mathbf{y}_{t+h}]}{\partial \mathbf{e}_{jt}}$ , therefore our impulse response matrix would be:

$$IRF(h) = \frac{\partial}{\partial \mathbf{e}'_t} \mathcal{P}_t[\mathbf{y}_{t+h}] = \boldsymbol{\Theta}_h = \begin{bmatrix} IRF_{11}(h) & \cdots & IRF_{1j}(h) \\ \vdots & \ddots & \vdots \\ IRF_{i1}(h) & \cdots & IRF_{ij}(h) \end{bmatrix}$$

What's more, there're some recursive relationships between  $\boldsymbol{\Theta}_i$  and  $\boldsymbol{\Theta}_j$ . First let's assume  $\alpha_0 = \mathbf{0}$  and  $\boldsymbol{\Theta}_0 = \mathbf{I}_m$ :

$$\begin{aligned}\mathbf{y}_0 &= \mathbf{e}_0 = \boldsymbol{\Theta}_0 \mathbf{e}_0 \\ \mathbf{y}_1 &= \mathbf{A}_1 \mathbf{y}_0 = \mathbf{A}_1 \boldsymbol{\Theta}_0 \mathbf{e}_0 = \boldsymbol{\Theta}_1 \mathbf{e}_0 \\ \mathbf{y}_2 &= \mathbf{A}_1 \mathbf{y}_1 + \mathbf{A}_2 \mathbf{y}_0 = \mathbf{A}_1 \boldsymbol{\Theta}_1 \mathbf{e}_0 + \mathbf{A}_2 \boldsymbol{\Theta}_0 \mathbf{e}_0 = \boldsymbol{\Theta}_2 \mathbf{e}_0 \\ \mathbf{y}_3 &= \mathbf{A}_1 \mathbf{y}_2 + \mathbf{A}_2 \mathbf{y}_1 + \mathbf{A}_3 \mathbf{y}_0 = \mathbf{A}_1 \boldsymbol{\Theta}_2 \mathbf{e}_0 + \mathbf{A}_2 \boldsymbol{\Theta}_1 \mathbf{e}_0 + \mathbf{A}_3 \boldsymbol{\Theta}_0 \mathbf{e}_0 = \boldsymbol{\Theta}_3 \mathbf{e}_0\end{aligned}$$

To summarize briefly, we can get the impulse response function of VAR at different stages as:

$$\begin{aligned}
\Theta_0 &= \mathbf{I}_m \\
\Theta_1 &= \mathbf{A}_1 \Theta_0 \\
\Theta_2 &= \mathbf{A}_1 \Theta_1 + \mathbf{A}_2 \Theta_0 \\
\Theta_3 &= \mathbf{A}_1 \Theta_2 + \mathbf{A}_2 \Theta_1 + \mathbf{A}_3 \Theta_0 \\
&\dots = \dots \\
\Theta_h &= \sum_{i=1}^h \mathbf{A}_i \Theta_{h-i} = \sum_{i=1}^h \mathbf{A}_i \left( \frac{\partial}{\partial \mathbf{e}_t'} \mathcal{P}_t [\mathbf{y}_{t+h-i}] \right)
\end{aligned} \tag{1}$$

Here we write the h-step impulse response function as a linear function of the previous impulse response functions. This is a typical form to compute impulse response function of in VAR in programming. We can also view the IRF of VAR from another equivalent perspective. From this perspective, we will introduce the explicit form of the h-step IRF of VAR.

Suppose  $\mathbf{y}_t$  is truly a VAR(p) process, then  $\mathbf{y}_{t+h}$  can be written as:

$$\begin{aligned}
\mathbf{y}_{t+h} &= \alpha_0 + \mathbf{A}_1 \mathbf{y}_{t+h-1} + \mathbf{A}_2 \mathbf{y}_{t+h-2} + \dots + \mathbf{A}_p \mathbf{y}_{t+h-p} + \mathbf{e}_{t+h} \\
&= \alpha_0 + \mathbf{A}_1 (\alpha_0 + \mathbf{A}_1 \mathbf{y}_{t+h-2} + \mathbf{A}_2 \mathbf{y}_{t+h-3} + \dots + \mathbf{A}_p \mathbf{y}_{t+h-p-1} + \mathbf{e}_{t+h-1}) \\
&\quad + \mathbf{A}_2 \mathbf{y}_{t+h-2} + \dots + \mathbf{A}_p \mathbf{y}_{t+h-p} + \mathbf{e}_{t+h} \\
&= \underbrace{\alpha_0 + \mathbf{A}_1 \alpha_0}_{\text{constant}} + (\mathbf{A}_1 \mathbf{A}_1 + \mathbf{A}_2) \mathbf{y}_{t+h-2} + \dots + \mathbf{A}_1 \mathbf{A}_p \mathbf{y}_{t+h-p-1} + \underbrace{\mathbf{A}_1 \mathbf{e}_{t+h-1} + \mathbf{e}_{t+h}}_{\boldsymbol{\mu}_{t+h-2}} \\
&= \dots \dots \dots \\
&= \mathbf{b}_0 + \sum_{j=1}^p \mathbf{B}_j \mathbf{y}_{t+1-j} + \underbrace{\sum_{i=1}^h \mathbf{C}_i \mathbf{e}_{t+i}}_{\boldsymbol{\mu}_t} \\
&= \mathbf{b}_0 + \mathbf{B}_1 \mathbf{y}_t + \sum_{j=2}^{p+1} \mathbf{B}_j \mathbf{y}_{t+1-j} + \boldsymbol{\mu}_t \\
&= \mathbf{b}_0 + \mathbf{B}_1 (\alpha_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \dots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{e}_t) + \sum_{j=2}^{p+1} \mathbf{B}_j \mathbf{y}_{t+1-j} + \boldsymbol{\mu}_t \\
&= \tilde{\mathbf{b}}_0 + \mathbf{B}_1 \mathbf{e}_t + \sum_{j=2}^{p+1} \tilde{\mathbf{B}}_j \mathbf{y}_{t+1-j} + \boldsymbol{\mu}_t
\end{aligned} \tag{2}$$

where  $\mathbf{B} = \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p)$

By the assumption we argued before ( $\mathbb{E}(\mathbf{e}_t) = 0, \mathbb{E}(\mathbf{y}_{t-l} \mathbf{e}_t') = 0, \mathbb{E}(\mathbf{e}_{t-l} \mathbf{e}_t') = 0$ ), we have:

$$\begin{aligned}
\text{Cov}(\mathbf{e}_t, \mathbf{y}_{t+1-j}) &= 0, \quad \forall j \in [2, p+1] \\
\text{Cov}\left(\mathbf{e}_t, \sum_{i=1}^h \mathbf{e}_{t+i}\right) &= \text{Cov}(\mathbf{e}_t, \boldsymbol{\mu}_t) = 0
\end{aligned}$$

Thus:

$$\text{IRF}_h = \frac{\partial}{\partial \mathbf{e}_t'} \mathcal{P}_t [\mathbf{y}_{t+h}] = \boldsymbol{\Theta}_h = \mathbf{B}_1 \quad (3)$$

Notice (3) is the same impulse response function as we deducted recursively in (1). Here we can see that deriving correct impulse responses functions from VARs can be extremely complicated (Hansen, 2000) .

In computer software, we estimate the h-step IRF through estimator of coefficient matrix, as  $\hat{\boldsymbol{\Theta}}_h = \hat{\mathbf{B}}_1 = \mathbf{f}(\hat{\mathbf{A}}_1, \dots, \hat{\mathbf{A}}_p)$ . We can achieve this by minimize the GLS (or LS) MSE of VAR(p).

