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ON THE VALIDITY OF THE ZEL'DOVICH APPROXIMATION

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ABSTRACT

The Zel'dovich approximation is often used to mimic the effects of nonlinear gravitational time evolution. We compare the predictions of the Zel'dovich approximation with those of the true nonlinear time evolution, for the probability distribution of mass density fluctuations averaged over a Gaussian ball of large radius, and for the large-scale streaming velocities of objects that do not trace the mass.

Subject headings: cosmic background radiation — early universe — galaxies: clustering

I. INTRODUCTION

Understanding the nature of the primordial fluctuations in the mass density, which gave rise to the large-scale structure of the universe, is one of the fundamental issues in physics today. Limits on the anisotropy of the microwave background radiation restrict these fluctuations to be small at early times. However, they do grow during the matter-dominated era because of a gravitational instability. If the length scales associated with the physics that generated these fluctuations are small compared with astrophysically relevant length scales, then it is likely that the primordial fluctuations will be scale-invariant (i.e., the probability distribution for the fluctuations in the mass density averaged over the horizon volume is independent of time). The simplest scale-invariant probability distribution for the mass density fluctuations is a Gaussian one, where the two-point correlation has a Zel'dovich power spectrum (Harrison 1970; Zel'dovich 1972; Peebles and Yu 1970).

Even if the primordial fluctuations are Gaussian, the nonlinear time evolution of the mass density fluctuations will ensure that at late times the mass density fluctuations are not Gaussian. A variety of methods have been used to understand the effects of the nonlinear time evolution. Often the Zel'dovich approximation (Zel'dovich 1970) is used to mimic the effects of nonlinear gravitational evolution. The Zel'dovich approximation takes the comoving coordinates of any mass point to be

$$\boldsymbol{x}(\boldsymbol{s}, t) = \boldsymbol{s} + [b(t)/a(t)]\boldsymbol{\alpha}(\boldsymbol{s}) , \tag{1}$$

where $a(t)$ is the Robertson-Walker scale factor, $b(t) = t^{2/3}a(t)$, and $\boldsymbol{s}$ is the initial comoving coordinate of the mass point. (The particles are assumed to be evenly distributed initially.) The Zel'dovich approximation reproduces linear perturbation theory [$\boldsymbol{\alpha}(\boldsymbol{s})$ characterizes the primordial fluctuations], as well as giving the correct description of the effects of the nonlinear time evolution (before orbit crossing), for one-dimensional perturbations.

In this paper, the validity of the Zel'dovich approximation as a model for the nonlinear time evolution of scale-invariant Gaussian primordial fluctuations will be examined. In § II the mass density fluctuations $\epsilon(R)$ averaged over a Gaussian ball with large radius R are considered. Assuming cold dark matter, predictions of the Zel'dovich approximation for the third and fourth moments of the probability distribution for $\epsilon(R)$ are compared with the predictions for these quantities given by the true nonlinear time evolution. We find that the Zel'dovich approximation obtains the correct sign for the connected parts of these moments, but underestimates their magnitudes by a factor that ranges roughly between 2 and 5.

In § III the predictions of the Zel'dovich approximation for the large-scale streaming velocities of objects that do not trace the mass are compared with those of the true nonlinear time evolution. In a model where the initial locations of objects are determined by the primordial fluctuations (e.g., objects occurring wherever the primordial mass density fluctuations are unusually large) and the objects subsequently move with the velocity of the mass, we find a qualitative difference between the predictions of the Zel'dovich approximation and the true nonlinear time evolution for large-scale streaming velocities. Concluding remarks are given in § IV.

