Fourier Analysis

[Lec 1]

Motivation:

Linear algebra shows us how arbitrary vectors in a vector space can be written as linear combinations of ~~ege~~ a set of basis vectors.

The idea is to extend this concept to functions. The resulting "coefficients" of the linear combinations of basis functions can then be manipulated to simplify calculations and gain insights into the behaviour of a given function.

We need some concepts & tools first:

1. To begin with, consider functions $f(x)$ defined on the range $-L \leq x \leq L$ [later we'll send $L \to \infty$]. {$f(x)$ may be complex}
2. Next, define a "dot product" (inner product) for such functions

$$\langle f | g \rangle \equiv \int_{-L}^{L} dx\, f^*(x)\, g(x)$$

which gives a norm

$$\| f^2 \| \equiv \langle f | f \rangle \equiv \int_{-L}^{L} dx\, |f(x)|^2$$

We will only consider functions for which this norm is finite ["square integrable"]

3. Now one can show that the ~~set of~~ functions $\cancel{\{e^{in\pi/L}\}}$ $\{e^{in\pi x/L}\}$.
where $n = 0, \pm 1, \pm 2, \ldots$
forms a complete & orthogonal set, in the sense that any square integrable function can be written as a linear combination of these.

More precisely, let $u_n(x) \equiv \frac{1}{\sqrt{2L}} e^{in\pi x/L}$

Then

$$\langle u_n | u_m \rangle = \delta_{nm} \qquad \text{[Orthogonality]}$$

& we can write

$$\left.\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \langle u_n | f \rangle \, u_n(x) \\ &= \sum_{n=-\infty}^{\infty} f_n \, u_n(x) \\ \text{with} \quad f_n \equiv \langle u_n | f \rangle &= \int_{-L}^{L} dx\, u_n^*(x) f(x) \end{aligned}\right\} \text{[Fourier Series]}$$

Just as in linear algebra, basis is not unique.
Eg., on the range $-1 \leq x \leq 1$ we can also use the Legendre Polynomials:

$$u_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x),$$

on the range $(-\infty, \infty)$ we can use Hermite polynomials, & so on.

Completeness implies

$$\sum_n u_n^*(x) u_n(x') = \delta_D(x'-x)$$ [closure relation]

↳ Dirac delta.

We discuss the proof of completeness later

Periodicity:

A function is periodic with period $2L$ if

$$f(x+2L) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

For a periodic $f(x)$ with period $2L$, the previous Fourier series is valid for all $x \in \mathbb{R}$

since

$$u_n(x+2L) = \frac{1}{\sqrt{2L}} e^{\frac{i n\pi}{L}(x+2L)} = u_n(x) \cdot e^{i \cdot 2n\pi} = u_n(x)$$

for all $x \in \mathbb{R}$

Say we have $F(x)$ defined only on some range $x \in [c,d]$. We can still define a useful Fourier series as follows:

Define $f(x) = \begin{cases} 0, & a < x < c \\ F(x), & c \leq x \leq d \\ 0, & d < x < b \end{cases}$

Let $b-a = 2L$.

Writing $f(x) = \sum_{n=-\infty}^{\infty} f_n u_n(x)$, we have

$$f_n = \int_a^b dx\, u_n^*(x) f(x) = \int_c^d dx\, u_n^*(x) F(x)$$

$f(x)$ over $\mathbb{R}$ will be periodic, but gives isolated copies of $F(x)$.

Fourier integral transform

This is more useful when describing functions that are not necessarily periodic and are defined over arbitrary ranges.

Formally, we can do this by sending $L \to \infty$ with some care.

Define $\quad k \equiv n\pi/L$

Consider a periodic function $f(x)$ with

$$f(x) = \sum_n f_n e^{in\pi x/L}$$
$$f_n = \int_{-L}^{L} \frac{dx}{2L} e^{-in\pi x/L} f(x)$$

} slightly different normalis^n than earlier

[Note that we have
$$\int_{-L}^{L} \frac{dx}{2L} e^{-im\pi x/L} f(x) = \sum_n f_n \int_{-L}^{L} \frac{dx}{2L} e^{\frac{i\pi x}{L}(n-m)}$$
$$= \sum_n f_n \delta_{mn} = f_m$$
So the change in normalis^n is consistent. See also below.]

