

# ADELES AND THE NUMBER THEORIST'S LANGLANDS

HONORS THESIS  
BY  
AARYAN SUKHADIA

SUPERVISED BY BRIAN CONRAD  
STANFORD UNIVERSITY

# ACKNOWLEDGEMENTS

TBC. Some key ones:

- Prof BC, for helping me choose topic and guiding me through some heinous material.
- Andrew Lee (classmate), for helping me work through some of the more nasty analysis bits.

# CONTENTS

Overview . . . . .	5
Format, Notation and Conventions . . . . .	6
<b>I Hello, Adele</b>	<b>10</b>
1 Adeles and Ideles . . . . .	10
1.1 The General Restricted Product . . . . .	11
1.2 Adeles, Ideles and Approximations . . . . .	16
1.3 Geometry of $\mathbb{A}_K/K$ . . . . .	18
1.4 Class Groups . . . . .	18
1.5 Characters on Adeles . . . . .	21
2 Something about Adelic points of schemes maybe?? . . . . .	21
<b>II The Thesis of Tate (a.k.a, <math>GL_1</math>)</b>	<b>22</b>
3 Idele Zeta Functions . . . . .	22
3.1 Local Zeta Integrals . . . . .	24
3.2 The Root and Riemann-Roch . . . . .	36
3.3 The Global Zeta Function . . . . .	36
4 Hecke's L-function . . . . .	37
5 Some Applications . . . . .	37
5.1 Dirichlet's Class Number Formula . . . . .	37
<b>III Modular Forms and Automorphic Representations (a.k.a <math>GL_2</math>)</b>	<b>38</b>
6 Automorphic Forms . . . . .	38
<b>IV The Conjectures of Langland (a.k.a <math>GL_n</math>)</b>	<b>39</b>
<b>Appendices</b>	<b>40</b>
A Fourier Theory of Locally Compact Groups . . . . .	40
A.1 Topologizing the dual . . . . .	41
A.2 Positive-Definite and Elementary Functions . . . . .	43
A.3 Fourier Inversion . . . . .	52
A.4 Pontryagin Duality . . . . .	57
B Local and Global Fields . . . . .	58
C Geometry of the Upper Half Plane . . . . .	59

C.1 Smth . . . . . 59

# INTRODUCTION

## OVERVIEW

Add smth here. Broadly: Expository work on adelic language, and what Langlands has to say about it (at least the NT side). We'll see how far into this we get before the thesis deadline.

Goal:

- Ch 1: Adeles - Motivation, basic properties, adelic points of a scheme maybe??
- Ch 2: Tate's Thesis, basically.

We follow the exposition of [RV98] and [Wei74]. - Ch 3: A primer on modular forms, and why they correspond to  $GL_2(A)$

- Ch 4: A (very) rough overview of what the Langlands conjectures say and some of the ideas that motivate(d) them.

May not fully reach Ch4 but good to keep as a dream for now.

# FORMAT, NOTATION AND CONVENTIONS

## Format

In terms of formatting, there will be four main types of colored environments in this text.

**DEFINITION 0.0.1.** A bright, almost-blaring red-backgrounded, red-striped box to house a vital Definition.

**EXAMPLE 0.0.2.** A purpled-striped environment to indicate an important or prototypical Example of a concept is contained in its confines.

**THEOREM 0.0.3.** A neutrally-toned gray-shaded box to house a Result, whether that be a theorem (such as this particular box), a proposition, etc.

**EXERCISE 0.0.A.** A cool green-colored box that houses an Exercise that (at least in the my experience) helps grasp or hone a core tool/technique. Exercises are meant to be reasonably doable by a reader who's paying attention, so if anything seems too hard re-read the subsection, and then e-mail me if it's still too hard.

As you might have deduced, Definitions, Examples and Results follow a [Subsection.Number] numbering scheme (except for corollaries, whose counters have the relevant result as their parent), all using a common counter, and Exercises follow a [Subsection.Letter] numbering scheme<sup>1</sup>.

As is standard in mathematical writing I've also filled this text with technically unnecessary but hopefully-enriching Remarks. Less-standard variants of asides I've used here are Sanity Checks (statements that should be intuitively obvious and verifiable in some epsilon time to an attentive reader) and Warnings (cautionary interventions about potentially confusing or unintuitive notation, terminology or concepts).

It is a corollary of Murphy's Law that this text will likely never be completely typo-free, but that doesn't mean it's something not to strive for. Please email me about any typographical, notational, mathematical or linguistic errors you might find.

---

<sup>1</sup>I'm of the opinion that if I ever require more than 26 exercises per subsection then either I'm offshoring too much to the reader or the subsection is too long.

## Assumed Knowledge

This text assumes some basic familiarity with elementary topology (including some elementary results on topological groups), measure theory, representation theory, functional analysis and algebraic number theory. Here are a bunch of quick litmus tests:

- Topology/Measure theory: Recall/understand that every locally compact group admits a left Haar measure.
- Representation theory: Recall/understand why any  $G$ -linear map between two irreducible representations is either the zero map or an algebraic isomorphism.
- Number Theory: Recall/understand the equivalent characterizations of Dedekind domains as (i) noetherian, one-dimensional, and integrally closed or (ii) unique factorization of ideals into primes, among others.
- Functional Analysis: Recall/understand why a scalar multiple of the identity is a normal operator.

The goal of this text is that if you're conceptually comfortable with some open neighborhood of the above results then it should be readable and understandable to you almost everywhere.

## Notation and Conventions

### Standard Mathematical Objects

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  will denote the set of Naturals, Integers, Rationals, Reals and Complex Numbers respectively. Naturals include 0, as we are not savages.
- $S^1$  denotes the circle, i.e the set of norm-one complex numbers.

### Algebraic Number Theory

- All rings are assumed to be unital and commutative.
- $R^\times$  will denote the set of invertible elements of a ring  $R$ .
- We use  $I \trianglelefteq R$  to say " $I$  is an ideal of the ring  $R$ ". If  $r_1, r_2, \dots$  is a set of elements in  $R$  we denote by  $\langle r_1, r_2, \dots \rangle$  the ideal generated by the  $r_i$ .
- For  $K$  a non-Archimedean local field with valuation  $v$ , we denote by  $\mathcal{O}_K$  the **ring of integers** (i.e elements of  $K$  with non-negative valuation). Moreover we denote by  $K_v$  the **completion** of  $K$  with respect to  $v$ , and  $\mathcal{O}_v$  the ring of integers of  $K_v$ .
- The symbol  $\mathfrak{p}$  ALWAYS refers to a prime ideal, and in the case that we are working in a DVR we denote its uniformizer by  $\pi$ . When necessary we add subscripts to indicate the relevant field we are working over.

### Functions

- $\mathcal{C}(X)$  denotes the set of continuous functions on a topological space  $X$ .
- $\text{Supp}(f)$  denotes the **support** (i.e pre-image of non-zero values) of a function.
- $\mathcal{C}_0(X)$  denotes the set of continuous functions on  $X$  that vanish at infinity (recall a function **vanishes at infinity** if  $f(x) \rightarrow 0$  as  $x \rightarrow \infty^2$ )
- $\mathcal{C}_c(X)$  denotes the set of continuous compactly supported functions on  $X$ . (Recall  $f$  is **compactly supported** its support has compact closure).

### Measures and Topological Groups

- The phrase "**almost everywhere**" means we are considering a set whose complement has measure zero.
- By a **locally compact group** we mean  $G$  is Hausdorff and every point has a compact neighborhood. Note we will ALWAYS assume the Hausdorff hypothesis.
- In many cases the locally compact groups we work with are also assumed to be abelian. We use the abbreviation "**LCA group**" in such cases.
- Recall that every locally compact group admits a (bi-invariant) **Haar measure**, which we often denote by  $dx$  or  $dy$  or  $dz$ , and so on.

---

<sup>2</sup>More precisely, the absolute value of  $x$ . We refine this notion in the text.



- Given a Borel (i.e measurable) set on  $A \subseteq G$  with Haar measure  $dx$ , for sake of clarity we denote by  $\text{Vol}(A, dx)$  the value  $dx(A)$ , i.e the measure of  $A$  with respect to  $dx$ .
- We use terms like **comeasure/covolume**/etc, to mean "measure/volume/etc of the complement".
- For  $G$  a topological group, we denote by  $\widehat{G}$  the **Pontryagin dual group** of continuous characters  $G \rightarrow S^1$ . See Appendix A for a detailed discussion.

### Misc Conventions

- When we have an infinite product we say "**almost everywhere**" to mean "in all but finitely many components".

## Chapter I

# HELLO, ADELE

*"In retrospect, we see now that the real numbers appear there as ... neither more nor less interesting to the arithmetician than its  $p$ -adic companions, and that there is at least one language and one technique, that of the **adeles**, for bringing them all together under one roof and making them cooperate for a common purpose."*

Preface to *Basic Number Theory*

ANDRÉ WEIL

## SECTION 1

### ADELES AND IDELES

**REMARK.** Some texts (e.g [Ber+13]) use the accented terms "adèles" and "idéles", but Chevalley introduced these terms more as a wordplay on "ideals," and so they don't necessarily exactly correspond to French words. Besides, I've never seen accents used for these terms by a French writer (see [Wei74] and [Lan94]), so I have chosen not to accent them in this text either.

We begin by computing the character group  $\widehat{\mathbb{Q}}$  of homomorphisms  $\mathbb{Q} \rightarrow S^1$ , a task often done with assumptions of deep algebraic number theory. We follow the more elementary method of [Con], and from it we motivate the object of our interest: adeles.

**WARNING.** Although  $\widehat{G}$  usually refers to group of *continuous characters* on  $G$  (see A.0.1), in this case we are NOT requiring that  $\chi \in \widehat{\mathbb{Q}}$  be a continuous map (unless of course we give  $\mathbb{Q}$  the discrete topology).

It is a standard result that the only continuous homomorphisms from  $\mathbb{R}$  to  $S^1$  are of the form

$$x \mapsto e^{i\lambda x}$$

for any  $\lambda \in \mathbb{R}$  (see the proof of A.0.2 for a more detailed discussion). It follows that the restriction of the above map to  $\mathbb{Q}$  is also a homomorphism to the circle. One might be tempted to think that this, too, characterizes all homomorphisms on  $\mathbb{Q}$ , but this is NOT the case.

[COMPLETE THIS]

## §1.1 The General Restricted Product

Let  $J$  be an indexing set (the reason for choosing  $J$  over  $I$  will become apparent later when we get introduced to the Ideles), suppose that for every  $v \in J$  we have a locally compact group indexed by  $G_v$ . We fix a finite subset  $J_\infty$  of  $J$  (which will later represent the infinite places), and suppose that for all  $v \notin J_\infty$  we have a compact open (and thus closed) subgroup  $H_v \leq G_v$ .

**DEFINITION 1.1.1.** The **restricted direct product** (also just called the **restricted product**) of the  $G_v$  with respect to the  $H_v$  is defined as

$$\prod_{v \in J} G_v := \left\{ (x_v) \in \prod_{v \in J} G_v : x_v \in H_v \text{ for all but finitely many } v \right\}$$

**REMARK (Notation).** The notation I've used for the restricted product (product and coproduct symbol overlayed on one another) is non-standard, but more distinct than the usual  $\prod'$ . I first saw this in Andrew Sutherland's lecture notes for 18.875 at MIT.

Let us denote this restricted product  $G$ . Note  $G \subseteq \prod G_v$  (the normal direct product of groups). We want to topologize  $G$ , but as it turns out, the subspace topology induced by the product topology on  $\prod G_v$  is insufficient. For reasons that will become clear later, we'll want sets like  $\prod H_v$  to be open, which won't be true in the subspace topology unless  $H_v = G_v$  for all but finitely many  $v \in J$ .

**DEFINITION 1.1.2.** The **restricted product topology** on  $\prod G_v$  is given by the basis

$$\left\{ \prod U_v : U_v \subseteq_{\text{open}} G_v, e \in U_v \text{ and } U_v = H_v \text{ for all but finitely many } v \right\}$$

**OBSERVATION.** Suppose we have a family of compact opens  $H'_v \subseteq G_v$  such that  $H'_v = H_v$  for all but finitely many  $v \in J$ . Then we note that  $\prod G_v$  wrt to the  $H_v$  is the same (both set-theoretically and topologically) as  $\prod G_v$  wrt to the  $H'_v$  (work this out rigorously if it's not clear!). It thus suffices to specify the  $H_v$  for all but finitely many  $v$ . A parallel observation we make is that for any finite positive integer  $n$  we have the topological isomorphism:

$$\begin{aligned} \prod_v G_v^n &\cong \prod_{v \in J} (\prod_v G_v) \\ (\dots, (g_{v,1}, \dots, g_{v,n}), \dots)_v &\mapsto ((g_{v,1})_v, \dots, (g_{v,n})_v)_v, \end{aligned} \tag{I.1}$$

where, naturally,  $G_v^n$  is the  $n$ -fold direct-product of  $G_v$  with itself.

We introduce a handy piece of notation that'll come in useful later. Let  $S$  be any finite subset of the index set  $J$  containing  $J_\infty$ . Then we define

$$G_S := \prod_{v \in S} G_v \times \prod_{v \notin S} H_v \tag{I.2}$$

For every  $S, T$  such that  $S \subseteq T$ , consider the inclusion maps  $i_{S,T} : G_S \hookrightarrow G_T$ . Then the restricted product  $G$  is in fact the colimit of this directed system, i.e:

$$G = \varinjlim_S G_S \tag{I.3}$$

**EXERCISE 1.1.A.** Verify the abstract-nonsense characterization of the restricted product given in Equation I.3 coincides (as a topological isomorphism) with the explicit description in 1.1.1.

**PROPOSITION 1.1.3.** Let  $G_v, H_v$  be as defined above and  $G$  be the restricted product. Then

- (a)  $G$  is a locally compact group.
- (b) A subset  $Y \subseteq G$  has compact closure iff  $Y \subseteq \prod K_v$  for some family of compact subsets  $K_v \subseteq G_v$  such that  $K_v = H_v$  for all but finitely many  $v$ .

*Proof.* **(a)**

We first verify (in perhaps a bit more rigor than required, but as good practice to get a feel for this construction) that it is indeed a topological group. The group structure is clear; for continuity of the group maps, recall any open set is of the form

$$\mathcal{U} := \prod_{v \in S} U_v \times \prod_{v \notin S} H_v$$

for some finite indexing set  $S$  and opens  $U_v \subseteq G_v$ . Since each  $H_v$  is a subgroup, the preimage of  $\mathcal{U}$  under the inversion map is:

$$\prod_{v \in S} (U_v)^{-1} \times \prod_{v \notin S} H_v,$$

which is also open since each  $G_v$  is a topological group, thereby giving us continuity of inversion.

For multiplication, the topological isomorphism in Equation I.1 gives us  $G \times G \cong \prod_v (G_v \times G_v)$ , allowing us to similarly look at preimages component by component. In particular, consider any element

$$\left( (k_v, k'_v)_{v \in S}, (k_v, k'_v)_{v \notin S} \right)$$

in the preimage of  $\mathcal{U}$  under the addition map. We note that by definition, away from some finite index set  $T \supseteq S$ , we must have that  $(k_v, k'_v) \in H_v \times H_v$  for all  $v \notin T$ , and clearly  $H_v \times H_v$  is in the pre-image of  $\mathcal{U}$ . For any  $v \in T$ , we know from the fact that  $G_v$  is a topological group that  $(k_v, k'_v)$  has some open neighborhood  $W_v \times W'_v$  contained in the pre-image of  $\mathcal{U}$ . Thus:

$$\left( (k_v, k'_v)_{v \in S}, (k_v, k'_v)_{v \notin S} \right) \in \prod_{v \in T} (W_v \times W'_v) \times \prod_{v \notin T} (H_v \times H_v) \subseteq \text{add}^{-1}(\mathcal{U}),$$

and so multiplication is continuous. Thus  $G$  is a topological group.

**WARNING.** Our work above seems (and may very well be) a tad pedantic with the details, and indeed several texts leave the fact of the restricted product being a topological group as a trivial observation. However it is worth being careful about constructions such as this; one might be tempted to think that if we have *any* directed system of topological groups with inclusion maps and define a colimit as in I.3, that the product must surely be a topological group. The editors of MIT's second-edition *Encyclopedic Dictionary of Mathematics* thought so too. However this turns out NOT the case: the colimit of the directed system of  $G_i := \mathbb{Q} \times \mathbb{R}^i$  does not have continuous addition; [TSH98] shows that we do get a nice output in this colimit construction when the groups are all locally compact.

We now show local compactness. Since each  $H_v$  is compact, Tychonoff tells us  $\prod_{v \in S} H_v$  is also compact. Thus  $G_S$  is the product of finitely many locally compact spaces with a compact space, and is thus itself locally compact. Moreover, the product topology on  $G_S$  is the SAME as the subspace topology induced by  $G$ , and since any  $(x_v) \in G$  is contained in such a  $G_S$ , we conclude  $G$  must be locally compact as well.

