Tate's Thesis:
Fourier Analysis on Number Fields

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Contents

Section 1

How to Read This Text

These notes assume some basic familiarity with elementary topology (including some elementary results on topological groups), measure theory, representation theory, functional analysis and algebraic number theory. Here are a bunch of quick litmus tests:

- Topology/Measure theory: Recall/understand that every locally compact group admits a left Haar measure.

- Representation theory: Recall/understand why any G-linear map between two irreducible representations is either the zero map or an algebraic isomorphism.

- Number Theory: Recall/understand the equivalent characterizations of Dedekind domains as (i) noetherian, one-dimensional, and integrally closed or (ii) unique factorization of ideals into primes, among others.

- Functional Analysis: Recall/understand why a scalar multiple of the identity is a normal operator.

The goal of these notes is that if you're conceptually comfortable with some open neighborhood of the above results then this text should be readable and understandable to you almost everywhere.

In terms of formatting, there will be four main types of colored boxes in this text.

> **Definition 1.0.1.** A bright, almost-blaring red-backgrounded, red-striped box to house a vital definition.

> **Example 1.0.2.** An equally-importantly-red-backgrounded but slightly less blaring purple-striped box to house an important or prototypical example of a concept.

> **Theorem 1.0.3.** A neutrally-toned gray-shaded box to house a result, whether that be a theorem (such as this particular box), a proposition, etc.

> **Exercise 1.0.A.** A cool green-colored box that houses an exercise that (at least in the author's experience) helps grasp or hone a core tool/technique. Exercises are meant to be reasonably doable by a reader who's paying attention, so if anything seems too hard re-read the subsection, and then e-mail me if it's still too hard.

As you might have deduced, Definitions, Examples and Results follow a [Subsection.Number] numbering scheme, all using a common counter, and Exercises follow a [Subsection.Letter] numbering scheme[1].
As is standard in mathematical writing I've also filled this text with technically unnecessary but hopefully-enriching asides under various names such as Remarks and Observations. A less-standard one used here is the Sanity Check, which is a result that should be intuitively obvious to the reader at that point, and verifiable in some epsilon time.

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[1] I'm of the opinion that if I ever require more than 26 exercises per subsection then either I'm offshoring too much to the reader or the subsection is too long.

It is a corollary of Murphy's Law that this text will likely never be completely typo-free, but that doesn't mean it's something not to strive for. Please email me about any typographical, notational, mathematical or lingustic errors you might find.

Section 2

Number Theory Background

Introduction

Here we build up the language required to discuss the

§2.1   Valuation Theory

Section 3

Pontryagin Duality

In this section we start to do how to do Fourier transforms on an abelian topological group G, via the *dual* $\widehat{G}$, the group of continuous homomorphisms $G \to S^1$.

> DEFINITION 3.0.1. The **Pontryagin Dual** of G, denoted $\widehat{G}$, is the group of all continuous complex homomorphisms $G \to S^1$ (where S^1 is given its standard subspace topology from $\mathbb{C}$), with multiplication as the group operation. Elements of $\widehat{G}$ are often referred to as **characters** of G.

> EXAMPLE 3.0.2. The dual group of the circle is the integers, i.e
> $$\widehat{S^1} \cong \mathbb{Z}$$

proof of 3.0.2. Recall the continuous homomorphism $\exp : \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$, the so-called *universal covering map*. For any continuous homomorphism $\phi : S^1 \to S^1$, we have the following diagram:

$$
\begin{array}{ccc}
 & \mathbb{R} & \\
\exp \downarrow & & \searrow \psi \\
S^1 & \xrightarrow{\phi} & S^1
\end{array}
$$

from which we see that any such ϕ must induce a continuous homomorphism $\psi = \phi \circ \exp : \mathbb{R} \to S^1$. Moreover, since the kernl of the universing covering map is exactly $\mathbb{Z}$, it follows ψ must also have kernel containing $\mathbb{Z}$.

A standard vector-space argument shows any continuous homomorphism $\mathbb{R} \to \mathbb{R}$ is of the form $x \mapsto \lambda x$ for $\lambda \in \mathbb{R}$. using the universal lift we deduce any continuous homomorphism $\mathbb{R} \to S^1$ is of the form $\psi_\lambda(x) = e^{2\pi i \lambda x}$, for $\lambda \in \mathbb{R}$. Since we want $\mathbb{Z}$ to be in the kernel we deduce $\lambda \in \mathbb{Z}$.

Thus we have:
$$\psi = e^{2\pi i n x} = \phi(e^{2\pi i x}) \implies \phi(y) = y^n$$

which was what we wanted to show. $\square$

SANITY CHECK. Having been introduced to the definition and an example, verify rigorously the (hopefully) intuitive identity for any topological groups G_1, G_2:

$$\widehat{G_1 \times G_2} \cong \widehat{G_1} \times \widehat{G_2}$$

§3.1 Topologizing the dual

We want to put a topology on $\widehat{G}$ that tells us if two functions are "close together". For a function $f \in \widehat{G}$, suppose we take a compact set $K \subseteq G$, and an open neighborhood $V \supseteq f(K)$ of the image of this compact set. Since S^1 is Hausdorff, $f(K)$ is a closed set. As such, we might expect that if we wiggle $f(K)$ around a little, it shoud still be

inside V. Thus we want to characterize a function g as being "close" to f if $g(K) \subseteq V$ as well. This motivates the following construction.

> **DEFINITION 3.1.1.** The **compact-open topology** on $\widehat{G}$ is the one generated by the following neighborhood base of the trivial character:
> $$\left\{ W(K,V) := \left\{ \chi \in \widehat{G} \ : \ \chi(K) \subseteq V \right\} \right\}_{K,V}$$
> where K ranges over all compact sets in G and V ranges over all neighborhoods of the identity in S^1.

> **PROPOSITION 3.1.2.** If G is discrete, then the compact-open topology on $\widehat{G}$ is the same as the topology of pointwise convergence.

Proof. Recall the topology of pointwise convergence is generated by sets of the form

$$S(g,V) := \left\{ \chi \in \widehat{G} \ : \ \chi(x) \in U \right\}_{g \in G; \ V \subseteq_{\text{open}} S^1}$$

For any G, this is finer than the compact-open topology, since for any compact $K \subseteq G$ we can write:

$$W(K,V) = \bigcup_{g \in K} S(g,V)$$

If G is discrete then every compact set is necessarily finite (and vice versa!), and thus $\{g\}$ is compact, so every subbasis element $S(g,U)$ is also a basis element of the compact-open topology. Thus the two topologies are equivalent. $\qquad\square$

We now introduce some notation that'll come in handy when doing technical analysis of $\widehat{G}$.

Notation

- For any $X \subseteq G$ (not necessarily a subgroup), then for $n \in \mathbb{N}$ we define

$$X^{(n)} := \left\{ \prod_{j=1}^{n} x_j \ : \ x_j \in X, \ j \in [1,n] \right\}$$

- Let $\phi : \mathbb{R} \to S^1$ be the universal cover projection map $r \mapsto e^{2\pi i r}$. Then for some $\epsilon \in (0,1]$, we define

$$N(\epsilon) := \phi\left(\left(\frac{-\epsilon}{3}, \frac{\epsilon}{3} \right) \right)$$

i.e $N(\epsilon)$ is some small symmetric arc around 1 in S^1.

> **LEMMA 3.1.3.** Take $m \in \mathbb{Z}^+$, and take $x \in \mathbb{C}$ such that $x^i \in N(1)$ for all $i \in [1,m]$. Then $x \in N(1/m)$.

Proof. In polar coordinates, every such x has radius 1 and angle some θ, and note that x^i has angle $i\theta$. We are assuming that $\theta, 2\theta, \ldots, m\theta$ are all contained in $(-2\pi/3, 2\pi/3) \pmod{2\pi}$. If $x \notin N(1/m)$, that means its associated argument θ is in the interval $[2\pi/3m, 2\pi(3m-1)/3m]$. In particular, this would mean that $m\theta$ would have argument between $2\pi/3$ and $4\pi/3$ inclusive, which is not in the open set $N(1)$, a contradiction. Thus $x \in N(1/m)$ $\qquad\square$

> COROLLARY 3.1.3.1. If U is a subset of G containing the identity then $\chi(U^{(m)}) \subseteq N(1) \implies \chi(U) \subseteq N(1/m)$.

> EXERCISE 3.1.A. Rigorously prove the above corollary.

> PROPOSITION 3.1.4. Suppose G is an abelian topological group. Then:
>
> (a) A homomorphism $\chi : G \to S^1$ is continuous (and hence in $\widehat{G}$) iff $\chi^{-1}(N(1))$ is a neighborhood of the identity in G.
>
> (b) The family $\{W(K, N(1))\}_{\text{compact } K \subseteq G}$ is a neighborhood base for the trivial character for the compact open topology of $\widehat{G}$.
>
> (c) G discrete $\implies \widehat{G}$ compact.
>
> (d) G compact $\implies \widehat{G}$ discrete.
>
> (e) G locally compact $\implies \widehat{G}$ locally compact.

Proof. **(a)**
If χ is continuous then since $1 \in N(1) \subseteq_{\text{open}} S^1$ we must have that $e \in U := \chi^{-1}(N(1)) \subseteq_{\text{open}} G$.
For the other direction, suppose $\chi(U) \subseteq N(1)$ for some neighborhood of the identity. Then by continuity of multiplication, for every $m \in \mathbb{Z}$ there must exist an open neighborhood V of the identity such that $V^{(m)} \subseteq U$. Then by 3.1.3.1 we must have $\chi(V) \subseteq N(1/m)$. Since the sets $\{N(1/m)\}_{m \in \mathbb{Z}_{>0}}$ form a neighborhood base around the identity of S^1, this fact suffices to prove continuity.

(b)
We want to show that for every compact $K_1 \subseteq$ and every positive integer m that there exists some compact $K \subseteq G$ such that

$$W(K, N(1)) \subseteq W(K_1, N(1/m))$$

Define $K := K_1^{(m)}$, which is the continuous image of the topological direct product K^m, and thus K itself is compact. If $\chi \in W(K, N(1))$ then, by construction, we know that $\chi(x), \chi(x)^2, \ldots, \chi(x)^m \in N(1)$ for all $x \in K_1$. By 3.1.3 we have that $\chi(x) \in N(1)$, from which it follows $\chi \in W(K_1, N(1/m))$, as desired.

(c)
If G is discrete then any group homomorphism $f : G \to S^1$ is continuous, so $\widehat{G} = \text{Hom}(G, S^1)$. Also recall from 3.1.2 that in this case $\widehat{G}$ has exactly the topology of pointwise convergence. With respect to this topology, note $\text{Hom}(G, S^1)$ is a closed subset of the space of $(S^1)^{|G|}$ (the set of all maps from G to S^1), which is a compact space by compactness of S^1. Thus $\text{Hom}(G, S^1) = \widehat{G}$ itself is compact, as desired.

