[Lec 7]

Green's Functions

For a linear system (linear filter), the impulse response function is called the Green's function.

We saw 1-d formal examples for time-dependent systems in Lec 2. In general, Green's functions can be defined for both space & time, in arbitrary dimensions & need not be restricted to scalar functions.

E.g.: Consider the Maxwell equation for the divergence of a static electric field in 3D

$$\nabla \cdot \vec{E} = \underbrace{\rho(\vec{r})}_{\text{charge density}}/\varepsilon_0$$

It is not difficult to see that the Green's function [impulse response] for this linear system is nothing but the electric field of a unit point charge at $\vec{r} = \vec{r}'$

$$\vec{G}(\vec{r}, \vec{r}') = \vec{E}_{(\text{point charge})}/q = \frac{1}{4\pi\varepsilon_0} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

which then says that

$$\vec{E}(\vec{r}) = \int d^3r' \, \rho(\vec{r}') \, \vec{G}(\vec{r}, \vec{r}')$$

$$\left[= (\vec{G} * \rho)(\vec{r}) \text{ since } \vec{G} = \vec{G}(\vec{r} - \vec{r}') \right]$$

Physically, this can be seen by ~~to~~ using the superposition principle to write the electric field due to a collection of point charges as

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$$

& then taking $q_i \to \rho(\vec{r}_i)\,\delta V_i$ & the continuum limit.

Mathematically, this follows if we can show that

$$\nabla\cdot\left(\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right) = 4\pi\,\delta_D(\vec{r}-\vec{r}\,')$$

which follows from Gauss' theorem.

- The Green's function in this example is therefore a function of space not time, in 3D not 1D, and is a vector field not a scalar.

- Notice that the translation invariance of the filter $(\nabla\cdot)$ means that $\vec{G}(\vec{r},\vec{r}\,') = \vec{G}(\vec{r}-\vec{r}\,')$

Having said this, we will focus on scalar functions of time $f(t)$ for now, & return to space & space+time dependence later.

Forced-damped harmonic oscillator [FDHO]

We'll use this example to illustrate several issues regarding boundary conditions & causality.

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t)/m$$

Boundary conditions:

Consider a generic linear filter (which is translation-invariant, although this is not necessary)

$$L\,u = F$$

The Green's function gives us a particular solution

$$u_p = G * F$$

To this we can can add any solution u_H of the homogeneous problem, so that

$$L\,u_H = 0$$

& get another solution u_G of $Lu = F$ where

$$u_G = u_H + u_p$$

→

Which homogeneous solution to choose (it could be $u_H = 0$) depends on

(a) What are the boundary conditions that u_G must satisfy and

(b) what b.c.'s are satisfied by the Green's function.

- In particular, it ~~was~~ is not possible in general to construct G such that we can set $u_H = 0$ & still satisfy (a).

This is only possible for "homogeneous" boundary conditions

$$a u + b \frac{\partial u}{\partial n} = 0$$

where $\frac{\partial u}{\partial n}$ is the derivative normal (perpendicular) to the boundary

In this situation, there is no external scale in the problem & it is worth calculating G such that $a G + b \frac{\partial G}{\partial n} = 0$

In all other cases, G can be calculated in the simplest possible way & then u_H can be chosen so that

$$u_H|_{bdry} = \underbrace{u_G|_{bdry}}_{\text{specified}} - G * F|_{bdry}$$

This was very abstract. Let's look at a concrete example with homogeneous b.c.'s.

Consider the FDHO with zero damping

$$\ddot{x} + \omega_0^2 x = F/m$$

- We want a solution that is "causal", i.e. the effect of $F(t=\tau)$ must only be felt at $t>\tau$.

The Green's function therefore satisfies

$$\ddot{G}(t,\tau) + \omega_0^2 G(t,\tau) = \frac{1}{m}\delta_D(t-\tau)$$

with the condition that $G(t,\tau) = 0$ for $t<\tau$

At $t>\tau$, this means

$$G(t,\tau) = A\cos(\omega_0(t-\tau)) + B\sin(\omega_0(t-\tau)) \quad , t>\tau.$$

Let us impose continuity of displacement at $t=\tau$ [else velocity is ∞]

$$\Rightarrow \quad A=0$$

To fix B, we need to know the velocity kick that is imparted by the impulse at $t=\tau$. This follows from integrating the diff. eqn between $\tau-\varepsilon$ & $\tau+\varepsilon$ & sending $\varepsilon\to 0$.