Note that the summation involves an increment $\Delta n = 1$, corresponding to.

$$\Delta k = \frac{\pi}{L}\Delta n = \frac{\pi}{L}.$$

~~To take the continuum limit~~

If $L \to \infty$, then $\Delta k \to 0$.

So
$$\sum_{n=-\infty}^{\infty} f_n e^{2n\pi x/L} \Delta n$$
$$= \sum_{k=-\infty}^{\infty} \Delta k \frac{L}{\pi} f_k e^{ikx}$$

$2L f_k \equiv \tilde{f}(k)$ — periodicity in x.

$$\to \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx}$$

[continuum limit]

2π → periodicity of phase kx.

So
$$\boxed{f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) \quad \& \quad \tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)}$$

[send $L \to \infty$ in defn of f_n]

This implies:
$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dx' e^{-ikx'} f(x')$$
$$= \int_{-\infty}^{\infty} dx' f(x') \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')}$$
$$\Rightarrow \boxed{\int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = 2\pi\, \delta_D(x-x')}$$

Previous gives continuum limit of closure relation, taking

$$\underset{\frac{1}{\sqrt{2L}} e^{i n\pi x/L}}{\underset{\parallel}{u_n(x)}} \longrightarrow u_k(x) \equiv \frac{1}{\sqrt{2\pi}} e^{ikx}$$

Orthogonality also follows simply:

$$\langle u_k | u_{k'} \rangle = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{i(k'-k)x} = \delta_D(k'-k),$$

consistent with previous relation

Scaling and sign convention:

We saw that Fourier series normalisation is not unique. The same applies for the Fourier transform.

In particular, if

$$f(x) = C \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k)$$

then

$$\tilde{f}(k) = \frac{1}{C} \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$$

What is important is to maintain definition of the Dirac delta through both forward & backward relns [orthogonality & completeness]

This is ~~main~~ accomplished by above.

Different authors have different conventions; each has advantages/disadvantages.

Most common choices:

$C = 1$ [these notes]

or $\sqrt{2\pi}$ [symmetrizes the fwd & bkwd integrals]

or 2π

Change in sign gives

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \hat{f}(k) \quad \& \quad \hat{f}(k) = \int_{-\infty}^{\infty} dx \, e^{ikx} f(x)$$

i.e., signs of fwd & bkwd transforms must be opposite.

We will use $\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{+ikx} \cdots$ for spatial transforms

and $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \cdots$ for time transforms

This will lead to expressions involving $i(kx - \omega t)$ which is constant for a wave travelling to the right.

Real vs. Complex:

Taking the complex conjugate of $f(x)$ we have

$$f^*(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \tilde{f}^*(k) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}^*(-k)$$

i.e, the Fourier transform of $f^*(x)$ is $\tilde{f}^*(-k)$

If $f(x)$ is real, then we must have

$$\tilde{f}^*(-k) = \tilde{f}(k)$$

Some useful Fourier conjugate ~~transform~~ pairs:

$f(x)$	$\tilde{f}(k)$
$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$	$e^{-k^2\sigma^2/2}$
$\frac{1}{x}$	$i\pi\,\mathrm{sgn}(-k)$
$\Theta(x) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(x)$	$\pi\,\delta_D(k) - i/k$
$A \cdot \Theta(R - \lvert x \rvert)$ ~~[smooth filter]~~	$2AR\,\mathrm{sinc}(kR)$
$\frac{a}{\pi}\frac{1}{x^2+a^2}$, $a>0$	$e^{-\lvert k \rvert a}$

Know these proofs!