(d)
Since the image of any group homomorphism is a subgroup of S^1, we have $\chi(G) \not\subseteq N(1)$ for any $\chi \in \widehat{G}$.
If G itself is compact, the set $W(G, N(1))$ must be an open set in $\widehat{G}$, which can only contain the trivial character, by what we showed. Thus the singleton of the identity is open in $\widehat{G}$, so it must be a discrete group.

(e)

FINISH LATER $\qquad\square$

§3.2 Functions of Positive Type

To motivate what a function of positive type is, we discuss some background on unitary representations. From now on let G be a locally compact group with a (left) Haar measure ds.

Let ρ be a representation of a topological group G in the space of unitary operators on a Hilbert space H. Fixing some $x \in H$, define a $\mathbb{C}$-valued function on G by:

$$\varphi(s) := \langle \rho(s)(x) \mid x \rangle$$

Take $s_1, \dots, s_n \in G$ and consider

$$A := \left[\varphi(s_j^{-1} s_i) \right] \in \mathrm{Mat}_{n \times n}(\mathbb{C})$$

We claim A is Hermitian and positive semidefinite. For the first point, note that since each $\rho(s)$ is a unitary operator,

$$
\begin{aligned}
A_{i,j} &= \left\langle \rho(s_j^{-1} s_i)(x) \mid x \right\rangle \\
&= \left\langle \rho(s_j)^* \rho(s_i)(x) \mid x \right\rangle \\
&= \left\langle \rho(s_i)(x) \mid \rho(s_j)(x) \right\rangle \\
&= \left\langle x \mid \rho(s_i)^* \rho(s_j)(x) \right\rangle \\
&= \left\langle x \mid \rho(s_i^{-1} s_j(x)) \right\rangle \\
&= \overline{\left\langle \rho(s_i^{-1} s_j)(x) \mid x \right\rangle} = \overline{A_{j,i}}
\end{aligned}
$$

Thus A is equal to its conjugate transpose, and is Hermitian. For positive semidefiniteness, take any $z = (z_i) \in \mathbb{C}^n$. Then:

$$
\begin{aligned}
\langle Az \mid z \rangle &= \sum_{i,j}^{n} \varphi(s_j^{-1} s_i) z_j \overline{z_i} \\
&= \sum_{i,j}^{n} \left\langle \rho(s_i)(x) \mid \rho(s_j)(x) \right\rangle z_j \overline{z_i} \\
&= \left\langle \sum_{j}^{n} \rho(s_j)(z_j x) \mid \sum_{i}^{n} \rho(s_i)(z_i x) \right\rangle \\
&= \left| \sum_{i}^{n} \rho(s_i)(z_i x) \right|^2 \geq 0
\end{aligned}
$$

Note on the third equality that there is no conjugacy required since A is Hermitian.

REMARK. Note A is actually Hermitian and positive semidifinite even in the non-locally compact case.

Let $\mathscr{C}_c(G)$ denote the set of $\mathbb{C}$-valued continuous functions on G with compact support. Recall for every $p \in$

$[1, \infty]$ that $\mathscr{C}_c(G) \subseteq L^p(G)$. Thus it is subject to the L^p-norm defined by:

$$\|f\|_p = \left(\int_G |f|^p \, ds \right)^{1/p}$$

for finite p, with $\|f\|_\infty$ being the essential supremum[2] of $|f|$. This norm also induces a topology, under which $\mathscr{C}_c(G)$ is dense in $L^p(G)$ [JUSTIFY MORE MAYBE??]

> **DEFINITION 3.2.1.** A Haar-measurable function $\phi \in L^\infty(G)$ is **positive definite** (or of **positive type**, we will oscillate between these two terminologies) if for any $f \in \mathscr{C}_c(G)$ we have:
>
> $$\iint \phi(s^{-1}t)f(s) \, ds \, \overline{f(t)} \, dt \ \geq 0$$
>
> where the integrals are taken over the entire group G.

> **LEMMA 3.2.2.** For ϕ a Haar-measurable function on G, define
>
> $$\psi(g, h) := \phi(g^{-1}h).$$
>
> Then ψ is Haar-measurable on $G \times G$.

FINISH THIS. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Note by 3.2.2 that Fubini's theorem applies to show that this double integral is defined.(Every locally compact group is the disjoint union of σ-compact spaces). Moreover, if $\text{Supp}(f)$ is contained in a compact subset K then the integrand in 3.2.1 has support contained in $K \times K \subseteq G \times G$.

Note that since $\phi \in L^\infty(G)$ that it is bounded above by $\|\phi\|_\infty$ on a set of comeasure zero. Thus the integral is bounded by:

$$\left| \iint \phi(s^{-1}t)f(s) \, ds \, \overline{f(t)} \, dt \ \right| \leq \|\phi\|_\infty \, (\sup |f| \cdot \mu(K))^2$$

Here $\mu(K)$ is the (finite!) Haar measure of K.

We describe some relations between these functions of positive type and Hilbert spaces and unitary representations.

If ϕ is a positive definite function then we can define a positive sesquilinear (i.e conjugate-symmetric) form on $\mathscr{C}_c(G)$ by:

$$\langle f_1 \mid f_2 \rangle_\phi := \iint \phi(s^{-1}t)f_1(s) \, ds \, \overline{f_2(t)} \, dt \, .$$

> **EXERCISE 3.2.A.** Verify the above equation is well-defined and finite for all $f_1, f_2 \in \mathscr{C}_c(G)$.

Set

$$W_\phi := \left\{ f \in \mathscr{C}_c(G) \, : \, \langle f \mid f \rangle_\phi = 0 \right\}$$

---

[2] recall the essential supremum is the supremum over a set of comeasure zero.

i.e the functions that are degenerate wrt the form defined by ϕ. Cauchy-Schwarz also tells us that W_ϕ is a subspace of $\mathscr{C}_c(G)$, and so we can form a quotient space $\mathscr{C}_c(G)/W_\phi$. By construction, $\langle | \rangle_\phi$ is a positive-definite Hermitian form on this quotient. Let V_ϕ denote the completion of this quotient with respect to this form, which thus acquires the structures of a Hilbert space.

For any $s \in G$ and any function f on G, we can define the function $L_s f$ by

$$L_s f(t) = f(s^{-1}t).$$

Note that if $f \in \mathscr{C}_c(G)$ then $L_s f \in \mathscr{C}_c(G)$, and moreover the map

$$G \to \text{End}(\mathscr{C}_c(G))$$
$$s \mapsto L_s$$

is a representation of G.

PROPOSITION 3.2.3. Let ϕ be a $\mathbb{C}$-valued positive-definite function on a locally compact group G. Then the mapping $s \mapsto L_s$ induces a unitary representation of G on the Hilbert space V_ϕ.

Proof. If ϕ is of positive type and $f \in \mathscr{C}_c(G)$ then:

$$\langle L_s f \mid L_s f \rangle_\phi = \iint \phi(t^{-1}u)f(s^{-1}t)\,dt\,\overline{f(s^{-1}u)}\,du$$
$$= \iint \phi((s^{-1}t)^{-1}(s^{-1}u))f(s^{-1}t)\,dt\,\overline{f(s^{-1}u)}\,du$$
$$= \iint \phi(t^{-1}u)f(t)\overline{f(u)}\,du$$
$$= \langle f \mid f \rangle_\phi$$

which proves that L_s is a unitary representation. $\square$

REMARK. It is also true that the above is a topological representation, the rigorous proof of which we omit here.

EXERCISE 3.2.B. Let ρ be a unitary representation of G on a Hilbert space V. For any $x \in V$, show the map

$$\phi : G \to \mathbb{C}$$
$$s \mapsto \langle x \mid \rho(s)(x) \rangle$$

is a function of positive type. This is essentially a converse of 3.2.3.

For further discussion of these functions of positive type, we recall the definition of a convolution.

DEFINITION 3.2.4. Let f, g be $\mathbb{C}$-valued Borel functions on a locally compact group G. Their **convolution**

$f * g$ is defined by:

$$f * g(t) := \int g(s^{-1}t)f(s)\,\mathrm{d}s = \int g(s^{-1})f(ts)\,\mathrm{d}s \,,$$

assuming this integral, taken over the full group, exists.

SANITY CHECK. Verify for $f \in \mathscr{C}_c(G)$ and $\phi \in L^\infty(G)$ that $f * \phi$ exists, and is continuous.

PROPOSITION 3.2.5. Let ϕ be a positive definite function on locally compact G. Then there exists some $x_\phi \in V_\phi$ such that

$$\phi(s) = \left\langle x_\phi \mid L_s x_\phi \right\rangle_\phi$$

almost everywhere for $s \in G$.

Proof. Let $\{\alpha\}$ be an indexing set for the open neighborhoods V_α of the identity of G. Since G is Hausdorff, we must have $\cap V_\alpha = \{e\}$. By Urysohn's Lemma, for every α we can construct continuous $g_\alpha : G \to \mathbb{R}_{>0}$ such that:

- Supp(g_α) is a compact subset of V_α.

- The identity $\int_G g_\alpha(s)\,\mathrm{d}s = 1$ holds.

If we write $\alpha \leq \beta$ whenever $V_\alpha \supseteq V_\beta$, then that makes $\{\alpha\}$ a directed set, which in turn gives us a net $\{g_\alpha(s)\,\mathrm{d}s\}$ of positive linear functionals on $\mathscr{C}_c(G)$ by:

$$f \mapsto \int_G f(s)g_\alpha(s)\,\mathrm{d}s$$

Note these converge (weakly) to the Dirac measure δ_e, which we recall is defined by

$$\delta_e(f) := f(e),$$

i.e evaluation at the identity. Consider the integral

$$\langle f \mid g_\alpha \rangle_\phi = \iint \phi(s^{-1}t)f(s)\,\mathrm{d}s\, g_\alpha(t)\,\mathrm{d}t = \int f * \phi(t) \cdot g_\alpha(t)\,\mathrm{d}t \,,$$

which exists, since $f * \phi$ is continuous and g_α has compact support. We can now define a linear form Φ on V_ϕ that on $\mathscr{C}_c(G)$ is given by:

$$\Phi(f) := \lim_\alpha \langle f \mid g_\alpha \rangle_\phi = \lim_\alpha \int f * \phi(t) \cdot g_\alpha(t)\,\mathrm{d}t \,.$$

Note for any $h \in \mathscr{C}_c(G)$ that $(f * \phi) \cdot h \in \mathscr{C}_c(G)$ as well. Thus the above limit exists since the g_α weakly converge to the Dirac measure, which also tells us

$$\Phi(f) = f * \phi(e) = \int \phi(s^{-1})f(s)\,\mathrm{d}s$$

Since V_ϕ is a Hilbert space, it's self-dual and therefore there exists $x_\phi \in V_\phi$ such that

$$\Phi(\xi) = \left\langle \xi \mid x_\phi \right\rangle_\phi$$

for all $\xi \in V_\phi$.