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} dt' \left[\ddot{G}(t';\tau) + \omega_0^2 G(t';\tau) \right] = \frac{1}{m}$$

$$\Rightarrow \quad \dot{G}\Big|_{\tau-\varepsilon}^{\tau+\varepsilon} \equiv [\dot{G}] = \frac{1}{m} .$$

$$\Rightarrow \quad B = 1/m\omega_0$$

$$\therefore \quad G(t,\tau) = \begin{cases} 0, & t < \tau \\ \dfrac{\sin(\omega_0(t-\tau))}{m\omega_0}, & t > \tau \end{cases}$$

$\rightarrow$ will lead to a causal solution $x(t)$

Let's now approach the same problem using Fourier analysis. For convenience let's switch on a non-zero damping term which will help clarify some conceptual points.

To be specific, we will assume the underdamped case $\omega_0^2 > \gamma^2$. A similar analysis can be done for the overdamped case.

The Green's function now satisfies

$$\ddot{G}(t,\tau) + 2\gamma \dot{G}(t,\tau) + \omega_0^2 G(t,\tau) = \frac{1}{m}\delta_D(t-\tau)$$

In Fourier domain,

$$G(t,\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{G}(\omega,\tau)$$

$$\delta_D(t-\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} e^{+i\omega\tau}$$

So that

$$m(\omega_0^2 - \omega^2 - 2i\omega\gamma)\,\tilde{G}(\omega,\tau) = e^{i\omega\tau}$$

or $$m\tilde{G}(\omega,\tau) = -\frac{e^{i\omega\tau}}{\omega^2 + 2i\omega\gamma - \omega_0^2}$$

which gives us

$$mG(t,\tau) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-\tau)}}{\omega^2 + 2i\omega\gamma - \omega_0^2} = mG(t-\tau)$$

Let $s = t-\tau$

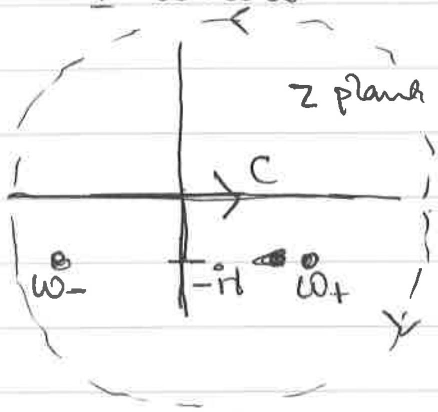
$$\therefore mG(s) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega s}}{\omega^2 + 2i\gamma\omega - \omega_0^2}$$

Consider the complex integral

$$\oint_C dz \frac{e^{-izs}}{z^2+2i\gamma z-\omega_0^2}$$

The integrand has poles at $z = \omega_\pm$ where

$$\omega_\pm = -i\gamma \pm \sqrt{\omega_0^2-\gamma^2}$$



So that both poles are in the lower half plane (LHP)

~~For $s < 0$, we~~

Consider the contour C which follows the real line & closes as a semi-circle either in the UHP or LHP depending on the value of s.

For $s < 0$, $e^{-izs} = e^{+iz|s|} \rightarrow e^{-\lambda|s|}$ when $z = i\lambda$

& hence it makes sense to close in the UHP where $\lambda > 0$.

Similarly for $s > 0$ we must close in the LHP.

Now notice that there are no poles in the UHP.
So for $s < 0$

$$0 = \oint_C dz \, \frac{e^{-izs}}{z^2+2i\gamma z - \omega_0^2} = \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega s}}{\omega^2+2i\gamma\omega-\omega_0^2} + \underbrace{\int_{\substack{\text{semicircle} \\ \text{at } \infty}}}_{0}$$

i.e $\quad G(t-\tau) = 0$ for $t < \tau$

In other words, analyticity [absence of poles] in the UHP guarantees causality.