For a stationary, stable VAR(p):  $\mathbf{y}_t = \boldsymbol{\alpha}_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \dots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{e}_t$  with  $\mathbf{y}_t = (y_{t,1}, y_{t,2}, \dots, y_{t,k})'$ , we write it as:

$$\mathbf{Y}_t = \mathbf{A}\mathbf{X} + \mathbf{U}$$

where,

$$\mathbf{Y}_t = \begin{bmatrix} y_{t,1} & y_{t,2} & \dots & y_{t,k} \end{bmatrix}^T, \quad \underbrace{\mathbf{A} = (\boldsymbol{\alpha}_0, \mathbf{A}_1, \dots, \mathbf{A}_p)}_{k \times (kp+k)}$$

$$\mathbf{X} = \underbrace{\begin{bmatrix} \mathbf{1}_{(1 \times k)} & \mathbf{Y}_{t-1}' & \dots & \mathbf{Y}_{t-p}' \end{bmatrix}^T}_{k(p+1) \times 1}, \quad \mathbf{U} = \begin{bmatrix} \mu_{t,1} & \mu_{t,2} & \dots & \mu_{t,k} \end{bmatrix}^T$$

Therefore,

$$\mathbf{Y}_t = \text{vec}(\mathbf{Y}_t) = (\mathbf{X}' \otimes \mathbf{I}_k) \bullet \text{vec}(\mathbf{A}) + \mathbf{U}$$

We have to choose  $\boldsymbol{\alpha} = \text{vec}(\mathbf{A})$  which minimizes:

$$\begin{aligned} S(\boldsymbol{\alpha}) &= \mathbf{U}'\mathbf{U} \\ &= (\mathbf{Y}_t - (\mathbf{X}' \otimes \mathbf{I}_k) \boldsymbol{\alpha})' (\mathbf{Y}_t - (\mathbf{X}' \otimes \mathbf{I}_k) \boldsymbol{\alpha}) \\ &= \mathbf{Y}_t' \mathbf{Y}_t - 2\boldsymbol{\alpha}' (\mathbf{X} \otimes \mathbf{I}_k) \mathbf{Y}_t + \boldsymbol{\alpha}' (\mathbf{X} \otimes \mathbf{I}_k) (\mathbf{X}' \otimes \mathbf{I}_k) \boldsymbol{\alpha} \\ &= \mathbf{Y}_t' \mathbf{Y}_t - 2\boldsymbol{\alpha}' (\mathbf{X} \otimes \mathbf{I}_k) \mathbf{Y}_t + \boldsymbol{\alpha}' (\mathbf{X} \mathbf{X}' \otimes \mathbf{I}_k) \boldsymbol{\alpha} \end{aligned}$$

Hence,

$$\frac{\partial S(\alpha)}{\partial \alpha} = 2 (X X' \otimes I_k) \alpha - 2 (X \otimes I_k) Y_t$$

Equating this term to zero using *First Order Condition* provides us with:

$$\begin{aligned} \hat{\alpha} &= \left( (X X')^{-1} \otimes I_k \right) \bullet (X \otimes I_k) Y_t \\ &= \left( (X X')^{-1} X \otimes I_k \right) \bullet \text{vec}(Y_t) \\ &= \text{vec} \left( Y_t \left( (X X')^{-1} X \right)' \right) \\ &= \text{vec} \left( (Y_t X') (X X')^{-1} \right) = \text{vec}(\hat{A}) \end{aligned}$$

Above, we conclude that our coefficient matrix estimator would be  $\hat{A} = (Y_t X') (X X')^{-1}$ . We put this estimator into explicit IRF form and achieve the impulse response function.

## 2.2 VAR with contemporaneously correlated error term

In the previous discussions, we only assumes  $e_t$  to be mean zero, uncorrelated with lagged  $y_{t-1}$ , and are serially uncorrelated. Under this condition, we define our impulse response function as  $IRF_{ij}(h) = \frac{\partial \mathcal{P}_t[y_{it+h}]}{\partial e_{jt}}$ . However, this definition may sounds problematic if  $e_{jt}$  and  $e_{it}$  are (in general) not independent, and consequently it would be improper to treat  $e_{jt}$  and  $e_{it}$  as simply fundamental "shocks".

Therefore, it would be a natural solution to orthogonalized the innovations so that they're uncorrelated. There are various ways to decompose the variance matrix  $\Sigma$ , a common way is to use a Cholesky decomposition. I can write the innovations as a function of the orthogonalized errors as:

$$e_t = B \epsilon_t$$

Therefore, we can decompose  $\Sigma$  into the product of an  $m \times m$  matrix  $B$  with its transpose. This set  $B$  to be a **lower triangular**.

$$\Sigma = B B', \quad B = \text{chol}(\Sigma) = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix}$$

Equivalently, all innovations related with the orthonalized error term can be writtern as:

$$\begin{aligned}
e_{1t} &= b_{11}\epsilon_{1t} \\
e_{2t} &= b_{21}\epsilon_{1t} + b_{22}\epsilon_{2t} \\
e_{3t} &= b_{31}\epsilon_{1t} + b_{32}\epsilon_{2t} + b_{33}\epsilon_{3t} \\
&\dots = \dots \\
e_{mt} &= b_{m1}\epsilon_{1t} + b_{m2}\epsilon_{2t} + \dots + b_{mm}\epsilon_{mt}
\end{aligned}$$

In this way, the orthonalized impulse response function would be:

$$\text{IRF}(h) = \frac{\partial}{\partial \epsilon'_t} \mathcal{F}_t [\mathbf{y}_{t+h}]$$

By Cholesky decomposition, we decompose the error terms into several uncorrelated shocks. However, as we can see from  $\mathbf{B}$ , there's an obvious drawback that we put too much restrictions on relationships of  $(e_{it}, e_{jt})_{\forall i \neq j}$ . In practice, the decomposition of  $\Sigma$  doesn't limit to Cholesky decomposition. A broader class of models can be written as:

$$\mathbf{H}\mathbf{e}_t = \mathbf{B}\epsilon_t$$

where the diagonal of  $\mathbf{H}$  are set to 1 due to normalizetion. Explicitly we can view this equation as:

$$\begin{aligned}
e_{1t} &= -h_{12}e_{2t} - h_{13}e_{3t} - \dots - h_{1p}e_{pt} + b_{11}\epsilon_{1t} + b_{12}\epsilon_{2t} + \dots + b_{1p}\epsilon_{pt} \\
e_{2t} &= -h_{21}e_{1t} - h_{23}e_{3t} - \dots - h_{2p}e_{pt} + b_{21}\epsilon_{1t} + b_{22}\epsilon_{2t} + \dots + b_{2p}\epsilon_{pt} \\
&\dots = \dots \\
e_{pt} &= -h_{p1}e_{1t} - h_{p2}e_{2t} - \dots - h_{pp}e_{p-1,t} + b_{p1}\epsilon_{1t} + b_{p2}\epsilon_{2t} + \dots + b_{pp}\epsilon_{pt}
\end{aligned} \tag{4}$$

This is the essence of Structural Vector Autoregressive models (SVARs). From (4), we can see that the key to decomposition is to identify the coefficients in  $\mathbf{H}$  and  $\mathbf{B}$ , which solution is not unique. The identification can be combined with specific economic conditions, and economists can articulate it on their own.

When defining the IRF of orthogonalized shocks, we can first write  $\mathbf{y}_t$  into its Wold decomposition forms:

$$\mathbf{y}_t = \boldsymbol{\mu} + \sum_{l=0}^{\infty} \boldsymbol{\Theta}_l \mathbf{e}_{t-l} = \sum_{l=0}^{\infty} \boldsymbol{\Theta}_l \mathbf{H}^{-1} \mathbf{B} \epsilon_{t-l}$$

In this way, our true IRF of orthogonalized shocks would be:

$$\text{IRF}(h) = \frac{\partial}{\partial \epsilon_t'} \mathcal{F}_t[\mathbf{y}_{t+h}] = \mathbf{\Theta}_h \mathbf{H}^{-1} \mathbf{B}$$

### 3. Review of Local Projection

#### 3.1 When the true model is VAR(p)

From discussion from above, it's obvious to conclude that the IRF form of VARs is a complicated iterated form, which associated with the specification and estimation of the unknown matrix  $\mathbf{A}$ ,  $\mathbf{H}$ , and  $\mathbf{B}$ . Excitingly, there is an alternative solution to achieve the same impulse response function, such as running a direct regression of  $\mathbf{y}_{t+h}$  with respect to  $(\mathbf{y}_t, \dots, \mathbf{y}_{t-p})$ , a method called *Local Projection* which introduced by Jordà in 2005. I will show the equivalence of IRF between VAR and Local Projection when the true data is generated by a VAR process.