We shall now see how x_ϕ behaves under the group representation $s \mapsto L_s$.

$$
\begin{aligned}
\left\langle \xi \mid L_s x_\phi \right\rangle_\phi &= \lim_\alpha \left\langle \xi \mid L_s g_\alpha \right\rangle_\phi \\
&= \lim_\alpha \iint \phi(t^{-1}u)\xi(t)\, dt\, g_\alpha(s^{-1}u)\, du \\
&= \lim_\alpha \iint \phi(t^{-1}su)\xi(t)\, dt\, g_\alpha(u)\, du \\
&= \int \phi(t^{-1}s)\xi(t)\, dt .
\end{aligned}
\tag{1}
$$

Swapping the arguments in Equation (1), and using conjugate-symmetry of Hermitian forms, we deduce that:

$$
\begin{aligned}
\left\langle L_s x_\phi \mid \xi \right\rangle_\phi &= \overline{\left\langle \xi \mid L_s x_\phi \right\rangle_\phi} \\
&= \int \overline{\phi(t^{-1}s)\xi(t)}\, dt \\
&= \int \phi(s^{-1}t)\overline{\xi(t)}\, dt ,
\end{aligned}
$$

and taking the case $s = e$ above we get:

$$
\left\langle x_\phi \mid \xi \right\rangle_\phi = \int \phi(t)\overline{\xi(t)}\, dt .
$$

By Equation (1), we can also deduce for any $\xi \in V_\phi$ and $h \in \mathscr{C}_c(G)$ that:

$$
\left\langle \xi \mid h \right\rangle_\phi = \iint \phi(s^{-1}t)\xi(s)\, ds\, \overline{h(t)}\, dt = \int \left\langle \xi \mid L_t x_\phi \right\rangle_\phi \overline{h(t)}\, dt ,
\tag{2}
$$

and by continuity this extends to all of V_ϕ.

OBSERVATION. In Equation (2), if we set $h = \xi$, note that if ξ is orthogonal to the $\mathbb{C}G$-submodule of V_ϕ generated by x_ϕ, then the integrand is zero for every $t \in G$, and thus $\left\langle \xi \mid \xi \right\rangle_\phi = 0$. Thus V_ϕ is generated (as a $\mathbb{C}G$-module) by x_ϕ. This fact will come in useful later.

To complete the proof of the main result, let us take $\xi = x_\phi$ in equation 2 and apply the above results to deduce that for any $\psi \in V_\phi$ we have:

$$
\int \phi(s)\overline{\psi(s)}\, ds = \left\langle x_\phi \mid \psi \right\rangle_\phi = \int \left\langle x_\phi \mid L_s x_\phi \right\rangle_\phi \overline{\psi(s)}\, ds .
$$

By equality of the two integrals above for every ψ, it follows that

$$
\phi(s) = \left\langle x_\phi \mid L_s x_\phi \right\rangle_\phi
$$

almost everywhere, as desired. $\qquad\square$

SANITY CHECK. For ϕ positive definite on locally compact G, verify that ϕ is equal to a continuous function almost everywhere. In fact, if ϕ IS continuous then the representation of 3.2.5 is true everywhere, not just on a comeasure-zero set.

COROLLARY 3.2.5.1. If ϕ as above is continuous, then:

(a) $\phi(e) \geq 0$.

(b) $\phi(e) = \sup_{s \in G} |\phi(s)|$.

(c) $\phi(s^{-1}) = \overline{\phi(s)}$ for every $s \in G$.

EXERCISE 3.2.C. Prove 3.2.5.1 rigorously. (Hint: Use the representation in 3.2.5 for (a), Cauchy-Schwarz for (b), and the fact that L_s is unitary for (c)).

§3.3   Elementary Functions

The positive-definite, continuous functions on G bounded by 1 in the L^∞ norm comprise an important subset of $L^\infty(G)$. We set

$$\mathscr{P}(G) := \left\{ \phi \in \mathscr{C}(G) \cap L^\infty(G) : \phi \text{ is positive-definite and } \|\phi\|_\infty \leq 1 \right\}$$

Related to this is the class of functions we are interested in.

DEFINITION 3.3.1. Define the set $\mathscr{E}(G)$ as the zero function unioned with any function satisfying:

(i) ϕ is continuous and of positive type.

(ii) $\phi(e) = 1$.

(iii) For every decomposition $\phi = \phi_1 + \phi_2$, where $\phi_1, \phi_2 \in \mathscr{P}(G)$, there exist $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ such that:

$$\phi_1 = \lambda_1 \phi, \quad \phi_2 = \lambda_2 \phi$$

The non-zero elements of $\mathscr{E}(G)$ are called **elementary functions**.

SANITY CHECK. Verify that $\mathscr{E}(G) \subseteq \mathscr{P}(G)$ (Hint: 3.2.5.1 might be helpful).

LEMMA 3.3.2. The sets $\mathscr{P}(G)$ and $\mathscr{E}(G)$ satisfy the following properties:

(a) $\mathscr{P}(G)$ is a convex, bounded subset of $L^\infty(G)$. It is also weakly compact as a subset of $L^1(G)^*$.

(b) Any convex, closed subset of $\mathscr{P}(G)$ containing its extreme points is all of $\mathscr{P}(G)$.

(c) The extreme points of $\mathscr{P}(G)$ are exactly the points of $\mathscr{E}(G)$.

Proof. **(a)**

By definition $\mathscr{P}(G)$ is bounded. Convexity follows from the fact that if we take a linear combination of two positive

definite functions then that function must be positive definite as well (if this isn't immediately clear, verify it!), and that if both the coefficients have norm ≤ 1 then the supremum of that sum must also have norm ≤ 1.

For the second statement, first note that if we have a sequence $\{\phi_n\}$ of elements in $\mathscr{P}(G)$ weakly converging to ϕ then by definition we have

$$\|\phi_n - \phi\|_\infty \to 0$$

and the fact that $\|\phi_n\|_\infty \leq 1$ for all n implies $\|\phi\|_\infty \leq 1$ as well. Moreover, recall the standard fact from functional analysis that we can identify $L^\infty(G)$ and $L^1(G)^*$ via an isometry. A sequence ϕ_n of functions in $\mathscr{P}(G)$ weaky converging to $\phi \in L^\infty(G)$ is equivalent to saying that:

$$\int f\phi_n \, ds \; \to \; \int f\phi \, ds \,,$$

from which it follows that:

$$\iint \phi_n(s^{-1}t)f(s) \, ds \; \overline{f(t)} \, dt \; \to \; \iint \phi(s^{-1}t)f(s) \, ds \; \overline{f(t)} \, dt$$

and thus positive-definiteness of all the ϕ_n also implies positive-definiteness of ϕ. Recall that this also implies ϕ is continuous.

WARNING. Technically ϕ is an equivalence class in $L^\infty(G)$ of functions that are measure-zero-different to a continuous function.

Thus $\mathscr{P}(G)$, as a subset of $L^1(G)^*$, is a closed subset of the unit ball, and thus compact under the weak-star topology via Alaoglu's theorem.

(b)

This is a special case of the Krein-Milman theorem.

(c)

FINISH THIS $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now we come to a result that brings together the theory of elementary functions with irreducible representations, using the theory we built in

> THEOREM 3.3.3. Let ϕ be continuous of positive type on G such that $\phi(e) = 1$. Then $\phi \in \mathscr{E}(G)$ if and only if the unitary representation $s \mapsto L_s$ of G in V_ϕ is irreducible.

Proof. For the forward direction, suppose ϕ is an elementary function. Let W be a closed, G-invariant subspace of V_ϕ with orthogonal complement $W^\perp$. Let Proj_W denote the orthogonal projection map onto W. Then since each L_s is unitary, we have the following commutative diagram:

$$
\begin{array}{ccc}
W \oplus W^\perp & \xrightarrow{\mathrm{Proj}_W} & W \\
{\scriptstyle L_s}\big\downarrow & & \big\downarrow{\scriptstyle L_s} \\
W \oplus W^\perp & \xrightarrow[\mathrm{Proj}_W]{} & W
\end{array}
$$

Thus, to show irreducibility of $s \mapsto L_s$ it suffices to show that if A is any orthogonal projection operator commuting with every L_s, then A is either the zero map or the identity map on V_ϕ.

Recall for any projection that $\langle Ax \mid y \rangle_\phi = \langle Ax \mid Ay \rangle_\phi$. It follows from this and our "representation result" in ?? then that:

$$
\begin{aligned}
\phi(s) &= \left\langle x_\phi \mid L_s x_\phi \right\rangle_\phi \\
&= \left\langle Ax_\phi \mid L_s x_\phi \right\rangle_\phi + \left\langle x_\phi - Ax_\phi \mid L_s x_\phi \right\rangle_\phi \\
&= \left\langle Ax_\phi \mid L_s Ax_\phi \right\rangle + \left\langle x_\phi - Ax_\phi \mid L_s(x_\phi - Ax_\phi) \right\rangle_\phi .
\end{aligned}
$$

By 3.2.B, we have decomposed ϕ into the sum of two functions of positive type, i.e two elements of $\mathscr{P}(G)$. Since we assumed ϕ to be elementary, by definition we must have:

$$
\left\langle Ax_\phi \mid L_s Ax_\phi \right\rangle = \left\langle Ax_\phi \mid AL_s x_\phi \right\rangle = \left\langle Ax_\phi \mid L_s x_\phi \right\rangle = \lambda \left\langle x_\phi \mid L_s x_\phi \right\rangle
$$

for all $s \in G$, where the first equality follows from A commuting with L_s and the second follows from the property of projection maps. Recall from our observation in the proof of 3.2.5 that V_ϕ is generated by x_ϕ as a $\mathbb{C}G$-module, so in fact the above identity holds over all of V_ϕ, which tells us $A = \lambda \cdot \mathrm{Id}_{V_\phi}$. Since A is a projection map however, it is idempotent, so $\lambda = 0$ (i.e A is the zero map) or $\lambda = 1$ (i.e A is the identity map), which completes the proof of the forward direction.