For $s > 0$, the contour is closed in the LHP & the integral gets contributions from both poles:

$$\oint_C dz \, \frac{e^{-izs}}{z^2+2i\gamma z-\omega_0^2} = \underset{\substack{\downarrow \\ \text{since} \\ \text{clockwise}}}{-} 2\pi i \left[\frac{e^{-is\omega_+}}{\omega_+ - \omega_-} + \frac{e^{-is\omega_-}}{\omega_- - \omega_+} \right]$$

$$= \frac{-2\pi i}{2\sqrt{\omega_0^2-\gamma^2}} \left[e^{-\gamma s} e^{-is\sqrt{\omega_0^2-\gamma^2}} - e^{-\gamma s} e^{+is\sqrt{\omega_0^2-\gamma^2}} \right]$$

$$= -\frac{2\pi e^{-\gamma s}}{\sqrt{\omega_0^2-\gamma^2}} \sin\left[s\sqrt{\omega_0^2-\gamma^2} \right]$$

$\longrightarrow$

So we have proved

$$\mathcal{G}(t,\tau) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega(t-\tau)}}{(\omega^2+2i\gamma\omega-\omega_0^2)} \cdot \frac{1}{m}$$

$$\cancel{= \Theta(t-\tau) \cdot e^{-\gamma(t}}$$

$$= \Theta(t-\tau) \frac{e^{-\gamma(t-\tau)}}{m\sqrt{\omega_0^2-\gamma^2}} \sin\left[\sqrt{\omega_0^2-\gamma^2}\,(t-\tau)\right]$$

For $\gamma \to 0$ this gives back $\Theta(t-\tau)\frac{\sin(\omega_0(t-\tau))}{m\omega_0}$ as before.

Note, however, that had we started with $\gamma = 0$, we would have an integral of the form $\oint dz \frac{e^{-izs}}{z^2-\omega_0^2}$ which has poles on the real axis.

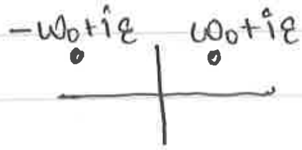
The effect of these poles depends on how we choose to cancel singularities & hence causality is not guaranteed in this case.

We can construct a causal Green's function by choosing to displace both poles into the LHP $[\pm\omega_0 \to \pm\omega_0 - i\varepsilon,\ \varepsilon > 0]$ & send $\varepsilon \to 0$ at the end.

This would give back our causal Green's function.

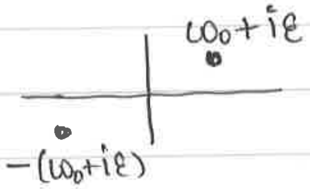
But we have other choices now:

E.g. shifting both poles into the UHP would give zero when $t > \tau$, which is acausal (but sometimes useful).



This gives the 'advanced' Green's function

There are two other choices as well



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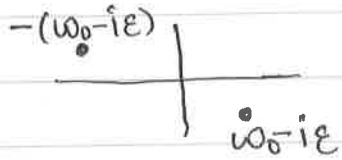


which turn out to be useful when studying propagating waves.

These ideas of analyticity in the complex plane become very important when studying causality in the context of QFT.

[Note that, as soon as we have singularities on the contours, the integral is not defined. Taking the principal value or shifting the poles gives ~~various~~ a variety of answers because these are all different techniques to regulate the divergence.]

[Lec 8]

Some further aspects of causal Green's functions

[Kramers-Kronig relations; low pass filtering]

As we saw, not all Green's functions are causal. Here we explore what constraints causality places on the Green's function

Consider a causal Green's function $\chi(t-\tau)$. This means $\chi(t-\tau)=0$ for $t<\tau$.

It's Fourier transform satisfies [Assuming $\tau=0$]

$$\chi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\chi}(\omega)$$

$\tilde{\chi}(\omega)$ is also called the "generalised susceptibility"

~~Since~~ From our previous discussion we know that $\chi(t)=0$ for $t<0$ implies that $\tilde{\chi}(\omega)$ must be analytic in the UHP.

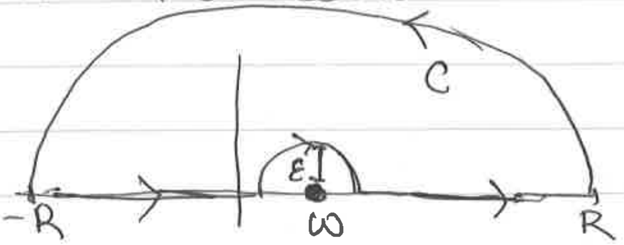
Now consider the complex function

$$\hat{f}(\omega') = \frac{\hat{\chi}(\omega')}{\omega'-\omega}$$

where ω' is a complex variable and ω is some fixed real number

This means $\tilde{f}(\omega')$ has a pole on the real axis at $\omega' = \omega$.