Furthermore, in real Macroeconomics analysis, it is unreasonable to assume the true DGP always follows the VAR process, therefore some linear or nonlinear misspecification may come into our sight. Under this consideration, traditional Local Projection can also be extended to a nonlinear form, thus showing more robustness in model misspecification.

First we discuss the condition when  $\mathbf{y}_t$  follows a VAR(p) form of  $\mathbf{y}_t = \alpha_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \dots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{e}_t$ . By definition of *Local Projection*, it can be viewed as a projection of  $\mathbf{y}_{t+h}$  onto its linear space generated by  $(\mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots, \mathbf{y}_{t-p})$  with  $\mathbf{y}_t = (y_{t,1}, y_{t,2}, \dots, y_{t,k})'$ . Therefore:

$$\begin{aligned} \mathbf{y}_{t+h} &= \gamma_0 + \mathbf{\Gamma}_1 \mathbf{y}_t + \dots + \mathbf{\Gamma}_{p+1} \mathbf{y}_{t-p} + \mathbf{u}_t \\ &= \gamma_0 + \mathbf{\Gamma}_1 (\alpha_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathbf{y}_{t-2} + \dots + \mathbf{A}_p \mathbf{y}_{t-p} + \mathbf{e}_t) + \dots + \mathbf{\Gamma}_{p+1} \mathbf{y}_{t-p} + \mathbf{u}_t \quad (5) \\ &= \tilde{\gamma}_0 + \mathbf{\Gamma}_1 \mathbf{e}_t + \sum_{j=2}^{p+1} \tilde{\mathbf{\Gamma}}_j \mathbf{y}_{t+1-j} + \mathbf{u}_t \end{aligned}$$

where

$$\mathbf{u}_t = \mathbf{y}_{t+h} - \mathbb{E}(\mathbf{y}_{t+h} | \mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots, \mathbf{y}_{t-p}), \quad \mathbb{E}(\mathbf{u}_t) = 0$$



We can match it with equation (2). For convenience I write equation (2) below again:

$$\begin{aligned} \mathbf{y}_{t+h} &= \tilde{\mathbf{b}}_0 + \mathbf{B}_1 \mathbf{e}_t + \sum_{j=2}^{p+1} \tilde{\mathbf{B}}_j \mathbf{y}_{t+1-j} + \boldsymbol{\mu}_t \\ &= \tilde{\mathbf{b}}_0 + \mathbf{B}_1 \mathbf{e}_t + \sum_{j=2}^{p+1} \tilde{\mathbf{B}}_j \mathbf{y}_{t+1-j} + \underbrace{\sum_{i=1}^h \mathbf{C}_i \mathbf{e}_{t+i}}_{\boldsymbol{\mu}_t} \end{aligned} \quad (6)$$

where

$$\boldsymbol{\mu}_t = \mathbf{C}_1 \mathbf{e}_{t+1} + \cdots + \mathbf{C}_h \mathbf{e}_{t+h} = \mathbf{y}_{t+h} - \mathbb{E}(\mathbf{y}_{t+h} | \mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \dots, \mathbf{y}_{t-p}), \quad \mathbb{E}(\boldsymbol{\mu}_t) = 0$$

From both equation (5) and equation (6), we can conclude that:

$$\begin{aligned} \mathbb{E}(\mathbf{e}'_t \boldsymbol{\mu}_t) &= 0 \\ \mathbb{E}(\mathbf{e}'_t \mathbf{u}_t) &= 0 \\ \mathbb{E}(\mathbf{e}'_t \mathbf{y}_{t+1-j}) &= 0, \quad \forall j \in [2, p+1] \end{aligned}$$

therefore using Frisch–Waugh–Lovell (FWL) theorem,

$$\mathbf{B}_1 = \boldsymbol{\Gamma}_1 = \frac{\partial}{\partial \mathbf{e}'_t} \mathcal{F}(\mathbf{y}_{t+h}) = \mathbb{E} \left( (\mathbf{y}_{t+h} \mathbf{e}'_t) (\mathbf{e}_t \mathbf{e}'_t)^{-1} \right) = \text{IRF}(h) \quad (7)$$

The direct intuition from equation (7) shows that the IRF( $h$ ) between VAR and LP (when the true model is VAR( $p$ )) are exactly the same. Since  $\mathbf{e}_t$  is unobservable, we cannot directly use (7) to calculate the IRF. Recall the explicit form of  $\hat{\mathbf{B}}_1$  is complicated, we can calculate the  $\widehat{\text{IRF}}(h)$  using  $\hat{\boldsymbol{\Gamma}}_1$  by running a simple regression of  $\mathbf{y}_{t+h}$  with respect to  $(\mathbf{y}_t, \dots, \mathbf{y}_{t-p})$  instead.

### 3.2 Extension of LP – when the true model is not VAR( $p$ )

Assume  $\mathbf{y}_t$  can be expressed as a function of its past white noise innovation,  $\Phi(\cdot)$ , and assume this function form is sufficiently well behaved, we can write *local projection* in a new form of  $\Phi(\mathbf{e}_t, \mathbf{e}_{t-1}, \mathbf{e}_{t-2}, \dots)$ . An example of this is as Jordà showed in 2005:

$$\mathbf{y}_{t+h} = \gamma_0 + \boldsymbol{\Gamma}_1 \mathbf{y}_t + \mathbf{D}_1 \mathbf{y}_t^2 + \mathbf{C}_1 \mathbf{y}_t^3 + \boldsymbol{\Gamma}_2 \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{p+1} \mathbf{y}_{t-p} + \mathbf{u}_t$$

As Jordà and Salyer (2003) argued, such flexible local projections actually perform more reasonable than VAR when approximating non-linear models. This conclusion can be understood by

a direct intuition. VAR (if not considered Threshold VAR) is a form of linear model, therefore the misspecification error will compound when the calculation of standard error from a non-linear model is more complicated than linear ones.

In conclusion, *Local Projection* provides direct method when estimating impulse responses, and is more robust to model misspecification than conventional methods.

## 4. Review of Equivalence of VAR and linear LP (Plagborg-Møller and Wolf, 2021)

In previous sections, this note gives review to the explicit form of IRF in VAR and local projections. When the true data generating process (DGP) follows a true VAR process, the IRF of VAR and local projections are exactly the same. By convention, we say that VARs are more efficient and the coefficients are easier to interpret, while LPs are more direct and robust to model misspecification. However, Stock and Watson (2018) pointed out that this opinion is in lack of formal analysis. Such further analysis was done by Plagborg-Møller and Wolf (2021) By not putting too much restrictions on the linearity or dimensionality of DGP, they concluded that linear LPs and VARs are not conceptually differently models.

In this section I sort out the core opinions in Plagborg-Møller and Wolf (2021) below. Their basic assumption is that the data  $\{\mathbf{w}_t\}$  are covariance stationary and purely non-deterministic, with absolutely summable Wold decomposition coefficients. What's more,  $\{\mathbf{w}_t\}$  is assumed to be a jointly Gaussian vector time series for notation simplicity. Here this note make a more readable review of their paper.

**Lemma 4.1.** *The ratio of  $LP(\infty)$  and  $VAR(\infty)$  impulse response functions are up to a constant, which implies the convertibility of those two methods.*

*Proof.* For  $LP(\infty)$ , the linear projection can be written as:

$$y_{t+h} = \mu_h + \delta'_h \mathbf{w}_t + \sum_{l=1}^{\infty} \delta'_{h,l} \mathbf{w}_{t-l} + v_{h,t}$$

where  $\mathbf{w}_t = (\mathbf{r}'_t, x_t, y_t, \mathbf{q}'_t)'$ , from which  $\mathbf{r}_t$  and  $\mathbf{q}_t$  are controlled variables while  $x_t$  and  $y_t$  are scalar time series, no predeterminedness assumptions are required. Here we mainly focus on the impulse response of  $y_t$  to  $x_t$ .

The reduced form of  $LP(\infty)$  can be written as:

$$y_{t+h} = \mu_h + \beta_h x_t + \gamma'_h \mathbf{r}_t + \sum_{l=1}^{\infty} \delta'_{h,l} \mathbf{w}_{t-l} + \epsilon_{h,t}$$

where the projection residual  $\epsilon_{h,t} = y_{t+h} - \mathbb{E}(y_{t+h}|x_t, \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t})$ ,  $Cov(\epsilon_{h,t}, x_t) = 0$  is under the assumption that  $w_t$  is a jointly Gaussian vector time series.