II. MASS DENSITY FLUCTUATIONS AVERAGED OVER A GAUSSIAN BALL

In this section we shall make predictions for the mass density fluctuations averaged over a Gaussian ball with a radius large compared with scales that have undergone very nonlinear evolution, but small compared with the horizon length. Therefore it is appropriate to treat the mass density as a self-gravitating Newtonian fluid at zero pressure. The equations of motion that govern the time evolution of the mass density fluctuations $\delta(\boldsymbol{x}, t)$ and the peculiar velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ are

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot (1 + \delta)\boldsymbol{v} = 0 , \tag{2a}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{\dot{a}}{a}\boldsymbol{v} + \frac{1}{a}(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{g} , \tag{2b}$$

$$\nabla \cdot \boldsymbol{g} = -4\pi G\langle\rho\rangle a\delta , \qquad \nabla \times \boldsymbol{g} = 0 . \tag{2c}$$

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In equations (2) t is the cosmic time. We adopt the convention that $t = 1$ is the present cosmic time and take the Robertson-Walker scale factor to be $a(t) = t^{2/3}$, so that at the present time comoving length scales and physical length scales coincide. It is assumed that $\Omega = 1$, which implies that $4\pi G\langle\rho\rangle = 2/3t^2$.

Equations (2) can be solved by first linearizing about $\delta = \boldsymbol{v} = 0$ and keeping only the fastest growing solution, then linearizing about this solution and again keeping the fastest growing solution, and so on. The resulting expressions for δ and $\boldsymbol{v}$ have the form

$$\delta(\boldsymbol{x}, t) = \sum_{n=1}^{\infty} \delta_n(\boldsymbol{x}) t^{2n/3} , \tag{3a}$$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{a(t)}{t} \sum_{n=1}^{\infty} \boldsymbol{v}_n(\boldsymbol{x}) t^{2n/3} , \qquad \nabla \times \boldsymbol{v}_n = 0 . \tag{3b}$$

The primordial fluctuations are characterized by $\delta_1(\boldsymbol{x})$. The equation of continuity gives $\delta_1(\boldsymbol{x}) = -(3/2)\nabla \cdot \boldsymbol{v}_1$. Equations (2) determine δ_n and $\boldsymbol{v}_n$, $n > 1$, in terms of δ_1. For the Fourier transforms of δ_n and $\boldsymbol{v}_n$ this relationship has the form

$$\tilde{\delta}_n(\boldsymbol{k}) = \int \frac{d\boldsymbol{q}_1}{(2\pi)^3} \cdots \int \frac{d\boldsymbol{q}_n}{(2\pi)^3} (2\pi)^3 \delta^3(\boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n - \boldsymbol{k}) P_n^{(s)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \tilde{\delta}_1(\boldsymbol{q}_1) \cdots \tilde{\delta}_1(\boldsymbol{q}_n) , \tag{4a}$$

$$\tilde{d}_n(\boldsymbol{k}) = \frac{1}{k^2} \int \frac{d\boldsymbol{q}_1}{(2\pi)^3} \cdots \int \frac{d\boldsymbol{q}_n}{(2\pi)^3} (2\pi)^3 \delta^3(\boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n - \boldsymbol{k}) Q_n^{(s)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \tilde{\delta}_1(\boldsymbol{q}_1) \cdots \tilde{\delta}_1(\boldsymbol{q}_n) , \tag{4b}$$

where we have introduced a velocity potential,

$$\boldsymbol{v}_n(\boldsymbol{x}) = \nabla d_m(\boldsymbol{x}) . \tag{5}$$

$P_n^{(s)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ and $Q_n^{(s)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ are homogeneous, symmetric, rotationally invariant functions of the wave vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ of degree zero. They embody the effects of the nonlinear time evolution on the primordial fluctuations. The primordial fluctuations are assumed to be Gaussian with the two-point correlation

$$\langle \tilde{\delta}_1(\boldsymbol{q}_1) \tilde{\delta}_1(\boldsymbol{q}_2) \rangle = (2\pi)^3 A(q_1) q_1 \delta^3(\boldsymbol{q}_1 + \boldsymbol{q}_2) . \tag{6}$$

In equation (6) $A(k)$ is a computable function of k that goes to a constant A as k goes to zero. For cold dark matter the k dependence of $A(k)$ arises because fluctuations that cross the horizon in the radiation-dominated era only grow logarithmically before the time of matter domination. This causes $A(k)$ to fall off as $[(\ln k)/k^2]^2$ for large k.