Consider the integral $\oint_C d\omega' \tilde{f}(\omega')$ with the contours



where the radius R of the semicircle will be sent to ∞ & the radius ε of the semicircle around ω will be sent to zero.

Clearly $\oint_C d\omega' \tilde{f}(\omega') = \oint_C d\omega' \dfrac{\tilde{\chi}(\omega')}{\omega' - \omega} = 0$

since C does not enclose any poles.

So we have

$$0 = \left(\int_{-R}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{R} + \int_{\substack{\text{semicircle} \\ \text{at } \omega \\ \text{[clockwise]}}} + \int_{\substack{\text{semicircle} \\ \text{of radius} \\ R \\ \text{[counterclockwise]}}} \right) d\omega' \frac{\tilde{\chi}(\omega')}{\omega' - \omega}$$

$$= \underbrace{P}_{\text{Principal value}} \int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}(\omega')}{\omega' - \omega} + \int_{\pi}^{0} i\varepsilon\, d\theta\, e^{i\theta} \frac{\tilde{\chi}(\omega + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}$$

$[\omega' = \omega + \varepsilon e^{i\theta}]$

$$= P \int_{-\infty}^{-\infty} d\omega' \frac{\tilde{\chi}(\omega')}{\omega' - \omega} + i \int_{\pi}^{0} d\theta\, \tilde{\chi}(\omega)$$

[we sent $\varepsilon \to 0$]

i.e. $$\boxed{P \int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}(\omega')}{\omega' - \omega} = i\pi \tilde{\chi}(\omega)} \longrightarrow$$

Or

$$\tilde{\chi}(\omega) = \frac{-i}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}(\omega')}{\omega'-\omega}$$

Writing $\tilde{\chi}(\omega) = \mathrm{Re}\,\tilde{\chi}(\omega) + i\,\mathrm{Im}\,\tilde{\chi}(\omega)$ we can easily show

$$\boxed{\begin{aligned} \mathrm{Re}\,\tilde{\chi}(\omega) &= \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\,\tilde{\chi}(\omega')}{\omega'-\omega} \\ \mathrm{Im}\,\tilde{\chi}(\omega) &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Re}\,\tilde{\chi}(\omega')}{\omega'-\omega} \end{aligned}}$$

↳ Kramers-Kronig relations or "dispersion" relations

[satisfied by any function that is analytic in the UHP.]

- Integrals of this type are also called Hilbert transforms

For the example of the damped Harmonic oscillator, we had the causal Green's function

$$\tilde{\chi}(\omega) = \frac{-1}{\omega^2-\omega_0^2+2i\gamma\omega}$$

$$= \frac{\omega_0^2-\omega^2+2i\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$$

On the real axis we have

$$\mathrm{Re}\,\tilde{\chi}(\omega) = \frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$

$$\mathrm{Im}\,\tilde{\chi}(\omega) = \frac{2\gamma\omega}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$

You should verify that the Kramers-Kronig relations are indeed satisfied.
Note that this will not be the case if $\gamma = 0$ [i.e if $\mathrm{Im}\,\tilde{\chi}(\omega) = 0$]

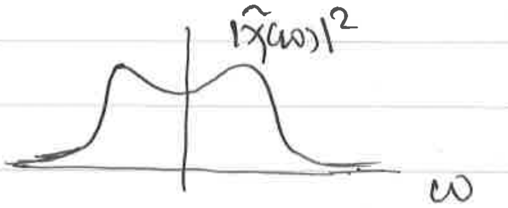
This is consistent with the fact that a nonzero γ introduces an arrow of time in the differential equation. [Note that $\mathrm{Im}\,\tilde{\chi}(\omega)$ is essentially the Fourier transform of the odd part of $\chi(t)$, which knows about the arrow of time.]

Finally, it is interesting to plot the power spectrum $|\tilde{\chi}(\omega)|^2$ for $\gamma \neq 0$

$$|\tilde{\chi}(\omega)|^2 = \frac{1}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \quad \text{with real } \omega.$$

For $|\omega| \to \infty$, $|\tilde{\chi}(\omega)|^2 \sim \frac{1}{\omega^4} \longrightarrow 0$

It turns out that this looks like



→ which is a passable low-pass filter.

This is interesting because the "ideal" low pass filter $\Theta(\omega_c - |\omega|)$ faces two problems:

(a) In time-domain it requires long time integrations in general due to the slow fall-off of $\mathrm{sinc}(\omega_0 s)$ $[s = t - \tau]$

(b) As a Green's function this filter is not causal $[\mathrm{sinc}(\omega_0 s) \neq 0$ for $s < 0]$

→ So the fact that $\tilde{\chi}(\omega)$ has an approximately finite band-width is interesting.