Using Frisch-Waugh-Lovell (FWL) theorem, the impulse response function of  $LP(\infty)$  is defined as follows:

$$\begin{aligned} \text{IRF}(h)_{LP(\infty)} &= \mathbb{E}(y_{t+h}|x_t = 1, \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t}) - \mathbb{E}(y_{t+h}|x_t = 0, \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t}) \\ &= \frac{\mathbb{E}(y_{t+h}\tilde{x}_t)}{\mathbb{E}(\tilde{x}_t^2)} = \frac{\mathbb{E}(y_{t+h}\tilde{x}_t) - \mathbb{E}(y_{t+h})\mathbb{E}(\tilde{x}_t)}{\mathbb{E}(\tilde{x}_t^2)} \\ &= \frac{Cov(y_{t+h}, \tilde{x}_t)}{\mathbb{E}(\tilde{x}_t^2)} = \beta_h \end{aligned} \quad (8)$$

where the projection residual of  $\tilde{x}_t$  to  $(\mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t})$  is defined as  $\tilde{x}_t = x_t - \mathbb{E}(x_t|\mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t})$ , therefore  $E(\tilde{x}_t) = 0$ . Notice that the  $\hat{\beta}_h$  would converge to the unbiased coefficient  $\beta_h$  of  $x_t$  to  $y_t$  **only if**  $x_t$  is uncorrelated with  $\epsilon_{h,t}$ , while this condition does not always hold in the real world since there might be omitted variable bias.

For a multivariate linear  $VAR(\infty)$ , the structural form of the projection can be written as:

$$\mathbf{w}_t = \mathbf{c} + \sum_{l=1}^{\infty} \mathbf{A}_l \mathbf{w}_{t-l} + \mathbf{u}_t$$

where the projection residual is defined as  $\mathbf{u}_t = \mathbf{w}_t - \mathbb{E}(\mathbf{w}_t|\{\mathbf{w}_\tau\}_{\tau < t}) = (\mathbf{u}'_{rt}, u_{xt}, u_{yt}, \mathbf{u}'_{qt})'$ .

Let  $\Sigma_u \equiv \mathbb{E}(u_t u_t')$  and define the Cholesky decomposition  $\Sigma_u = \mathbf{B}\mathbf{B}'$ , we can get the structural form of VAR:

$$\left( \mathbf{I} - \sum_{l=1}^{\infty} \mathbf{A}_l L^l \right) \mathbf{w}_t = \mathbf{A}(L) \mathbf{w}_t = \mathbf{c} + \mathbf{B}\boldsymbol{\eta}_t$$

Set  $\mathbf{C}(L) = \Sigma_u \mathbf{C}_L L^L = \mathbf{A}(L)^{-1}$ , and set  $\tilde{\mathbf{c}} = \mathbf{c} \cdot \mathbf{A}(L)^{-1}$ . Then:

$$\mathbf{w}_t = \tilde{\mathbf{c}} + \mathbf{C}(L) \mathbf{B} \boldsymbol{\eta}_t, \quad \begin{bmatrix} \mathbf{r}_t \\ x_t \\ y_t \\ \mathbf{q}_t \end{bmatrix} = \tilde{\mathbf{c}} + \mathbf{C}(L) \mathbf{B} \begin{bmatrix} \boldsymbol{\eta}_{rt} \\ \eta_{xt} \\ \eta_{yt} \\ \boldsymbol{\eta}_{qt} \end{bmatrix}$$

As a result, since  $\boldsymbol{\eta}_t$  is a vector of uncorrelated fundamental shocks and  $\eta_{xt}$  follows a normal distribution by assumption, the impulse response function of VAR is:

$$\text{IRF}(h)_{VAR(\infty)} = \mathbf{C}(h)_{(n_r+2, \bullet)} \mathbf{B}_{(\bullet, n_r+1)} = \frac{\mathbb{E}(y_{t+h} \eta_{xt})}{\mathbb{E}(\eta_{xt}^2)} = \text{Cov}(y_{t+h}, \eta_{xt})$$

Since  $\mathbf{B} \boldsymbol{\eta}_t = \mathbf{u}_t$  where  $\mathbf{B}$  is a lower triangle,  $\eta_{xt}$  and  $\boldsymbol{\eta}_{rt}$  are uncorrelated. We can have:

$$\begin{aligned} u_{rt}(i) &= \mathbf{B}_{(i,1:i)} \boldsymbol{\eta}_{rt}(1:i) + \underbrace{\mathbf{B}_{(i,i+1:r)} \boldsymbol{\eta}_{rt}(i+1:r) + \mathbf{B}_{(n_r+1,n_r+1)} \eta_{xt} + \mathbf{B}_{(n_r+1,n_r+2)} \eta_{yt} + \dots}_{0} \\ u_{xt} &= \mathbf{B}_{(n_r+1,1:n_r)} \boldsymbol{\eta}_{rt} + \mathbf{B}_{(n_r+1,n_r+1)} \eta_{xt} + \underbrace{\mathbf{B}_{(n_r+1,n_r+2)} \eta_{yt} + \mathbf{B}_{(n_r+1,n_r+3:n)} \boldsymbol{\eta}_{qt}}_{0} \end{aligned}$$

Therefore, for  $\forall i \in [1, r]$ ,  $\text{Cov}(u_{rt}(i), \eta_{xt}) = 0$ . Actually, as we can find from the deduction above, this outcome is based on a strict assumption, that we have to use the Cholesky decomposition, and would fail otherwise if  $\boldsymbol{\mu}_{rt}$  and  $\eta_{xt}$  are correlated.

$$\begin{aligned} \mathbf{B}_{(n_r+1,n_r+1)} \eta_{xt} &= u_{xt} - \mathbb{E}(u_{xt} | \boldsymbol{\eta}_{rt}) \\ &= u_{xt} - \mathbb{E}(u_{xt} | \mathbf{u}_{rt}) \\ &= \tilde{u}_{xt} \\ \eta_{xt} &= \frac{\tilde{u}_{xt}}{\mathbf{B}_{(n_r+1,n_r+1)}} \end{aligned}$$

Since  $\tilde{x}_t = x_t - \mathbb{E}(x_t | \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t})$ ,  $u_{xt} = x_t - \mathbb{E}(x_t | \{\mathbf{w}_\tau\}_{\tau < t})$ , therefore the relationship between  $\tilde{u}_{xt}$  and  $\tilde{x}_t$  is:

$$\begin{aligned}
u_{xt} - \tilde{x}_t &= \mathbb{E}(x_t | \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t}) - \mathbb{E}(x_t | \{\mathbf{w}_\tau\}_{\tau < t}) \\
&= \mathbb{E}[u_{xt} + \mathbb{E}(x_t | \{\mathbf{w}_\tau\}_{\tau < t}) | \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t}] - \mathbb{E}(x_t | \{\mathbf{w}_\tau\}_{\tau < t}) \\
&= \mathbb{E}(u_{xt} | \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t}) + \mathbb{E}(x_t | \{\mathbf{w}_\tau\}_{\tau < t}) - \mathbb{E}(x_t | \{\mathbf{w}_\tau\}_{\tau < t}) \\
&= \mathbb{E}(u_{xt} | \mathbf{r}_t, \{\mathbf{w}_\tau\}_{\tau < t}) \\
&= \mathbb{E}(u_{xt} | \boldsymbol{\mu}_{rt}, \{\mathbf{w}_\tau\}_{\tau < t}) = \mathbb{E}(\mu_{xt} | \boldsymbol{\mu}_{rt}) \\
&= u_{xt} - \tilde{u}_{xt}
\end{aligned}$$

Therefore,

$$\tilde{x}_t = \tilde{u}_{xt}, \quad \mathbf{B}_{n_r+1, n_r+1}^2 = E(\tilde{u}_{xt}^2) = E(\tilde{x}_t^2)$$

Therefore the impulse response function for  $\text{VAR}(\infty)$  is:

$$\text{IRF}_{\text{VAR}(\infty)}(h) = \text{Cov}(y_{t+h}, \eta_{x,t}) = \frac{\text{Cov}(y_{t+h}, \tilde{x}_t)}{\sqrt{E(\tilde{x}_t^2)}} \quad (9)$$

Compare (8) and (9), we draw the conclusion that:

$$\text{IRF}_{\text{LP}(\infty)}(h) = \text{IRF}_{\text{VAR}(\infty)}(h) \times \sqrt{E(\tilde{x}_t^2)} \quad (10)$$

□

Recall that in section 3, we mentioned that if the true DGP follows a  $\text{VAR}(p)$  process, then the impulse response functions by linear local projection and  $\text{VAR}(p)$  are same. This is a special case of what we conclude from equation (10). If the true model is  $\text{VAR}(\infty)$ , then  $E(\tilde{u}_{xt}^2) = E(\tilde{x}_t^2) = 1$  and the constant term in equation (10) would become 1.