**(b)**

If  $Y$  is contained in some product of compact sets  $K = \prod K_v$  (which is compact by Tychonoff) its closure  $\bar{Y}$  is a closed subset of a compact subset of a Hausdorff space  $G$ , and thus  $\bar{Y}$  is compact too.

For the other direction, suppose  $\bar{Y}$  is a compact subset of  $G$ . Since the sets  $G_S$  defined above cover  $G$ , a finite number of  $G_S$  must cover  $\bar{Y}$ . Note a union of a finite number of  $G_S$  is just a set of the form  $G_{\mathcal{S}}$ , for  $\mathcal{S}$  being the union of all the  $S$  such that  $G_S$  is in the cover of  $\bar{Y}$ .

Consider the (continuous!) projection maps  $\rho_v : G \rightarrow G_v$ . Thus each  $\rho_v(\bar{Y}) \subseteq G_v$  is compact. And since all but finitely many components of  $G_{\mathcal{S}}$  are equal to  $H_v$ , the image  $\rho_v(\bar{Y}) \subseteq H_v$  for all but finitely many  $v$ .

Thus  $\bar{Y}$ , and therefore  $Y$ , is contained in a set of the desired form.  $\square$

**OBSERVATION.** For each  $v$  we have a natural topological embedding  $G_v \hookrightarrow G$  into the  $v$ -th component of the restricted product. Thus we naturally identify  $G_v$  with a closed subgroup of  $G$ .

We now proceed to flesh out the theory of measures on this restricted product.

**PROPOSITION 1.1.4.** For all but finitely many  $v$ , suppose  $dg_v$  be a (left) Haar measure on  $G_v$  normalized so that  $\text{Vol}(H_v, dg_v) = 1$ . Then there is a unique Haar measure  $dg$  on  $G$  such that the restriction  $dg_S$  of  $dg$  to any  $G_S$  is exactly the product measure.

*Proof.* Choose such a set  $S$  satisfying the conditions of finiteness and containin  $J_{\infty}$ , and define  $dg_S$  as the product of the measures  $dg_v$ . Note the measure of the compact group  $\prod_{v \in S} H_v$  is indeed finite, by how we've normalized things. It is a verification of axioms to deduce the rest that  $dg_S$  is indeed a Haar measure on  $G_S$ .

Suppose we have another finite set of indices  $T \supseteq S$ . Then, by construction, it follows that  $dg_S$  coincides with the restriction of  $dg_T$  to  $G_S \subseteq G_T$ .

Since  $G$  is locally compact, we know it has a (unique up to scaling!) Haar measure, which restricts to a Haar measure on any  $G_S$ . Fixing any  $S$ , we can scale our Haar measure on  $G$  so that when it restricts to  $G_S$  it exactly coincides with  $dg_S$ .  $\square$

Thus we write

$$dg = \prod_v dg_v \tag{I.4}$$

for the left Haar measure on  $G$ , oftentimes referred to as the measure induced by the component measures.

As in an AP Calc class, now that we've covered the "differentials", we move onto integration.

**PROPOSITION 1.1.5.** Let  $f$  be an integrable function on  $G$ . Then

$$\int_G f(g) dg = \lim_S \int_{G_S} f(g_S) dg_S$$

**EXERCISE 1.1.B.** Verify the above.

Our next result will be that integrals behave fairly well with respect to products. We first introduce a bunch of notation.

Let  $S_0$  be a finite set of indices containing  $J_\infty$  and the  $v$  for which  $\text{Vol}(H_v, dg_v) \neq 1$ .

Suppose for each  $v$  we have a continuous integrable  $f_v$  on  $G_v$  such that  $f_v|_{H_v} = 1$  for all  $v \notin S_0$ .

For  $g = (g_v) \in G$ , define

$$f(g) = \prod_v f_v(g_v)$$

**PROPOSITION 1.1.6.** With the notation above, the function  $f$  is well-defined and continuous. For any  $S \supseteq S_0$ , we have

$$\int_{G_S} f(g_S) dg_S = \prod_{v \in S} \int_{G_v} f_v(g_v) dg_v$$

*Proof.* Since  $g = (g_v)$  is in the restricted direct product, then  $g_v \in H_v$  for all but finitely many  $v$ . Combining this with the fact that  $f_v|_{H_v} = 1$  for almost all  $v$ , we deduce  $\prod f_v(g_v)$  is a finite product, and thus well-defined.

To show  $f$  is continuous, consider a base for  $G$  consisting of sets of the form

$$\prod U_v \times \prod H_v$$

where the first product contains all the (finitely many!) components of  $G$  where  $f_v$  is non-trivial. Thus  $f$  can be computed locally as a finite product of continuous functions, and is thus locally continuous, and thus globally.

The fact that the integral behaves well with products is a consequence of definitions itself, and the fact that  $f$  is a well-defined function.  $\square$

**COROLLARY 1.1.7.** With the notation as in the above result,

$$\int_G f(g) dg = \prod_v \int_{G_v} f_v(g_v) dg_v$$

and  $f \in L^1(G)$  provided that the RHS is a finite product.

*Proof.* By 1.1.5, we know  $f$  is integrable iff  $\lim_S \int_{G_S} f(g_S) dg_S < \infty$ , which is true iff

$$\lim_S \prod_{v \in S} \int_{G_v} f_v(g_v) dg_v = \prod_v \int_{G_v} f_v(g_v) dg_v < \infty,$$

by the equation in 1.1.6.  $\square$

**PROPOSITION 1.1.8.** Suppose all the  $G_v$  are abelian, and suppose we have functions  $\{f_v\}$  on  $G_v$  such that they are the characteristic functions for  $H_v$  for all but finitely many  $v$ . Then  $f$ 's Fourier transform is integrable and given by

$$\widehat{f}(g) = \prod_v \widehat{f}_v(g_v)$$

*Proof.* Firstly note that even in the non-abelian case, by 1.1.7, we have

$$\int_G f(g) dg = \prod_v \int_{G_v} f_v(g_v) dg_v = \prod_{v \in S} \int_{G_v} f_v(g_v) dg_v$$

for some finite set  $S$ , since  $f_v$  is the characteristic function of  $H_v$ . Thus we have a finite product of continuous functions, and thus integrable.

In the abelian case, let  $\chi = (\chi_v)$  denote a character of  $G$ , and define

$$h = \prod_v (h_v := f_v \chi_v)$$

Since  $\chi$  is unitary,  $h$  must be integrable, and applying the identity in 1.1.7, we get our desired result in terms of the Fourier transform.  $\square$

From now on assume all groups are abelian. We want to build a measure on the Pontryagin dual  $\widehat{G}$  that is dual to  $dg$  in the sense of the Fourier inversion theorem (A.3.3). For each  $v$ , let

$$d\chi_v = \widehat{dg_v}$$

denote the dual measure to  $dg_v$  on  $\widehat{G}_v$ . Note for any  $f_v \in L^1(G_v)$ , we have that

$$\widehat{f}_v(\chi_v) = \int_{G_v} f_v(g_v) \overline{\chi_v}(g_v) dg_v$$

by definition. If  $f_v$  is the characteristic function of  $H_v$  (which is visibly integrable and positive definite) then

$$\widehat{f}_v(\chi_v) = \int_{H_v} \chi_v(g_v) dg_v = \begin{cases} \text{Vol}(H_v, dg_v), & \chi_v|_{H_v} = 1 \\ 0, & \text{otherwise,} \end{cases}$$

since the integral of a non-trivial character over a group is zero. In other words, if  $H_v^*$  is the subgroup of  $\widehat{G}_v$  of characters trivial on  $H_v$ , then  $\widehat{f}_v(\chi_v)$  is the characteristic function of  $H_v^*$  scaled by  $\text{Vol}(H_v, dg_v)$ . Thus, computing the fourier inversion formula, for  $y \in H_v$  we have:

$$f_v(y) = \int_{\widehat{G}_v} \widehat{f}_v(\chi_v) \chi_v(y) d\chi_v = \text{Vol}(H_v, dg_v) \int_{H_v^*} d\chi_v = \text{Vol}(H_v, dg_v) \text{Vol}(H_v^*, d\chi_v) = 1,$$

where the last equality follows from our choice of  $f_v$  as the characteristic function of  $H_v$  and  $y \in H_v$ . Recall in our construction of  $dg$  that  $dg_v$  assigns volume 1 to  $H_v$  for all but finitely many  $v$ . The above equation then tells us

$\text{Vol}(H_v, d\chi_v) = 1$  almost everywhere as well, and so we can define

$$d\chi := \prod_v d\chi_v \quad (I.5)$$

as in I.4, and this is also well-defined.

**PROPOSITION 1.1.9.** The measure  $d\chi$ , as defined in I.5, is dual to  $dg$ , defined in I.4, in the sense that:

$$f(g) = \int_{\hat{G}} \hat{f}(\chi) \chi(g) d\chi$$

for all  $f \in V^1(G)$ .

*Proof.* It suffices [WHY] to check on a functions that are product functions of the form

$$f = \prod_v f_v,$$

where  $f_v$  is the characteristic function of  $H_v$  for almost all  $v$ . For any input  $g$ , the LHS above is  $\prod f_v(g_v)$  by definition, and using the fact that each  $d\chi_v$  is dual to  $dg_v$ , we get:

$$f_v(g_v) = \int_{\hat{G}_v} \hat{f}_v(\chi_v) \chi_v(g_v) d\chi_v,$$

and multiplying them together (while applying Fubini) gives us our desired result.  $\square$

## §1.2 Adeles, Ideles and Approximations

Let  $K$  be a global field, and let  $K_v$  denote its completion at a place  $v$ . With respect to addition, note  $K_v$  is a locally compact additive group. For all finite places  $v$ , recall  $K_v$  admits its local ring of integers  $\mathcal{O}_v$  as an open compact subgroup.

**DEFINITION 1.2.1.** The **adele group**  $\mathbb{A}_K$  of a global field  $K$  is the restricted direct product of  $K_v$  with respect to the open subgroups  $\mathcal{O}_v$ . Since this admits a natural ring structure as well, it is also referred to as the **adele ring**.

Note we have an embedding

$$\begin{aligned} K &\hookrightarrow \mathbb{A}_K \\ x &\mapsto (x, x, \dots) \end{aligned}$$

This is well-defined since  $K$  always has a natural embedding in  $K_v$  for any place  $v$ , and by unique factorization, every element is a local integer for all but finitely many  $v$ .

Similarly, we can consider  $K^\times$ , the locally compact group of units of  $K$  wrt to multiplication. Then the local units  $\mathcal{O}_v^\times$  are an open compact subgroup of each  $K_v^\times$ .



**DEFINITION 1.2.2.** The **idele group**  $\mathbb{I}_K$  of a global field  $K$  is the restricted direct product of the  $K_v^\times$  with respect to  $\mathcal{O}_v^\times$ .

We have a similar embedding

$$\begin{aligned} K^\times &\hookrightarrow \mathbb{I}_K \\ x &\mapsto (x, x, \dots) \end{aligned}$$

which is again well-defined.

The induced topologies on the adèle and idele group are called the **adèlic** and **idèlic** topologies, respectively.

**WARNING.** Although  $\mathbb{I}_K$  is algebraically isomorphic to  $A_K^\times$ , this is not a topological embedding. For example, taking  $K = \mathbb{Q}$  and  $S$  being any finite collection of primes, and let  $U_p$  be any neighborhood of 1 in  $\mathbb{Q}_p$  for  $p \in S$ . Then

$$\left( \prod_{p \in S} N_p \times \prod_{p \notin S} \mathbb{Z}_p \right) \cap \mathbb{I}_{\mathbb{Q}} \not\subseteq \mathbb{R}^\times \times \prod_{p < \infty} \mathbb{Z}_p^\times = A_{\mathbb{Q}}^\times$$

Note we can select some  $(x_p) \in \mathbb{I}_{\mathbb{Q}}$  such that  $x_p \in \mathbb{Z}_p \setminus \mathbb{Z}_p^\times$  for some  $p \notin S$ , which therefore excludes it from being in  $A_{\mathbb{Q}}^\times$  on the RHS. In particular, the idèlic topology is finer than the induced subspace topology.

Let  $S_\omega$  denote the infinite places of  $K$ . We denote by  $A_\omega$  the open subgroup  $A_{S_\omega} \subseteq A_K$ , i.e the elements of the adèle group where all the components at the finite places are integral.

**THEOREM 1.2.3 (Approximation).** For every global field  $K$ ,

$$A_K = K + A_\omega$$

where  $K$  is identified with its natural diagonal embedding.

*Proof.* Given some  $x = (x_v) \in A_K$ , we want to find some  $y \in K$  such that  $x_v - y \in \mathcal{O}_v$  for all  $v$ .

By definition,  $x_v \in \mathcal{O}_v$  for all but finitely many places  $v$ , each corresponding to a prime  $\mathfrak{p}_v \trianglelefteq \mathcal{O}_v$ . Letting  $p_v \in \mathcal{O}_K$  be an element of  $\mathfrak{p}$ -adic valuation 1 in each of these ideals, note that there must exist some  $n_v \in \mathbb{N}$  such that  $p_v^{n_v} x_v \in \mathcal{O}_v$ . Set

$$q := \prod_{v \in T} p_v^{n_v}$$

where  $T$  is the (finite!) set of finite places where  $x_v \notin \mathcal{O}_v$ . By CRT, we can find some  $\lambda \in \mathcal{O}_K$  such that  $qx_v - \lambda \in \mathfrak{p}_v^{n_v}$  for all  $v \in T$ . Letting  $y = \lambda/q$ , then by construction  $x_v - y \in \mathcal{O}_v$  for all  $v$ , as desired. [FILL IN MORE DETAIL LATER]  $\square$

**COROLLARY 1.2.4.** With the notation above,  $K \cap A_\omega = \mathcal{O}_K$

**SANITY CHECK.** Verify the above corollary.

### §1.3 Geometry of $\mathbb{A}_K/K$

We explore the behavior of the adeles under base-change.

**LEMMA 1.3.1.** Take  $E/K$  a finite extension of global fields, and let  $\{u_1, \dots, u_n\}$  be a  $K$ -basis of  $E$ . Then the natural map

$$\begin{aligned} \alpha : \prod_{j=1}^n \mathbb{A}_K &\rightarrow \mathbb{A}_E \\ ((x_{v,j})_v)_j &\mapsto \sum_j u_j (x_{v,j})_v \end{aligned}$$

is an isomorphism of topological groups.

FINISH LATER. □

**THEOREM 1.3.2.** Any global field is a discrete, cocompact subgroup of its adèle ring.

*Proof.* Let  $K_0$  denote the smallest global subfield of  $K$  (i.e  $\mathbb{Q}$  or  $\mathbb{F}_p(t)$ ), and let  $n := [K : K_0]$ . Then by 1.3.1, we have the following commutative diagram:

$$\begin{array}{ccc} \prod_{j=1}^n \mathbb{A}_{K_0} & \longrightarrow & \mathbb{A}_K \\ \uparrow & & \uparrow \\ \prod_{j=1}^n K_0 & \longrightarrow & K \end{array}$$

where the horizontal maps are topological isomorphisms. Thus  $\mathbb{A}_K/K$  is compact iff  $(\mathbb{A}_{K_0}/K_0)^n$  is compact, which is true iff  $\mathbb{A}_{K_0}/K_0$  is compact.

Thus it suffices to consider  $K = K_0$ , in which case we have exactly one infinite place. Define a subset  $C$  of the adèle group by

$$C := \left\{ x \in \mathbb{A}_K : |x_\infty|_\infty \leq \frac{1}{2} \text{ and } |x_v|_v \leq 1, \forall v \neq \infty \right\}$$

If we show that  $C \cap K = \{0\}$  and  $\mathbb{A}_K = C + K$ , we will be done, since  $C \cong \mathbb{A}_K/K$ .

If  $x \in K \cap C$ , then we must have  $x \in \mathcal{O}_K$ , since  $x \in \mathcal{O}_v$  for every place  $v$ . If  $K = \mathbb{Q}$  then this condition ( $x \in \mathbb{Z}$ ) combined with the condition that  $|x_\infty|_\infty \leq 1/2$  forces  $x = 0$ . If  $K = \mathbb{F}_p(t)$ , then let the infinite place be the one defined by  $1/t$ . Then  $|x|_\infty = p^{-\text{ord}_{1/t}(x)}$ , which is  $\geq 1$  for anything in  $\mathcal{O}_K = \mathbb{F}_p[t]$ . Thus  $C \cap K = \{0\}$ .

[FINISH TEDIOUS DETAILS HERE] □

[FINISH STUFF HERE]

### §1.4 Class Groups

**DEFINITION 1.4.1.** For a global field  $K$ , the quotient

$$C_K := \mathbb{I}_K / K^\times$$

is called the **idele class group** of  $K$ .