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[Lec 2]

Convolution & correlation

$$(f * g)(x) \equiv \int_{-\infty}^{\infty} dy\, f(x-y)\, g(y) \equiv \text{convolution of } f(x) \,\&\, g(x)$$

[g smoothed w/ kernel f, or vice-versa]

~~Introduce~~

Fourier transforming this,

$$\widetilde{(f*g)}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} dy\, f(x-y)\, g(y)$$

$$= \int_{-\infty}^{\infty} dy\, g(y) \int_{-\infty}^{\infty} dx\, e^{-ik(x-y)} f(x-y)\, e^{-iky}$$

$$= \int_{-\infty}^{\infty} dy\, e^{-iky} g(y) \int_{-\infty}^{\infty} dx'\, e^{-ikx'} f(x') \qquad [x' = x-y]$$

$$= \tilde{g}(k)\, \tilde{f}(k)$$

i.e, the Fourier transform of a convolution is the product of the individual Fourier transforms

Correlation of $a(t)$ & $b(t)$ defined as

$$\xi_{ab}(\tau) \equiv \int_{-\infty}^{\infty} dt\, a(t+\tau)\, b^*(t)$$

~~Fourier transforming,~~

$$\cancel{\int_{-\infty}^{\infty} d\tau\, e^{+i\omega\tau}\, \xi_{ab}(\tau) = \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} \int_{-\infty}^{\infty} dt\, a(t-\tau)\, b(t)}$$

$$\cancel{\int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}}$$

Fourier transforming

$$\int_{-\infty}^{\infty} d\tau\, e^{+i\omega\tau}\, \xi_{ab}(\tau) =$$

$$= \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} \int_{-\infty}^{\infty} dt\, a(t+\tau)\, b^*(t)$$

$$= \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega'(t+\tau)}\, \tilde{a}(\omega) \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} e^{-i\omega'' t}\, \tilde{b}^*(-\omega'')$$

$$= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi}\, \tilde{a}(\omega')\, \tilde{b}^*(-\omega'') \times \int_{-\infty}^{\infty} d\tau\, e^{i(\omega-\omega')\tau} \int_{-\infty}^{\infty} dt\, e^{-i(\omega'+\omega'')t}$$

$$= \int_{-\infty}^{\infty} d\omega'\, \tilde{a}(\omega') \int_{-\infty}^{\infty} d\omega''\, \tilde{b}^*(-\omega'')\, \delta_D(\omega-\omega')\, \delta_D(\omega'+\omega'')$$

$$= \tilde{a}(\omega)\, \tilde{b}^*(\omega)$$

i.e $\quad \tilde{\xi}_{ab}(\omega) = \tilde{a}(\omega)\, \tilde{b}^*(\omega)$

~~Parseval's theorem~~

Auto-correlation $\quad \xi_{aa}(\tau) = \int_{-\infty}^{\infty} dt\, a(t+\tau)\, a^*(t)$

$\longrightarrow \quad \tilde{\xi}_{aa}(\omega) = |\tilde{a}(\omega)|^2 = \text{power spectrum}$

~~Now, $\int_{-\infty}^{\infty} d\omega\, |\tilde{a}(\omega)|^2 = \int dt$~~ $\longrightarrow$

Now,

$$\int_{-\infty}^{\infty} d\omega \, |\tilde{a}(\omega)|^2$$

$$= \int_{-\infty}^{\infty} d\omega \, \tilde{\xi}_{aa}(\omega)$$

$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} \xi_{aa}(\tau)$$

$$= \int_{-\infty}^{\infty} d\tau \, \xi_{aa}(\tau) \cdot 2\pi \delta_D(\tau)$$

$$= 2\pi \, \xi_{aa}(\tau=0)$$

$$\therefore \int_{-\infty}^{\infty} d\omega |\tilde{a}(\omega)|^2 = 2\pi \int_{-\infty}^{\infty} dt \, |a(t)|^2 \longrightarrow \text{Parseval's theorem}$$

[can be thought as energy conservation if $a(t) \sim$ voltage across resistance, or electric field of propagating wave; also as conservation of probability in QM]