For the reverse direction, suppose that the representation $s \mapsto L_s$ is irreducible. We have to show that, for any decomposition of $\phi = \phi_1 + \phi_2$ into a sum of functions from $\mathscr{P}(G)$, that each ϕ_i is of the form $\lambda_i \phi$. Note for each $f \in \mathscr{C}_c(G)$ that

$$
\begin{aligned}
\langle f \mid f \rangle_\phi &= \iint \phi(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \\
&= \iint \left(\phi_1(s^{-1}t) + \phi_2(s^{-1}t) \right) f(s) \, ds \, \overline{f(t)} \, dt \\
&= \iint \phi_1(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \; + \iint \phi_2(s^{-1}t) f(s) \, ds \, \overline{f(t)} \, dt \\
&= \langle f \mid f \rangle_{\phi_1} + \langle f \mid f \rangle_{\phi_2} ,
\end{aligned}
$$

and so we get the key inequality

$$
\langle f \mid f \rangle_{\phi_1} \le \langle f \mid f \rangle_\phi . \tag{3}
$$

Thus any compactly-supported continuous function degenerate wrt $\langle | \rangle_\phi$ is also degenerate wrt $\langle | \rangle_{\phi_1}$. Thus ϕ also defines a Hermitian form on the (completed) quotient V_ϕ. There must exist a continuous, positive-definite $A \in \mathrm{End}(V_\phi)$ such that

$$
\langle A\xi \mid \psi \rangle_\phi = \langle \xi \mid \psi \rangle_{\phi_1}
$$

for all $\xi, \psi \in V_\phi$. In particular, we must also have

$$
\left\langle Ax_\phi \mid L_s x_\phi \right\rangle_\phi = \left\langle x_\phi \mid L_s x_\phi \right\rangle_{\phi_1}
$$

for all $s \in G$. We claim that A commutes with all the L_s. To see this, note that:

$$
\begin{aligned}
\left\langle AL_s x_\phi \mid \psi \right\rangle_\phi &= \langle L_s \phi \mid \psi \rangle_{\phi_1} \\
&= \langle \phi \mid L_{s^{-1}} \psi \rangle_{\phi_1} \quad \text{by the fact that } L_s \text{ is unitary,} \\
&= \langle A\phi \mid L_{s^{-1}} \psi \rangle_\phi \\
&= \langle L_s A\phi \mid \psi \rangle_\phi
\end{aligned}
$$

for all $\phi, \psi \in \mathscr{C}_C(G)$, and thus A must commute with L_s in V_ϕ.

We claim that:

$$
\left\langle x_\phi \mid L_s x_\phi \right\rangle_{\phi_1} = \phi_1(s)
$$

everywhere in G. To see this, recall that the key construction of the proof of the representation in 3.2.5 involved the convergence of the net of functionals $\{g_\alpha\}$. The inequality in Equation (3) tells us that convergence in the norm induced by ϕ also implies convergence in the norm induced by ϕ_1, and from this the rest of the proof implies the same equality for the ϕ_1-induced norm.

Putting everything together then, we have:

$$
\phi_1(s) = \left\langle x_\phi \mid L_s x_\phi \right\rangle_{\phi_1} = \left\langle Ax_\phi \mid L_s x_\phi \right\rangle_\phi
$$

A consequence of Schur's Lemma specific to the case of unitary representations (which we hand-wave for now) tells us that, provided A commutes with every L_s (which we've shown), A must be a scalar multiple of Id_{V_ϕ}. Thus,

$$
\phi_1(s) = \left\langle \lambda x_\phi \mid L_s x_\phi \right\rangle_\phi = \lambda \phi(s)
$$

for all $s \in G$. The same argument follows for ϕ_2, which completes our proof that ϕ is elementary. $\qquad \square$

We conclude this section with a very nice characterization of elementary functions in the abelian case.

> THEOREM 3.3.4. If G is locally compact abelian, then the elementary functions on G are precisely the continuous characters $G \to S^1$.

Proof. First off, a character χ on G is bounded and $L^\infty(G)$. To show it is positive type, we see that:

$$
\begin{aligned}
\iint \chi(s^{-1}t) f(s) \, \mathrm{d}s \, \overline{f(t)} \, \mathrm{d}t &= \iint \overline{\chi(s)} \chi(t) f(s) \, \mathrm{d}s \, \overline{f(t)} \, \mathrm{d}t \\
&= \int \overline{\chi(s)} f(s) \, \mathrm{d}s \int \chi(t) \overline{f(t)} \, \mathrm{d}t \\
&= \left| \int \chi(s) f(s) \, \mathrm{d}s \right|^2 \geq 0
\end{aligned}
$$

for all $f \in \mathscr{C}_c(G)$. Moreover, necessarily we must have $\chi(e) = 1$ for any character. Using our previous result 3.3.3, all we have left to show is, for all ϕ positive definite with value 1 at the identity, that ϕ is a character iff the representation of G in V_ϕ is irreducible.

For one direction, suppose ϕ is a character. Then, for any $f \in \mathscr{C}_c(G)$ we have:

$$\langle f \mid f \rangle_\phi = \left| \int \overline{\phi(s)} f(s) \, \mathrm{d}s \right|^2$$

This induces a function $f \mapsto \langle f \mid f \rangle_\phi$, and since its image lies in $\mathbb{C}$, its kernal must have codimension one. Thus the quotient V_ϕ is one-dimensional and therefore must be an irreucible representation.

For the reverse direction, assume the representation of G in V_ϕ is irreducible. Then by another consequence of Schur's Lemma (which we again handwave for now), the representation $s \mapsto L_s$ is one-dimensional. It follows that $L_s(\xi) = \lambda(s)\xi$ for all $\xi \in V_\phi$, where $\lambda(s)$ depends continuously on s. Moreover, recall from 3.2.3 that each L_s is unitary, so we must have $|\lambda(s)| = 1$ for all s, and thus λ is a character (continuous map to S^1) of G. Putting it together with our result from 3.2.5 and our assumption that $\phi(e) = 1$, we get:

$$\phi(s) = \left\langle x_\phi \mid L_s x_\phi \right\rangle_\phi = \overline{\lambda(s)} \left\langle x_\phi \mid x_\phi \right\rangle_\phi = \overline{\lambda(s)}\phi(e) = \overline{\lambda(s)}$$

and thus ϕ must also be a continuous character, as desired. $\qquad\square$

§3.4   Fourier Inversion

Throughout this subsection we let G be a locally compact abelian group (we'll use the acronym LCAG for this) with bi-invariant Haar measure $\mathrm{d}x$, with continuous character group $\widehat{G}$. We'll Our main tool for establishing Pontryagin duality is the Fourier Inversion formula.

DEFINITION 3.4.1. For $f \in L^1(G)$, we define its **Fourier transform** as:

$$\widehat{f} : \widehat{G} \to \mathbb{C}$$
$$\chi \mapsto \int_G f(y)\overline{\chi(y)} \, \mathrm{d}y$$

SANITY CHECK. Verify that $\widehat{f}$ is well-defined, and that $\left| \widehat{f}(\chi) \right| \leq \|f\|_1$.

EXAMPLE 3.4.2. In the case $G = \mathbb{R}$, with the addition operation, we can identify each $t \in \mathbb{R}$ with the character $s \mapsto e^{ist}$. The Fourier transform then gives us:

$$\widehat{f}(t) = \int_\mathbb{R} f(s)e^{-ist} \, \mathrm{d}s$$

which is the classical Fourier transform.

WARNING. Pedantically speaking, the Fourier transform in the above example is a function on $\widehat{\mathbb{R}}$ rather than $\mathbb{R}$.

Let $V(G)$ denote the complex span of the continuous functions of positive type on G, and define

$$V^1(G) := V(G) \cap L^1(G).$$

THEOREM 3.4.3 (Fourier Inversion). There exists a Haar measure $d\chi$ on $\widehat{G}$ such that, for all $f \in V^1(G)$,

$$f(y) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(y) \, d\chi \, .$$

Moreover, the Fourier transform $f \mapsto \widehat{f}$ identifies $V^1(G)$ with $V^1(\widehat{G})$.

DEFINITION 3.4.4. The measure $d\chi$ is called the **dual measure** of dx, the given Haar measure on G.

To prove the existence of the dual measure, we discuss some more properties of convolution.

PROPOSITION 3.4.5 (Properties of Convolution). Let f, g be $\mathbb{C}$-valued Borel functions on LCAG G. Then:

(a) If the convolution $f * g(x)$ exists for some $x \in G$, then so does $g * f(x)$, and they are equal to one another.

(b) If $f, g \in L^1(G)$, then $f * g(x)$ exists for almost all $x \in G$. Moreover, $f * g \in L^1(G)$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 \, .$$

(c) If $f, g, h \in L^1(G)$ then $(f * g) * h = f * (g * h)$.

Proof. **(a)**
By translation-invariance of the Haar measure on G, we can replace y by yx in the convolution integrand to get:

$$\begin{aligned}
f * g(x) &= \int g(y^{-1}x) f(y) \, dy \\
&= \int g(y^{-1}) f(yx) \, dy \\
&= g * f(x),
\end{aligned}$$

where the last equality follows from the fact that the Haar measure is inversion-invariant for Borel subsets of G.

(b)
Consider the homeomorphism

$$\alpha : G \times G \xrightarrow{\sim} G \times G$$
$$(x, y) \mapsto (yx, y).$$

Note the inverse map α^{-1} sends $(x, y) \mapsto (y^{-1}x, y)$. For any open $U \subseteq \mathbb{C}$, we note that $\alpha(f^{-1}(U) \times G)$ is a Borel subset of $G \times G$. By construction, we observe that $(x, y) \in \alpha(f^{-1}(U) \times G)$ iff $y^{-1}x \in f^{-1}(U)$. Thus the map

$$(x, y) \mapsto f(y^{-1}x)$$

is a Borel function on $G \times G$, and therefore so is the map

$$(x, y) \mapsto f(y^{-1}x)g(y),$$

since the product of Borel functions is also Borel. Since both f and g are L^1-functions, we have

$$\iint \left| f(y^{-1}x) \right| \, \mathrm{d}x \, \left| g(y) \right| \, \mathrm{d}y \, < \infty.$$

Thus Fubini's theorem applies here to give us

$$\iint \left| f(y^{-1}x)g(y) \right| \, \mathrm{d}y \, \mathrm{d}x \, = \|f\|_1 \, \|g\|_1 \, .$$

It follows that $|f| * |g|$ is an L^1-function and is thus finite almost everywhere. Thus $f * g$ must also be L^1, and the norm inequality also follows from this.