– Recall that LCR circuits give us a realisation of a damped harmonic oscillator.
→ This offers a practical way of building causal low-pass filters.

Diffusion equation

This is another, slightly different application of the Green's function.

Consider the equation

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

where $u(t,\vec{x})$ is defined in D-dimensional space.

We would like to solve this equation for $t > 0$, with the initial condition

$$u(t=0,\vec{x}) = u_0(\vec{x})$$

- This equation is called the diffusion eqn in random walk problems & the heat equation in thermodynamics. It also appears in magnetohydrodynamics (MHD) when studying magnetic fields in a plasma.
- The equation also allows for source/sink terms that can appear on the right handside.
- The constant $\alpha > 0$ is called the diffusion constant, or the heat conductivity
- Notice that the equation is not invariant under $t \to -t$. [So we might expect a causal Green's function through contour integrn]

Let us solve this using the Green's function approach.

The Green's function satisfies

$$\frac{\partial}{\partial t} G(t,\vec{x}) - \alpha \nabla^2 G(t,\vec{x}) = \delta_D(t)\,\delta_D(\vec{x})$$

[For convenience we have taken the impulse to occur at the origin of both time & space. It's straightfwd to relax this.]

Fourier transforming from $\vec{x} \to \vec{k}$ we get

$$\frac{\partial}{\partial t} \tilde{G}(t,\vec{k}) + \alpha k^2 \tilde{G}(t,\vec{k}) = \delta_D(t)$$

where $G(t,\vec{x}) = \int \frac{d^D k}{(2\pi)^D} e^{i\vec{k}\cdot\vec{x}}\, \tilde{G}(t,\vec{k})$
& $k^2 = \sum_{j=1}^{D} k_j^2$

Fourier transforming in time we get

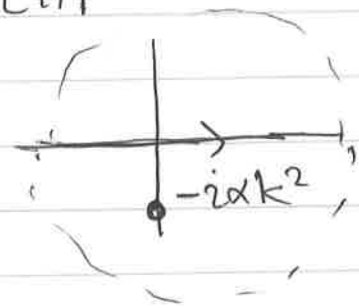
$$-i\omega \tilde{G}(\omega,\vec{k}) + \alpha k^2 \tilde{G}(\omega,\vec{k}) = 1$$

where $\tilde{G}(t,\vec{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\, \tilde{G}(t,\vec{k})$

$$\Rightarrow \quad \tilde{G}(\omega,\vec{k}) = \frac{1}{-i\omega + \alpha k^2} = \frac{i}{\omega + i\alpha k^2}$$

$\therefore \quad \tilde{G}(t,\vec{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i e^{-i\omega t}}{\omega + i\alpha k^2} = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega + i\alpha k^2}$

Since $\alpha > 0$ & $k^2 > 0$, the integrand has a single, simple pole at $\omega = -i\alpha k^2$ in the LHP



As before, for $t<0$ we must close in the UHP where the integrand is analytic.

So $\tilde{G}(t,\vec{k}) = 0$ for $t<0$.

For $t>0$ we close in the LHP and get:

$$\underset{[t>0]}{\tilde{G}(t,\vec{k})} = \frac{i}{2\pi} \oint_{\substack{\text{closed} \\ \text{in LHP}}} d\omega \frac{e^{-i\omega t}}{\omega + i\alpha k^2} = \frac{i}{2\pi}(-2\pi i) \times \underset{\omega = -i\alpha k^2}{\text{residue at}}$$

$$= e^{-it(-i\alpha k^2)}$$
$$= e^{-\alpha t k^2}$$

$\therefore \quad \tilde{G}(t,\vec{k}) = \Theta(t)\, e^{-\alpha t k^2}$ is the causal Green's function in $\vec{k}$-space [Note rotational invariance]

One more Fourier transform will give us $\tilde{G}(t,\vec{x})$ $\longrightarrow$

$$G(t,\vec{x}) = \int \frac{d^D k}{(2\pi)^D} e^{i\vec{k}\cdot\vec{x}} \tilde{G}(t,\vec{k})$$

$$= \int \frac{d^D k}{(2\pi)^D} e^{i\vec{k}\cdot\vec{x}} \Theta(t) e^{-\alpha t k^2} \qquad [k^2 = \sum_j k_j^2]$$

$$= \Theta(t) \prod_{j=1}^{D} \int_{-\infty}^{\infty} \frac{dk_j}{2\pi} e^{ik_j x_j} e^{-\alpha t k_j^2}$$