Goroff *et al.* (1986) have introduced a diagrammatic notation for the computation of the connected correlations of δ and $\boldsymbol{v}$. They found that the connected correlations are dominated at small wavenumbers by the tree graphs. Although the perturbative expansions for δ and $\boldsymbol{v}$ do not converge for arbitrarily large times, because of orbit crossing, the predictions of the tree graphs for the connected correlations of δ and $\boldsymbol{v}$ at small wavenumbers probably remain valid; they are completely insensitive to physical phenomena associated with large wavenumbers. The connected correlations of the mass density fluctuations averaged over a Gaussian ball of large radius R,

$$\epsilon(R) \equiv \frac{\int d\boldsymbol{x} e^{-x^2/R^2} \delta(\boldsymbol{x}, t)}{\int d\boldsymbol{x} e^{-x^2/R^2}} = \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{-k^2R^2/4} \tilde{\delta}(\boldsymbol{k}, t) , \tag{7}$$

are determined by the correlations of the mass density fluctuations as all the external wave vectors become small. The tree graphs yield the following expressions for the connected parts of the first four moments of the probability distribution for $\epsilon(R)$ (at the present cosmic time):

$$\langle \epsilon(R)^2 \rangle = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \exp(-k^2R^2/2) A(k) k , \tag{8}$$

$$\langle \epsilon(R)^3 \rangle = \int \frac{d\boldsymbol{k}_1}{(2\pi)^3} \int \frac{d\boldsymbol{k}_2}{(2\pi)^3} \exp[-(k_1^2 + k_2^2 + \boldsymbol{k}_1 \cdot \boldsymbol{k}_2)R^2/2] 6A(k_1)A(k_2)k_1k_2 P_2^{(s)}(\boldsymbol{k}_1, \boldsymbol{k}_2) , \tag{9}$$

$$\begin{aligned} \langle \epsilon(R)^4 \rangle_c = \int \frac{d\boldsymbol{k}_1}{(2\pi)^3} \int \frac{d\boldsymbol{k}_2}{(2\pi)^3} \int \frac{d\boldsymbol{k}_3}{(2\pi)^3} \exp[-(k_1^2 + k_2^2 + k_3^2 + \boldsymbol{k}_1 \cdot \boldsymbol{k}_3 + \boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \boldsymbol{k}_2 \cdot \boldsymbol{k}_3)R^2/2] 24A(k_2)A(k_3)k_2k_3 \\ \times [A(k_1)k_1 P_3^{(s)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) + 2A(|\boldsymbol{k}_2 + \boldsymbol{k}_1|)|\boldsymbol{k}_2 + \boldsymbol{k}_1| P_2^{(s)}(\boldsymbol{k}_2, -(\boldsymbol{k}_1 + \boldsymbol{k}_2)) P_2^{(s)}(\boldsymbol{k}_3, (\boldsymbol{k}_1 + \boldsymbol{k}_2))] . \end{aligned} \tag{10}$$

With cold dark matter (Bardeen *et al.* 1986)

$$A(k) = \left[\frac{\ln(1 + 2.34q)}{2.34q}\right]^2 A(1 + 3.89q + (16.1q)^2 + (5.49q)^3 + (6.71q)^4)^{-1/2} , \tag{11}$$

where $q = (k/h^2)$ Mpc^{-1} (as is appropriate to an $\Omega = 1$ universe with a Hubble constant of $100h$ km s^{-1} Mpc^{-1}. Equation (8) is the primordial two-point correlation. For very large radius the k dependence of the function $A(k)$ is irrelevant, and the integral in equation (8) can be performed analytically, yielding

$$\langle \epsilon(R)^2 \rangle = \frac{A}{\pi^2 R^4} \qquad \text{as } R \to \infty . \tag{12}$$