(c)

By work similar to our proof of part (b), Fubini's theorem applies here.

$$
\begin{aligned}
f * (g * h)(x) &= \int f(y^{-1}x)(g * h)(y) \, \mathrm{d}y \\
&= \int f(y^{-1}x) \int g(z^{-1}y)h(z) \, \mathrm{d}z \, \mathrm{d}y \\
&= \iint f(y^{-1}x)g(z^{-1}y)h(z) \, \mathrm{d}y \, \mathrm{d}z \\
&= \iint f(y^{-1}z^{-1}x)g(y)h(z) \, \mathrm{d}y \, \mathrm{d}z \, , \text{ by replacing } y \text{ with } yz, \\
&= \int f * g(z^{-1}x)h(z) \, \mathrm{d}z \\
&= (f * g) * h(x),
\end{aligned}
$$

as desired. $\qquad\square$

OBSERVATION. The above properties of convolution essentially tell us that $L^1(G)$ is a Banach algebra with respect to convolution. If G is discrete, then the characteristic function of the identity also maps $L^1(G)$ unital. It is also a fact (that we do not prove here) that the converse is true: i.e if $L^1(G)$ is unital wrt convolution then G must be discrete.

EXERCISE 3.4.A. Show that the Fourier transform of the convolution $f * g$ is exactly the product of Fourier transforms $\widehat{f}\widehat{g}$.

PROPOSITION 3.4.6. Let B denote the Banach algebra $L^1(G)$, and let $\widehat{B} := \mathrm{hom}_{\mathbb{C}}(B, \mathbb{C}) \setminus \{0\}$ be the non-zero complex characters of B. For any $\chi \in \widehat{G}$ and $f \in B$, define

$$v_\chi(f) = \widehat{f}(\chi) = \int f(y)\overline{\chi(y)} \, \mathrm{d}y \, .$$

Then $v_\chi \in \widehat{B}$ for all χ. Moreover, the map

$$\widehat{G} \to \widehat{B}$$

$$\chi \mapsto v_\chi$$

is a bijection.

Proof. Linearity of v_χ follows from the direct calculation:

$$v_\chi(\lambda f + g) = \int (\lambda f + g)(y)\overline{\chi(y)}\,dy$$

$$= \lambda \int f(y)\overline{\chi(y)}\,dy \; + \int g(y)\overline{\chi(y)}\,dy$$

$$= \lambda v_\chi(f) + v_\chi(g)$$

for any $\lambda \in \mathbb{C}$. Exercise 3.4.A shows that v_χ is compatible with the convolution operation on B. Moreover, since each χ takes values of norm 1, v_χ is not identically zero on $L^1(G)$.

To show injectivity of the map $\widehat{G} \to \widehat{B}$, suppose for some $\chi, \chi' \in G$ that

$$v_\chi(f) = \int f(y)\overline{\chi(y)}\,dy \; = \int f(y)\overline{\chi'(y)}\,dy \; = v_{\chi'}(f)$$

for all $f \in B$. Then χ, χ' must agree on a set of comeasure zero on G. Since they are both continuous characters, we must have $\chi = \chi'$, proving injectivity.

It now remains to show that every nonzero character of B is of the form v_χ for some $\chi \in \widehat{G}$. Let $\psi : B \to \mathbb{C}$ be a non-trivial algebra homomorphism. Standard Gelfand theory tells us that ψ is a functional on $L^1(G) = B$ of norm ≤ 1 under the Gelfand topology. By the duality of L^∞ and L^1 there must exist some $\phi \in L^\infty(G)$ with the same norm such that

$$\psi(f) = \int f(x)\phi(x)\,dx$$

for all $f \in B$. Our goal is thus to show that ϕ is a continuous character, from which we deduce $\psi = v_{\phi^{-1}}$, which will prove the map in the theorem statement gives us our bijection between $\widehat{G}$ and $\widehat{B}$. Our strategy will be to prove ϕ is multiplicative on G, and combining this with the fact that it has L^∞-norm bounded by 1 will force $|\phi(x)| = 1$ for all $x \in G$, and therefore will imply ϕ is a character.

Recall the representation of G given by $L_s f(x) = f(s^{-1}x)$. We compute:

$$\int \psi(f)\phi(y)g(y)\,dy \; = \psi(f)\psi(g)$$

$$= \psi(f \star g)$$

$$= \iint f(y^{-1}x)g(y)\,dy\,\phi(x)\,dx$$

$$= \int L_y f(x)\phi(x)\,dx\,g(y)\,dy$$

$$= \int \psi(L_y f)g(y)\,dy\,.$$

This equality of integrals tells gives us the equality

$$\psi(f)\phi(y) = \psi(L_y f)$$

for almost all $y \in G$. The RHS above is continuous in y [WHY??? EXPLAIN MORE] and thus the LHS must be too (a deduction we can make by the assumption that ψ is non-zero), which tells us ϕ is a continuous function. To show ϕ must be multiplicative, we apply Equation (3.4) repeatedly to get:

$$\psi(f)\phi(xy) = \psi(L_{xy} f) = \psi(L_x L_y f) = \psi(L_y f)\phi(x) = \psi(f)\phi(x)\phi(y),$$

which implies ϕ is multiplicative, as we wanted (again, we must use the fact that ψ is nonzero here). Thus ϕ is a continuous character and $\psi = v_{\overline{\phi}}$, completing our proof. $\qquad\square$

To prove more serious results about dual measures on

§3.5   Pontryagin Duality

Section 4

Adeles and Ideles

§4.1 Background: Defining Restricted Products

Let J be an indexing set (the reason for choosing J over I will become apparent later when we get introduced to the Ideles), suppose that for every $v \in J$ we have a locally compact group indexed by G_v. We fix a finite subset J_∞ of J (which will later represent the infinite places), and suppose that for all $v \notin J_\infty$ we have a compact open (and thus closed) subgroup $H_v \leq G_v$.

> **DEFINITION 4.1.1.** The **restricted direct product** of the G_v with respect to the H_v is defined as
>
> $$\coprodprod_{v \in J} G_v := \left\{ (x_v) \in \prod G_v \ : \ x_v \in H_v \text{ for all but finitely many } v \right\}$$

REMARK. The notation I've used for the restricted product (product and coproduct symbol overlayed on one another) is non-standard, but more distinct than the usual $\prod'$, in my opinion. I first saw this in Andrew Sutherland's lecture notes for 18.875 course at MIT, and I've also seen it in Keith Conrad's notes.

Let us denote this restricted product G. Note $G \subseteq \prod G_v$ (the normal direct product of groups). We want to topologize G, but as it turns out, the subspace topology induced by the product topology on $\prod G_v$ is insufficient. For reasons that will become clear later, we'll want sets like $\prod H_v$ to be open, which won't be true in the subspace topology unless $H_v = G_v$ for all but finitely many $v \in J$.

> **DEFINITION 4.1.2.** The **restricted product topology** on $\coprodprod G_v$ is given by the basis
>
> $$\mathcal{B} := \left\{ \prod U_v \ : \ U_v \subseteq_{\text{open}} G_v, \ e \in U_v \text{ and } U_v = H_v \text{ for all but finitely many } v \right\}$$

SANITY CHECK. Suppose we have a family of compact opens $H_v' \subseteq G_v$ such that $H_v' = H_v$ for all but finitely many $v \in J$. Then confirm that $\coprodprod G_v$ wrt to the H_v is the same (both set-theoretically and topologically) as $\coprodprod G_v$ wrt to the H_v'. It thus suffices to specify the H_v for all but finitely many v.

We introduce a handy piece of notation that'll come in useful later. Let S be any finite subset of the index set J containing J_∞. Then we define

$$G_S := \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$$

For every S, T such that $S \subseteq T$, consider the inclusion maps $i_{S,T} : G_S \hookrightarrow G_T$. Then with respect to this directed system, we have:

$$G = \varinjlim_S G_S$$

EXERCISE 4.1.A. Verify the above statement rigorously.

PROPOSITION 4.1.3. Let G_v, H_v be as defined above and G be the restricted product. Then

(a) G is a locally compact group.

(b) A subset $Y \subseteq G$ has compact closure iff $Y \subseteq \prod K_v$ for some family of compact subsets $K_v \subseteq G_v$ such that $K_v = H_v$ for all but finitely many v.

Proof. **(a)**
Since each H_v is compact, Tychonoff tells us $\prod_{v \notin S} H_v$ is also compact. Thus G_S is the product of finitely many locally compact spaces with a compact space, and is thus itself locally compact. Moreover, the product topology on G_S is the SAME as the subspace topology induced by G, and since any $(x_v) \in G$ is contained in such a G_S, we conclude G must be locally compact as well.

(b)
If Y is contained in some product of compact sets $K = \prod K_v$ (which is compact by Tychonoff) its closure $\overline{Y}$ is a closed subset of a compact subset of a Hausdorff space G, and thus $\overline{Y}$ is compact too.
For the other direction, suppose $\overline{Y}$ is a compact subset of G. Since the sets G_S defined above cover G, a finite number of G_S must cover $\overline{Y}$. Note a union of a finite number of G_S is just a set of the form $G_{\mathscr{S}}$, for $\mathscr{S}$ being the union of all the S such that G_S is in the cover of $\overline{Y}$.
Consider the (continuous!) projection maps $\rho_v : G \to G_v$. Thus each $\rho_v(\overline{Y}) \subseteq G_v$ is compact. And since all but finitely many components of $G_{\mathscr{S}}$ are equal to H_v, the image $\rho_v(\overline{Y}) \subseteq H_v$ for all but finitely many v.
Thus $\overline{Y}$, and therefore Y, is contained in a set of the desired form. $\square$

§4.2   Measures on Restricted Products

Inline with the notation from the previous section, let G be the restricted direct product of locally compact G_v wrt a family of compact subgroups H_v, for $v \notin J_\infty$. For any finite subset S containing J_∞, let G_S be the same as defined before.

PROPOSITION 4.2.1. For all but finitely many v, suppose $\mathrm{d}g_v$ be a (left) Haar measure on G_v normalized so that $\int_{H_v} \mathrm{d}g_v = 1$. Then there is a unique Haar measure $\mathrm{d}g$ on G such that the restriction $\mathrm{d}g_S$ of $\mathrm{d}g$ on any G_S is exactly the product measure.

Proof. Choose such a set S and define $\mathrm{d}g_S$ as the product of the measures $\mathrm{d}g_v$. Note the measure of the compact group $\prod_{v \notin S} H_v$ is indeed finite, by how we've normalized things. It is a verification of axioms to deduce the rest that $\mathrm{d}g_S$ is indeed a Haar measure on G_S.
Suppose we have another finite set of indices $T \supseteq S$. Then, by construction it follows that $\mathrm{d}g_S$ coincides with the restriction of $\mathrm{d}g_T$ to $G_S \leq G_T$.

Since G is locally compact, we know it has a Haar measure, which restructs to a Haar measure on any G_S. Fixing any S, we can define [ASK QUESTION HERE ABOUT EXISTENCE OF THING THAT RESTRICTS APPROPRIATELY] $\square$

Thus we write

$$\mathrm{d}g \;=\; \prod_v \mathrm{d}g_v$$

for the left Haar measure on G, oftened referred to as the measure induced by the component measures.