**WARNING.** As seen in the previous section,  $\mathbb{A}_K/K$  is compact. One might be led to think  $\mathbb{I}_K/K^\times$  is also compact but this is NOT the case, as we'll see later.

To analyze this, we first standardize the absolute value functions we're using.

**DEFINITION 1.4.2.** For  $k$  a local field, the **normalized** absolute value  $|\cdot|_k$  on  $k$  is defined as follows:

- (a) For  $k = \mathbb{R}$  we set  $|\cdot|_k$  to be the usual absolute value function.
- (b) For  $k = \mathbb{C}$  we set  $|z|_k := z\bar{z}$ , which is the square of the usual absolute value function on  $\mathbb{C}$ .
- (c) For  $k$  non-Archimedean with uniformizer  $\pi$  and residue order  $q$ , we set  $|\pi|_k := q^{-1}$ . (Note this extends uniquely to  $k$ ).

Recall that if  $k'/k$  is a finite extension of local fields then  $|x|_{k'} = |N_{k'/k}(x)|_k$ .

**DEFINITION 1.4.3.** Let  $K$  be a global field, and let  $|\cdot|_v$  denote the normalized absolute value on the completion  $K_v$ . Then the **idelic absolute value** is defined by:

$$\begin{aligned} |\cdot|_{\mathbb{A}_K} : \mathbb{I}_K &\rightarrow \mathbb{R}_{>0} \\ x = (x_v)_v &\mapsto \prod_v |x_v|_v \end{aligned}$$

**SANITY CHECK.** Why is the product defined above well-defined?

**THEOREM 1.4.4 (Artin's Product Formula).** Let  $K$  be a global field. Then for every  $x \in K^\times$  we have  $|x|_{\mathbb{A}_K} = 1$

*Proof.* Let  $E/K$  be finite separable. Then

$$\begin{aligned} |x|_{\mathbb{A}_E} &= \prod_{u \in \mathcal{P}_K} \prod_{v \in \mathcal{P}_E, v|u} |x|_v \\ &= \prod_{u \in \mathcal{P}_K} \prod_{v \in \mathcal{P}_E, v|u} |N_{E_v/K_u}(x)|_u \\ &= \prod_u |N_{E/K}(x)|_u \end{aligned}$$

[ASK QUESTION ABOUT MISSING EXPONENT ABOVE] □

We are now ready to define a subgroup of the ideles for which the embedding of  $K$  IS cocompact.

**DEFINITION 1.4.5.** For  $K$  a global field, the **ideles of norm one** are:

$$\mathbb{I}_K^1 := \ker(|\cdot|_{\mathbb{A}_K})$$

and the **norm-one idele class group** is the quotient:

$$C_K^1 := \mathbb{I}_K^1 / K^\times$$

Note by the Artin product formula (1.4.4) that  $K^\times \subseteq \mathbb{I}_K^1$ , so this quotient is well-defined. We also have the SES

$$1 \rightarrow C_K^1 \rightarrow C_K \rightarrow |\mathbb{I}_K|_{\mathbb{A}_K} \rightarrow 1.$$

**THEOREM 1.4.6.** The norm-one idele class-group is always compact.

*Proof.* Recall from the proof of 1.3.2 that there exists some compact subset  $Y \subseteq \mathbb{A}_K$  such that  $\mathbb{A}_K = K + Y$ . Fix a Haar measure  $\mu$  on  $\mathbb{A}_K$  (which exists as it's a locally compact group), then  $\mu(Y) < \infty$ . Choose a compact subset  $Z \subseteq \mathbb{A}_K$  such that  $\mu(Z) > \mu(Y)$ . Consider the following two subsets:

$$Z_1 := \{z_1 - z_2 : z_1, z_2 \in Z\}$$

$$Z_2 := \{z_1 z_2 : z_1, z_2 \in Z\}$$

By continuity of addition and multiplication these two sets are also compact. Since  $K$  is discrete in  $\mathbb{A}_K$ , we have  $K \cap Z_2$  is finite, with non-zero elements  $x_1, \dots, x_r$ . [LOTS OF EXERCISE NONSENSE READ IN DETAIL and COMPLETE LATER]  $\square$

For a given finite set  $S$  of places of  $K$  containing  $S_\infty$ , the Archimedean places, it is useful to have  $S$ -versions of the aforementioned groups.

**WARNING (SOME WEIRD SHIT ABOUT NOTATION EXCLUDING INFINITE PLACES).**

**DEFINITION 1.4.7.** Take  $S$  a finite set of places of a global field  $K$  containing the Archimedean places. Then:

(a) The  **$S$ -adeles** of  $K$  are defined as:

$$\mathbb{A}_{K,S} := \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$$

(b) The **ring of  $S$ -integers** is defined as:

$$\mathcal{O}_{K,S} := K \cap \mathbb{A}_{K,S}$$

i.e the elements of  $K$  that are  $v$ -integral for all  $v \notin S$ .

(c) The  **$S$ -ideles** are defined as:

$$\mathbb{I}_{K,S} := \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

(d) The  **$S$ -ideles of norm one** defined as:

$$\mathbb{I}_{K,S}^1 := \mathbb{I}_K^1 \cap \mathbb{I}_{K,S}$$

OBSERVATION.

EXERCISE 1.4.A. Prove the above observation.

## §1.5 Characters on Adeles

[Insert stuff here]

We now proceed to a slightly more advanced task

## SECTION 2

# SOMETHING ABOUT ADELIC POINTS OF SCHEMES MAYBE??

## Chapter II

# THE THESIS OF TATE (A.K.A, $GL_1$ )

*"In hindsight, Tate's work may be viewed as ... a fundamental reference and starting point for anyone interested in the modern theory of automorphic representations."*

---

Ch. 6 of *An Introduction to the Langlands Program*

STEPHEN S. KUDLA

## SECTION 3

### IDELE ZETA FUNCTIONS

A lot of the major strides in classical analytic number theory were made in the 19th and 20th century, where it was discovered that a lot of information about the structure of prime numbers could be deduced by studying the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is absolutely convergent for complex  $s$  such that  $\Re(s) > 1$ . This function admits an analytic continuation to the entire complex plane, save for a simple pole at  $s = 1$ , and satisfies the functional equation:

$$\xi(s) = \xi(1-s),$$

where the Riemann-xi function is defined as

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

Much of the early work for  $\zeta(s)$  for real  $s$  was done by Euler, who established the classic Euler product expansion

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

A generalization of the zeta function can be done via Dirichlet series. Given any multiplicative sequence  $\{a_n\}$  (i.e.  $\gcd(m, n) = 1 \implies a_m a_n = a_{mn}$ ) we can create the series

$$L(s) := \sum \frac{a_n}{n^s}.$$

**EXAMPLE 3.0.1 (Dirichlet  $L$ -function).** Recall a **Dirichlet character** with modulus  $m \in \mathbb{Z}_{>0}$  is a completely multiplicative function  $\chi : \mathbb{Z} \rightarrow \mathbb{C}$  that is  $m$ -periodic, nonzero on any input coprime to  $m$  and zero otherwise. Given a Dirichlet character  $\chi$ , its **Dirichlet  $L$ -series** is given by:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Studying the analytic properties of the various  $L$ -series is vital for decoding properties of prime numbers in the integers. In particular, it is famously used to show Dirichlet's theorem that any arithmetic series with coprime base and difference contains infinitely many primes.

As a generalization, we can craft similar  $L$ -functions over number fields.

**EXAMPLE 3.0.2 (Generalized Dedekind Zeta Function).** Let  $K$  be a number field, and let  $\chi$  be a character of standard ideal class group  $\text{Cl}(K)$ , then we can construction the Dirichlet series:

$$L(s, \chi) = \sum_{(0) \neq \mathfrak{a} \leq \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where  $N(\mathfrak{a})$  is the standard ideal norm taken over  $\mathbb{Q}$  as the base field. When  $\chi = 1$  this is known as the **Dedekind zeta function**.

It turns out the residues of the analytic continuations these generalized Dedekind zeta functions encode information about the class numbers of the their number fields.

One can generalize the above even further to consider crafting a series  $L(s, \chi)$  for any  $\chi$  that is a continuous character of the idele class group  $C_K$ . It is a famous result of Hecke that  $L(s, \chi)$  has analytic continuation and satisfies an appropriate functional equation for any number field  $K$  and any idele class character  $\chi$ . The original formulation of this result involved a complicated computation with these things called theta functions. Moreover in the functional equation a so-called *root number*  $W(\chi)$  associated to each  $\chi$  kept popping up, with no good explanation for its value.

Under the guidance of Emil Artin, in his 1950 thesis John Tate made use of Fourier analysis on adèle groups to reprove the results of Hecke and establish these "local" functional equations that gave an "explicit factorization" of the root number.

Because of the limits of space and time we do not go into Hecke's original approach via theta functions here, but In his thesis, Tate worked with some ad-hoc function spaces over local and global fields. We standardize our workings over the so-called *Schwarz-Bruhat functions* on local fields. Over an Archimedean number field (i.e  $\mathbb{R}$  or  $\mathbb{C}$ ), recall a **Schwarz** function (more commonly known as a *rapidly-decreasing function*) is a smooth  $\mathbb{C}$ -valued

function  $f$  such that

$$\lim_{x \rightarrow \infty} p(x)f(x) = 0$$

for all polynomials  $p(x)$ .

**DEFINITION 3.0.3.** A **Schwarz-Bruhat** function on a local field  $F$  is a Schwarz function if  $F$  is Archimedean and a smooth<sup>a</sup>, compactly-supported function in  $F$  is non-Archimedean. We denote by  $S(F)$  the complex vector space of Schwarz-Bruhat functions on  $F$ .

<sup>a</sup>A function on a non-Archimedean local field is smooth if it is locally constant.

We use these functions because they guarantee that taking their integral over the entire group is well-defined.

### §3.1 Local Zeta Integrals

Before analyzing the global zeta function of a function on an adèle group of a global field, we first understand the induced zeta functions on each of the local components. The goal of this subsection will be to introduce so-called *local  $L$ -factors* associated to a character  $\chi$  (and its dual under the Fourier transform  $\chi^\vee$ ) of a local field. In particular, these local  $L$ -factors will be realized as the gcd of some local zeta functions. We start with a slew of notations and definitions, so buckle up.

Let  $F$  be a local field with absolute value  $|\cdot|$  and Haar measure  $dx$ . Let  $X(F^\times) := \text{Hom}_{\text{cont}}(F^\times, \mathbb{C}^\times)$  be the continuous group homomorphisms into the complex numbers. The elements  $\chi \in X(F^\times)$  will be referred to as **quasi-characters** of  $F^\times$ .

**REMARK.** Sometimes the elements of  $X(F^\times)$  are just called **characters** instead, and continuous homomorphisms into  $S^1$  (i.e elements of  $\widehat{F^\times}$ , with the notation of A) are called **unitary characters**. We use the other convention because it is consistent with Tate's own formulation in his original thesis and because it gives us an excuse to say and write the prefix "quasi". Also note that we use the hyphenation convention with "quasi".

When  $F$  is non-Archimedean, let  $\mathcal{O}_F$  denote its ring of integers,  $\mathfrak{p}$  its maximal ideal with uniformizer  $\pi$  and residue field  $\mathbb{F}_q$ . Recall that we can write  $F^\times \cong U_F \times S_F$ , where  $U_F$  is the (compact!) subgroup of absolute-value-one elements, and  $S_F = |F^\times|$ , i.e. the image of  $F^\times$  in the valuation map<sup>1</sup>.

We make the observation that any quasi-character  $\chi \in X(F^\times)$  factors into a product:

$$\chi = \mu |\cdot|^s,$$

where  $s \in \mathbb{C}$  and  $\mu$  is the pullback of a character on  $U_F$  (defined uniquely by restriction of  $\chi$ !). This is because compactness of  $U_F$  forces any quasi-character on it to have image in  $S^1$ , and the only quasi-characters on  $S_F$  are maps  $t \mapsto t^s$ .

**DEFINITION 3.1.1.** For any quasi-character  $\chi = \mu |\cdot|^s \in X(F^\times)$ , we call  $\Re(s)$  the **exponent** of  $\chi$ . Furthermore, we say  $\chi$  is **unramified** if  $\mu = 1$ .

<sup>1</sup>recall this is  $\mathbb{R}_{>0}$  in the Archimedean case and  $q^{\mathbb{Z}}$  otherwise



**WARNING.** Recall that for any  $q \in \mathbb{R}_{>0}$  we have the formula:

$$q^i = \cos(\ln(q)) + i \sin(\ln(q)).$$

In particular, in the non-Archimedean case the value of  $s$  is NOT uniquely determined (e.g  $i$  and  $(1 + 2\pi)i$  would correspond to the same map), which is why we take the exponent to be the real part, which IS uniquely determined.

We now define the local  $L$ -factors  $L(\chi)$ .

(1) In the non-Archimedean case, we set:

$$L(\chi) := \begin{cases} (1 - \chi(\pi_F))^{-1}, & \text{if } \chi \text{ is unramified,} \\ 1, & \text{otherwise.} \end{cases} \quad (\text{II.1})$$

(2) In the Archimedean case we recall by Gelfand-Mazur that  $F = \mathbb{C}$  or  $\mathbb{R}$ .

(i) If  $F = \mathbb{C}$ , note any  $\chi$  is of the form:

$$\chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$$

for  $s \in \mathbb{C}$  and  $n \in \mathbb{Z}$  uniquely defined. Here we use our result in A.0.2 to characterize the characters of  $U_{\mathbb{C}} = S^1$  and writing our domain in polar coordinates automatically makes  $r$  equal to the absolute value of our complex input. We then set:

$$L(\chi_{s,n}) := \Gamma_{\mathbb{C}}(s + \frac{|n|}{2}) := (2\pi)^{-(s + \frac{|n|}{2})} \Gamma(s + \frac{|n|}{2}). \quad (\text{II.2})$$

(ii) If  $F = \mathbb{R}$  the  $U_F = \{\pm 1\}$ . Here any quasi-character is of the form  $\chi = \mu |\cdot|^s$ , where  $\mu$  and  $s$  are uniquely defined, and  $\mu$  is either trivial or the sign homomorphism  $\text{sgn} : x \mapsto x/|x|$ . We then set:

$$L(\chi) := \begin{cases} \Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2), & \mu = 1, \\ \Gamma_{\mathbb{R}}(s + 1), & \mu = \text{sgn}. \end{cases} \quad (\text{II.3})$$

**DEFINITION 3.1.2.** If  $\chi \in X(F^\times)$  and  $s \in \mathbb{C}$  then  $\chi |\cdot|^s$  is also a quasi-character, and we use the following notation for its local  $L$ -factor:

$$L(s, \chi) := L(\chi |\cdot|^s).$$

Moreover we define the **shifted dual** of  $\chi$  by:

$$\chi^\vee := \chi^{-1} |\cdot|,$$

so that

$$L((\chi |\cdot|^s)^\vee) = L(1 - s, \chi^{-1}).$$

We now state a result that we will later extend to adèle groups.

**PROPOSITION 3.1.3.** Any local field  $F$  is isomorphic (as a LCAG) to  $\widehat{F} = \text{Hom}_{\text{cont}}(F, S^1)$ . For any fixed  $\psi \in \widehat{F}$ , the map

$$a \mapsto \psi_a := \psi(a \cdot -)$$

is an topological group isomorphism.

**EXERCISE 3.1.A.** Prove 3.1.3.

In such a case of a *self-dual* local field  $F$ , we can also speak of its Haar measure  $dx$  being self-dual, with the dual defined in A.3.4. Define

$$d^*x := c \cdot \frac{dx}{|x|} \quad (\text{II.4})$$

for some  $c \in \mathbb{R}_{>0}$ , which we always set to 1 in the Archimedean case. Then  $d^*x$  is a Haar measure on  $F^\times$ . Fixing an additive character  $\psi \in \widehat{F}$ , for any  $f \in S(F)$  we can define its Fourier transform by:

$$\widehat{f}(y) := \int_F f(x) \psi(xy) dx. \quad (\text{II.5})$$

Note this almost the same formulation as in A.3.1, though we drop the conjugation in the second factor of the integrand, as a simple matter of notational convenience to make our lives easier in coming calculations. The function  $\widehat{f}$  is well-defined and in-fact again lies in  $S(F)$ , though it depends on our choice of  $\psi$  and  $dx$ . In Tate's original thesis he normalizes his Haar measure to be self-dual relative to  $\psi$ , from which he obtained the nice identity  $f(x) = \widehat{\widehat{f}}(x)$ . We avoid this normalization for the local non-Archimedean case, as it allows for more flexibility when we actually want to apply our theory more concretely.