The expression for $P_2^{(s)}$ is so simple that the asymptotic form of equation (9) at large radius can also be derived analytically. Using (Peebles 1980)

$$P_2^{(s)}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{5}{7} + \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{2k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{2}{7}\left(\frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{k_1 k_2}\right)^2 , \tag{13}$$

we find that

$$\langle \epsilon(R)^3 \rangle = \frac{32A^2}{7\pi^4 R^8}\left(\frac{11\pi}{12\sqrt{3}} - 1\right) \quad \text{as } R \to \infty . \tag{14}$$

Goroff *et al.* (1986) have evaluated numerically the integrals in equations (8)–(10) using the expressions for $P_2^{(s)}$ and $P_3^{(s)}$ that follow from equations (2).

The purpose of this section is to compare the predictions of the Zel'dovich approximation for the first four connected moments of the probability distribution for $\epsilon(R)$ with the predictions of the true nonlinear gravitational evolution (determined by eqs. [2]) for these quantities. In the Zel'dovich approximation any mass point has comoving coordinates

$$\boldsymbol{x}(\boldsymbol{s}, t) = \boldsymbol{s} + t^{2/3}\boldsymbol{\alpha}(\boldsymbol{s}) . \tag{15}$$

In equation (15) $\boldsymbol{s}$ is the initial comoving coordinate. Writing the peculiar velocity as

$$\boldsymbol{v} \equiv a(t)\frac{d\boldsymbol{x}}{dt} = \frac{2}{3}t^{1/3}\boldsymbol{p} , \tag{16}$$

it is evident from equation (15) that $\boldsymbol{p} = \boldsymbol{\alpha}(\boldsymbol{s})$. To derive an expression for the mass density fluctuations in the Zel'dovich approximation, it is convenient to introduce a phase-space density distribution; $f(\boldsymbol{x}, \boldsymbol{p}; t)d\boldsymbol{x}d\boldsymbol{p}$ is the number of particles (which we assume to have mass m) in the 6-dimensional phase-space volume $d\boldsymbol{x}d\boldsymbol{p}$ at time t. Since the particle with comoving coordinate $\boldsymbol{x}$ at time t originated at the comoving coordinate

$$\boldsymbol{s} = \boldsymbol{x} - t^{2/3}\boldsymbol{p} , \tag{17}$$

the appropriate phase-space distribution for particles that were evenly distributed initially is

$$f(\boldsymbol{x}, \boldsymbol{p}; t) = \frac{\langle \rho \rangle}{m}\delta^3(\boldsymbol{p} - \boldsymbol{\alpha}(\boldsymbol{s})) , \tag{18}$$

with $\boldsymbol{s}$ given by equation (17). Before orbit crossing we can expand the Dirac δ-function, yielding

$$f(\boldsymbol{x}, \boldsymbol{p}; t) = \frac{\langle \rho \rangle}{m}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\alpha_{i_1}\cdots\alpha_{i_n}\frac{\partial}{\partial p_{i_1}}\cdots\frac{\partial}{\partial p_{i_n}}\delta^3(\boldsymbol{p}) . \tag{19}$$

Putting this into the equation for the mass density,

$$\rho(\boldsymbol{x}, t) = m\int f(\boldsymbol{x}, \boldsymbol{p}; t)d\boldsymbol{p} , \tag{20}$$

and integrating by parts, we have

$$\frac{\rho(\boldsymbol{x}, t)}{\langle \rho \rangle} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n/3}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_n}}[\alpha_{i_1}(\boldsymbol{x})\cdots\alpha_{i_n}(\boldsymbol{x})] . \tag{21}$$