Our next goal is integration.

PROPOSITION 4.2.2. Let f be an integrable function on G. Then

$$\int_G f(g)\,\mathrm{d}g \;=\; \lim_S \int_{G_S} f(g_S)\,\mathrm{d}g_S$$

EXERCISE 4.2.A. Verify the above.

Our next result will be that integrals behave fairly well with respect to products. We first introduce a bunch of notation.

Let S_0 be a finite set of indices containing J_∞ and the v for which $\mathrm{Vol}(H_v, \mathrm{d}g_v) \neq 1$.

Suppose for each v we have a continuous integrable f_v on G_v such that $f_v\,|_{H_v} = 1$ for all $v \notin S_0$.

For $g = (g_v) \in G$, define

$$f(g) = \prod_v f_v(g_v)$$

PROPOSITION 4.2.3. With the notation above, the function f is well-defined and continuous. For any $S \supseteq S_0$, we have

$$\int_{G_S} f(g_S)\,\mathrm{d}g_S = \prod_{v \in S} \int_{G_v} f_v(g_v)\,\mathrm{d}g_v$$

Proof. Since $g = (g_v)$ is in the restricted direct product, then $g_v \in H_v$ for all but finitely many v. Combining this with the fact that $f_v\,|_{H_v} = 1$ for almost all v, we deduce $\prod f_v(g_v)$ is a finite product, and thus well-defined.

To show f is continuous, consider a base for G consisting of sets of the form

$$\prod U_v \times \prod H_v$$

where the first product contains all the (finitely many!) components of G where f_v is non-trivial. Thus f can be computed locally as a finite product of continuous functions, and is thus locally continuous, and thus globally. The fact that the integral behaves well with products is a consequence of definitions itself, and the fact that f is a well-defined function. $\square$

COROLLARY 4.2.3.1. With the notation as in the above result,

$$\int_G f(g)\,\mathrm{d}g \;=\; \prod_v \int_{G_v} f_v(g_v)\,\mathrm{d}g_v$$

and $f \in L^1(G)$ provided that the RHS is a finite product.

Proof. By 4.2.2, we know f is integrable iff $\lim_S \int_{G_S} f(g_S) \, \mathrm{d}g_S < \infty$, which is true iff

$$\lim_S \prod_{v \in S} \int_{G_v} f_v(g_v) \, \mathrm{d}g_v = \prod_v \int_{G_v} f_v(g_v) \, \mathrm{d}g_v < \infty,$$

by the equation in 4.2.3. $\qquad\square$

PROPOSITION 4.2.4. Suppose all the G_v are abelian, and suppose we have functions $\{f_v\}$ on G_v such that they are the characteristic functions for H_v for all but finitely many v. Then f's Fourier transform is integrable and given by

$$\widehat{f}(g) = \prod_v \widehat{f_v}(g_v)$$

Proof. Firstly note that even in the non-abelian case, by 4.2.3.1, we have

$$\int_G f(g) \, \mathrm{d}g = \prod_v \int_{G_v} f_v(g_v) \, \mathrm{d}g_v = \prod_{v \in S} \int_{G_v} f_v(g_v) \, \mathrm{d}g_v$$

for some finite set S, since f_v is the characteristic function of H_v. Thus we have a finite product of continuous functions, and thus integrable.

In the abelian case, let $\chi = (\chi_v)$ denote a character of G, and define

$$h = \prod_v (h_v := f_v \chi_v)$$

Since χ is unitary, h must be integrable, and applying the identity in 4.2.3.1, we get our desired result in terms of the Fourier transform. $\qquad\square$

From now on assume all groups are abelian. We want to build a measure on the Pontryagin dual $\widehat{G}$ that is dual to $\mathrm{d}g$ in the sense of the Fourier inversion theorem.

[FINISH THIS PART HERE]

§4.3 Adeles, Ideles and Approximations

Let K be a global field, and let K_v denote its completion at a place v. With respect to addition, note K_v is a locally compact additive group. For all finite places v, recall K_v admits its local ring of integers $\mathcal{O}_v$ as an open compact subgroup.

DEFINITION 4.3.1. The **adele group** $\mathbb{A}_K$ of a global field K is the restricted direct product of K_v with respect to the open subgroups $\mathcal{O}_v$. Since this admits a natural ring structure as well, it is also referred to as the **adele ring**.

Note we have an embedding

$$K \hookrightarrow \mathbb{A}_K$$
$$x \mapsto (x, x, \ldots)$$

This is well-defined since K always has a natural embedding in K_v for any place v, and by unique factorization, every element is a local integer for all but finitely many v.

Similarly, we can conisder $K^\times$, the locally compact group of units of K wrt to multiplication. Then the local units $\mathcal{O}_v^\times$ are an open compact subgroup of each $K_v^\times$.

> **Definition 4.3.2.** The **idele group** $\mathbb{I}_K$ of a global field K is the restricted direct product of the $K_v^\times$ with respect to $\mathcal{O}_v^\times$.

We have a similar embedding

$$K^\times \hookrightarrow \mathbb{I}_K$$
$$x \mapsto (x, x, \dots)$$

which is again well-defined.

The induced topologies on the adele and idele group are called the **adelic** and **idelic** topologies, respectively.

Remark. Note that $\mathbb{I}_K \cong \mathbb{A}_K^\times$ as an algebraic isomorphism. However, this is not a topological embedding. For example, taking $K = \mathbb{Q}$ and S being any finite collection of primes, and let U_p be any neighborhood of 1 in $\mathbb{Q}_p$ for $P \in S$. Then

$$\left(\prod_{p \in S} N_p \times \prod_{p \notin S} \mathbb{Z}_p \right) \cap \mathbb{I}_\mathbb{Q} \not\subseteq \mathbb{R}^\times \times \prod_{p < \infty} \mathbb{Z}_p^\times = \mathbb{A}_\mathbb{Q}^\times$$

Note we can select some $(x_p) \in \mathbb{I}_\mathbb{Q}$ such that $x_p \in \mathbb{Z}_p \setminus \mathbb{Z}_p^\times$ for some $p \notin S$, which therefore excludes it from being in $\mathbb{A}_\mathbb{Q}^\times$ on the RHS.

Thus the idelic topology is finer than the induced subspace topology.

Let S_ω denote the infinite places of K. We denote by $\mathbb{A}_\omega$ the open subgroup $\mathbb{A}_{S_\omega} \subseteq \mathbb{A}_k$, i.e the elements of the adele group where all the components at the finite places are integral.

> **Theorem 4.3.3 (Approximation).** For every global field K,
>
> $$\mathbb{A}_K = K + A_\omega$$
>
> where K is identified with its natural diagonal embedding.

Proof. Given some $x = (x_v) \in \mathbb{A}_K$, we want to find some $y \in K$ such that $x_v - y \in \mathcal{O}_v$ for all v.

By definition, $x_v \in \mathcal{O}_v$ for all but finitely many places v, each corresponding to a prime $\mathfrak{p}_v \trianglelefteq \mathcal{O}_v$. Letting $p_v \in \mathcal{O}_K$ be an element of $\mathfrak{p}$-adic valuation 1 in each of these ideals, note that there must exist some $n_v \in \mathbb{N}$ such that $p_v^{n_v} x_v \in \mathcal{O}_v$. Set

$$q := \prod_{v \in T} p_v^{n_v}$$

where T is the (finite!) set of finite places where $x_v \notin \mathcal{O}_v$. By CRT, we can find some $\lambda \in \mathcal{O}_K$ such that $qx_v - \lambda \in \mathfrak{p}_v^{n_v}$ for all $v \in T$. Letting $y = \lambda/q$, then by construction $x_v - y \in \mathcal{O}_v$ for all v, as desired. [FILL IN MORE DETAIL LATER] $\qquad \square$

COROLLARY 4.3.3.1. With the notation above, $K \cap \mathbb{A}_\omega = \mathcal{O}_K$

SANITY CHECK. Verify the above corollary.

§4.4   Geometry of $\mathbb{A}_K/K$

We explore the behavior of the adeles under base-change.

LEMMA 4.4.1. Take E/K a finite extension of global fields, and let $\{u_1, \dots u_n\}$ be a K-basis of E. Then the natural map

$$\alpha : \prod_{j=1}^{n} \mathbb{A}_K \to \mathbb{A}_E$$

$$((x_{v,j})_v)_j \mapsto \sum_j u_j(x_{v,j})v$$

is an isomorphism of topological groups.

FINISH LATER. □

THEOREM 4.4.2. Any global field is a discrete, cocompact subgroup of its adele ring.

Proof. Let K_0 denote the smallest global subfield of K (i.e $\mathbb{Q}$ or $\mathbb{F}_p(t)$]), and let $n := [K : K_0]$. Then by 4.4.1, we have the following commutative diagram:

$$\begin{array}{ccc} \prod_{j=1}^{n} \mathbb{A}_{K_0} & \longrightarrow & \mathbb{A}_K \\ \uparrow & & \uparrow \\ \prod_{j=1}^{n} K_0 & \longrightarrow & K \end{array}$$

where the horizontal maps are topological isomorphisms. Thus $\mathbb{A}_K/K$ is compact iff $(\mathbb{A}_{K_0}/K_0)^n$ is compact, which is true iff $\mathbb{A}_{K_0}/K_0$ is compact.

Thus it suffices to consider $K = K_0$, in which case we have exactly one infinite place. Define a subset C of the adele group by

$$C := \left\{ x \in \mathbb{A}_K : |x_\infty|_\infty \leq \frac{1}{2} \text{ and } |x_v|_v \leq 1, \ \forall v \neq \infty \right\}$$

If we show that $C \cap K = \{0\}$ and $\mathbb{A}_K = C + K$, we will be done, since $C \cong \mathbb{A}_K/K$.

If $x \in K \cap C$, then we must have $x \in \mathcal{O}_K$, since $x \in \mathcal{O}_v$ for every place v. If $K = \mathbb{Q}$ then this condition ($x \in \mathbb{Z}$) combined with the condition that $|x_\infty|_\infty \leq 1/2$ forces. $x = 0$. If $K = \mathbb{F}_p(t)$, then let the infinite place be the one defined by $1/t$. Then $|x|_\infty = p^{-\operatorname{ord}_{1/t}(x)}$, which is ≥ 1 for anything in $\mathcal{O}_K = \mathbb{F}_p[t]$. Thus $C \cap K = \{0\}$.

[FINISH TEDIOUS DETAILS HERE] □

[FINISH STUFF HERE]

§4.5 Class Groups

DEFINITION 4.5.1. For a global field K, the quotient

$$C_K := \mathbb{I}_K/K^\times$$

is called the **idele class group** of K.