**DEFINITION 3.1.4.** Given  $f \in S(F)$  and  $\chi \in X(F^\times)$ , we define the associated **local zeta function** to be:

$$Z(f, \chi) := \int_{F^\times} f(x) \chi(x) d^*x.$$

The following is the main result of this subsection.

**THEOREM 3.1.5.** Let  $F$  be a local field, and take  $f \in S(F)$  and  $\chi = \mu |\cdot|^s \in X(F^\times)$  with exponent  $\sigma = \Re(s)$ . Then:

- (a) If  $\sigma > 0$  then the local zeta function  $Z(f, \chi)$  is absolutely convergent.
- (b) If  $\sigma \in (0, 1)$  then the following functional equation holds:

$$Z(\widehat{f}, \chi^\vee) = \gamma(\chi, \psi, dx) Z(f, \chi)$$

for some  $\gamma$  independent of  $f$  and meromorphic as a function of  $s$ .

(c) There exists a factor  $\varepsilon(\chi, \psi, dx)$  nonzero for all inputs and satisfies:

$$\gamma(\chi, \psi, dx) = \varepsilon(\chi, \psi, dx) \frac{L(\chi^\vee)}{L(\chi)}$$

**OBSERVATION.** Note that part (a) tells us that  $Z(f, \chi)$  is absolutely convergent to the right of 0 on the complex plane, from which it follows by construction that  $Z(\hat{f}, \chi^\vee)$  is absolutely convergent to the left of 1. Using this domain overlap along with part (b), we see that this theorem automatically yields a meromorphic continuation of the local zeta function.

*proof of (a) in 3.1.5.* To show  $Z(f, \chi)$  is absolutely convergent, using our relationship between our Haar measures in Equation II.4 and the fact that  $\mu$  is unitary, it suffices to show that:

$$I(f, \sigma) = c \int_{F \setminus \{0\}} |f(x)| |x|^{\sigma-1} dx < \infty.$$

In the Archimedean case, finiteness of  $I(f, \sigma)$  follows from the fact that  $f$  is a Schwarz function, and thus rapidly decaying as  $|x|$  approaches infinity. Moreover, around zero since  $\sigma > 0$  the  $|x|^{\sigma-1}$  part is integrable, so we are happy.

In the non-Archimedean case,  $f$  is locally constant with compact support, it must factor through some finite group quotient of the form  $\mathfrak{p}^m/\mathfrak{p}^n$ , for some integers  $m \leq n$  (here  $\mathfrak{p}^k = \langle \pi^k \rangle$  represents the fractional ideal of  $\mathcal{O}_F$  generated by the  $k$ -th power of its uniformizer). By linearity and translation-invariance of the Haar measure, it suffices to check finiteness of the integral for functions  $f$  that are just the characteristic functions of the various ideals  $\mathfrak{p}^j$ .

Using unique factorization in any fractional ideal, we can decompose it into a disjoint union:

$$\mathfrak{p}^j \setminus \{0\} = \bigsqcup_{k=j}^{\infty} \pi^k \mathcal{O}_F^\times.$$

Thus our integral becomes:

$$\begin{aligned} I(f, \sigma) &= c \int_{F \setminus \{0\}} |f(x)| |x|^{\sigma-1} dx \\ &= \int_{F^\times} |f(x)| |x|^\sigma d^*x \\ &= \sum_{k \geq j} \int_{\pi^k \mathcal{O}_F^\times} |x|^\sigma d^*x \\ &= \sum_{k \geq j} \text{Vol}(\mathcal{O}_F^\times, d^*x) q^{-k\sigma} \\ &= \text{Vol}(\mathcal{O}_F^\times, d^*x) \frac{q^{-j\sigma}}{1 - q^{-\sigma}} \end{aligned}$$

which is finite when  $\sigma > 0$ , as desired. □

To prove the next part of our main theorem, we use a lemma that was key in Tate's original thesis.

**LEMMA 3.1.6.** Select some auxiliary function  $g \in S(F)$ . Then for all quasi-characters  $\chi$  with exponent  $\sigma \in (0, 1)$ , we have:

$$Z(f, \chi)Z(\widehat{g}, \chi^\vee) = Z(\widehat{f}, \chi^\vee)Z(g, \chi).$$

*Proof.* Expanded out, the LHS of the Lemma statement is the following product of integrals:

$$Z(f, \chi)Z(\widehat{g}, \chi^\vee) = \int_{F^\times} f(x)\chi(x) d^*x \cdot \int_{F^\times} \widehat{g}(y)\chi^{-1}(y)|y| d^*y,$$

both of which are absolutely convergent by part (a) of 3.1.5. We can turn this into a double integral over the product space by:

$$\iint_{F^\times \times F^\times} f(x)\widehat{g}(y)\chi(xy^{-1})|y| d^*x d^*y.$$

Using invariance of the product measure under the "shearing" automorphism  $(x, y) \mapsto (x, xy)$ , we obtain:

$$\iint_{F^\times \times F^\times} f(x)\widehat{g}(xy)\chi(y^{-1})|xy| d^*x d^*y,$$

which, by Fubini's theorem, is equal to:

$$\int \left( \int f(x)\widehat{g}(xy)|x| d^*x \right) \chi(y^{-1})|y| d^*y.$$

To prove the lemma it remains to show that the inner integral is symmetric in  $f$  and  $g$ . Expanding out the Fourier transform of  $g$  (recalling our convention in Equation II.5 and transferring it to an integral over  $F$  using Equation II.4), we get that the inner integral is equal to:

$$c \iint_{F \times F} f(x)g(z)\psi(xyz) dz dx.$$

Note there is a (slight) abuse of notation here as technically we shouldn't be integrating over elements in the product containing a 0, but we ignore it as a set of measure zero. We switch the order of the derivatives via Fubini to obtain:

$$c \iint_{F \times F} f(x)g(z)\psi(xyz) dx dz = c \int_F g(z)\widehat{f}(zy) dz = \int_{F^\times} g(z)\widehat{f}(zy)|z| d^*z$$

which gives us the desired result. □

*proof of (b) in 3.1.5.* Our functional equation in Lemma 3.1.6 tells us that:

$$\gamma(\chi, \psi, dx) := \frac{Z(\widehat{f}, \chi^\vee)}{Z(f, \chi)}$$

is independent of our choice of  $f \in S(F)$ . Note that:

$$\chi^{lor} = (\mu|\cdot|^s)^\vee = \mu^{-1}|\cdot|^{1-s}$$

By part (a) of the main theorem, we deduce that  $Z(\widehat{f}, \chi^\vee)$  is absolutely convergent when  $\sigma = \Re(s) < 1$ , and we have already seen that the denominator of  $\gamma$  is absolutely convergent for all positive  $\sigma$ , so the functional equation in terms of  $\gamma$  holds for all  $\sigma \in (0, 1)$  as desired.

To show  $\gamma$  is meromorphic as a function of  $s$ , we turn to the proof of part (c) of the theorem.  $\square$

*proof of (c) in 3.1.5 in the Archimedean case.* As shown in the proof of part (b) of the theorem, the value of the  $\gamma$  function is independent of our choice of  $f \in S(F)$ . Thus we may choose a special function, or family of functions, for each of the possible cases of  $F$  to make our computations convenient. We will select a standard measure  $dx$  for each case, which is self-dual for an appropriate choice of  $\psi$ , and then extend our results to arbitrary  $dx$  and  $\psi$  afterwards.

**Case 1:  $F = \mathbb{R}$**

We take  $dx$  to be the usual Lebesgue measure, and set our additive character to be:

$$\psi(x) = e^{-2\pi ix}.$$

As we noted in our discussion of the local  $L$ -factors, any  $\chi \in X(\mathbb{R}^\times)$  is of the form  $|\cdot|^s$  or  $\text{sgn}|\cdot|^s$ .

Sub-Case (i):  $\chi = |\cdot|^s$

Take  $f(x) = e^{-\pi x^2}$ , which is visibly in  $S(\mathbb{R})$ . Then recalling that we set the scaling constant in the measure transformation of Equation II.4 to 1, we get:

$$Z(f, \chi) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s d^*x = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} dx.$$

Using a substitution  $u = \pi x^2$  (hence  $du = 2\pi x dx$ ), the integral becomes:

$$\pi^{-s/2} \int_0^\infty e^{-u} u^{s/2-1} du = \pi^{-s/2} \Gamma(s/2).$$

and we see from Equation II.3 that this is exactly our local  $L$ -factor, so in fact  $Z(f, \chi) = L(\chi)$  for all characters of this form. Next we claim that  $f$  is self-dual via the Fourier transform, i.e.

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi ixy} dx = f(x).$$

We are omitting the details of the final equality but this is a well-known identity in classical functional analysis, and provable via a simple contour integral. Thus, we have:

$$Z(\widehat{f}, \chi^\vee) = \int_{\mathbb{R}^\times} \widehat{f}(x) \chi^\vee(x) d^*x = \int_{\mathbb{R}^\times} f(x) \chi^\vee(x) d^*x = Z(f, \chi^\vee)$$

which is equal to  $L(\chi^\vee)$  by our previous calculation in this proof. Thus if  $\chi = |\cdot|^s$ , we have:

$$\begin{aligned} \gamma(\chi, \psi, dx) &= \frac{L(\chi^\vee)}{L(\chi)} \\ \implies \epsilon(\chi, \psi, dx) &= 1. \end{aligned} \tag{II.6}$$

Sub-Case (ii):  $\chi = \text{sgn}|\cdot|^s$

In this case we choose  $f(x) = xe^{-\pi x^2}$ , which is again immediately verifiable as rapidly decaying. Expanding the

local zeta function we get:

$$\begin{aligned}
 Z(f, \chi) &= \int_{\mathbb{R}^\times} x e^{-\pi x^2} \frac{x}{|x|} |x|^s d^*x \\
 &= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^{s+1} d^*x \\
 &= \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) \\
 &= \Gamma_{\mathbb{R}}(s+1)
 \end{aligned}$$

which we again recall from Equation II.3 as by definition our real local  $L$ -factor  $L(\chi)$ .

Via another contour integral that we handwave, it can be shown that the Fourier transform of this function is of the form:

$$\widehat{f}(y) = i y e^{-\pi y^2} = i f(y)$$

from which it follows

$$Z(\widehat{f}, \chi^\vee) = \int_{\mathbb{R}^\times} i f(y) \chi^\vee d^*x = i L(\chi^\vee),$$

and thus by a similar calculation to the last sub-case we deduce:

$$\varepsilon(\chi, \psi, dx) = i. \quad (\text{II.7})$$

### Case 2: $F = \mathbb{C}$

Here we let our measure on  $\mathbb{C}$  be  $dz d\bar{z} = 2 dx dy$ , or double the standard Lebesgue measure. In polar coordinates this means  $dz = 2 dr d\theta$ , which also implies  $d^*z = (2/r) dr d\theta$ . We choose this as it is self-dual with respect to the additive quasi-character

$$\psi(z) := e^{-2\pi i(z+\bar{z})}.$$

We also adjust the standard norm on  $\mathbb{C}$  to agree with the module, i.e

$$|z| := z\bar{z}.$$

Since  $\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$ , any multiplicative quasi-character must be of the form:

$$\chi_{s,n} := (r e^{i\theta} \mapsto r^s e^{in\theta})$$

for  $s \in \mathbb{C}$  and  $n \in \mathbb{Z}$  uniquely defined. We choose our class of functions to be:

$$f_n(z) := \begin{cases} \frac{1}{2\pi} (\bar{z})^n e^{-2\pi z\bar{z}}, & n \geq 0, \\ \frac{1}{2\pi} z^{-n} e^{-2\pi z\bar{z}}, & n < 0. \end{cases}$$

We note that  $f$  is visibly a Schwartz function for any  $n$ . Computing the local zeta function for non-negative  $n$ , we get:

$$Z(f_n, \chi_{s,n}) = \int_{\mathbb{C}^\times} f_n(z) \chi_{s,n}(z) d^*z$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{\mathbb{C}^\times} (\bar{z})^n e^{-2\pi z \bar{z}} (z \bar{z})^s e^{i n \arg(z)} d^* z \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty (r e^{-i\theta})^n e^{-2\pi r^2} (r^2)^s e^{i n \theta} (2/r) dr d\theta \\
&= \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty r^{2s+n-1} e^{-2\pi r^2} dr d\theta \\
&= (2\pi)^{-(s+\frac{n}{2})} \int_0^\infty e^{-2\pi r^2} (2\pi r)^{s+\frac{n}{2}-1} 4\pi r dr \\
&= (2\pi)^{-(s+\frac{n}{2})} \int_0^\infty e^{-t} t^{s+\frac{n}{2}-1} dt \\
&= (2\pi)^{-(s+\frac{n}{2})} \Gamma(s + \frac{n}{2}) = L(\chi_{s,n})
\end{aligned}$$

where we recall our definition of the local  $L$ -factor in II.2. A similar calculation also shows that  $Z(f_n, \chi_{s,n}) = L(\chi_{s,n})$  when  $n < 0$ .

It remains to compute the "dual" local zeta function. We do this via a mixture of handwaving and clever observations. First we notice that  $\chi_{n,s}^\vee = \chi_{1-s,-n}$  for any  $s$  and  $n$ . Moreover, we via yet another hand-waved complex integral, we deduce that

$$\widehat{f}_n(z) = \frac{1}{2\pi} i^{|n|} f_{-n}(z)$$

for all  $n$ . Thus, using the fact that our computation was linear in the exponents  $s$  and  $n$ , we deduce that

$$Z(\widehat{f}_n, \chi_{s,n}^\vee) = i^{|n|} L(\chi_{s,n}^\vee).$$

The reader is welcome to rigorously verify this calculation. In any event, as a consequence we get that:

$$\begin{aligned}
\gamma(\chi_{s,n}, \psi, dx) &= i^{|n|} \frac{L(\chi_{s,n}^\vee)}{\chi_{s,n}} \\
\implies \varepsilon(\chi_{s,n}, \psi, dx) &= i^{|n|},
\end{aligned}$$

completing the complex case.

To conclude our analysis on the Archimedean local fields, we note by looking at all the (nowhere-vanishing!) local epsilon factors computed in II.6, II.7 and ??, and their corresponding local  $L$ -factors that  $\gamma(\chi, \psi, dx)$  is indeed meromorphic as a function of  $s$ , completing our proof.  $\square$

Before we proceed complete the proof of the final part of the theorem for the non-Archimedean case, we need to build some more tools and lemmas to use.

**DEFINITION 3.1.7.** Given a ring  $R$  with an additive character  $\psi : R \rightarrow S^1$  and a multiplicative character  $\chi : R^\times \rightarrow S^1$ , we define their associated **Gauss sum** to be:

$$g(\chi, \psi) := \int_{R^\times} \chi(u) \psi(u) d^* u.$$

**REMARK.** Gauss sums turn out to be a hugely convenient notation that show up in a host of different applications, from theta-functions to cyclotomic fields. In fact, Weil made use of these Gauss-sums as a key step in his diagonal variety point-counting argument to motivate the famed Weil Conjectures in his original 1936 paper.

We now build the notion of a conductor of a (quasi-)character. As is standard we work with a local field  $F$  with ring of integers  $\mathcal{O}_F$ , prime ideal  $\mathfrak{p}$ , and uniformizer  $\pi$ .

**DEFINITION 3.1.8.** Let  $\psi$  be an additive quasi-character on  $F$ . Then the **conductor** of  $\psi$  is the ideal  $\mathfrak{p}^m$ , where

$$m := \min \left\{ r \in \mathbb{Z} : \psi|_{\mathfrak{p}^r} = 1 \right\},$$

i.e the conductor is the largest fractional ideal of  $\mathcal{O}_F$  on which  $\psi$  is trivial. In the case that  $\psi$  is trivial we set the conductor to be  $\mathcal{O}_F$  itself.

Now suppose  $\chi$  is a multiplicative quasi-character on  $F^\times$ . Then its **conductor** is  $\mathfrak{p}^n$ , where

$$n := \min \left\{ r \in \mathbb{N} : \chi|_{1+\mathfrak{p}^r} = 1 \right\},$$

i.e the conductor is the largest possible unit subgroup of  $\mathcal{O}_F$  on which  $\chi$  is trivial.

Sometimes the conductors are defined as the integers  $m$  and  $n$  themselves rather than the ideals, but we avoid this convention.

**REMARK (Terminology).** The origin of the term "conductor" comes from Dedekind, who denoted by "*Führer*" (whose meaning was closer to "guide", "attendant" or "conductor" in the original German, rather than its modern associations) an ideal of an order in a number field, which is why classical texts denote conductors using  $\mathfrak{f}$ . It drifted via osmosis to other fields in number theory, and its use in relation to characters and representations was popularized in large part by Artin and the eponymous Artin conductor of local field extensions.

Our goal will be to use this language and notation of conductors and Gauss sums to simplify the integrands of the local-zeta functions. The main tool is the following key lemma.