The $n = 1$ term corresponds to linear perturbation theory, and it implies that $\delta_1(\boldsymbol{x}) = -\boldsymbol{\nabla} \cdot \boldsymbol{\alpha}$ or, equivalently, $\boldsymbol{\alpha} = (3/2)\boldsymbol{v}_1$. Fourier-transforming equation (21) gives the result that the Zel'dovich approximation corresponds to assigning the functions $P_n^{(s)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ the values

$$P_n^{(s)}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = \frac{1}{n!}\frac{(\boldsymbol{k} \cdot \boldsymbol{q}_1)}{q_1^2}\cdots\frac{(\boldsymbol{k} \cdot \boldsymbol{q}_n)}{q_n^2} , \tag{22}$$

where $\boldsymbol{k} = \boldsymbol{q}_1 + \cdots + \boldsymbol{q}_n$.

Using these values, we have evaluated the predictions that the Zel'dovich approximation makes for the first four moments of the probability distribution of the mass density fluctuations averaged over a large Gaussian ball. Since the Zel'dovich approximation gives the correct evolution of the mass density fluctuations in linear perturbation theory, the second moment agrees with the true dynamical evolution. The ratio of the third moment in the Zel'dovich approximation to the true third moment is presented in Figure 1. The Zel'dovich approximation obtains the correct sign for this moment but is about a factor of 2 smaller than the true value. The asymptotic value of the ratio plotted in Figure 1 can be determined analytically by comparing equation (14) with the corresponding expression in the Zel'dovich approximation (ZA) (derived by using eq. [22] for $P_2^{(s)}$ in eq. [9]):

$$\langle \epsilon(R)^3 \rangle_{\mathrm{ZA}} = \frac{8A^2}{\pi^4 R^8}\left\{\frac{2\pi}{3\sqrt{3}} - 1\right\} \quad \text{as } R \to \infty . \tag{23}$$