WARNING. As seen in the previous section, $\mathbb{A}_K/K$ is compact. One might be led to think $\mathbb{I}_K/K^\times$ is also compact but this is NOT the case, as we'll see later.

To analyze this, we first standardize the absolute value functions we're using.

DEFINITION 4.5.2. For k a local field, the **normalized** absolute value $|\cdot|_k$ on k is defined as follows:

(a) For $k = \mathbb{R}$ we set $|\cdot|_k$ to be the usual absolute value function.

(b) For $k = \mathbb{C}$ we set $|z|_k := z\bar{z}$, which is the square of the usual absolute value function on $\mathbb{C}$.

(c) For k non-Archimedean with uniformizer π and residue order q, we set $|\pi|_k := q^{-1}$. (Note this extends uniquely to k).

Recall that if k'/k is a finite extension of local fields then $|x|_{k'} = \left| N_{k'/k}(x) \right|_k$.

DEFINITION 4.5.3. Let K be a global field, and let $|\cdot|_v$ denote the normalized absolute value on the completion K_v. Then the **idelic absolute value** is defined by:

$$|\cdot|_{\mathbb{A}_K} : \mathbb{I}_K \to \mathbb{R}_{>0}$$
$$x = (x_v)_v \mapsto \prod_v |x_v|_v$$

SANITY CHECK. Why is the product defined above well-defined?

THEOREM 4.5.4 (Artin's Product Formula). Let K be a global field. Then for every $x \in K^\times$ we have $|x|_{\mathbb{A}_K} = 1$

Proof. Let E/K be finite separable. Then

$$|x|_{\mathbb{A}_E} = \prod_{u \in \mathscr{P}_K} \prod_{v \in \mathscr{P}_E, v|u} |x|_v$$
$$= \prod_{u \in \mathscr{P}_K} \prod_{v \in \mathscr{P}_E, v|u} \left| N_{E_v/K_u}(x) \right|_u$$
$$= \prod_u \left| N_{E/K}(x) \right|_u$$

[ASK QUESTION ABOUT MISSING EXPONENT ABOVE] □

We are now ready to define a subgroup of the ideles for which the embedding of K IS cocompact.

DEFINITION 4.5.5. For K a global field, the **ideles of norm one** are:

$$\mathbb{I}_K^1 := \ker(|\cdot|_{\mathbb{A}_K})$$

and the **norm-one idele class group** is the quotient:

$$C_K^1 := \mathbb{I}_K^1/K^\times$$

Note by the Artin product formula (4.5.4) that $K^\times \subseteq \mathbb{I}_K^1$, so this quotient is well-defined. We also have the SES

$$1 \to C_K^1 \to C_K \to |\mathbb{I}_K|_{\mathbb{A}_K} \to 1.$$

THEOREM 4.5.6. The norm-one idele class-group is always compact.

Proof. Recall from the proof of 4.4.2 that there exists some compact subset $Y \subseteq \mathbb{A}_K$ such that $\mathbb{A}_K = K + Y$. Fix a Haar measure μ on $\mathbb{A}_K$ (which exists as it's a locally compact group), then $\mu(Y) < \infty$. Choose a compact subset $Z \subseteq \mathbb{A}_K$ such that $\mu(Z) > \mu(Y)$. Consider the following two subsets:

$$Z_1 := \{z_1 - z_2 \ : \ z_1, z_2 \in Z\}$$
$$Z_2 := \{z_1 z_2 \ : \ z_1, z_2 \in Z\}$$

By continuity of addition and multiplication these two sets are also compact. Since K is discrete in $\mathbb{A}_K$, we have $K \cap Z_2$ is finite, with non-zero elements $x_1, \ldots, x_r$. [LOTS OF EXERCISE NONSENSE READ IN DETAIL and COMPLETE LATER] □

For a given finite set S of places of K containing S_∞, the Archimedean places, it is useful to have S-versions of the aforementioned groups.

WARNING (SOME WEIRD SHIT ABOUT NOTATION EXCLUDING INFINITE PLACES).

DEFINITION 4.5.7. Take S a finite set of places of a global field K containing the Archimedean places. Then:

(a) The S-**adeles** of K are defined as:
$$\mathbb{A}_{K,S} := \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$$

(b) The **ring of S-integers** is defined as:
$$\mathcal{O}_{K,S} := K \cap \mathbb{A}_{K,S}$$

   i.e the elements of K that are v-integral for all $v \notin S$.

(c) The S-**ideles** are defined as:
$$\mathbb{I}_{K,S} := \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

(d) The *S*-**ideles of norm one** defined as:

$$\mathbb{I}^1_{K,S} := \mathbb{I}^1_K \cap \mathbb{I}_{K,S}$$

OBSERVATION.

EXERCISE 4.5.A. Prove the above observation.

§4.6   Practical Example: Characters on $\mathbb{Q}$ and $C_{\mathbb{Q}}$

We begin by computing the character group $\widehat{\mathbb{Q}}$ of homomorphisms $\mathbb{Q} \to S^1$.

WARNING. Despite previous instances where we've used the notation $\widehat{G}$, in this case we are NOT requiring that $\chi \in \widehat{\mathbb{Q}}$ be a continuous map (unless of course we give $\mathbb{Q}$ the discrete topology).

It is a standard result from algebraic topology that the only continuous homomorphisms from $\mathbb{R}$ to S^1 are of the form

$$x \mapsto e^{i\lambda x}$$

for any $\lambda \in \mathbb{R}$. It follows that the restriction of the above map to $\mathbb{Q}$ is also a homomorphism to the circle. One might be tempted to think that this, too, characterizes all homomorphisms, but this is NOT the case.

[Insert stuff here]

We now proceed to a slightly more advanced task

Section 5

Tate's Thesis and Applications

Introduction

A lot of the major strides in classical analytic number theory were made in the 19th and 20th century, where it was discovered that a lot of information about the structure of prime numbers could be deduced by studying the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is absolutely convergent for complex s such that $\mathfrak{Re}(s) > 1$. This function admits an analytic continuation to the entire complex plane, save for a simple pole at $s = 1$, and satisfies the functional equation:

$$\xi(s) = \xi(1 - s),$$

where the Riemann-xi function is defined as

$$\xi(s) := \frac{1}{2}s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s.$$

Much of the early work for $\zeta(s)$ for real s was done by Euler, who established the classic Euler product expansion

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

A generalization of the zeta function can be done via Dirichlet series. Given any multiplicative sequence $\{a_n\}$ (i.e $\gcd(m, n) = 1 \implies amn = a_m a_n$) we can create the series

$$L(s) := \sum \frac{a_n}{n^s}.$$

EXAMPLE 5.0.1 (Dirichlet L-function). Recall a **Dirichlet character** with modulus $m \in \mathbb{Z}_{>0}$ is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ that is m-periodic, nonzero on any input coprime to m and zero otherwise. Given a Dirichlet character χ, its **Dirichlet L-series** is given by:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

These L-series are vital to the proof of Dirichlet's theorem on primes in arithmetic progressions.

EXAMPLE 5.0.2 (Generalized Dedekind Zeta Function). Let K be a number field, and let χ be a character of

standard ideal class group $\text{Cl}(K)$, then we can construction the Dirichlet series:

$$L(s, \chi) = \sum_{(0) \neq \mathfrak{a} \trianglelefteq \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where $N(\mathfrak{a})$ is the standard ideal norm taken over $\mathbb{Q}$ as the base field. When $\chi = 1$ this is known as the **Dedekind zeta function**.

One can generalize the above even further to consider crafting a series $L(s, \chi)$ for any χ that is a continuous character of the idele class group C_K. It is a famous result of Hecke that $L(s, \chi)$ has analytic continuation and satisfies an appropriate functional equation for any number field K and any idele class character χ. The original formulation of this result involved a complicated computation with these things called theta functions. Moreover in the functional equation a so-called *root number* $W(\chi)$ associated o each χ kept popping up, with no good explanation for its value.

Under the guidance of Emil Artin, in his 1950 thesis John Tate made use of Fourier analysis on adele groups to reprove the results of Hecke and establish these "local" functional equations that gave an "explicit factorization" of the root number.

In his thesis, Tate worked with some ad-hoc function spaces over local and global fields. We standardize our workings over the so-called *Schwarz-Bruhat functions* on local fields. Over an Archimedean number field (i.e $\mathbb{R}$ or $\mathbb{C}$), recall a **Schwarz** function (more commonly known as a *rapidly-decreasing function*) is a smooth $\mathbb{C}$-valued function f such that

$$\lim_{x \to \infty} p(x) f(x) = 0$$

for all polynomials $p(x)$.

DEFINITION 5.0.3. A **Schwarz-Bruhat** function on a local field F is a Schwarz function if F is Archimedean and a smooth[a], compactly-supported function in F is non-Archimedean. We denote by $S(F)$ the complex vector space of Schwarz-Bruhat functions on F.

___
[a]A function on a non-Archimedean local field is smooth if it is locally constant.

We use these functions because they guarantee that taking their integral over the entire group is well-defined.

§5.1   Local Zeta Functions

The goal of this section will be to introduce this thing called the *local L-factor* associated to a character χ, and in particular to realize it as the gcd of some "local zeta integrals", which will be objects that [WHAT IS THE MOTIVATION??]. We start with a slew of notations and definitions, so buckle up.

Let F be a local field with absolute value $|\cdot|$ and Haar measure $\text{d}x$. Let $X(F^\times) := \text{Hom}_{cont}(F^\times, \mathbb{C}^\times)$ be the continuous group homomorphisms into the complex numbers. The elements $\chi \in X(F^\times)$ will be referred to as **characters** of $F^\times$.

REMARK. Sometimes the elements of $X(F^\times)$ are called **quasi-characters** instead, with characters being reserved for continuous homomorphisms into the circle (in fact, this is the convention used by Tate in his original thesis). For our purposes we refer to the latter cases as **unitary characters**. Using the notation of Section 3, unitary

characters would be the elements of $\widehat{F^\times}$. We use this convention as the term "character" used more generally for a multiplicative homomorphism into the unit group of an arbitrary field.

When F is non-Archimedean, let $\mathcal{O}_F$ denote its ring of integers, $\mathfrak{p}_F$ (or just $\mathfrak{p}$ if it's clear from context) its maximal ideal, π_F its uniformizer and $\mathbb{F}_q$ its residue field. Recall that we can write $F^\times \cong U_F \times S_F$, where U_F is the group of unit-absolute-value elements of F, and $S_F = |F^\times|$, i.e. the image of $F^\times$ in the valuation map[3]. We make the observation that any $\chi \in X(F^\times)$ factors into a product:

$$\chi = \mu |\cdot|^s,$$

where $s \in \mathbb{C}$ and μ is the pullback of a unitary character on U_F (defined uniquely by restriction of χ!). This is because compactness of U_F forces any character on it to be unitary, and the only characters on S_F are maps $t \mapsto t^s$.