**LEMMA 3.1.9.** Let  $\chi$  be a multiplicative character on  $\mathcal{O}_F^\times$  and  $\psi$  be an additive character, and suppose they have conductors  $\mathfrak{p}^n$  and  $\mathfrak{p}^m$  respectively. Then:

- (a) If  $m < n$  then  $g(\chi, \psi) = 0$ .
- (b) If  $m = n$  then  $|g(\chi, \psi)|^2 = c \operatorname{Vol}(\mathcal{O}_F, dx) \operatorname{Vol}(1 + \mathfrak{p}^n, d^*x)$ .
- (c) If  $m > n$  then  $|g(\chi, \psi)|^2 = c \operatorname{Vol}(\mathcal{O}_F, dx) (\operatorname{Vol}(1 + \mathfrak{p}^n, d^*x) - \operatorname{Vol}(1 + \mathfrak{p}^{m-1}, d^*x))$ .

*Proof.* Case 1:  $m < n$

We decompose  $\mathcal{O}_F^\times$  into the disjoint union of (finitely many!) cosets of the unit subgroups  $U_m := 1 + \mathfrak{p}^m$ . Note for any  $a, b \in \mathcal{O}_F$  we have

$$\psi(a(1 + \pi^m b)) = \psi(a)\psi(a\pi^m b) = \psi(a),$$



by definition of the conductor  $\mathfrak{p}^m = \langle \pi^m \rangle$ . The Gauss sum thus becomes:

$$g(\chi, \psi) = \sum_{\mathcal{O}_F^\times / U_m} \left( \psi(a) \chi(a) \int_{U_m} \chi(u) d^* u \right).$$

However since  $m < n$ , the subgroup  $U_m$  is a strict superset of  $U_n$ , and so, by definition of the conductor, the character  $\chi$  is non-trivial on  $U_m$ . Recall the integral of a character over a compact subgroup is zero, which in turn makes the entire Gauss sum zero, as desired. This proves (i).

#### Case 2: $m \geq n$

Expanding out the expression and making use of Fubini, we get:

$$\begin{aligned} |g(\chi, \psi)|^2 &= g(\chi, \psi) \overline{g(\chi, \psi)} \\ &= \int_{\mathcal{O}_F^\times} \chi(x) \psi(x) d^* x \cdot \overline{\int_{\mathcal{O}_F^\times} \chi(y) \psi(y) d^* y} \\ &= \iint_{\mathcal{O}_F^\times} \chi(xy^{-1}) \psi(x - y) d^* x d^* y. \end{aligned}$$

Making the substitution  $xy^{-1} \mapsto z$  the above integral becomes

$$\int_{\mathcal{O}_F^\times} \chi(z) h(z) d^* z,$$

where

$$h(z) := \int_{\mathcal{O}_F^\times} \psi(yz - y) d^* y = c \int_{\mathcal{O}_F^\times} \psi(y(z - 1)) dy.$$

Note we can switch the Haar measure in the second equality using Equation II.4 and the fact that  $|y| = 1$  for all  $y \in \mathcal{O}_F^\times$ . Making use of the fact that  $\mathcal{O}_F$  is a local ring, we get:

$$h(z) = c \left( \int_{\mathcal{O}_F} \psi(y(z - 1)) dy - \int_{\mathfrak{p}} \psi(y(z - 1)) dy \right).$$

Note that if  $v_{\mathfrak{p}}(z - 1) < m - 1$  then in the above expression  $\psi$  takes values outside of its conductor in both of the integrals, and thus is a non-trivial character over  $\mathcal{O}_F$ . If  $v_{\mathfrak{p}}(z - 1) = m - 1$  then  $\psi$  is trivial only over  $\mathfrak{p}$ , not  $\mathcal{O}_F$ , and otherwise it is trivial over the entire ring. These observations give us:

$$h(z) = \begin{cases} 0, & v_{\mathfrak{p}}(z - 1) < m - 1 \\ -c \frac{\text{Vol}(\mathcal{O}_F, dx)}{q}, & v_{\mathfrak{p}}(z - 1) = m - 1 \\ c \text{Vol}(\mathcal{O}_F, dx) \left(1 - \frac{1}{q}\right), & v_{\mathfrak{p}}(z - 1) \geq m. \end{cases}$$

All that remains is to substitute these cases back into the integral  $\int \chi(z) h(z) d^* z$  to show the desired equalities in part (ii) and part (iii), which we leave as an exercise.  $\square$

**EXERCISE 3.1.B.** Finish the last part of the proof of 3.1.9.

*proof of (c) in 3.1.5 in the non-Archimedean char 0 case.* Here  $F$  is a finite extension of  $\mathbb{Q}_p$  for  $p$  a prime number. We

first define our "canonical" additive character  $\psi_p$ .  
finish this

Next we classify our multiplicative characters. Recall from our discussion in the prelude to Definition 3.1.1 that any such character of  $F^\times$  is of the form:

$$\chi_{s,n} := (x \mapsto |x|^s \omega(x/|x|)),$$

where  $\omega$  is a unitary character with conductor  $\mathfrak{p}^n$ . Note the indices  $s, n$  do not completely determine the character  $\chi_{s,n}$ , as there is some choice of  $\omega$  to be made, but this notation will end up sufficing for the computations ahead. Note also that in the unramified case (i.e  $n = 0$ ), the unitary character  $\omega$  is trivial. This fact will be used later.

Now the class of functions we work with. Supposing our additive character  $\psi$  have conductor  $\mathfrak{p}^m$ , we define:

$$f(x) := \begin{cases} \psi(x), & x \in \mathfrak{p}^{m-n} \\ 0, & \text{otherwise.} \end{cases}$$

This is a Schwartz-Bruhat function as it is compactly-supported by definition and the canonical character  $\psi$  is trivial on  $\mathbb{Z}_p$ , so it is also locally constant. We now compute the local zeta functions  $Z(f, \chi_{s,n})$  in two cases:  $n$  zero, and  $n$  positive.

### **Case 1: $n = 0$**

This means that  $\omega$  is trivial everywhere, i.e  $\chi_{s,0}$  is an unramified character. We recall  $\psi$  is trivial on  $\mathfrak{p}^m$ , and therefore  $f$  is zero outside of the conductor of  $\psi$ . Using similar methods as in the proof of 3.1.9, we compute:

$$\begin{aligned} Z(f, \chi_{s,0}) &= \int_{F^\times} f(x) \chi_{s,0}(x) d^*x \\ &= \int_{\mathfrak{p}^m \setminus \{0\}} \psi(x) |x|^s \omega(x/|x|) d^*x \\ &= \int_{\mathfrak{p}^m \setminus \{0\}} |x|^s d^*x \\ &= \sum_{k \geq m} \left( \int_{\pi^k \mathcal{O}_F^\times} |x|^s d^*x \right) \\ &= \text{Vol}(\mathcal{O}_F^\times, d^*x) \sum_{k \geq m} q^{-ks} \\ &= q^{-ms} \text{Vol}(\mathcal{O}_F^\times, d^*x) L(\chi_{s,0}), \end{aligned} \tag{II.8}$$

where we recall our definition of the local  $L$ -factor in the unramified case in Equation II.1.

### **Case 2: $n > 0$**

Recalling that  $f$  is zero outside of  $\mathfrak{p}^{m-n}$ , by similar logic as in the  $n = 0$  case we get:

$$Z(f, \chi_{s,n}) = \sum_{k \geq m-n} \left( \int_{\pi^k \mathcal{O}_F^\times} \psi(x) \omega(x/|x|) d^*x \right)$$

$$\begin{aligned}
&= \sum_{k \geq m-n} \left( \int_{\mathcal{O}_F^\times} \psi(\pi^k u) \omega(\pi^k u) / \omega(\pi^k) d^* u \right) \\
&= \sum_{k \geq m-n} q^{-ks} \left( \int_{\mathcal{O}_F^\times} \psi(\pi^k u) \omega(u) d^* u \right) \\
&= \sum_{k \geq m-n} q^{-ks} g(\omega, \psi_{\pi^k}).
\end{aligned}$$

In the last line the Gauss sum notation comes as a typographical godsave for us, where  $\psi_{\pi^k}(x) := \psi(x\pi^k)$ . We note that the conductor of  $\psi_{\pi^k}$  is  $m - k$ , so in particular if  $k > m - n$  it contains the conductor of  $\omega$ . From 3.1.9 we thus get:

$$Z(f, \chi_{s,n}) = q^{(n-m)s} g(\omega, \psi_{\pi^{m-n}}) = q^{(n-m)s} c \text{Vol}(\mathcal{O}_F, dx) \text{Vol}(1 + \mathfrak{p}^n, d^* x) \quad (\text{II.9})$$

which is nothing but a non-trivial exponential function in  $s$  with no zeroes or poles, which is what we want since the local  $L$ -factor is 1 in this case.

Now we need to compute the local zeta function of the dual. Note by definition that:

$$\widehat{f}(y) = \int_F f(x) \psi(xy) dx = \int_{\mathfrak{p}^{m-n}} \psi(x(y+1)) dx. \quad (\text{II.10})$$

We proceed with similar casework to before.

**Case 1:  $n = 0$**

Since the conductor of  $\psi$  is  $\mathfrak{p}^m$ , we observe from the integral in Equation II.10 that  $\widehat{f}(y)$  is the integral of  $\psi(x(y+1))$  taken over  $\mathfrak{p}^m$ . The character is non-trivial on  $\mathfrak{p}^m$ , and thus  $\widehat{f}(y)$  is zero, exactly when  $y \notin \mathcal{O}_F$ . Thus in this case we have:

$$\widehat{f}(y) = \begin{cases} \text{Vol}(\mathfrak{p}^m, dx), & y \in \mathcal{O}_F \\ 0, & \text{otherwise,} \end{cases}$$

i.e we have a scaled characteristic function of the ring of integers. Using this, we compute:

$$\begin{aligned}
Z(\widehat{f}, \chi_{s,0}^\vee) &= \int_{\mathcal{O}_F \setminus \{0\}} \widehat{f}(x) \chi_{s,0}^\vee(x) d^* x \\
&= \text{Vol}(\mathfrak{p}^m, dx) \sum_{k \geq 0} \left( \int_{\mathcal{O}_F^\times} |\pi^k x|^{1-s} d^* x \right) \\
&= \text{Vol}(\mathfrak{p}^m, dx) \text{Vol}(\mathcal{O}_F^\times, d^* x) \frac{1}{1 - q^{-(1-s)}} \\
&= \text{Vol}(\mathfrak{p}^m, dx) \text{Vol}(\mathcal{O}_F^\times, d^* x) L(\chi_{s,0}^\vee)
\end{aligned} \quad (\text{II.11})$$

which is what we wanted. Combing II.8 and II.11, we get that for  $\chi = \chi_{s,0}$  we have:

$$\begin{aligned}
r(\chi, \psi, dx) &= q^{ms} \text{Vol}(\mathfrak{p}^m, dx) \frac{L(\chi^\vee)}{L(\chi)} \\
\implies \varepsilon(\chi, \psi, dx) &= q^{ms} \text{Vol}(\mathfrak{p}^m, dx).
\end{aligned} \quad (\text{II.12})$$

**Case 2:  $n > 0$**

Examining II.5 again, we see that the integrand  $\psi(x(y+1))$  is trivial over  $\mathfrak{p}^{m-n}$  exactly when  $y+1 \in \mathfrak{p}^n$ , and

otherwise it is a trivial character and thus  $\widehat{f}(y) = 0$ . We deduce then that  $\widehat{f}$  is the characteristic function of  $\mathfrak{p}^n - 1$  scaled by the factor  $\text{Vol}(\mathfrak{p}^{m-n}, dx)$ . Noting that  $\mathfrak{p}^n - 1 \subseteq \mathcal{O}_F^\times$ , we computing the dual local zeta function:

$$\begin{aligned}
 Z(\widehat{f}, \chi_{s,n}^\vee) &= \int_{\mathfrak{p}^n - 1} \widehat{f}(x) \chi_{s,n}^\vee(x) d^*x \\
 &= \text{Vol}(\mathfrak{p}^{m-n}, dx) \int_{\mathfrak{p}^n - 1} |x|^{1-s} \overline{\omega}(x) d^*x \\
 &= q^{n-m} \text{Vol}(\mathcal{O}_F, dx) \int_{1+\mathfrak{p}^n} \overline{\omega}(x) d^*x \\
 &= q^{n-m} \text{Vol}(\mathcal{O}_F, dx) \text{Vol}(1 + \mathfrak{p}^n, d^*x) \omega(-1),
 \end{aligned} \tag{II.13}$$

where the last line follows from the fact that the conductor of  $\omega$  is equal to the conductor of its conjugate, so it is trivial on all of  $\text{Vol}(1 + \mathfrak{p}^n, d^*x)$ . We also note that this is constant in  $s$ , which is what we want, as we recall from II.1 that the local  $L$ -factor of a non-unramified character is 1. Combining II.9 and II.13 for  $\chi = \chi_{s,n}$ , we get:

$$\varepsilon(\chi, \psi, dx) = \gamma(\chi, \psi, dx) = \frac{q^{(m-n)(s-1)} \omega(-1)}{c g(\omega, \psi_{\pi^{m-n}})} = \frac{1}{c} q^{(m-n)(s-1)} g(\overline{\omega}, \psi_{\pi^{m-n}}). \tag{II.14}$$

**SANITY CHECK.** If it's not already clear, the final equality above made use of the following string of identities that any Gauss sum satisfies:

$$1/g(\chi, \psi) = \overline{g(\chi, \psi)} = \chi(-1) g(\overline{\chi}, \psi).$$

Verify that each one of them makes sense.

We now conclude our analysis. The equality involving the epsilon factor and relevant local  $L$ -factors have been shown in II.12 and II.14, from all of which  $\gamma(\chi, \psi, dx)$  is visibly meromorphic, completing our proof.  $\square$

## §3.2 The Root and Riemann-Roch

## §3.3 The Global Zeta Function

### DEFINITION 3.3.1.

We first present a large multi-part lemma on some basic properties of adelic characters.

**LEMMA 3.3.2.** We fix a global field  $K$  with standard character  $\psi_K$  on the adèle group  $\mathbb{A}_K$ . Then:

- (a)  $\mathbb{A}_K$  is self-dual via the map  $y \mapsto \psi_{K,y}$ .
- (b) The character  $\psi_K$  is trivial on  $K \subseteq \mathbb{A}_K$  and thus induces a character on  $\mathbb{A}_K/K$ .
- (c) We have the following isomorphism:

$$\begin{aligned}
 \widehat{\mathbb{A}_K/K} &\xrightarrow{\sim} K \\
 \psi_{K,x} &\mapsto x
 \end{aligned} \tag{II.15}$$

(d) For any character  $\psi$  of  $\mathbb{A}_K/K$ , the component  $\psi_v$  has conductor  $\mathcal{O}_v$  almost everywhere.

## SECTION 4

# HECKE'S L-FUNCTION

## SECTION 5

# SOME APPLICATIONS

### §5.1 Dirichlet's Class Number Formula

## Chapter III

# MODULAR FORMS AND AUTOMORPHIC REPRESENTATIONS (A.K.A $GL_2$ )

## SECTION 6

### AUTOMORPHIC FORMS

We follow the notes of [Sut24].

## Chapter IV

# THE CONJECTURES OF LANGLAND (A.K.A $GL_n$ )

# APPENDICES

## APPENDIX A

### FOURIER THEORY OF LOCALLY COMPACT GROUPS

[Probably cut down on this a LOT, mostly wrote this up to help me keep track of the content.]

#### Introduction

In this section we start to do how to do Fourier transforms on an abelian topological group  $G$ , via the *dual*  $\widehat{G}$ , the group of continuous homomorphisms  $G \rightarrow S^1$ .

**DEFINITION A.0.1.** The **Pontryagin<sup>a</sup> Dual** of  $G$ , denoted  $\widehat{G}$ , is the group of all continuous complex homomorphisms  $G \rightarrow S^1$  (where  $S^1$  is given its standard subspace topology from  $\mathbb{C}$ ), with multiplication as the group operation. Elements of  $\widehat{G}$  are often referred to as **characters** of  $G$ .

<sup>a</sup>A Russian name, sometimes also anglicized as Pontrjagin or Pontriagin

**EXAMPLE A.0.2.** The dual group of the circle is the integers, i.e

$$\widehat{S^1} \cong \mathbb{Z}$$

*proof of A.0.2.* Recall the continuous homomorphism  $\exp : \mathbb{R} \rightarrow S^1$  given by  $x \mapsto e^{2\pi i x}$ , the so-called *universal covering map*. For any continuous homomorphism  $\phi : S^1 \rightarrow S^1$ , we have the following diagram:

$$\begin{array}{ccc} \mathbb{R} & & \\ \exp \downarrow & \searrow \psi & \\ S^1 & \xrightarrow{\phi} & S^1 \end{array}$$

from which we see that any such  $\phi$  must induce a continuous homomorphism  $\psi = \phi \circ \exp : \mathbb{R} \rightarrow S^1$ . Moreover, since the kernel of the universal covering map is exactly  $\mathbb{Z}$ , it follows  $\psi$  must also have kernel containing  $\mathbb{Z}$ .