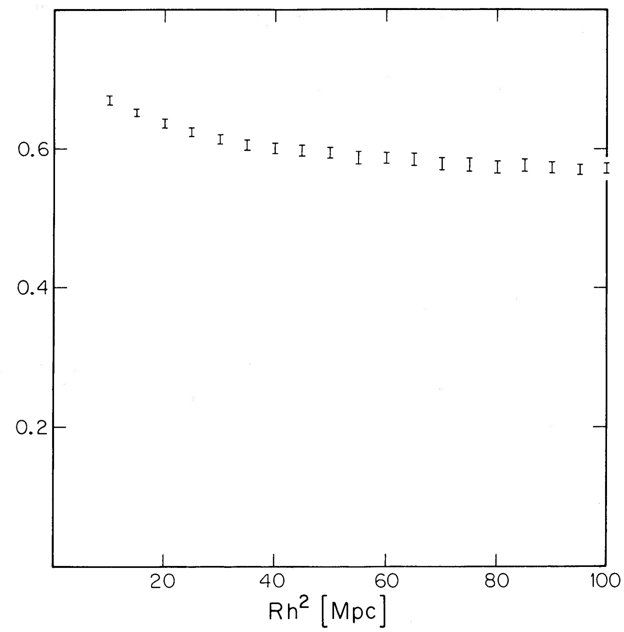

FIG. 1

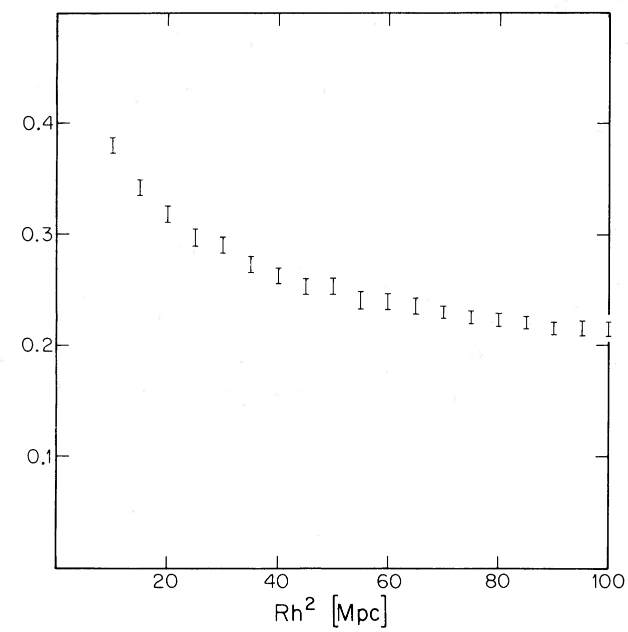

FIG. 2

FIG. 1.—Plot of $\langle\epsilon(R)^3\rangle$, computed in the Zel'dovich approximation, divided by the true value of $\langle\epsilon(R)^3\rangle$ versus Rh^2 (in units of Mpc). The error bars are 1 σ errors incurred in a Monte Carlo evaluation of the integrals in eq. (9).

FIG. 2.—Plot of $\langle\epsilon(R)^4\rangle_c$, computed in the Zel'dovich approximation, divided by the true value of $\langle\epsilon(R)^4\rangle_c$ versus Rh^2 (in units of Mpc). The error bars are 2 σ errors incurred in a Monte Carlo evaluation of the integrals in eq. (10).

Figure 2 shows the ratio of the Zel'dovich approximation's prediction for the connected part of the fourth moment of the probability distribution for $\epsilon(R)$ to its value predicted by the true nonlinear evolution. The Zel'dovich approximation gets the correct sign but underestimates the magnitude of this moment by a factor that ranges roughly between 2 and 5 as R goes from $10h^{-2}$ Mpc to $100h^{-2}$ Mpc.

III. LARGE-SCALE DEVIATIONS FROM THE HUBBLE FLOW

The average, over a very large region (with a fuzzy boundary), of the peculiar velocities $\boldsymbol{v}$ of a class of objects with number density n is determined by the Fourier transform of $(1/\bar{n})n\boldsymbol{v}$ ($\bar{n}$ denotes the average number density) at small wave vectors. For definiteness, imagine that the locations of the objects are determined initially by local properties of the primordial mass density fluctuations and the objects subsequently move with the velocity of the mass. For example, an initial number density (Kaiser 1984; Politzer and Wise 1984)

$$n_0(\boldsymbol{x}) = C \exp\left[T\delta_f(\boldsymbol{x})/\langle\delta_f^2(0)\rangle^{1/2}\right] , \tag{24}$$

where $\delta_f(\boldsymbol{x})$ denotes the primordial fluctuations filtered on the comoving scale that collapses to form the objects,

$$\delta_f(\boldsymbol{x}) = \int d\boldsymbol{y} W(\boldsymbol{x}-\boldsymbol{y})\delta_1(\boldsymbol{y}) , \tag{25}$$

corresponds to objects that form preferentially where filtered primordial fluctuations are greater than $T[\langle\delta_f^2(0)\rangle]^{1/2}$. In this model the number density of the objects at a later time is determined by the equation of continuity

$$\frac{\partial n}{\partial t} + \frac{1}{a(t)} \nabla \cdot (n\boldsymbol{v}) = 0 . \tag{26}$$

Since the spatial dependence of the number density is not small at early times, it cannot be treated as a perturbation. However, the time dependence of this number density is small at early times, since the velocity of the mass (which appears in eq. [26]) is small. The expression for the peculiar velocity given in equation (3b), together with the equation of continuity (26), implies that the number density has an expansion of the form

$$n(\boldsymbol{x}, t) = \sum_{j=0}^{\infty} n_j(\boldsymbol{x}) t^{2j/3} , \tag{27}$$

where, for $j \geq 1$,

$$n_j(\boldsymbol{x}) = -\frac{3}{2j} \sum_{k=1}^{j} \nabla \cdot [\boldsymbol{v}_k(\boldsymbol{x}) n_{j-k}(\boldsymbol{x})] . \tag{28}$$