Each integral in the product is of the form

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-k^2\sigma^2/2} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

with $\sigma^2 = 2\alpha t$

So

$$\boxed{G(t,\vec{x}) = \Theta(t) \frac{1}{(4\pi\alpha t)^{D/2}} e^{-r^2/4\alpha t}}$$

$\boxed{\text{Causal Green's function of diffusion eqn}}$ where $r^2 = \sum_{j=1}^{D} x_j^2$

This is a spherically symmetric D-dimensional Gaussian with a time-dependent width $\sigma = \sqrt{2\alpha t}$

[The $\sqrt{t}$ dependence of the standard deviation should be familiar from the problem of Brownian motion.]

Recall we want to solve $\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$ for $t > 0$
with $u(t=0, \vec{x}) = u_0(\vec{x})$

Claim: We can use the Green's function to write

$$u(t, \vec{x}) = (G * u_0)(t, \vec{x}) = \int d^D x' \, G(t, \vec{x} - \vec{x}') u_0(\vec{x}')$$

i.e., a spatial convolution of G & u_0.
[This clarifies why G is also called the propagator.]

The easiest way to verify this is by directly checking whether the equation & initial conditions are satisfied.

We have

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = \int d^D x' \left[\frac{\partial}{\partial t} G(t, \vec{x} - \vec{x}') - \alpha \nabla^2 G(t, \vec{x} - \vec{x}') \right] \times u_0(\vec{x}')$$

$$= \int d^D x' \, \delta_D(t) \, \delta_D(\vec{x} - \vec{x}') \, u_0(\vec{x}')$$

$$= \delta_D(t) \, u_0(\vec{x})$$

$$= 0 \text{ for } t > 0$$

$\longrightarrow$ So the eq^n is satisfied

Also $\lim_{t \to 0^+} G(t, \vec{x} - \vec{x}') = \delta_D(\vec{x} - \vec{x}')$ [Gaussian with width $\sigma \to 0$]

So $\lim_{t \to 0^+} u(t, \vec{x}) = \int d^D x' \, \delta_D(\vec{x} - \vec{x}') \, u_0(\vec{x}')$

$$= u_0(\vec{x}) \text{ as required.}$$

[Lec 9]

Helmholtz equation & Wave equation

Start with the wave equation

$$-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \nabla^2 u = S(t,\vec{x})$$ with some source S.

We'd like to solve this using the Green's function approach. Along the way we'll also see solutions of the Helmholtz eq^n.

The Green's function satisfies

$$-\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} + \nabla^2 G = \delta_D(t)\,\delta_D(\vec{x})$$ [Assuming a pulse at the origin]

Fourier transforming in time gives

$$+\frac{\omega^2}{c^2}\tilde{G}(\omega,\vec{x}) + \nabla^2\tilde{G}(\omega,\vec{x}) = \delta_D(\vec{x})$$

where $G(t,\vec{x}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\,\tilde{G}(\omega,\vec{x})$

Define the constant (wrt $\vec{x}$)
$k \equiv \omega/c$.

$$\therefore \quad \nabla^2\tilde{G}(\omega,\vec{x}) + k^2\tilde{G}(\omega,\vec{x}) = \delta_D(\vec{x})$$

$\nabla^2 u + k^2 u = F(\vec{x})$, with $u = u(\vec{x})$ is the Helmholtz eq^n. So $\tilde{G}(\omega,\vec{x})$ is the Green's function for the Helmholtz eq^n, labelled by $\omega = ck$.

Fourier transforming in $\vec{x}$ [with conjugate variable $\vec{q}$] gives

$$(k^2 - q^2)\,\tilde{G}(\omega,\vec{q}) = 1 \quad \text{where} \quad \tilde{G}(\omega,\vec{x}) = \int \frac{d^D q}{(2\pi)^D}\, e^{i\vec{q}\cdot\vec{x}}\, \tilde{G}(\omega,\vec{q})$$

$$\& \quad q^2 = \sum_{j=1}^{D} q_j^2$$

i.e $\tilde{G}(\omega,\vec{q}) = \dfrac{-1}{q^2 - k^2}$

So that

$$\boxed{\tilde{G}(\omega,\vec{x}) = -\int \frac{d^D q}{(2\pi)^D}\, \frac{e^{i\vec{q}\cdot\vec{x}}}{q^2 - k^2}} \longrightarrow$$

Green's function for Helmholtz equation.