A standard vector-space argument shows any continuous homomorphism  $\mathbb{R} \rightarrow \mathbb{R}$  is of the form  $x \mapsto \lambda x$  for  $\lambda \in \mathbb{R}$ . using the universal lift we deduce any continuous homomorphism  $\mathbb{R} \rightarrow S^1$  is of the form  $\psi_\lambda(x) = e^{2\pi i \lambda x}$ , for  $\lambda \in \mathbb{R}$ .



Since we want  $\mathbb{Z}$  to be in the kernel we deduce  $\lambda \in \mathbb{Z}$ .

Thus we have:

$$\psi = e^{2\pi i n x} = \phi(e^{2\pi i x}) \implies \phi(y) = y^n$$

which was what we wanted to show. □

**EXERCISE A.0.A.** Using a similar argument to above, show that  $\widehat{\mathbb{Z}} \cong S^1$ .

Combining A.0.2 and A.0.A, we get the wonderful identities:

$$\widehat{S^1} \cong S^1 \text{ and } \widehat{\mathbb{Z}} \cong \mathbb{Z}.$$

It turns out this duality holds true for ANY topological group that is locally compact. Proving this result, and developing the theory of Fourier analysis over arbitrary topological groups along the way to do so, will be the main goal of this section.

**REMARK.** Lev Pontryagin's foundational work on this duality result initially imposed some auxiliary conditions of second-countability and compactness; it was later generalized to locally compact groups by van Kampen and Weil. Pontryagin's other contributions were largely in topology (he is often referred to as one of the co-founders of cobordism theory), which is especially surprising considering he was blind since the age of 14, and so was making huge strides in the study of shapes and spaces without being able to draw or see!

## §A.1 Topologizing the dual

We want to put a topology on  $\widehat{G}$  that tells us if two functions are "close together". For a function  $f \in \widehat{G}$ , suppose we take a compact set  $K \subseteq G$ , and an open neighborhood  $V \supseteq f(K)$  of the image of this compact set. Since  $S^1$  is Hausdorff,  $f(K)$  is a closed set. As such, we might expect that if we wiggle  $f(K)$  around a little, it should still be inside  $V$ . Thus we want to characterize a function  $g$  as being "close" to  $f$  if  $g(K) \subseteq V$  as well. This motivates the following construction.

**DEFINITION A.1.1.** The **compact-open topology** on  $\widehat{G}$  is the one generated by the following neighborhood base of the trivial character:

$$\left\{ W(K, V) := \left\{ \chi \in \widehat{G} : \chi(K) \subseteq V \right\} \right\}_{K, V}$$

where  $K$  ranges over all compact sets in  $G$  and  $V$  ranges over all neighborhoods of the identity in  $S^1$ .

**PROPOSITION A.1.2.** If  $G$  is discrete, then the compact-open topology on  $\widehat{G}$  is the same as the topology of pointwise convergence.

*Proof.* Recall the topology of pointwise convergence is generated by sets of the form

$$S(g, V) := \left\{ \chi \in \widehat{G} : \chi(x) \in V \right\}_{g \in G; V \subseteq_{\text{open}} S^1}$$

For any  $G$ , this is finer than the compact-open topology, since for any compact  $K \subseteq G$  we can write:

$$W(K, V) = \bigcup_{g \in K} S(g, V)$$

If  $G$  is discrete then every compact set is necessarily finite (and vice versa!), and thus  $\{g\}$  is compact, so every subbasis element  $S(g, U)$  is also a basis element of the compact-open topology. Thus the two topologies are equivalent.  $\square$

We now introduce some notation that'll come in handy when doing technical analysis of  $\widehat{G}$ .

- For any  $X \subseteq G$  (not necessarily a subgroup), then for  $n \in \mathbb{N}$  we define

$$X^{(n)} := \left\{ \prod_{j=1}^n x_j : x_j \in X, j \in [1, n] \right\}$$

- Let  $\phi : \mathbb{R} \rightarrow S^1$  be the universal cover projection map  $r \mapsto e^{2\pi i r}$ . Then for some  $\epsilon \in (0, 1]$ , we define

$$N(\epsilon) := \phi \left( \left( \frac{-\epsilon}{3}, \frac{\epsilon}{3} \right) \right)$$

i.e  $N(\epsilon)$  is some small symmetric arc around 1 in  $S^1$ .

**LEMMA A.1.3.** Take  $m \in \mathbb{Z}^+$ , and take  $x \in \mathbb{C}$  such that  $x^i \in N(1)$  for all  $i \in [1, m]$ . Then  $x \in N(1/m)$ .

*Proof.* In polar coordinates, every such  $x$  has radius 1 and angle some  $\theta$ , and note that  $x^i$  has angle  $i\theta$ . We are assuming that  $\theta, 2\theta, \dots, m\theta$  are all contained in  $(-2\pi/3, 2\pi/3) \pmod{2\pi}$ . If  $x \notin N(1/m)$ , that means its associated argument  $\theta$  is in the interval  $[2\pi/3m, 2\pi(3m-1)/3m]$ . In particular, this would mean that  $m\theta$  would have argument between  $2\pi/3$  and  $4\pi/3$  inclusive, which is not in the open set  $N(1)$ , a contradiction. Thus  $x \in N(1/m)$   $\square$

**COROLLARY A.1.4.** If  $U$  is a subset of  $G$  containing the identity then  $\chi(U^{(m)}) \subseteq N(1) \implies \chi(U) \subseteq N(1/m)$ .

**EXERCISE A.1.A.** Rigorously prove the above corollary.

**PROPOSITION A.1.5.** Suppose  $G$  is an abelian topological group. Then:

- A homomorphism  $\chi : G \rightarrow S^1$  is continuous (and hence in  $\widehat{G}$ ) iff  $\chi^{-1}(N(1))$  is a neighborhood of the identity in  $G$ .
- The family  $\{W(K, N(1))\}_{\text{compact } K \subseteq G}$  is a neighborhood base for the trivial character for the compact open topology of  $\widehat{G}$ .
- $G$  discrete  $\implies \widehat{G}$  compact.
- $G$  compact  $\implies \widehat{G}$  discrete.

(e)  $G$  locally compact  $\implies \widehat{G}$  locally compact.

*Proof.* (a)

If  $\chi$  is continuous then since  $1 \in N(1) \subseteq_{\text{open}} S^1$  we must have that  $e \in U := \chi^{-1}(N(1)) \subseteq_{\text{open}} G$ .

For the other direction, suppose  $\chi(U) \subseteq N(1)$  for some neighborhood of the identity. Then by continuity of multiplication, for every  $m \in \mathbb{Z}$  there must exist an open neighborhood  $V$  of the identity such that  $V^{(m)} \subseteq U$ . Then by A.1.4 we must have  $\chi(V) \subseteq N(1/m)$ . Since the sets  $\{N(1/m)\}_{m \in \mathbb{Z}_{>0}}$  form a neighborhood base around the identity of  $S^1$ , this fact suffices to prove continuity.

(b)

We want to show that for every compact  $K_1 \subseteq G$  and every positive integer  $m$  that there exists some compact  $K \subseteq G$  such that

$$W(K, N(1)) \subseteq W(K_1, N(1/m))$$

Define  $K := K_1^{(m)}$ , which is the continuous image of the topological direct product  $K^m$ , and thus  $K$  itself is compact. If  $\chi \in W(K, N(1))$  then, by construction, we know that  $\chi(x), \chi(x)^2, \dots, \chi(x)^m \in N(1)$  for all  $x \in K_1$ . By A.1.3 we have that  $\chi(x) \in N(1)$ , from which it follows  $\chi \in W(K_1, N(1/m))$ , as desired.

(c)

If  $G$  is discrete then any group homomorphism  $f : G \rightarrow S^1$  is continuous, so  $\widehat{G} = \text{Hom}(G, S^1)$ . Also recall from A.1.2 that in this case  $\widehat{G}$  has exactly the topology of pointwise convergence. With respect to this topology, note  $\text{Hom}(G, S^1)$  is a closed subset of the space of  $(S^1)^{|G|}$  (the set of all maps from  $G$  to  $S^1$ ), which is a compact space by compactness of  $S^1$ . Thus  $\text{Hom}(G, S^1) = \widehat{G}$  itself is compact, as desired.

(d)

Since the image of any group homomorphism is a subgroup of  $S^1$ , we have  $\chi(G) \not\subseteq N(1)$  for any  $\chi \in \widehat{G}$ .

If  $G$  itself is compact, the set  $W(G, N(1))$  must be an open set in  $\widehat{G}$ , which can only contain the trivial character, by what we showed. Thus the singleton of the identity is open in  $\widehat{G}$ , so it must be a discrete group.

(e)

FINISH LATER

□

## §A.2 Positive-Definite and Elementary Functions

To motivate what a function of positive type is, we discuss some background on unitary representations. From now on let  $G$  be a locally compact group with a (left) Haar measure  $ds$ .

Let  $\rho$  be a representation of a topological group  $G$  in the space of unitary operators on a Hilbert space  $H$ . Fixing some  $x \in H$ , define a  $\mathbb{C}$ -valued function on  $G$  by:

$$\varphi(s) := \langle \rho(s)x \mid x \rangle$$

Take  $s_1, \dots, s_n \in G$  and consider

$$A := [\varphi(s_j^{-1}s_i)] \in \text{Mat}_{n \times n}(\mathbb{C})$$

We claim  $A$  is Hermitian and positive semidefinite. For the first point, note that since each  $\rho(s)$  is a unitary operator,

$$\begin{aligned}
 A_{i,j} &= \langle \rho(s_j^{-1}s_i)(x) \mid x \rangle \\
 &= \langle \rho(s_j)^* \rho(s_i)(x) \mid x \rangle \\
 &= \langle \rho(s_i)(x) \mid \rho(s_j)(x) \rangle \\
 &= \langle x \mid \rho(s_i)^* \rho(s_j)(x) \rangle \\
 &= \langle x \mid \rho(s_i^{-1}s_j)(x) \rangle \\
 &= \overline{\langle \rho(s_i^{-1}s_j)(x) \mid x \rangle} = \overline{A_{j,i}}
 \end{aligned}$$

Thus  $A$  is equal to its conjugate transpose, and is Hermitian. For positive semidefiniteness, take any  $z = (z_i) \in \mathbb{C}^n$ . Then:

$$\begin{aligned}
 \langle Az \mid z \rangle &= \sum_{i,j} \varphi(s_j^{-1}s_i) z_j \overline{z_i} \\
 &= \sum_{i,j} \langle \rho(s_i)(x) \mid \rho(s_j)(x) \rangle z_j \overline{z_i} \\
 &= \left\langle \sum_j \rho(s_j)(z_j x) \mid \sum_i \rho(s_i)(z_i x) \right\rangle \\
 &= \left| \sum_i \rho(s_i)(z_i x) \right|^2 \geq 0
 \end{aligned}$$

Note on the third equality that there is no conjugacy required since  $A$  is Hermitian.

**REMARK.** Note  $A$  is actually Hermitian and positive semidefinite even in the non-locally compact case.

Let  $\mathcal{C}_c(G)$  denote the set of  $\mathbb{C}$ -valued continuous functions on  $G$  with compact support. Recall for every  $p \in [1, \infty]$  that  $\mathcal{C}_c(G) \subseteq L^p(G)$ . Thus it is subject to the  $L^p$ -norm defined by:

$$\|f\|_p = \left( \int_G |f|^p \, ds \right)^{1/p}$$

for finite  $p$ , with  $\|f\|_\infty$  being the essential supremum<sup>1</sup> of  $|f|$ . This norm also induces a topology, under which  $\mathcal{C}_c(G)$  is dense in  $L^p(G)$  [JUSTIFY MORE MAYBE??]

**DEFINITION A.2.1.** A Haar-measurable function  $\phi \in L^\infty(G)$  is **positive definite** (or of **positive type**, we will oscillate between these two terminologies) if for any  $f \in \mathcal{C}_c(G)$  we have:

$$\iint \phi(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \geq 0$$

where the integrals are taken over the entire group  $G$ .

<sup>1</sup>recall the essential supremum is the supremum over a set of comeasure zero.

**LEMMA A.2.2.** For  $\phi$  a Haar-measurable function on  $G$ , define

$$\psi(g, h) := \phi(g^{-1}h).$$

Then  $\psi$  is Haar-measurable on  $G \times G$ .

*FINISH THIS.* □

Note by A.2.2 that Fubini's theorem applies to show that this double integral is defined. (Every locally compact group is the disjoint union of  $\sigma$ -compact spaces). Moreover, if  $\text{Supp}(f)$  is contained in a compact subset  $K$  then the integrand in A.2.1 has support contained in  $K \times K \subseteq G \times G$ .

Note that since  $\phi \in L^\infty(G)$  that it is bounded above by  $\|\phi\|_\infty$  on a set of comeasure zero. Thus the integral is bounded by:

$$\left| \iint \phi(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \right| \leq \|\phi\|_\infty (\sup |f| \cdot \mu(K))^2$$

Here  $\mu(K)$  is the (finite!) Haar measure of  $K$ .

We describe some relations between these functions of positive type and Hilbert spaces and unitary representations.

If  $\phi$  is a positive definite function then we can define a positive sesquilinear (i.e conjugate-symmetric) form on  $\mathcal{C}_c(G)$  by:

$$\langle f_1 | f_2 \rangle_\phi := \iint \phi(s^{-1}t) f_1(s) \, ds \, \overline{f_2(t)} \, dt .$$

**EXERCISE A.2.A.** Verify the above equation is well-defined and finite for all  $f_1, f_2 \in \mathcal{C}_c(G)$ .

Set

$$W_\phi := \left\{ f \in \mathcal{C}_c(G) : \langle f | f \rangle_\phi = 0 \right\}$$

i.e the functions that are degenerate wrt the form defined by  $\phi$ . Cauchy-Schwarz also tells us that  $W_\phi$  is a subspace of  $\mathcal{C}_c(G)$ , and so we can form a quotient space  $\mathcal{C}_c(G)/W_\phi$ . By construction,  $\langle | \rangle_\phi$  is a positive-definite Hermitian form on this quotient. Let  $V_\phi$  denote the completion of this quotient with respect to this form, which thus acquires the structures of a Hilbert space.

For any  $s \in G$  and any function  $f$  on  $G$ , we can define the function  $L_s f$  by

$$L_s f(t) = f(s^{-1}t).$$

Note that if  $f \in \mathcal{C}_c(G)$  then  $L_s f \in \mathcal{C}_c(G)$ , and moreover the map

$$\begin{aligned} G &\rightarrow \text{End}(\mathcal{C}_c(G)) \\ s &\mapsto L_s \end{aligned}$$

is a representation of  $G$ .

**PROPOSITION A.2.3.** Let  $\phi$  be a  $\mathbb{C}$ -valued positive-definite function on a locally compact group  $G$ . Then the mapping  $s \mapsto L_s$  induces a unitary representation of  $G$  on the Hilbert space  $V_\phi$ .

*Proof.* If  $\phi$  is of positive type and  $f \in \mathcal{C}_c(G)$  then:

$$\begin{aligned} \langle L_s f \mid L_s f \rangle_\phi &= \iint \phi(t^{-1}u) f(s^{-1}t) \, dt \, \overline{f(s^{-1}u)} \, du \\ &= \iint \phi((s^{-1}t)^{-1}(s^{-1}u)) f(s^{-1}t) \, dt \, \overline{f(s^{-1}u)} \, du \\ &= \iint \phi(t^{-1}u) f(t) \overline{f(u)} \, du \\ &= \langle f \mid f \rangle_\phi \end{aligned}$$

which proves that  $L_s$  is a unitary representation. □

**REMARK.** It is also true that the above is a topological representation, the rigorous proof of which we omit here.

**EXERCISE A.2.B.** Let  $\rho$  be a unitary representation of  $G$  on a Hilbert space  $V$ . For any  $x \in V$ , show the map

$$\begin{aligned} \phi : G &\rightarrow \mathbb{C} \\ s &\mapsto \langle x \mid \rho(s)(x) \rangle \end{aligned}$$

is a function of positive type. This is essentially a converse of A.2.3.

For further discussion of these functions of positive type, we recall the definition of a convolution.

**DEFINITION A.2.4.** Let  $f, g$  be  $\mathbb{C}$ -valued Borel functions on a locally compact group  $G$ . Their **convolution**  $f * g$  is defined by:

$$f * g(t) := \int g(s^{-1}t) f(s) \, ds = \int g(s^{-1}) f(ts) \, ds,$$

assuming this integral, taken over the full group, exists.