The first few terms in the expansion for nv are

$$(nv)(x, t) = \frac{a(t)}{t} [n_0(x)v_1(x)t^{2/3} + n_1(x)v_1(x)t^{4/3} + n_0(x)v_2(x)t^{4/3} + O(t^2)] . \tag{29}$$

Writing

$$v_2 = -\tfrac{3}{2}(v_1 \cdot \nabla)v_1 + w_2 , \tag{30}$$

and using equation (28), the expression for nv becomes

$$(nv)(x, t) = \frac{a(t)}{t} \{n_0(x)v_1(x)t^{2/3} - \tfrac{3}{2}\nabla \cdot [n_0(x)v_1(x)v_1(x)]t^{4/3} + n_0(x)w_2(x)t^{4/3} + O(t^2)\} . \tag{31}$$

The first term in equation (31) only involves linear perturbation theory. The second term is not important for the large-scale average of nv, since it is a total derivative. It is the third term that gives a deviation of the large-scale streaming velocity of objects from that of linear perturbation theory. Using equations (2), it is straightforward to see that

$$w_2 = \tfrac{9}{14}[(v_1 \cdot \nabla)v_1 - (\nabla \cdot v_1)v_1] . \tag{32}$$

Note that w_2 vanishes for one-dimensional primordial fluctuations.

In the model where the initial number density has the form given in equation (24), the leading nontrivial perturbative contribution to the two-point correlation of $(1/\bar{n})nv$ can easily be evaluated. We find

$$\frac{1}{\bar{n}^2} \langle (\tilde{nv})(k_1, t)(\tilde{nv})(k_2, t)\rangle = \left[\frac{a(t)}{t}\right]^2 t^{4/3}\langle \tilde{v}_1(k_1)\tilde{v}_1(k_2)\rangle\left[1 + \frac{4}{7}T\frac{\langle \delta_1(0)\delta_f(0)\rangle}{\langle \delta_f^2(0)\rangle^{1/2}}t^{2/3} + O(t^{4/3})\right] \tag{33}$$

at small k_1. Typically $\langle \delta_1(0)\delta_f(0)\rangle/\langle \delta_f^2(0)\rangle^{1/2}$ is greater than $\langle \delta_f^2(0)\rangle^{1/2}$.

For the objects to have collapsed by today ($t = 1$), $T\langle \delta_f^2(0)\rangle^{1/2}$ cannot be small. In particular, the spherical model suggests that $T\langle \delta_f^2(0)\rangle^{1/2} > 1.7$. Equation (33) then implies that the mean square value of the large-scale streaming velocity of the objects is at least twice as big as that of the mass. Of course, with such a large correction there is no reason to expect that higher order terms in the perturbative expansion, which have been neglected in equation (33), are unimportant.

We have seen that the large-scale streaming velocities of objects whose initial positions are determined by local properties of the primordial mass density fluctuations, and subsequently move with the velocity of the mass, are expected to be significantly different (Grinstein *et al.* 1987) from the large-scale streaming velocity of the mass (which is dominated by linear perturbation theory). The main purpose of this section is to contrast this with what happens when the objects move not with the true velocity of the mass but rather with the velocity of the mass in the Zel'dovich approximation. The velocity of the mass (using Eulerian coordinates) in the Zel'dovich approximation can be determined by combining the expression for the mass density in equation (21) with the equation of continuity (2a). At the lowest nontrivial order of perturbation theory, the velocity in the Zel'dovich approximation is given by equation (30) with w_2 equal to zero. Thus, in the first nontrivial order of perturbation theory, the Zel'dovich approximation has the large-scale streaming velocity of the objects equal to that of the mass.