In practice, we can scale  $\tilde{x}_t$  to have variance 1, thus the linear LP and VAR impulse response function coincide. Be aware that we make no assumption about the true DGP model for the equality in (10). The real model can as well be a non-parametric model, and the validity about the equivalence of linear LPs and VARs still holds.

**Lemma 4.2.** *The non-negative integers  $h$ , if  $p$  satisfies  $h \leq p$ , the IRF ratio of  $\text{LP}(p)$  and  $\text{VAR}(p)$  is approximately up to a constant, which implies the convertibility of those two methods.*

*Proof.* For  $\text{LP}(p)$ , the linear projection can be written as:

$$y_{t+h} = \mu_h + \beta_h x_t + \gamma'_h \mathbf{r}_t + \sum_{l=1}^p \delta'_{h,l} \mathbf{w}_{t-l} + \epsilon_{h,t}, \quad \mathbf{w}_t = (r'_t, x_t, y_t, q_t')$$

And the impulse response function is:

$$\begin{aligned} \text{IRF}_{LP(p)}(h) &= \mathbb{E} \left( y_{t+h} | x_t = 1, \mathbf{r}_t, \{\mathbf{w}_\tau\}_{p \leq \tau < t} \right) - \mathbb{E} \left( y_{t+h} | x_t = 0, \mathbf{r}_t, \{\mathbf{w}_\tau\}_{p \leq \tau < t} \right) \\ &= \frac{\mathbb{E}(y_{t+h} \tilde{x}_t(p))}{\mathbb{E}(\tilde{x}_t^2(p))} = \frac{\mathbb{E}(y_{t+h} \tilde{x}_t) - \mathbb{E}(y_{t+h}) \mathbb{E}(\tilde{x}_t(p))}{\mathbb{E}(\tilde{x}_t^2(p))} \\ &= \frac{\text{Cov}(y_{t+h}, \tilde{x}_t(p))}{\mathbb{E}(\tilde{x}_t^2(p))} = \beta_h \end{aligned}$$

where the projection residual of  $\tilde{x}_t(p)$  to  $(\mathbf{r}_t, \{\mathbf{w}_\tau\}_{p \leq \tau < t})$  is defined as  $\tilde{x}_t(p) = x_t(p) - \mathbb{E}(x_t(p) | \mathbf{r}_t, \{\mathbf{w}_\tau\}_{p \leq \tau < t})$ , therefore  $\mathbb{E}(\tilde{x}_t(p)) = 0$ .

For VAR(p), the linear projection can be written as:

$$\mathbf{w}_t = \mathbf{c} + \sum_{l=1}^p \mathbf{A}_l \mathbf{w}_{t-l} + \mathbf{u}_t$$

where the projection residual is defined as  $\mathbf{u}_t = \mathbf{w}_t - \mathbb{E}(\mathbf{w}_t | \{\mathbf{w}_\tau\}_{p \leq \tau < t}) = (\boldsymbol{\mu}'_{rt}, \mu_{xt}, \mu_{yt}, \boldsymbol{\mu}'_{qt})'$ .

Decompose  $\boldsymbol{\mu}_t$  using Cholesky decomposition, therefore,

$$\left( \mathbf{I} - \sum_{l=1}^p \mathbf{A}_l L^l \right) \mathbf{w}_t = \mathbf{A}^{(p)} \mathbf{w}_t = \mathbf{c} + \mathbf{B}^{(p)} \bar{\boldsymbol{\eta}}_t, \quad \mathbf{w}_t = \tilde{\mathbf{c}} + \mathbf{C}^{(p)}(L) \mathbf{B}^{(p)} \bar{\boldsymbol{\eta}}_t$$

Define  $\text{Cov}^p(y_{t+h}, x_t)$  as the covariance calculated from VAR(p) model. In this way, the impulse response function for VAR(p) has a similar form as VAR( $\infty$ ), which is:

$$\text{IRF}_{VAR(p)}(h) = \mathbf{C}^{(p)}(h)_{(n_r+2, \bullet)} \mathbf{B}^{(p)}_{(\bullet, n_r+1)} = \frac{\mathbb{E}(y_{t+h} \bar{\eta}_{xt})}{\mathbb{E}(\bar{\eta}_{xt}^2)} = \text{Cov}^p(y_{t+h}, \bar{\eta}_{xt})$$

Since  $\mathbf{B} \bar{\boldsymbol{\eta}}_t = \mathbf{u}_t$  and  $\bar{\eta}_{x,t}$  and  $\bar{\eta}_{rt}$  are uncorrelated, we can have:

$$u_{xt} = \mathbf{B}_{n_r+1, 1:n_r}(p) \bar{\eta}_{rt} + \mathbf{B}_{n_r+1, n_r+1}(p) \bar{\eta}_{xt}$$

and similarly,

$$\begin{aligned}
\mathbf{B}_{n_r+1, n_r+1}(p) \bar{\eta}_{x,t} &= u_{x,t} - \mathbb{E}(u_{x,t} | \bar{\boldsymbol{\eta}}_{rt}) \\
&= u_{x,t} - \mathbb{E}(u_{x,t} | \mathbf{u}_{rt}) \\
&= \tilde{u}_{x,t} = \tilde{x}_t(p) \\
&= x_t - \sum_{l=0}^p \boldsymbol{\rho}_l(p)' \mathbf{w}_{t-l}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{B}_{n_r+1, n_r+1}(p)^2 &= \mathbb{E}(\tilde{x}_t(p)^2) \\
\bar{\eta}_{x,t} &= \frac{\tilde{x}_t(p)}{\mathbf{B}_{n_r+1, n_r+1}(p)} = \frac{\tilde{x}_t(p)}{\sqrt{\mathbb{E}(\tilde{x}_t(p)^2)}}
\end{aligned}$$

And finally get the impulse response function of VAR(p):

$$\begin{aligned}
\text{IRF}_{\text{VAR}(p)}(h) &= \text{Cov}^p(y_{t+h}, \bar{\eta}_{x,t}) = \frac{\text{Cov}^p(y_{t+h}, \tilde{x}_t(p))}{\sqrt{\mathbb{E}(\tilde{x}_t(p)^2)}} \\
&= \text{IRF}_{\text{LP}(p)}(h) \times \sqrt{\mathbb{E}(\tilde{x}_t(p)^2)} + \frac{\text{Cov}^p(y_{t+h}, \tilde{x}_t(p)) - \text{Cov}(y_{t+h}, \tilde{x}_t(p))}{\sqrt{\mathbb{E}(\tilde{x}_t(p)^2)}}
\end{aligned}$$

Notice that the first term equals to 0 if  $h \leq p$ , follows the argument in Brockwell and Davis (2009). Therefore:

$$\begin{aligned}
\text{Cov}^p(y_{t+h}, \tilde{x}_t(p)) - \text{Cov}(y_{t+h}, \tilde{x}_t(p)) &= \sum_{l=0}^p \underbrace{\{\text{Cov}(y_{t+h}, w_{t-l}) - \text{Cov}^p(y_{t+h}, w_{t-l})\}}_{=0 \text{ when } (t+h)-(t-l) \leq p} \rho_l(p) \\
&= \sum_{l=p-h+1}^p \{\text{Cov}(y_{t+h}, w_{t-l}) - \text{Cov}^p(y_{t+h}, w_{t-l})\} \rho_l(p)
\end{aligned}$$