Mathematically, this is the statement that the Fourier transform operator preserves the norm

$$\langle f | f \rangle \equiv \int_{-\infty}^{\infty} dx \, |f^*(x)|^2$$

through the relation

$$[\, |\tilde{f}\rangle = F|f\rangle \qquad \langle \tilde{f}| = \langle f | F^\dagger \,]$$

$$\langle f | f \rangle = \frac{1}{2\pi} \langle \tilde{f} | \tilde{f} \rangle = \langle f | \frac{1}{2\pi} F^\dagger F | f \rangle$$

So that $\frac{1}{2\pi} F^\dagger F$ is the identity, making $\frac{1}{\sqrt{2\pi}} F$ unitary.

Linear filters

Consider a system which takes an input $i(t)$ & returns an output $o(t)$.
E.g., a forced harmonic oscillator described by

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F/m$$

takes an input force $F(t)$ & generates an output $x(t)$ as a response to this force.

~~The filter in this case is the differ~~
Visually, we can imagine

$$i(t) \longmapsto \boxed{\text{Filter}} \longmapsto o(t)$$

A filter is linear if ~~an input~~ the linear combination of inputs

$$c_1 i_1(t) + c_2 i_2(t)$$

generates a response

$$c_1 O_1(t) + c_2 O_2(t)$$

where $O_1(t)$ & $O_2(t)$ are the responses to $i_1(t)$ & $i_2(t)$ respectively.

Define the impulse response function $g(t,\tau)$ as the output of the system at time t due to an impulsive input $\delta_D(t-\tau)$ centered at $t=\tau$.

Why is this useful?
- Consider any input $i(t)$ & write it as

$$i(t) = \int_{-\infty}^{\infty} d\tau\, \delta_D(t-\tau)\, i(\tau)$$

which says that this input is a linear combination of impulses centered at ~~t~~ ~~[illegible]~~ various times τ.

The linearity of the filter now means that if we know the response $g(t,\tau)$ ~~of~~ to the impulse $\delta_D(t-\tau)$, then we can construct the full response $o(t)$ as the linear combination

$$o(t) = \int_{-\infty}^{\infty} d\tau\, g(t,\tau)\, i(\tau)$$

~~[Compare~~

Now assume that the system (i.e filter) itself does not change with time.
[In the example above, this would mean that β & ω_0 are constants]

In other words, the filter is invariant under time translations.

This means the impulse response function only depends on the difference $t-\tau$.

$$g(t,\tau) = g(t-\tau)$$

In this case

$$o(t) = \int_{-\infty}^{\infty} d\tau\, g(t-\tau)\, i(\tau)$$
$$= (g * i)(t)$$

This means we can use Fourier analysis to construct the output

$$\tilde{o}(\omega) = \tilde{g}(\omega)\, \tilde{i}(\omega)$$

These concepts will return when discussing Green's functions.

[Jump to derivatives]

Poisson Summation formula [Discuss after derivatives & higher dimensions]

Define the "Dirac Comb"

$$C_D(x) \equiv \sum_{n=-\infty}^{\infty} \delta_D(x-nL) \quad , \quad L > 0$$

This is clearly periodic with period L & consists of an infinite series of spikes.
Take the fundamental interval to be $[-L/2, L/2]$

Since $C_D(x)$ is periodic, we can write it as a Fourier series:

$$C_D(x) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n x/L}$$

[2 because period is L not 2L]

where

$$C_n = \frac{1}{L}\int_{-L/2}^{L/2} dx\, e^{-2\pi i n x/L}\, C_D(x)$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} dx \sum_m \delta_D(x-mL)\, e^{-2\pi i n x/L}$$

[Only m=0 survives]

$$= \frac{1}{L}\int_{-L/2}^{L/2} dx\, \delta_D(x)\, e^{-2\pi i n x/L}$$

$$= 1/L$$

So
$$C_D(x) = \sum_{n=-\infty}^{\infty} \delta_D(x-nL) = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{2\pi i n x/L}$$

and the infinite sum of δ_D's is the same as an infinite sum over oscillating terms, which is possible due to destructive interference.