It is straightforward to show that this can be generalized to all orders of perturbation theory. Let $g(x, p; t)dxdp$ be the number of objects in the 6-dimensional phase-space volume $dxdp$. In the Zel'dovich approximation the phase-space distribution function g is given by

$$g(x, p; t) = n_0(s)\delta^3(p - \alpha(s)) , \tag{34}$$

with s given by equation (17). Applying the same methods that were introduced in § II, to compute

$$(nv)(x, t) = \frac{2}{3}\frac{a(t)}{t}t^{2/3}\int dp\, p g(x, p; t) , \tag{35}$$

we find that nv is given by

$$(nv)(x, t) = \frac{2}{3}\frac{a(t)}{t}\sum_{n=0}^{\infty} t^{2(n+1)/3}\frac{(-1)^n}{n!}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_n}}[n_0(x)\alpha(x)\alpha_{i_1}(x)\cdots\alpha_{i_n}(x)] , \tag{36}$$

where $\alpha = (3/2)v_1$.

Since the terms which go beyond linear perturbation theory are total derivatives, they are unimportant for the average of nv over large scales. In the Zel'dovich approximation objects whose initial positions are determined by the primordial fluctuations, and subsequently move with the velocity of the mass, have a large-scale streaming velocity that is dominated by linear perturbation theory. This occurs because in the Zel'dovich approximation an object's velocity changes with time only by a multiplicative factor of $t^{1/3}$. Hence the average of (nv) over a large region changes (apart from the factor of $t^{1/3}$) only because objects flow through the surface of the region.

We have also examined the large-scale streaming velocities of objects with number densities of the form $n \propto \rho^k$, $k = 2, 3, \ldots$, where ρ is the *evolved* mass density field. In this case, at low orders in perturbation theory, the Zel'dovich approximation gives the result that the large-scale average of $(1/\bar{n})nv$ is the same as that of v, while the true nonlinear time evolution gives the result that they are different. (We believe that this persists to all orders in perturbation theory, because one could find a number density n_0, which was determined by local properties of the primordial fluctuations, that evolved into n.)

IV. CONCLUDING REMARKS

Gravitational perturbation theory can be used to predict moments of the probability distribution for the mass density fluctuations averaged over a Gaussian ball of large radius. In this paper we compared the predictions of gravitational perturbation theory for (the connected part of) the third and fourth moments with those of the Zel'dovich approximation. The Zel'dovich approximation is successful in reproducing the correct sign for these moments, but it significantly underestimates their magnitudes. This is not surprising, since the Zel'dovich approximation neglects gravitational effects that cause the mass points' velocities to deviate from their initial directions.

At early times other features of the mass density fluctuations can be computed meaningfully in gravitational perturbation theory and compared with the Zel'dovich approximation. For example, the true nonlinear time evolution gives (Peebles 1980)

$$\langle \delta^3(0, t) \rangle = \tfrac{34}{7} \langle \delta_1^2(0) \rangle^2 t^{8/3} + O(t^4) , \tag{37}$$

while the Zel'dovich approximation gives

$$\langle \delta^3(0, t) \rangle = 4 \langle \delta_1^2(0) \rangle^2 t^{8/3} + O(t^4) . \tag{38}$$

Here the agreement between the Zel'dovich approximation and the true nonlinear time evolution is quite good.

In general, the large-scale streaming velocities of objects that do not trace the mass are different from the large-scale streaming velocity of the mass (Grinstein *et al.* 1987). Using a model where the objects' initial locations are determined by local properties of the primordial mass density fluctuations, and the objects subsequently move with the velocity of the mass, we showed that this discrepancy can be significant. However, if the objects do not move with the true velocity of the mass, but rather move with the velocity of the mass in the Zel'dovich approximation, then their large-scale streaming velocity is the same as that of the mass. A similar phenomenon occurs in at least some cases where the objects' number density is determined by local properties of the *evolved* mass density fluctuations. The Zel'dovich approximation may give very misleading estimates for the large-scale streaming velocities of objects that do not trace the mass.

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