However, as Plagborg-Møller and Wolf (2021) argues in their paper, empirically relevant DGP often have  $\rho_l(p) \approx 0$  for long lag  $l$ . Therefore, for  $h \ll p$ , we can typically observe the equivalence IRF between LP(p) and VAR(p) in the short horizon:

$$\text{IRF}_{\text{VAR}(p)}(h) \approx \sqrt{\mathbb{E}(\tilde{x}_t(p)^2)} \times \text{IRF}_{\text{LP}(p)}(h), \quad h \ll p$$

□

In the research of Macroeconomics, the use of external instruments is growing at a rapid pace (Stock and Watson, 2018). Sometimes we are interested in an exogenous shock, such as the shock of monetary policy measured by change of federal fund futures, which cannot be affected by other structural shocks or by lags of itself. An extension of VAR and local projections lies in the interest to instrumental variable, which is an observable variable  $z_t$  that is correlated with  $\epsilon_{zt}$  but not with other shocks. Notice that under the scheme of SVAR with first-ordered external instrument, and if we decompose the structural shocks using Cholesky decomposition, then the external shock is uncorrelated with other shocks by definition. Follow such consideration, we may want to discover the equivalence of VAR and local projection with external instrumental variables (LP-IV).

**Lemma 4.3.** *The impulse response function of Local Projection Instrumental Variable (LP-IV) estimation is equivalent to estimating a SVAR with the instrument ordered first.*

*Proof.* Suppose  $z_t$  is the instrumental variable of  $x_t$  that follows the previous definition of  $\mathbf{w}_t = (\mathbf{r}_t, x_t, y_t, \mathbf{q}_t)'$ , then the new data set including external instrumental variable can be defined as  $\mathbf{W}_t = (z_t, \mathbf{w}_t)'$ .

The first-stage linear LP-IV projection would be:

$$x_t = \mu_F + \beta_F z_t + \sum_{l=1}^{\infty} \delta'_{F,l} \mathbf{W}_{t-1} + \epsilon_{F,t}$$

The second-stage "reduced-form" LP-IV projection can be written as:

$$y_{t+h} = \mu_{S,h} + \beta_{S,h} z_t + \sum_{l=1}^{\infty} \delta'_{S,h,l} \mathbf{W}_{t-1} + \epsilon_{S,h,t}$$

The corresponding impulse response function (IRF) for the two stage local projection would be:

$$\begin{aligned} \text{IRF}_{\text{SVAR}}(x_t, z_t) &= \text{IRF}_{\text{f-LPIV}}(x_t, z_t) \times \sqrt{E(\tilde{z}_t^2)} \\ \text{IRF}_{\text{SVAR}}(y_{t+h}, z_t) &= \text{IRF}_{\text{s-LPIV}}(y_{t+h}, z_t) \times \sqrt{E(\tilde{z}_t^2)} \end{aligned} \tag{11}$$

where

$$\begin{aligned} \tilde{z}_t &= z_t - \mathbb{E}(z_t | \{\mathbf{W}_\tau\}_{\tau < t}) \\ &= \alpha \epsilon_{1,t} + v_t \end{aligned}$$



therefore  $\sqrt{E(\tilde{z}_t^2)} = \sqrt{\alpha^2 + \sigma_v^2}$ .

Let the first equation in (11) to be divided by the second term:

$$\frac{\text{IRF}_{\text{SVAR}}(y_{t+h}, z_t)}{\text{IRF}_{\text{SVAR}}(x_t, z_t)} = \frac{\text{IRF}_{\text{s-LPIV}}(y_{t+h}, z_t)}{\text{IRF}_{\text{f-LPIV}}(x_t, z_t)} = \text{IRF}_{\text{LPIV}}(y_{t+h}, x_t) \quad (12)$$

The second equation in (12) follows the conclusion in Stock and Watson (2018). Similarly, we expect same equivalence to be hold when the infinite lag is truncated by lag  $p$ .

□

## 5. Empirical Examples

In this section, I conduct a similar empirical test with different data from Plagborg-Møller and Wolf (2021). In the recent paper from Jarociński and Karadi (2020), they use the co-movement of two sets of high frequency data (federal fund rate and S&P 500 stock market index) to decompose announcement surprise into monetary policy shock and central bank information effect. The high frequency surprise variables they used are three-month fed funds futures, and the S&P 500 stock market index; and five controlled variables include one year treasury yield as low-frequency monetary policy indicator, monthly average log S&P 500 prices as stock price index, log growth of industrial production as real economic activities (use real GDP for robustness test), log growth of CPI as price level (use GDP deflator in log levels for robustness test), and finally excess bond premium as indicator of financial conditions. The data spans from December 1990 to December 2016.

The variables in this empirical test are:

**Table 1:** Summary Statistics

| <i>Endogenous Variable</i>   |  |
|--|--|
| x  | one year government bond rate  |
| y  | excess bond premium  |
| $q_1$  | stock price index: log S&P 500   |
| $q_2$  | output growth (real GDP, Robustness Test: log growth of Industrial Production) |
| $q_3$  | price level (GDP deflator, Robustness Test: log growth of CPI)                 |
| <i>Exogenous Shock (but treated as endogenous instrument in this case)</i> |  |
| z  | changes of three-month fed funds futures around policy announcements           |

In this section I will follow the notation from Plagborg-Møller and Wolf (2021) to make clear of the example. Be aware that the illustration listed below is the essence of coding. I define:

$$\mathbf{W}_t = \left( z_t, x_t, y_t, \underbrace{q_{1,t}, q_{2,t}, q_{3,t}}_{\mathbf{q}'_t} \right)', \quad \boldsymbol{\eta}_t = \left( \eta_{zt}, \eta_{xt}, \eta_{yt}, \underbrace{\eta_{q_1,t}, \eta_{q_2,t}, \eta_{q_3,t}}_{\boldsymbol{\eta}'_{qt}} \right)'$$

Therefore, the multivariate linear SVAR( $p$ ) form would be:

$$\mathbf{W}_t = \mathbf{c} + \sum_{l=1}^p \mathbf{A}_l \mathbf{W}_{t-l} + \mathbf{B} \boldsymbol{\eta}_t, \quad \mathbf{B} = \begin{pmatrix} b_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ b_{61} & \cdots & b_{66} \end{pmatrix}$$

Suppose there's an initial shock and  $\eta_{zt} = 1$ , therefore the initial influence to all variables in  $\mathbf{W}_t$  equals the first column of  $\mathbf{B}$ . Follow the SVAR's recursive form of achieving IRF, the corresponding impulse response function matrix to this shock would be:

$$\begin{aligned} \text{IRF} &= \begin{pmatrix} b_{11} & \mathbf{A}_1(1, :)\text{IRF}(:, 1) & \mathbf{A}_1(1, :)\text{IRF}(:, 2) + \mathbf{A}_2(1, :)\text{IRF}(:, 1) & \cdots \\ b_{21} & \mathbf{A}_1(2, :)\text{IRF}(:, 1) & \mathbf{A}_1(2, :)\text{IRF}(:, 2) + \mathbf{A}_2(2, :)\text{IRF}(:, 1) & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ b_{61} & \mathbf{A}_1(6, :)\text{IRF}(:, 1) & \mathbf{A}_1(6, :)\text{IRF}(:, 2) + \mathbf{A}_2(6, :)\text{IRF}(:, 1) & \cdots \end{pmatrix} \\ &= \left( \frac{\partial \mathbf{z}(t, \dots, t+h)}{\partial \eta_{zt}} \quad \frac{\partial \mathbf{x}(t, \dots, t+h)}{\partial \eta_{zt}} \quad \frac{\partial \mathbf{y}(t, \dots, t+h)}{\partial \eta_{zt}} \quad \cdots \quad \frac{\partial \mathbf{q}_3(t, \dots, t+h)}{\partial \eta_{zt}} \right)' \end{aligned}$$

therefore

$$\frac{\text{IRF}_{\text{SVAR}}(y_t \cdots y_{t+h}, z_t)}{\text{IRF}_{\text{SVAR}}(x_t, z_t)} = \frac{\partial \mathbf{y}(t, \dots, t+h)}{\partial \eta_{zt}} / \frac{\partial x(t)}{\partial \eta_{zt}} \quad (13)$$