This can also be written as

$$\sum_{n=-\infty}^{\infty} \delta_D(n - x/L) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x/L}$$

Now consider Fourier transform of $C_D(x)$

$$\tilde{C}_D(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, C_D(x)$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, \delta_D(x - nL)\, e^{-ikx}$$

$$= \sum_{n=-\infty}^{\infty} e^{-iknL}$$

$$= \sum_{n=-\infty}^{\infty} e^{+iknL} \qquad [n \to -n]$$

i.e

$$\tilde{C}_D(k) = \sum_{n=-\infty}^{\infty} e^{2\pi i n (kL/2\pi)}$$

$$= \sum_{n=-\infty}^{\infty} \delta_D(n - kL/2\pi)$$

$$= \tilde{C}_D(k) = \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} \delta_D\left(k - \frac{2\pi n}{L}\right)$$

So the Fourier transform of the Dirac comb, of period L is another Dirac comb of period $2\pi/L$.

Now consider the convolution theorem with one function as the Dirac comb:

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, \tilde{f}(k)\, \tilde{C}_D(k) = \int_{-\infty}^{\infty} dy\, f(x-y)\, C_D(y)$$

Set $x=0$ & use the previous results

$$\frac{2\pi}{L} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}(k)\, \delta_D\!\left(k - \frac{2\pi n}{L}\right)$$
$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dy\, f(-y)\, \delta_D(y - nL)$$

$$\therefore \boxed{\frac{1}{L} \sum_{n=-\infty}^{\infty} \tilde{f}\!\left(\frac{2\pi n}{L}\right) = \sum_{n=-\infty}^{\infty} f(nL)}$$ [$n \to -n$ on RHS]

Poisson summation formula.

→ [See also Exc. 2 or 3]

+ Useful in series summation & sol^{n}s of diffusion eqn etc

E.g., recall $f(x) = \frac{a}{\pi} \frac{1}{x^2+a^2} \Rightarrow \tilde{f}(k) = e^{-a|k|}$, $a>0$

This gives

$$\frac{1}{L} \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|/L} = \frac{a}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2L^2+a^2} = \frac{a}{\pi L^2} \sum \frac{1}{n^2+(a/L)^2}$$

So if $\lambda \equiv a/L$, $\displaystyle\sum_{n=-\infty}^{\infty} e^{-2\pi\lambda|n|} = \frac{\lambda}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2+\lambda^2}$

$$\therefore \sum_{n=-\infty}^{\infty} \frac{1}{n^2+\lambda^2} = \frac{\pi}{\lambda}\left[1 + 2\sum_{n=1}^{\infty} e^{-2\pi\lambda n}\right]$$
$$= \frac{\pi}{\lambda}\left[1 + \frac{2e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\right]$$

Manipulating this & using $\cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2}$

& $\sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2}$

gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+\lambda^2} = \frac{\pi}{\lambda} \coth(\pi\lambda)$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\lambda^2} = \frac{\pi}{2\lambda} \left[\coth\pi\lambda - \frac{1}{\pi\lambda}\right]$$

& , e.g. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$

$$= \lim_{\lambda\to 0} \frac{\pi}{2\lambda} \left[\frac{1}{\pi\lambda} + \frac{\pi\lambda}{3} + O(\lambda^2) - \frac{1}{\pi\lambda}\right]$$

$$= \pi^2/6$$

Fourier transform of derivatives: ☆ [Lec 2]

$$g(x) = \frac{df}{dx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) \frac{d}{dx} e^{ikx}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \cdot ik\tilde{f}(k)$$

i.e $\tilde{g}(k) = ik\tilde{f}(k)$

→ generalises to $\boxed{\widetilde{f^{(n)}}(k) = (ik)^n \tilde{f}(k)}$

[Lec 2]

Fourier transform in higher dimensions:

In D-dimensions,

$$\tilde{f}(\vec{k}) \equiv \int d^D x \, e^{-i\vec{k}\cdot\vec{x}} f(\vec{x}) \qquad \text{[vector notation]}$$

or

$$\tilde{f}(\underset{\sim}{k}) = \int d^D x \, e^{-i\underset{\sim}{k}^T \underset{\sim}{x}} f(\underset{\sim}{x}) \qquad \text{[matrix notation]}$$

$$\int d^D x \longrightarrow \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_D$$

Inverse relation:

$$f(\vec{x}) = \int \frac{d^D k}{(2\pi)^D} e^{i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k})$$ & similarly matrix notn.