**SANITY CHECK.** Verify for  $f \in \mathcal{C}_c(G)$  and  $\phi \in L^\infty(G)$  that  $f * \phi$  exists, and is continuous.

**PROPOSITION A.2.5.** Let  $\phi$  be a positive definite function on locally compact  $G$ . Then there exists some  $x_\phi \in V_\phi$  such that

$$\phi(s) = \langle x_\phi \mid L_s x_\phi \rangle_\phi$$

almost everywhere for  $s \in G$ .

*Proof.* Let  $\{\alpha\}$  be an indexing set for the open neighborhoods  $V_\alpha$  of the identity of  $G$ . Since  $G$  is Hausdorff, we must have  $\cap V_\alpha = \{e\}$ . By Urysohn's Lemma, for every  $\alpha$  we can construct continuous  $g_\alpha : G \rightarrow \mathbb{R}_{>0}$  such that:

- $\text{Supp}(g_\alpha)$  is a compact subset of  $V_\alpha$ .
- The identity  $\int_G g_\alpha(s) \, ds = 1$  holds.

If we write  $\alpha \leq \beta$  whenever  $V_\alpha \supseteq V_\beta$ , then that makes  $\{\alpha\}$  a directed set, which in turn gives us a net  $\{g_\alpha(s) ds\}$  of positive linear functionals on  $\mathcal{C}_c(G)$  by:

$$f \mapsto \int_G f(s) g_\alpha(s) ds$$

Note these converge (weakly) to the Dirac measure  $\delta_e$ , which we recall is defined by

$$\delta_e(f) := f(e),$$

i.e evaluation at the identity. Consider the integral

$$\langle f | g_\alpha \rangle_\phi = \iint \phi(s^{-1}t) f(s) ds g_\alpha(t) dt = \int f * \phi(t) \cdot g_\alpha(t) dt,$$

which exists, since  $f * \phi$  is continuous and  $g_\alpha$  has compact support. We can now define a linear form  $\Phi$  on  $V_\phi$  that on  $\mathcal{C}_c(G)$  is given by:

$$\Phi(f) := \lim_\alpha \langle f | g_\alpha \rangle_\phi = \lim_\alpha \int f * \phi(t) \cdot g_\alpha(t) dt.$$

Note for any  $h \in \mathcal{C}_c(G)$  that  $(f * \phi) \cdot h \in \mathcal{C}_c(G)$  as well. Thus the above limit exists since the  $g_\alpha$  weakly converge to the Dirac measure, which also tells us

$$\Phi(f) = f * \phi(e) = \int \phi(s^{-1}) f(s) ds$$

Since  $V_\phi$  is a Hilbert space, it's self-dual and therefore there exists  $x_\phi \in V_\phi$  such that

$$\Phi(\xi) = \langle \xi | x_\phi \rangle_\phi$$

for all  $\xi \in V_\phi$ .

We shall now see how  $x_\phi$  behaves under the group representation  $s \mapsto L_s$ .

$$\begin{aligned} \langle \xi | L_s x_\phi \rangle_\phi &= \lim_\alpha \langle \xi | L_s g_\alpha \rangle_\phi \\ &= \lim_\alpha \iint \phi(t^{-1}u) \xi(t) dt g_\alpha(s^{-1}u) du \\ &= \lim_\alpha \iint \phi(t^{-1}su) \xi(t) dt g_\alpha(u) du \\ &= \int \phi(t^{-1}s) \xi(t) dt. \end{aligned} \tag{1}$$

Swapping the arguments in Equation (1), and using conjugate-symmetry of Hermitian forms, we deduce that:

$$\begin{aligned} \langle L_s x_\phi | \xi \rangle_\phi &= \overline{\langle \xi | L_s x_\phi \rangle_\phi} \\ &= \int \overline{\phi(t^{-1}s) \xi(t)} dt \\ &= \int \phi(s^{-1}t) \overline{\xi(t)} dt, \end{aligned}$$

and taking the case  $s = e$  above we get:

$$\langle x_\phi \mid \xi \rangle_\phi = \int \phi(t) \overline{\xi(t)} dt.$$

By Equation (1), we can also deduce for any  $\xi \in V_\phi$  and  $h \in \mathcal{C}_c(G)$  that:

$$\langle \xi \mid h \rangle_\phi = \iint \phi(s^{-1}t) \xi(s) ds \overline{h(t)} dt = \int \langle \xi \mid L_t x_\phi \rangle_\phi \overline{h(t)} dt, \quad (2)$$

and by continuity this extends to all of  $V_\phi$ .

**OBSERVATION.** In Equation (2), if we set  $h = \xi$ , note that if  $\xi$  is orthogonal to the  $\mathbb{C}G$ -submodule of  $V_\phi$  generated by  $x_\phi$ , then the integrand is zero for every  $t \in G$ , and thus  $\langle \xi \mid \xi \rangle_\phi = 0$ . Thus  $V_\phi$  is generated (as a  $\mathbb{C}G$ -module) by  $x_\phi$ . This fact will come in useful later.

To complete the proof of the main result, let us take  $\xi = x_\phi$  in equation 2 and apply the above results to deduce that for any  $\psi \in V_\phi$  we have:

$$\int \phi(s) \overline{\psi(s)} ds = \langle x_\phi \mid \psi \rangle_\phi = \int \langle x_\phi \mid L_s x_\phi \rangle_\phi \overline{\psi(s)} ds.$$

By equality of the two integrals above for every  $\psi$ , it follows that

$$\phi(s) = \langle x_\phi \mid L_s x_\phi \rangle_\phi$$

almost everywhere, as desired. □

**SANITY CHECK.** For  $\phi$  positive definite on locally compact  $G$ , verify that  $\phi$  is equal to a continuous function almost everywhere. In fact, if  $\phi$  IS continuous then the representation of A.2.5 is true everywhere, not just on a comeasure-zero set.

**COROLLARY A.2.6.** If  $\phi$  as above is continuous, then:

- (a)  $\phi(e) \geq 0$ .
- (b)  $\phi(e) = \sup_{s \in G} |\phi(s)|$ .
- (c)  $\phi(s^{-1}) = \overline{\phi(s)}$  for every  $s \in G$ .

**EXERCISE A.2.C.** Prove A.2.6 rigorously. (Hint: Use the representation in A.2.5 for (a), Cauchy-Schwarz for (b), and the fact that  $L_s$  is unitary for (c)).

The positive-definite, continuous functions on  $G$  bounded by 1 in the  $L^\infty$  norm comprise an important subset of  $L^\infty(G)$ . We set

$$\mathcal{P}(G) := \{ \phi \in \mathcal{C}(G) \cap L^\infty(G) : \phi \text{ is positive-definite and } \|\phi\|_\infty \leq 1 \}$$

Related to this is the class of functions we are interested in.



**DEFINITION A.2.7.** Define the set  $\mathcal{E}(G)$  as the zero function unioned with any function satisfying:

- (i)  $\phi$  is continuous and of positive type.
- (ii)  $\phi(e) = 1$ .
- (iii) For every decomposition  $\phi = \phi_1 + \phi_2$ , where  $\phi_1, \phi_2 \in \mathcal{P}(G)$ , there exist  $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$  such that:

$$\phi_1 = \lambda_1 \phi, \quad \phi_2 = \lambda_2 \phi$$

The non-zero elements of  $\mathcal{E}(G)$  are called **elementary functions**.

**SANITY CHECK.** Verify that  $\mathcal{E}(G) \subseteq \mathcal{P}(G)$  (Hint: A.2.6 might be helpful).

**LEMMA A.2.8.** The sets  $\mathcal{P}(G)$  and  $\mathcal{E}(G)$  satisfy the following properties:

- (a)  $\mathcal{P}(G)$  is a convex, bounded subset of  $L^\infty(G)$ . It is also weakly compact as a subset of  $L^1(G)^*$ .
- (b) Any convex, closed subset of  $\mathcal{P}(G)$  containing its extreme points is all of  $\mathcal{P}(G)$ .
- (c) The extreme points of  $\mathcal{P}(G)$  are exactly the points of  $\mathcal{E}(G)$ .

*Proof.* **(a)**

By definition  $\mathcal{P}(G)$  is bounded. Convexity follows from the fact that if we take a linear combination of two positive definite functions then that function must be positive definite as well (if this isn't immediately clear, verify it!), and that if both the coefficients have norm  $\leq 1$  then the supremum of that sum must also have norm  $\leq 1$ .

For the second statement, first note that if we have a sequence  $\{\phi_n\}$  of elements in  $\mathcal{P}(G)$  weakly converging to  $\phi$  then by definition we have

$$\|\phi_n - \phi\|_\infty \rightarrow 0$$

and the fact that  $\|\phi_n\|_\infty \leq 1$  for all  $n$  implies  $\|\phi\|_\infty \leq 1$  as well. Moreover, recall the standard fact from functional analysis that we can identify  $L^\infty(G)$  and  $L^1(G)^*$  via an isometry. A sequence  $\phi_n$  of functions in  $\mathcal{P}(G)$  weakly converging to  $\phi \in L^\infty(G)$  is equivalent to saying that:

$$\int f \phi_n \, ds \rightarrow \int f \phi \, ds,$$

from which it follows that:

$$\iint \phi_n(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \rightarrow \iint \phi(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt$$

and thus positive-definiteness of all the  $\phi_n$  also implies positive-definiteness of  $\phi$ . Recall that this also implies  $\phi$  is continuous.

**WARNING.** Technically  $\phi$  is an equivalence class in  $L^\infty(G)$  of functions that are measure-zero-different to a continuous function.

Thus  $\mathcal{P}(G)$ , as a subset of  $L^1(G)^*$ , is a closed subset of the unit ball, and thus compact under the weak-star topology via Alaoglu's theorem.

**(b)**

This is a special case of the Krein-Milman theorem.

**(c)**

FINISH THIS □

Now we come to a result that brings together the theory of elementary functions with irreducible representations, using the theory we built in

**THEOREM A.2.9.** Let  $\phi$  be continuous of positive type on  $G$  such that  $\phi(e) = 1$ . Then  $\phi \in \mathcal{E}(G)$  if and only if the unitary representation  $s \mapsto L_s$  of  $G$  in  $V_\phi$  is irreducible.

*Proof.* For the forward direction, suppose  $\phi$  is an elementary function. Let  $W$  be a closed,  $G$ -invariant subspace of  $V_\phi$  with orthogonal complement  $W^\perp$ . Let  $\text{Proj}_W$  denote the orthogonal projection map onto  $W$ . Then since each  $L_s$  is unitary, we have the following commutative diagram:

$$\begin{array}{ccc} W \oplus W^\perp & \xrightarrow{\text{Proj}_W} & W \\ L_s \downarrow & & \downarrow L_s \\ W \oplus W^\perp & \xrightarrow{\text{Proj}_W} & W \end{array}$$

Thus, to show irreducibility of  $s \mapsto L_s$  it suffices to show that if  $A$  is any orthogonal projection operator commuting with every  $L_s$ , then  $A$  is either the zero map or the identity map on  $V_\phi$ .

Recall for any projection that  $\langle Ax \mid y \rangle_\phi = \langle Ax \mid Ay \rangle_\phi$ . It follows from this and our "representation result" in ?? then that:

$$\begin{aligned} \phi(s) &= \langle x_\phi \mid L_s x_\phi \rangle_\phi \\ &= \langle Ax_\phi \mid L_s x_\phi \rangle_\phi + \langle x_\phi - Ax_\phi \mid L_s x_\phi \rangle_\phi \\ &= \langle Ax_\phi \mid L_s Ax_\phi \rangle_\phi + \langle x_\phi - Ax_\phi \mid L_s(x_\phi - Ax_\phi) \rangle_\phi. \end{aligned}$$

By A.2.B, we have decomposed  $\phi$  into the sum of two functions of positive type, i.e two elements of  $\mathcal{P}(G)$ . Since we assumed  $\phi$  to be elementary, by definition we must have:

$$\langle Ax_\phi \mid L_s Ax_\phi \rangle = \langle Ax_\phi \mid AL_s x_\phi \rangle = \langle Ax_\phi \mid L_s x_\phi \rangle = \lambda \langle x_\phi \mid L_s x_\phi \rangle$$

for all  $s \in G$ , where the first equality follows from  $A$  commuting with  $L_s$  and the second follows from the property of projection maps. Recall from our observation in the proof of A.2.5 that  $V_\phi$  is generated by  $x_\phi$  as a CG-module, so in fact the above identity holds over all of  $V_\phi$ , which tells us  $A = \lambda \cdot \text{Id}_{V_\phi}$ . Since  $A$  is a projection map however, it is idempotent, so  $\lambda = 0$  (i.e  $A$  is the zero map) or  $\lambda = 1$  (i.e  $A$  is the identity map), which completes the proof of the forward direction.

For the reverse direction, suppose that the representation  $s \mapsto L_s$  is irreducible. We have to show that, for any decomposition of  $\phi = \phi_1 + \phi_2$  into a sum of functions from  $\mathcal{P}(G)$ , that each  $\phi_i$  is of the form  $\lambda_i \phi$ . Note for each  $f \in \mathcal{C}_c(G)$  that

$$\begin{aligned} \langle f | f \rangle_\phi &= \iint \phi(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \\ &= \iint (\phi_1(s^{-1}t) + \phi_2(s^{-1}t)) f(s) \, ds \, \overline{f(t)} \, dt \\ &= \iint \phi_1(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt + \iint \phi_2(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \\ &= \langle f | f \rangle_{\phi_1} + \langle f | f \rangle_{\phi_2}, \end{aligned}$$

and so we get the key inequality

$$\langle f | f \rangle_{\phi_1} \leq \langle f | f \rangle_\phi. \quad (3)$$

Thus any compactly-supported continuous function degenerate wrt  $\langle | \rangle_\phi$  is also degenerate wrt  $\langle | \rangle_{\phi_1}$ . Thus  $\phi$  also defines a Hermitian form on the (completed) quotient  $V_\phi$ . There must exist a continuous, positive-definite  $A \in \text{End}(V_\phi)$  such that

$$\langle A\xi | \psi \rangle_\phi = \langle \xi | \psi \rangle_{\phi_1}$$

for all  $\xi, \psi \in V_\phi$ . In particular, we must also have

$$\langle Ax_\phi | L_s x_\phi \rangle_\phi = \langle x_\phi | L_s x_\phi \rangle_{\phi_1}$$

for all  $s \in G$ . We claim that  $A$  commutes with all the  $L_s$ . To see this, note that:

$$\begin{aligned} \langle AL_s x_\phi | \psi \rangle_\phi &= \langle L_s \phi | \psi \rangle_{\phi_1} \\ &= \langle \phi | L_{s^{-1}} \psi \rangle_{\phi_1} \text{ by the fact that } L_s \text{ is unitary,} \\ &= \langle A\phi | L_{s^{-1}} \psi \rangle_\phi \\ &= \langle L_s A\phi | \psi \rangle_\phi \end{aligned}$$

for all  $\phi, \psi \in \mathcal{C}_c(G)$ , and thus  $A$  must commute with  $L_s$  in  $V_\phi$ .

We claim that:

$$\langle x_\phi | L_s x_\phi \rangle_{\phi_1} = \phi_1(s)$$

everywhere in  $G$ . To see this, recall that the key construction of the proof of the representation in A.2.5 involved the convergence of the net of functionals  $\{g_\alpha\}$ . The inequality in Equation (3) tells us that convergence in the norm induced by  $\phi$  also implies convergence in the norm induced by  $\phi_1$ , and from this the rest of the proof implies the same equality for the  $\phi_1$ -induced norm.

Putting everything together then, we have:

$$\phi_1(s) = \langle x_\phi | L_s x_\phi \rangle_{\phi_1} = \langle Ax_\phi | L_s x_\phi \rangle_\phi$$

A consequence of Schur's Lemma specific to the case of unitary representations (which we hand-wave for now)

tells us that, provided  $A$  commutes with every  $L_s$  (which we've shown),  $A$  must be a scalar multiple of  $\text{Id}_{V_\phi}$ . Thus,

$$\phi_1(s) = \langle \lambda x_\phi \mid L_s x_\phi \rangle_\phi = \lambda \phi(s)$$

for all  $s \in G$ . The same argument follows for  $\phi_2$ , which completes our proof that  $\phi$  is elementary.  $\square$

We conclude this section with a very nice characterization of elementary functions in the abelian case.

**THEOREM A.2.10.** If  $G$  is locally compact abelian, then the elementary functions on  $G$  are precisely the continuous characters  $G \rightarrow S^1$ .