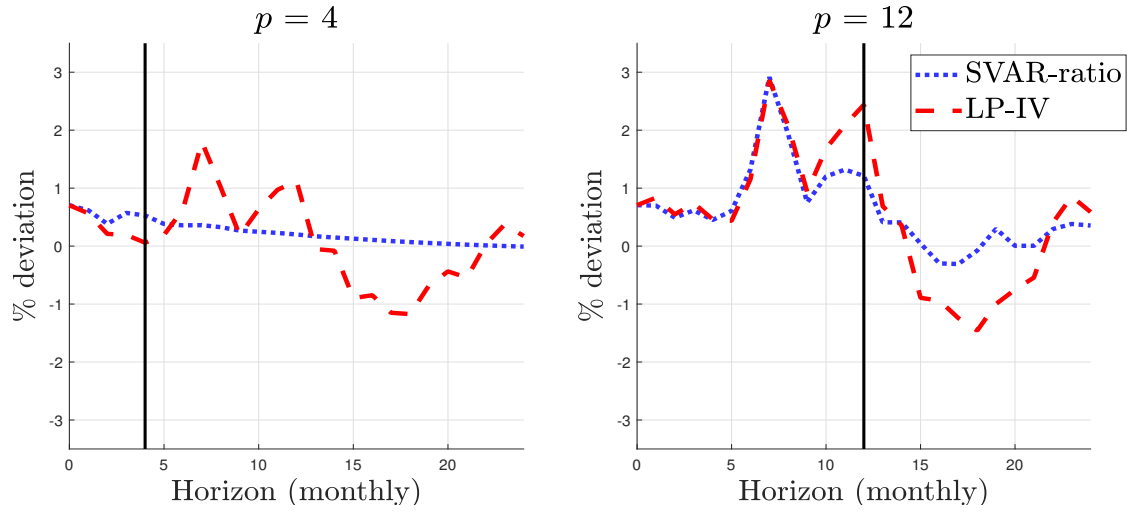
By the same way, the two-step reduced form of linear Local Projection with instrumental variable can defined as:

$$\begin{cases} x_t = \mu_F + \beta_F z_t + \sum_{l=1}^p \boldsymbol{\delta}'_{F,l} \mathbf{W}_{t-l} + \epsilon_{F,t} \\ y_{t+h} = \mu_{S,h} + \beta_{S,h} z_t + \sum_{l=1}^p \boldsymbol{\delta}'_{S,h,l} \mathbf{W}_{t-l} + \epsilon_{S,h,t} \end{cases}$$

and the impulse response function of linear LPIV( $p$ ) is:

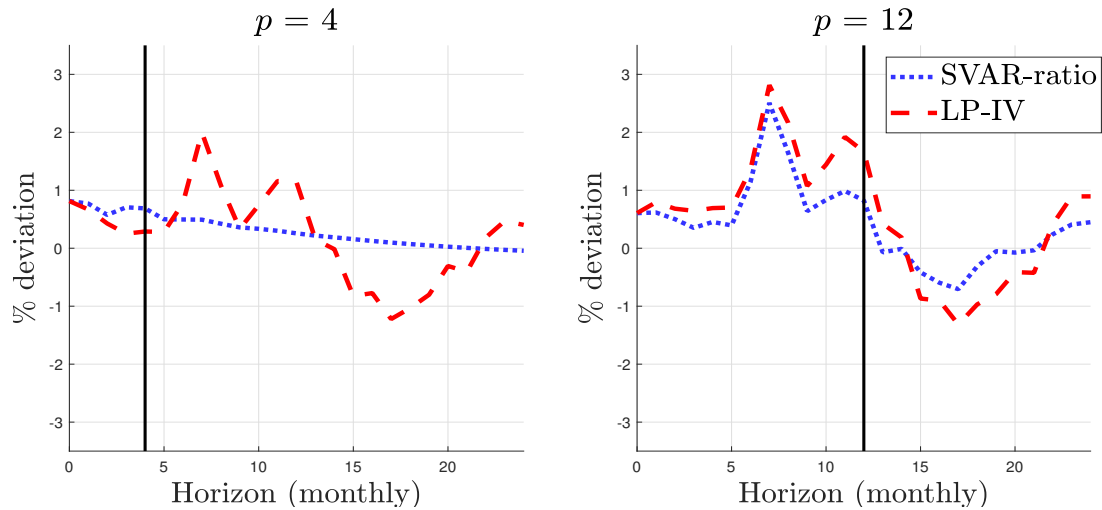
$$\frac{\text{IRF}_{\text{s-LPIV}}(y_{t+h}, z_t)}{\text{IRF}_{\text{f-LPIV}}(x_t, z_t)} = \text{IRF}_{\text{LPIV}}(y_{t+h}, x_t) \quad (14)$$

In the following graphs, we expect (13) and (14) to be equal.



**Figure 1.** Estimated IRF of the Excess Bond Premium to a monetary policy shock

In figure (1), the lags are set to 4 and 12 respectively, same as the empirical test in Plagborg-Møller and Wolf (2021). We can see the impulse response functions of SVAR and LP-IV are close to each other when horizon is smaller than lag, and deviate from each other when the horizon extends.



**Figure 2.** Robustness test: Estimated IRF of the Excess Bond Premium to a monetary policy shock

Figure (2) is a robustness test. I replace the log growth of industrial production and of CPI with monthly real GDP and GDP deflator. Specifically, I use a Kalman filter to distribute quarterly real GDP and GDP deflator series across months using method from Stock and Watson (2010).

More related articles can be viewed in Li, Plagborg-Møller, and Wolf (Working Papers), Plagborg-Møller (2019), Montiel Olea and Plagborg-Møller (Forthcoming), Plagborg-Møller and Wolf (2021).

## References

- Brockwell, Peter J, and Richard A Davis. 2009. *Time series: theory and methods*. Springer science & business media.
- Hansen, Bruce E. 2000. "Sample Splitting and Threshold Estimation." *Econometrica* 68 (3): 575–603.
- Jarociński, Marek, and Peter Karadi. 2020. "Deconstructing Monetary Policy Surprises—The Role of Information Shocks." *American Economic Journal: Macroeconomics* 12 (2): 1–43.
- Jordà, Òscar. 2005. "Estimation and Inference of Impulse Responses by Local Projections." *American Economic Review* 95 (1): 161–182.
- Jordà, Òscar, and Kevin D. Salyer. 2003. "The response of term rates to monetary policy uncertainty." *Finance and the Macroeconomy, Review of Economic Dynamics* 6 (4): 941–962.
- Li, Dake, Mikkel Plagborg-Møller, and Christian K. Wolf. Working Papers. "Local Projections vs. VARs: Lessons From Thousands of DGPs." Matlab code (GitHub).
- Montiel Olea, José Luis, and Mikkel Plagborg-Møller. Forthcoming. "Local Projection Inference is Simpler and More Robust Than You Think." Code and data (GitHub) Video of online seminar (YouTube), *Econometrica*.
- Nakamura, Emi, and Jón Steinsson. 2018. "Identification in Macroeconomics." *Journal of Economic Perspectives* 32 (3): 59–86.
- Plagborg-Møller, Mikkel. 2019. "Bayesian Inference on Structural Impulse Response Functions." Replication files (Matlab code and data), *Quantitative Economics* 10 (1): 145–184.
- Plagborg-Møller, Mikkel, and Christian K. Wolf. 2021. "Local Projections and VARs Estimate the Same Impulse Responses." *Econometrica* 89 (2): 955–980.
- Ramey, V.A. 2016. "Chapter 2 - Macroeconomic Shocks and Their Propagation," edited by John B. Taylor and Harald Uhlig, 2:71–162. *Handbook of Macroeconomics*. Elsevier.
- Sims, Christopher A. 1980. "Macroeconomics and Reality." *Econometrica* 48 (1): 1–48.
- Stock, James H, and Mark W Watson. 2010. "Dynamic Factor Models." In *Oxford Handbook of Economic Forecasting*, edited by Michael P. Clements and David F. Henry. Oxford: Oxford University Press.
- . 2018. *Identification and Estimation of Dynamic Causal Effects in Macroeconomics Using External Instruments*. Working Paper, Working Paper Series 24216. National Bureau of Economic Research, January.