Dirac delta generalises to

$$\int \frac{d^D k}{(2\pi)^D} e^{i\underset{\sim}{k}^T(\underset{\sim}{x}-\underset{\sim}{x}')} = \delta_D(\underset{\sim}{x}-\underset{\sim}{x}') \equiv \prod_{i=1}^{D} \delta_D(x_i - x_i')$$

Vector & matrix notations will be used interchangeably, depending on convenience.

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Symmetries in higher dimensions:

- Radial symmetry $f(\vec{x}) = f(|\vec{x}|)$

- 2D:

$$\tilde{f}(\vec{k}) = \int d^2x \, e^{-i\vec{k}\cdot\vec{x}} f(\vec{x})$$

$$= \int_0^{2\pi} d\phi \int_0^{\infty} d\rho \cdot \rho \, f(\rho) \, e^{-ik\rho\cos\phi} \quad , \; k \equiv |\vec{k}|$$

[choosing direction of $\vec{k}$ as $\phi = 0$]

$$= \int_0^{\infty} d\rho \, \rho \, f(\rho) \underbrace{\int_0^{2\pi} d\phi \, e^{-ik\rho\cos\phi}}_{= J_0(k\rho)\cdot 2\pi}$$

$$= \tilde{f}(|\vec{k}|)$$

$\doteq$

$\therefore$ ~~$\tilde{f}(|\vec{k}|) =$~~

$$\doteq \tilde{f}(k) = 2\pi \int_0^{\infty} d\rho \cdot \rho \, J_0(k\rho) \, f(\rho)$$

Inverse relⁿ gives

$$f(\vec{x}) = f(\rho) = \frac{1}{(2\pi)^2} \int_0^{\infty} dk \cdot k \int_0^{2\pi} d\phi_k \, e^{ik\rho\cos\phi_k} \, \tilde{f}(k)$$

$$= \frac{1}{2\pi} \int_0^{\infty} dk \, k \, J_0(k\rho) \, \tilde{f}(k)$$

$\longrightarrow$ Zeroth order Hankel transform

– 3D: $f(\vec{x}) = f(r)$

$$\tilde{f}(\vec{k}) = \int_0^{2\pi} d\phi \int_0^{\infty} dr\, r^2 f(r) \int_{-1}^{1} d(\cos\theta)\, e^{-ikr\cos\theta}$$

$$= 2\pi \int_0^{\infty} dr \cdot r^2 f(r) \cdot \frac{1}{-ikr} \left[e^{-ik\mu} \Big|_{\mu=-1}^{1} \right]$$

$$= \tilde{f}(k) = 4\pi \int_0^{\infty} dr\, r^2 f(r) \left(\frac{\sin kr}{kr} \right)$$

Derivatives also generalise to higher dimensions

$$\nabla f(\vec{x}) \rightarrow \widetilde{\nabla f}(\vec{k}) = i\vec{k}\, \tilde{f}(\vec{k})$$

$$\nabla^2 f(\vec{x}) \rightarrow \widetilde{\nabla^2 f}(\vec{k}) = -k^2 \tilde{f}(\vec{k}) \quad \& \text{ so on}$$

→ Useful in solving, e.g., Poisson equation

$$\nabla^2 \Phi(\vec{x}) = 4\pi G \rho(\vec{x})$$

$$\Rightarrow \tilde{\Phi}(\vec{k}) \propto -k^2 \tilde{\rho}(\vec{k})$$

Typically these transforms must be performed numerically: method of choice → DFT