The solution depends on the value of D, and on the chosen boundary conditions.

Knowing this solⁿ, the Green's function of the wave eqn is then

$$\boxed{G(t,\vec{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \tilde{G}(\omega,\vec{x})}$$

which may also require specifying a contour [i.e boundary conditions]

Unlike the heat equation, it now turns out to be easier to handle $D=1$ & $D=3$ than $D=2$. So let's do $D=1, 3$ & 2 in that order.

$D=1$

$$\tilde{G}(\omega, x) = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqx}}{q^2-k^2}$$

The integrand has poles on the real axis at $q = \pm|k|$ [Assume $k>0$ for now]

Notice that no matter how we ~~close~~ choose the contours, the pole at $q=+|k|$ will contribute a residue $\sim e^{+ikx}$ & similarly $q=-|k|$ will give $\sim e^{-ikx}$.

When combining with the time-domain Fourier transform, these will respectively give $e^{i(kx-\omega t)}$ & $e^{-i(kx+\omega t)}$

Constant for right-moving wave $x = ct + \text{const}$

Constant for left moving wave $x = -ct + \text{const}$

[Recall $\omega = ck$]

So suppose we want to describe a wave that is moving away from the origin. Then clearly we would like to pick the residue at $q=+k$ for $x>0$ & at $q=-k$ for $x<0$.

Recall that for $x>0$ we must close in the UHP & for $x<0$ in the LHP.

This means we will pick the correct solution if we let $k \to k+i\varepsilon$ $[\varepsilon>0]$ & send $\varepsilon \to 0$ at the end. This will give us

Closing in the UHP [for $x>0$] will pick the $q=k+i\varepsilon$ pole & closing in the LHP [for $x<0$] the $q=-k-i\varepsilon$ pole, as required.

So we find

$$\tilde{G}(\omega,x) = -\frac{1}{2\pi}\cdot 2\pi i \begin{cases} \dfrac{e^{i(k+i\varepsilon)x}}{2(k+i\varepsilon)}, & x>0 \\ \underset{\text{Clockwise}}{-}\dfrac{e^{-i(k+i\varepsilon)x}}{(-2(k+i\varepsilon))}, & x<0 \end{cases}$$

$$\xrightarrow[\varepsilon\to 0]{} -\frac{i}{2k}e^{ik|x|}$$

[When $k<0$, same form can be obtained by using $k\to k-i\varepsilon$, $\varepsilon>0$.]

$$\therefore \tilde{G}_{1D}(\omega,x) = -(i/2k)\,e^{ik|x|}$$

And hence $$G(t,x) = \frac{c}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikct}\,e^{ik|x|}\left(-\frac{i}{2k}\right)$$

$$= -\frac{ic}{4\pi}\int_{-\infty}^{\infty}\frac{dk}{k}\,e^{ik(|x|-ct)}$$

$$\cancel{= -\frac{ic}{4\pi}\cdot i\pi\,\mathrm{sgn}(|x|-ct)}$$

$$\therefore \cancel{G_{1D}(t,x) = \frac{c}{4}\,\mathrm{sgn}(|x|-ct)}$$

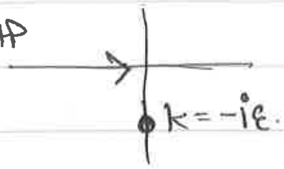
We'd like to impose another boundary condition, that $G(t,x) \to 0$ for $t<0$. ~~as $t \to \infty$~~. This can be done by shifting the pole at $k=0$ to the LHP by setting $k \to k+i\varepsilon$, $\varepsilon > 0$.

So $$G(t,x) = -\frac{ic}{4\pi}\int_{-\infty}^{\infty} \frac{dk}{(k+i\varepsilon)}\, e^{ik(|x|-ct)}\, e^{-\varepsilon(|x|-ct)}$$

$\mapsto$ close in UHP

$$= -\frac{ic}{4\pi}\begin{cases} 0\,, & |x|>ct \\ -2\pi i\,, & |x|<ct \end{cases}$$

$\hookrightarrow$ close in LHP



$$= -\frac{c}{2}\,\Theta(t-|x|/c)$$

i.e, the Green's function is non-zero only inside the expanding wavefront $|x| = ct$, as befits a causal response. Interestingly the solution is a non-zero constant inside the wavefront.