## Appendix: Structural Estimation

We use a canonical DSGE model to explore the case of an efficient steady state of the monetary policy problem. When nominal rigidities coexist with real imperfections, the flexible price equilibrium allocation is generally inefficient. In that case, it is no longer optimal for the central bank to seek to replicate that allocation. On the other hand, any deviation of economic activity from its natural (i.e., flexible price) level generates variations in inflation, with consequent relative price distortions.

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha_x x_t^2) \right\} \quad (15)$$

where  $x_t \equiv y_t - y_t^e$  denotes the welfare-relevant output gap, i.e., the deviation between (log) output  $y_t$  and its efficient level  $y_t^e$ . As before,  $\pi_t \equiv p_t - p_{t-1}$  denotes the rate of inflation between periods  $t-1$  and  $t$ . Coefficient  $\alpha_x$  represents the weight of output gap fluctuations (relative to inflation) in the loss function, and is given by  $\alpha_x = \frac{\kappa}{\varepsilon}$  where  $\kappa$  is the coefficient on  $x_t$  in the New Keynesian Phillips curve, and  $\varepsilon$  is the elasticity of substitution between goods. More generally, and stepping beyond the welfare-theoretic justification for (15), one can interpret  $\alpha_x$  as the weight attached by the central bank to deviations of output from its efficient level (relative to price stability) in its own loss function, which does not necessarily have to coincide with the household's. Gap  $\tilde{y}_t$  in the NKPC relationship. This yields the following structural equation for inflation

$$\pi_t = \beta E_t \{ \pi_{t+1} \} + \kappa x_t + u_t \quad (16)$$

where  $u_t \equiv \kappa (y_t^e - y_t^n)$ . Hence the central bank will seek to minimize target (15) with constraint (16). While (16) is the only constraint needed in order to determine the equilibrium path for output and inflation under the optimal policy, implementation of that policy requires the use of an additional condition linking those variables with the monetary policy instrument, i.e., the interest rate.

$$x_t = -\frac{1}{\sigma} (i_t - E_t \{ \pi_{t+1} \} - r_t^e) + E_t \{ x_{t+1} \} \quad (17)$$

After having analyzed the optimal policy under discretion, next is the case of a central bank that is assumed to be able to commit, with full credibility, to a policy plan. In the context of the model, such a plan consists of a specification of the desired levels of inflation and the output gap at all possible dates and states of nature, current and future. More specifically, the monetary authority is assumed to choose a state-contingent sequence  $\{x_t, \pi_t\}_{t=0}^{\infty}$  that minimizes

$$\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \alpha_x x_t^2) \quad (18)$$

subject to the sequence of constraints

$$\pi_t = \beta E_t \{ \pi_{t+1} \} + \kappa x_t + u_t \quad (19)$$

In order to solve the previous problem it is useful to write down the associated Lagrangian, which is given by

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{1}{2} (\pi_t^2 + \alpha_x x_t^2) + \gamma_t (\pi_t - \kappa x_t - \beta \pi_{t+1}) \right] \quad (20)$$

Combining the two optimality conditions to eliminate the Lagrange multipliers yields

$$x_0 = -\frac{\kappa}{\alpha_x} \pi_0 \quad (21)$$

and

$$x_t = x_{t-1} - \frac{\kappa}{\alpha_x} \pi_t \quad (22)$$

for  $t = 1, 2, 3, \dots$ . Consider the rule

$$i_t = r_t^e + \phi_\pi \pi_t \quad (23)$$

where  $\phi_\pi \equiv (1 - \rho_u) \frac{\kappa \sigma}{\alpha_x} + \rho_u$ .

As aforementioned, the model can be defined as follows:

1. Definition efficient interest rate

$$r_t^e = \sigma (y_{t-1}^e - y_t^e)$$

2. Definition efficient output

$$y_t^{nat} = a \psi_t^{n,ya}$$

3. Definition linking various output gaps

$$\tilde{y}_t = x_t + (y_t^e - y_t^{nat})$$

4. New Keynesian Phillips Curve

$$\pi_t = \beta \pi_{t-1} + \kappa x_t + u_t$$

5. Dynamic IS Curve

$$x_t = -\frac{1}{\sigma (i_t - \pi_{t-1} - r_t^e)} + x_{t-1}$$

6. Definition natural rate of interest

$$r_t^{nat} = \sigma \psi_t^{n,ya} (a_{t-1} - a_t)$$

7. Definition real interest rate

$$r_t^r = i_t - \pi_{t-1}$$

8. Natural output

$$y_t^{nat} = \phi a_t$$

9. Definition output gap

$$\tilde{y} = y_t - y_t^{nat}$$

10. cost push shock

$$u_t = \rho_u u_{t-1} + \varepsilon_{u,t}$$

11. TFP shock

$$a_t = \rho_a a_{t-1} + \varepsilon_{a,t}$$

12. Production function

$$y_t = a_t + (1 - \alpha) * n_t$$

13. Money growth

$$\Delta m = 4(y_t - y_{t-1} - \eta(i_t - i_{t-1}) + \pi_t)$$

14. Annualized nominal interest rate

$$i_t^{ann} = 4i_t$$

15. Annualized real interest rate

$$r_t^{r,ann} = 4r_t^r$$

16. Annualized natural interest rate

$$r_t^{nat,ann} = 4r_t^{nat}$$

17. Annualized inflation

$$\pi_t^{ann} = 4\pi_t$$

18. Definition price level

$$\pi_t = p_t - p_{t-1}$$

19. Interest Rate Rule that implements optimal solution

$$i_t = r_t^e + \phi_\pi \pi_t$$

**Table 2: Summary Statistics**

| <i>Endogenous Variable</i>  |   |                 |   |
|-----------------------------|---|-----------------|---|
| Symbol                      | Meaning   | Symbol          | Meaning   |
| $\pi$                       | inflation   | $\tilde{y}$     | output gap  |
| $y^{nat}$                   | natural output                                      | $y$             | output  |
| $r^e$                       | efficient interest rate                             | $y^e$           | efficient output                                  |
| $x$                         | welfare-relevant output gap                         | $r^{nat}$       | natural interest rate                             |
| $r^r$                       | real interest rate                                  | $i$             | nominal interest rate                             |
| $n$                         | hours worked  | $\Delta m$      | money growth                                      |
| $u$                         | AR(1) cost push shock process                       | $a$             | AR(1) technology shock process                    |
| $r^{r,ann}$                 | annualized real interest rate                       | $i^{ann}$       | annualized nominal interest rate                  |
| $r^{nat,ann}$               | annualized natural interest rate                    | $\pi^{ann}$     | annualized inflation rate                         |
| $p$                         | price level   |                 |   |
| <i>Exogenous Shocks</i>     |   |                 |   |
| $\varepsilon_a$             | technology shock                                    | $\varepsilon_u$ | monetary policy shock                             |
| <i>Parameter Settings</i>   |   |                 |   |
| $\beta = 0.99$              | discount factor                                     | $\rho_a = 0.9$  | autocorrelation technology shock                  |
| $\rho_u = 0$                | autocorrelation cost push shock                     | $\sigma = 1$    | log utility                                       |
| $\phi = 1$                  | unitary Frisch elasticity                           | $\phi_y = .5/4$ | output feedback Taylor Rule                       |
| $\eta = 4$                  | semi-elasticity of money demand                     | $\epsilon = 6$  | demand elasticity                                 |
| $\theta = 2/3$              | Calvo parameter                                     |                 |   |
| <i>Composite Parameters</i> |   |                 |   |
| $\Omega$                    | $(1 - \alpha)/(1 - \alpha + \alpha\epsilon)$        | $\psi_{n,ya}$   | $(1 + \phi)/(\sigma(1 - \alpha) + \phi + \alpha)$ |
| $\lambda$                   | $(1 - \theta) \cdot (1 - \beta\theta)/\theta\Omega$ | $\kappa$        | $\lambda(\sigma + (\phi + \alpha)/(1 - \alpha))$  |
| $\alpha_x$                  | $\kappa/\epsilon$                                   | $\phi_\pi$      | $\kappa\sigma(1 - \rho_u)/\alpha_x + \rho_u$      |