Mathematical Methods I

Assignment 4: Green's Functions

Total marks: 25

All questions are compulsory. Please hand in your answer sheets to one of the tutors no later than **5pm Friday 4 September 2015**.

1 Forced damped harmonic oscillator *(12 marks)*

Consider the differential equation

$$\ddot{x}(t) + 2\gamma\,\dot{x}(t) + \omega_0^2\,x(t) = F(t)$$

with γ and ω_0 constants and $\omega_0 > \gamma > 0$ (i.e., an underdamped forced harmonic oscillator). We want to solve this with the initial conditions $x(0) = 0 = \dot{x}(0)$.

1. If $\phi_1(t)$ and $\phi_2(t)$ are two linearly independent solutions of the homogeneous equation, then show that the quantity $G(t,s)$ given by

$$G(t,s) = \theta(t-s)\left[\frac{\phi_1(s)\phi_2(t) - \phi_2(s)\phi_1(t)}{W(s)}\right]\theta(s) \tag{1}$$

   is a causal Green's function of the system satisfying $G(0,s) = 0 = \dot{G}(0,s)$, where W is the Wronskian of ϕ_1 and ϕ_2.

2. Solve the homogenous equation and find ϕ_1 and ϕ_2 (considering only real solutions).

3. Plug these into equation (1) to derive the expression for the corresponding Green's function and compare this with the result derived in class which used Fourier transforms.

4. For a general forcing term $F(t)$, write down the solution of the differential equation which satisfies the required initial conditions.

2 Convolution properties *(3 marks)*

Prove the following properties of the convolution operation $(*)$ in D dimensions, where A, B and C are functions of the D-dimensional variable $\vec{x}$:

1. Linearity

$$((A+B)*C)(\vec{x}) = (A*C)(\vec{x}) + (B*C)(\vec{x})$$

2. Associativity

$$(A*(B*C))(\vec{x}) = ((A*B)*C)(\vec{x})$$

3 Green's function as a propagator *(10 marks)*

Consider the D-dimensional Gaussian $G_{\Delta t}(\vec{x})$ with Fourier transform $\tilde{G}_{\Delta t}(\vec{k}) = \exp(-\alpha k^2 \Delta t)$ where $k^2 = \vec{k} \cdot \vec{k}$ and α and Δt are positive constants. Consider the recurrence relation for a sequence of outputs $\{O_{(j)}(\vec{x})\}$ given by

$$O_{(j)}(\vec{x}) = (G_{\Delta t} * O_{(j-1)})(\vec{x}) \quad \text{for } j \geq 2 \quad ; \quad O_{(1)}(\vec{x}) = (G_{\Delta t} * I)(\vec{x}) \,,$$

where $I(\vec{x})$ is some (square-integrable) input function. You may want to use results from the previous problem below.

1. Show that this recurrence constitutes a linear filter.

2. Show that we can equivalently write

$$O_{(j)}(\vec{x}) = (G_{j\Delta t} * I)(\vec{x})$$

   where $G_{j\Delta t}(\vec{x})$ is simply $G_{\Delta t}(\vec{x})$ with Δt replaced with $j\Delta t$.

3. Show that in the limit $\Delta t \to 0$, $j \to \infty$ with $t \equiv j\Delta t$ remaining finite, the resulting function $O(t, \vec{x}) = \lim_{j\to\infty, \Delta t \to 0} O_{(j)}(\vec{x})$ satisfies the diffusion equation

$$\frac{\partial O}{\partial t} = \alpha \nabla^2 O \,,$$

   with the initial condition $O(0, \vec{x}) = I(\vec{x})$, where ∇^2 is the D-dimensional Laplacian operator.