*Proof.* First off, a character  $\chi$  on  $G$  is bounded and  $L^\infty(G)$ . To show it is positive type, we see that:

$$\begin{aligned} \iint \chi(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt &= \iint \overline{\chi(s)} \chi(t) f(s) \, ds \, \overline{f(t)} \, dt \\ &= \int \overline{\chi(s)} f(s) \, ds \int \chi(t) \overline{f(t)} \, dt \\ &= \left| \int \chi(s) f(s) \, ds \right|^2 \geq 0 \end{aligned}$$

for all  $f \in \mathcal{C}_c(G)$ . Moreover, necessarily we must have  $\chi(e) = 1$  for any character. Using our previous result A.2.9, all we have left to show is, for all  $\phi$  positive definite with value 1 at the identity, that  $\phi$  is a character iff the representation of  $G$  in  $V_\phi$  is irreducible.

For one direction, suppose  $\phi$  is a character. Then, for any  $f \in \mathcal{C}_c(G)$  we have:

$$\langle f \mid f \rangle_\phi = \left| \int \overline{\phi(s)} f(s) \, ds \right|^2$$

This induces a function  $f \mapsto \langle f \mid f \rangle_\phi$ , and since its image lies in  $\mathbb{C}$ , its kernel must have codimension one. Thus the quotient  $V_\phi$  is one-dimensional and therefore must be an irreducible representation.

For the reverse direction, assume the representation of  $G$  in  $V_\phi$  is irreducible. Then by another consequence of Schur's Lemma (which we again handwave for now), the representation  $s \mapsto L_s$  is one-dimensional. It follows that  $L_s(\xi) = \lambda(s)\xi$  for all  $\xi \in V_\phi$ , where  $\lambda(s)$  depends continuously on  $s$ . Moreover, recall from A.2.3 that each  $L_s$  is unitary, so we must have  $|\lambda(s)| = 1$  for all  $s$ , and thus  $\lambda$  is a character (continuous map to  $S^1$ ) of  $G$ . Putting it together with our result from A.2.5 and our assumption that  $\phi(e) = 1$ , we get:

$$\phi(s) = \langle x_\phi \mid L_s x_\phi \rangle_\phi = \overline{\lambda(s)} \langle x_\phi \mid x_\phi \rangle_\phi = \overline{\lambda(s)} \phi(e) = \overline{\lambda(s)}$$

and thus  $\phi$  must also be a continuous character, as desired.  $\square$

### §A.3 Fourier Inversion

Throughout this subsection we let  $G$  be a locally compact abelian group (we'll use the acronym LCAG for this) with bi-invariant Haar measure  $dx$ , with continuous character group  $\widehat{G}$ . Our main tool for establishing Pontryagin duality is the Fourier Inversion formula.

**DEFINITION A.3.1.** For  $f \in L^1(G)$ , we define its **Fourier transform** as:

$$\begin{aligned}\widehat{f} : \widehat{G} &\rightarrow \mathbb{C} \\ \chi &\mapsto \int_G f(y) \overline{\chi(y)} \, dy\end{aligned}$$

**SANITY CHECK.** Verify that  $\widehat{f}$  is well-defined, and that  $|\widehat{f}(\chi)| \leq \|f\|_1$ .

**EXAMPLE A.3.2.** In the case  $G = \mathbb{R}$ , with the addition operation, we can identify each  $t \in \mathbb{R}$  with the character  $s \mapsto e^{ist}$ . The Fourier transform then gives us:

$$\widehat{f}(t) = \int_{\mathbb{R}} f(s) e^{-ist} \, ds$$

which is the classical Fourier transform.

**WARNING.** Pedantically speaking, the Fourier transform in the above example is a function on  $\widehat{\mathbb{R}}$  rather than  $\mathbb{R}$ .

Let  $V(G)$  denote the complex span of the continuous functions of positive type on  $G$ , and define

$$V^1(G) := V(G) \cap L^1(G).$$

**THEOREM A.3.3 (Fourier Inversion).** There exists a Haar measure  $d\chi$  on  $\widehat{G}$  such that, for all  $f \in V^1(G)$ ,

$$f(y) = \int_{\widehat{G}} \widehat{f}(\chi) \chi(y) \, d\chi.$$

Moreover, the Fourier transform  $f \mapsto \widehat{f}$  identifies  $V^1(G)$  with  $V^1(\widehat{G})$ .

**DEFINITION A.3.4.** The measure  $d\chi$  is called the **dual measure** of  $dx$ , the given Haar measure on  $G$ .

To prove the existence of the dual measure, we discuss some more properties of convolution.

**PROPOSITION A.3.5 (Properties of Convolution).** Let  $f, g$  be  $\mathbb{C}$ -valued Borel functions on LCAG  $G$ . Then:

- (a) If the convolution  $f * g(x)$  exists for some  $x \in G$ , then so does  $g * f(x)$ , and they are equal to one another.
- (b) If  $f, g \in L^1(G)$ , then  $f * g(x)$  exists for almost all  $x \in G$ . Moreover,  $f * g \in L^1(G)$  and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

- (c) If  $f, g, h \in L^1(G)$  then  $(f * g) * h = f * (g * h)$ .

*Proof.* (a)

By translation-invariance of the Haar measure on  $G$ , we can replace  $y$  by  $yx$  in the convolution integrand to get:

$$\begin{aligned} f * g(x) &= \int g(y^{-1}x)f(y) \, dy \\ &= \int g(y^{-1})f(yx) \, dy \\ &= g * f(x), \end{aligned}$$

where the last equality follows from the fact that the Haar measure is inversion-invariant for Borel subsets of  $G$ .

(b)

Consider the homeomorphism

$$\begin{aligned} \alpha : G \times G &\xrightarrow{\sim} G \times G \\ (x, y) &\mapsto (yx, y). \end{aligned}$$

Note the inverse map  $\alpha^{-1}$  sends  $(x, y) \mapsto (y^{-1}x, y)$ . For any open  $U \subseteq \mathbb{C}$ , we note that  $\alpha(f^{-1}(U) \times G)$  is a Borel subset of  $G \times G$ . By construction, we observe that  $(x, y) \in \alpha(f^{-1}(U) \times G)$  iff  $y^{-1}x \in f^{-1}(U)$ . Thus the map

$$(x, y) \mapsto f(y^{-1}x)$$

is a Borel function on  $G \times G$ , and therefore so is the map

$$(x, y) \mapsto f(y^{-1}x)g(y),$$

since the product of Borel functions is also Borel. Since both  $f$  and  $g$  are  $L^1$ -functions, we have

$$\iint |f(y^{-1}x)| \, dx \, |g(y)| \, dy < \infty.$$

Thus Fubini's theorem applies here to give us

$$\iint |f(y^{-1}x)g(y)| \, dy \, dx = \|f\|_1 \|g\|_1.$$

It follows that  $|f| * |g|$  is an  $L^1$ -function and is thus finite almost everywhere. Thus  $f * g$  must also be  $L^1$ , and the norm inequality also follows from this.

(c)

By work similar to our proof of part (b), Fubini's theorem applies here.

$$\begin{aligned}
 f * (g * h)(x) &= \int f(y^{-1}x)(g * h)(y) dy \\
 &= \int f(y^{-1}x) \int g(z^{-1}y)h(z) dz dy \\
 &= \iint f(y^{-1}x)g(z^{-1}y)h(z) dy dz \\
 &= \iint f(y^{-1}z^{-1}x)g(y)h(z) dy dz, \text{ by replacing } y \text{ with } yz, \\
 &= \int f * g(z^{-1}x)h(z) dz \\
 &= (f * g) * h(x),
 \end{aligned}$$

as desired. □

**OBSERVATION.** The above properties of convolution essentially tell us that  $L^1(G)$  is a Banach algebra with respect to convolution. If  $G$  is discrete, then the characteristic function of the identity also maps  $L^1(G)$  unital. It is also a fact (that we do not prove here) that the converse is true: i.e if  $L^1(G)$  is unital wrt convolution then  $G$  must be discrete.

**EXERCISE A.3.A.** Show that the Fourier transform of the convolution  $f * g$  is exactly the product of Fourier transforms  $\widehat{f}\widehat{g}$ .

**PROPOSITION A.3.6.** Let  $B$  denote the Banach algebra  $L^1(G)$ , and let  $\widehat{B} := \text{hom}_{\mathbb{C}}(B, \mathbb{C}) \setminus \{0\}$  be the non-zero complex characters of  $B$ . For any  $\chi \in \widehat{B}$  and  $f \in B$ , define

$$v_{\chi}(f) = \widehat{f}(\chi) = \int f(y)\overline{\chi(y)} dy.$$

Then  $v_{\chi} \in \widehat{B}$  for all  $\chi$ . Moreover, the map

$$\begin{aligned}
 \widehat{B} &\rightarrow \widehat{B} \\
 \chi &\mapsto v_{\chi}
 \end{aligned}$$

is a bijection.

*Proof.* Linearity of  $v_{\chi}$  follows from the direct calculation:

$$\begin{aligned}
 v_{\chi}(\lambda f + g) &= \int (\lambda f + g)(y)\overline{\chi(y)} dy \\
 &= \lambda \int f(y)\overline{\chi(y)} dy + \int g(y)\overline{\chi(y)} dy \\
 &= \lambda v_{\chi}(f) + v_{\chi}(g)
 \end{aligned}$$

for any  $\lambda \in \mathbb{C}$ . Exercise A.3.A shows that  $v_{\chi}$  is compatible with the convolution operation on  $B$ . Moreover, since each  $\chi$  takes values of norm 1,  $v_{\chi}$  is not identically zero on  $L^1(G)$ .

To show injectivity of the map  $\widehat{G} \rightarrow \widehat{B}$ , suppose for some  $\chi, \chi' \in G$  that

$$v_\chi(f) = \int f(y) \overline{\chi(y)} dy = \int f(y) \overline{\chi'(y)} dy = v_{\chi'}(f)$$

for all  $f \in B$ . Then  $\chi, \chi'$  must agree on a set of comeasure zero on  $G$ . Since they are both continuous characters, we must have  $\chi = \chi'$ , proving injectivity.

It now remains to show that every nonzero character of  $B$  is of the form  $v_\chi$  for some  $\chi \in \widehat{G}$ . Let  $\psi : B \rightarrow \mathbb{C}$  be a non-trivial algebra homomorphism. Standard Gelfand theory tells us that  $\psi$  is a functional on  $L^1(G) = B$  of norm  $\leq 1$  under the Gelfand topology. By the duality of  $L^\infty$  and  $L^1$  there must exist some  $\phi \in L^\infty(G)$  with the same norm such that

$$\psi(f) = \int f(x) \phi(x) dx$$

for all  $f \in B$ . Our goal is thus to show that  $\phi$  is a continuous character, from which we deduce  $\psi = v_{\phi^{-1}}$ , which will prove the map in the theorem statement gives us our bijection between  $\widehat{G}$  and  $\widehat{B}$ . Our strategy will be to prove  $\phi$  is multiplicative on  $G$ , and combining this with the fact that it has  $L^\infty$ -norm bounded by 1 will force  $|\phi(x)| = 1$  for all  $x \in G$ , and therefore will imply  $\phi$  is a character.

Recall the representation of  $G$  given by  $L_s f(x) = f(s^{-1}x)$ . We compute:

$$\begin{aligned} \int \psi(f) \phi(y) g(y) dy &= \psi(f) \psi(g) \\ &= \psi(f * g) \\ &= \iint f(y^{-1}x) g(y) dy \phi(x) dx \\ &= \int L_y f(x) \phi(x) dx g(y) dy \\ &= \int \psi(L_y f) g(y) dy. \end{aligned}$$

This equality of integrals tells gives us the equality

$$\psi(f) \phi(y) = \psi(L_y f)$$

for almost all  $y \in G$ . The RHS above is continuous in  $y$  [WHY??? EXPLAIN MORE] and thus the LHS must be too (a deduction we can make by the assumption that  $\psi$  is non-zero), which tells us  $\phi$  is a continuous function.

To show  $\phi$  must be multiplicative, we apply Equation (A.3) repeatedly to get:

$$\psi(f) \phi(xy) = \psi(L_{xy} f) = \psi(L_x L_y f) = \psi(L_y f) \phi(x) = \psi(f) \phi(x) \phi(y),$$

which implies  $\phi$  is multiplicative, as we wanted (again, we must use the fact that  $\psi$  is nonzero here). Thus  $\phi$  is a continuous character and  $\psi = v_{\phi^{-1}}$ , completing our proof.  $\square$



We now analyze the space of Fourier transforms with a ring structure. We define:

$$\widehat{A} := \left\{ \widehat{f} : f \in L^1(G) \right\}$$

to be the *ring of Fourier transforms* on  $G$  (sometimes also denoted  $\widehat{A}(G)$ , if the underlying group isn't already clear from context). By A.3.6, we know that

**DEFINITION A.3.7.** The **transform topology** on  $\widehat{G}$  is the weakest topology that makes all the  $\widehat{f} \in \widehat{A}$  continuous.

**SANITY CHECK.** Why is  $\widehat{A}$  closed under multiplication of elements?

## §A.4 Pontryagin Duality

**APPENDIX B****LOCAL AND GLOBAL FIELDS**

## APPENDIX C

# GEOMETRY OF THE UPPER HALF PLANE

Introduction mainly follows [DS06] and [MM06].

### §C.1 Smth

The group of  $2 \times 2$  complex invertible matrices act on the Riemann sphere  $\mathbb{P}_{\mathbb{C}}^1$  via the following map:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d},$$

and the class of such maps are known as **fractional linear transformations**.

**DEFINITION C.1.1.** The **upper half plane** is what it sounds like, namely the set:

$$\mathfrak{H} := \{z \in \mathbb{C} : \Im(z) > 0\}.$$

We handwave the following elementary result:

**THEOREM C.1.2.** (a) For any  $z \in \mathfrak{H}$  there exists  $\alpha \in \mathrm{SL}_2(\mathbb{R})$  such that  $\alpha i = z$ .

(a) We have the following chain of isomorphisms:

$$\mathrm{GL}_2^+(\mathbb{R})/\mathbb{R}^\times \cong \mathrm{SL}_2(\mathbb{R})/\{\pm 1\} \cong \mathrm{Aut}(\mathfrak{H}).$$

(a) The  $\mathrm{SL}_2(\mathbb{R})$ -stabilizer of  $i \in \mathfrak{H}$  is  $\mathrm{SO}_2(\mathbb{R})$ .

Recalling that when group acting on the space is ALSO locally compact and second countable, then the algebraic orbit-stabilizer isomorphism is in fact a homeomorphism. This gives us the following corollary.

**COROLLARY C.1.3.** The map

$$\alpha \mathrm{SO}_2(\mathbb{R}) \mapsto \alpha_i$$

is a homeomorphism  $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \xrightarrow{\sim} \mathfrak{H}$ .

Thus we can represent elements  $z \in \mathfrak{H}$  as right cosets  $\beta \mathrm{SO}_2(\mathbb{R})$  for  $\beta \in \mathrm{SL}_2(\mathbb{R})$ . Thus, the action of  $\mathrm{SL}_2(\mathbb{R})$  on  $H$  can be computed by

$$\alpha \cdot z = \alpha \beta \mathrm{SO}_2(\mathbb{R}),$$

i.e left matrix multiplication rather than fractional linear transformation.

# REFERENCES

- [Ber+13] J. Bernstein et al. *An Introduction to the Langlands Program*. Birkhäuser Boston, 2013. ISBN: 9780817682262.
- [Con] K. Conrad. “The Character Group of  $Q$ ”. URL: <https://kconrad.math.uconn.edu/blurbs/gradnumthy/characterQ.pdf>.
- [DS06] F. Diamond and J. Shurman. *A First Course in Modular Forms*. Graduate Texts in Mathematics. Springer New York, 2006. ISBN: 9780387272269.
- [Lan94] S. Lang. *Algebraic Number Theory*. Graduate Texts in Mathematics. Springer, 1994. ISBN: 9780387942254.
- [MM06] T. Miyake and Y. Maeda. *Modular Forms*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2006. ISBN: 9783540295938.
- [RV98] D. Ramakrishnan and R.J. Valenza. *Fourier Analysis on Number Fields*. Graduate Texts in Mathematics. Springer New York, 1998. ISBN: 9780387984360. URL: [https://books.google.com/books?id=g\\_K0HqWU538C](https://books.google.com/books?id=g_K0HqWU538C).
- [Sut24] Andrew Sutherland. *Lecture notes for 18.876 (Graduate Number Theory II)*. 2024. URL: <https://math.mit.edu/classes/18.786/2024/lectures.html>.
- [TSH98] Nobuhiko Tatsuuma, Hiroaki Shimomura, and Takeshi Hirai. “On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms”. In: *Journal of Mathematics of Kyoto University* 38.3 (1998), pp. 551 –578. doi: 10.1215/kjm/1250518067. URL: <https://doi.org/10.1215/kjm/1250518067>.
- [Wei74] Andréi Weil. *Basic Number Theory*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer New York, 1974. ISBN: 9783540069355.