Let's do $D=3$ next.

$$\tilde{G}(\omega,\vec{x}) = -\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{q}\cdot\vec{x}}}{q^2-k^2}$$ [change to polar variables]

$$= -\frac{1}{4\pi^2}\int_0^\infty dq \frac{q^2}{q^2-k^2}\int_{-1}^{1} d\mu\, e^{iqr\mu}$$

where $r=|\vec{x}|$ & we chose the direction of $\vec{x}$ as the q_z-axis

$$= -\frac{1}{4\pi^2}\int_0^\infty dq \frac{q^2}{q^2-k^2}\cdot\frac{(e^{iqr}-e^{-iqr})}{iqr}$$

$$= -\frac{1}{8\pi^2}\int_{-\infty}^{\infty} \frac{dq\cdot q^2}{(q^2-k^2)}\left(\frac{e^{iqr}-e^{-iqr}}{iqr}\right)$$

[since integrand is even]

$$= -\frac{1}{8\pi^2}\int_{-\infty}^{\infty} \frac{dq\cdot q}{(q^2-k^2)}\frac{(e^{iqr}-e^{-iqr})}{(ir)}$$

~~As before,~~ $$= -\frac{1}{8\pi^2 ir}\left[\int_{-\infty}^{\infty}\frac{dq\, q\, e^{iqr}}{(q^2-k^2)} - \int_{-\infty}^{\infty}\frac{dq\, q\, e^{-iqr}}{(q^2-k^2)}\right]$$

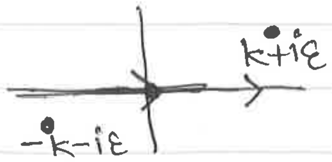
Very similar to 1-D case, except that we have two integrals & there's an extra q in the integrand numerator. Poles are still at ~~$q=\pm k$~~ $q=\pm k$. As before, we want to pick out e^{+ikr} for $k>0$, which can again be done by sending ~~$k\to k+\epsilon$~~ $k\to k+i\epsilon$, $\epsilon>0$ in both integrals

[And similarly $k\to k-i\epsilon$ if $k<0$]

$\longrightarrow$

Assume $k > 0$

First integral gives



$$\int_{-\infty}^{\infty} \frac{dq \cdot q \exp(iqr)}{(q-k-i\varepsilon)(q+k+i\varepsilon)}$$

$$= 2\pi i \times \mathrm{Res}(q = k+i\varepsilon) \rightarrow \text{[closed in UHP]}$$

$$= 2\pi i \cdot \frac{(k+i\varepsilon)}{2(k+i\varepsilon)} e^{i(k+i\varepsilon)r}$$

$$= \pi i e^{ikr} \quad [\varepsilon \to 0]$$

& Second gives

$$-\int_{-\infty}^{\infty} \frac{dq \cdot q \cdot \exp(-iqr)}{(q-k-i\varepsilon)(q+k+i\varepsilon)}$$

$$= +2\pi i \times \mathrm{Res}(q = -k-i\varepsilon)$$

Clockwise

$$= 2\pi i \frac{(-k-i\varepsilon)}{2(-k-i\varepsilon)} e^{-i(-k-i\varepsilon)r}$$

$$= +\pi i e^{ikr} \quad [\varepsilon \to 0]$$

[Same form is obtained for $k<0$ if we send $k \to k - i\varepsilon, \varepsilon > 0$]

So
$$\boxed{\tilde{G}(\omega, \vec{x}) = -\frac{1}{4\pi r} e^{ikr}} \rightarrow \text{3D}$$

& hence
$$G(t, \vec{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left(-\frac{1}{4\pi r}\right) e^{i\omega r/c}$$

$$= \boxed{G(t, \vec{x}) = -\frac{1}{4\pi r} \delta_D(t - r/c)} \rightarrow \text{retarded propagator}$$

$\hookrightarrow$ expanding wavefront.

Using this in the inhom. wave eqn gives

$$u(t, \vec{r}) = -\frac{1}{4\pi} \int d^3r' \frac{S(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|} \rightarrow \text{Retarded propagation}$$

$$= \int dt' \int d^3r' G(t-t', \vec{r}-\vec{r}') \times S(t', \vec{r}')$$