> **DEFINITION 5.1.1.** For any character $\chi = \mu |\cdot|^s \in X(F^\times)$, we call $\mathfrak{Re}\,(s)$ the **exponent** of χ.
> Furthermore, we say χ is **unramified** if $\mu = 1$.

WARNING. Recall that for any $q \in \mathbb{R}_{>0}$ we have the formula:

$$q^i = \cos(\ln(q)) + i\sin(\ln(q)).$$

In particular, in the non-Archimedean case the value of s is NOT uniquely determined (e.g i and $(1 + 2\pi)i$ would correspond to the same map). However, the real part of s IS uniquely determined, so the exponent is well-defined.

We now define the local L-factors $L(\chi)$.

(1) In the non-Archimedean case, we set:

$$L(\chi) := \begin{cases} (1 - \chi(\pi_F))^{-1}, & \text{if } \chi \text{ is unramified,} \\ 1, & \text{otherwise.} \end{cases} \tag{4}$$

(2) In the Archimedean case we recall by Gelfand-Mazur that $F = \mathbb{C}$ or $\mathbb{R}$.

    (i) If $F = \mathbb{C}$, note any χ is of the form:

$$\chi_{s,n} \,:\, re^{i\theta} \mapsto r^s e^{in\theta}$$

    for $s \in \mathbb{C}$ and $n \in \mathbb{Z}$ uniquely defined. Here we use our result in 3.0.2 to characterize the unitary characters of $U_\mathbb{C} = S^1$ and writing our domain in polar coordinates automatically makes r equal to the absolute value of our complex input. We then set:

$$L(\chi_{s,n}) := \Gamma_\mathbb{C}(s + \frac{|n|}{2}) := (2\pi)^{-\left(s + \frac{|n|}{2}\right)} \Gamma(s + \frac{|n|}{2}). \tag{5}$$

    (ii) If $F = \mathbb{R}$ the $U_F = \{\pm 1\}$. Note any character is of the form $\chi = \mu |\cdot|^s$, where μ and s are uniquely defined,

---

[3]recall this is $\mathbb{R}_{>0}$ in the Archimedean case and $q^\mathbb{Z}$ otherwise

and μ is either trivial or the sign character $\mathrm{sgn} : x \mapsto x/|x|$. We then set:

$$L(\chi) := \begin{cases} \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2), & \mu = 1, \\ \Gamma_{\mathbb{R}}(s+1), & \mu = \mathrm{sgn}. \end{cases} \tag{6}$$

DEFINITION 5.1.2. If $\chi \in X(F^{\times})$ and $s \in \mathbb{C}$ then $\chi|\cdot|^s$ is also a character, and we use the following notation for its local L-factor:

$$L(s, \chi) := L(\chi|\cdot|^s).$$

Moreover we define the **shifted dual** of χ by:

$$\chi^{\vee} := \chi^{-1}|\cdot|,$$

so that

$$L\left((\chi|\cdot|^s)^{\vee}\right) = L\left(1 - s, \chi^{-1}\right).$$

We now state a result that we will later extend to adele groups.

PROPOSITION 5.1.3. Any local field F is isomorphic (as a LCAG) to $\widehat{F} = \mathrm{Hom}_{cont}(F, S^1)$. For any fixed $\psi \in \widehat{F}$, the map

$$a \mapsto \psi_a := \psi(a \cdot -)$$

is an topological group isomorphism.

EXERCISE 5.1.A. Prove 5.1.3. [CHANGE THIS this is way too lengthy to be an exercise]

In such a case of a *self-dual* local field F, we can also speak of its Haar measure $\mathrm{d}x$ being self-dual, with the dual defined in 3.4.4. Define

$$\mathrm{d}^*x := c \cdot \frac{\mathrm{d}x}{|x|} \tag{7}$$

for some $c \in \mathbb{R}_{>0}$, which we always set to 1 in the Archimedean case. Then d^*x is a Haar measure on $F^{\times}$.
Fixing an additive (unitary) character $\psi \in \widehat{F}$, for any $f \in S(F)$ we can define its Fourier transform by:

$$\widehat{f}(y) := \int_F f(x)\psi(xy)\,\mathrm{d}x. \tag{8}$$

Note this almost the same formulation as in 3.4.1, though we drop the conjugation in the second factor of the integrand, as a simple matter of notational convenience to make our lives easier in coming calculations. The function $\widehat{f}$ is well-defined and in-fact again lies in $S(F)$, though it depends on our choice of ψ and $\mathrm{d}x$. In Tate's original thesis he normalizes his Haar measure to be self-dual relative to ψ, from which he obtained the nice identity $f(x) = \widehat{\widehat{f}}(x)$. We avoid this normalization for the local non-Archimedean case (WHY??).

Definition 5.1.4. Given $f \in S(F)$ and $\chi \in X(F^\times)$, we define the associated **local zeta function** to be:

$$Z(f, \chi) := \int_{F^\times} f(x)\chi(x)\, \mathrm{d}^* x .$$

The following is the main result of this subsection.

Theorem 5.1.5. Let F be a local field, and take $f \in S(F)$ and $\chi = \mu|\cdot|^s \in X(F^\times)$ with exponent $\sigma = \mathfrak{Re}\,(s)$. Then:

(a) If $\sigma > 0$ then the local zeta function $Z(f, \chi)$ is absolutely convergent.

(b) If $\sigma \in (0, 1)$ then the following functional equation holds:

$$Z(\widehat{f}, \chi^\vee) = \gamma(\chi, \psi,\, \mathrm{d}x\,)Z(f, \chi)$$

for some γ independent of f and meromorphic as a function of s.

(c) There exists a factor $\varepsilon(\chi, \psi,\, \mathrm{d}x\,)$ nonzero for all inputs and satisfies:

$$\gamma(\chi, \psi,\, \mathrm{d}x\,) = \varepsilon(\chi, \psi,\, \mathrm{d}x\,)\frac{L(\chi^\vee)}{L(\chi)}$$

proof of (a) in 5.1.5. To show $Z(f, \chi)$ is absolutely convergent, using our relationship between our Haar measures in Equation 7 and the fact that μ is unitary, it suffices to show that:

$$I(f, \sigma) = c \int_{F\setminus\{0\}} |f(x)||x|^{\sigma-1}\, \mathrm{d}x\ < \infty.$$

In the Archimedean case, finiteness of $I(f, \sigma)$ follows from the fact that f is a Schwarz function, and thus rapidly decaying as $|x|$ approaches infinity. Moreover, around zero since $\sigma > 0$ the $|x|^{\sigma-1}$ part is integrable, so we are happy.

In the non-Archimedean case, f is locally constant with compact support, it must factor through some finite group quotient of the form $\langle \pi_F^m \rangle / \langle \pi_F^n \rangle$, for some integers $m \leq n$ (here $\langle \pi_F^k \rangle$ represents the fractional ideal of $\mathcal{O}_F$ generated by the k-th power of its uniformizer). By linearity and translation-invariance of the Haar measure, it suffices to check finiteness of the integral for functions f that are just the characteristic functions of the various ideals $\langle \pi_F^j \rangle$.

Using unique factorizaition in any fractional ideal, we can decompose it into a disjoint union:

$$\langle \pi_F^j \rangle \setminus \{0\} = \bigsqcup_{k=j}^{\infty} \pi_F^k \mathcal{O}_F^\times.$$

Thus our integral becomes:

$$\begin{aligned}
I(f, \sigma) &= c \int_{F\setminus\{0\}} |f(x)||x|^{\sigma-1}\, \mathrm{d}x \\
&= \int_{F^\times} |f(x)||x|^\sigma\, \mathrm{d}^* x
\end{aligned}$$

$$= \sum_{k \geq j} \int_{\pi_F^k \mathcal{O}_F^\times} |x|^\sigma \, \mathrm{d}^* x$$

$$= \sum_{k \geq j} \mathrm{Vol}(\mathcal{O}_F^\times, \mathrm{d}^* x) q^{-k\sigma}$$

$$= \mathrm{Vol}(\mathcal{O}_F^\times, \mathrm{d}^* x) \frac{q^{-j\sigma}}{1 - q^{-\sigma}}$$

which is finite when $\sigma > 0$, as desired. □

To prove the next part of our main theorem, we use a lemma that was key in Tate's original thesis.

> **LEMMA 5.1.6.** Select some auxiliary function $g \in S(F)$. Then for all characters χ with exponent $\sigma \in (0, 1)$, we have:
> $$Z(f, \chi) Z(\widehat{g}, \chi^\vee) = Z(\widehat{f}, \chi^\vee) Z(g, \chi).$$

Proof. Expanded out, the LHS of the Lemma statement is the following product of integrals:

$$Z(f, \chi) Z(\widehat{g}, \chi^\vee) = \int_{F^\times} f(x) \chi(x) \, \mathrm{d}^* x \; \cdot \; \int_{F^\times} \widehat{g}(y) \chi^{-1}(y) |y| \, \mathrm{d}^* y \,,$$

both of which are absolutely convergent by part (a) of 5.1.5. We can turn this into a double integral over the product space by:

$$\iint_{F^\times \times F^\times} f(x) \widehat{g}(y) \chi(xy^{-1}) |y| \, \mathrm{d}^* x \, \mathrm{d}^* y \,.$$

Using invariance of the product measure under the "shearing" automorphism $(x, y) \mapsto (x, xy)$, we obtain:

$$\iint_{F^\times \times F^\times} f(x) \widehat{g}(xy) \chi(y^{-1}) |xy| \, \mathrm{d}^* x \, \mathrm{d}^* y \,,$$

which, by Fubini's theorem, is equal to:

$$\int \left(\int f(x) \widehat{g}(xy) |x| \, \mathrm{d}^* x \right) \chi(y^{-1}) |y| \, \mathrm{d}^* y \,.$$

To prove the lemma it remains to show that the inner integral is symmetric in f and g. Expanding out the Fourier transform of g (recalling our convention in Equation 8 and transferring it to an integral over F using Equation 7), we get that the inner integral is equal to:

$$c \iint_{F \times F} f(x) g(z) \psi(xyz) \, \mathrm{d}z \, \mathrm{d}x \,.$$

Note there is a (slight) abuse of notation here as technically we shouldn't be integrating over elements in the product containing a 0, but we ignore it as a set of measure zero. We switch the order of the derivatives via Fubini to obtain:

$$c \iint_{F \times F} f(x) g(z) \psi(xyz) \, \mathrm{d}x \, \mathrm{d}z \; = c \int_F g(z) \widehat{f}(zy) \, \mathrm{d}z \; = \int_{F^\times} g(z) \widehat{f}(zy) |z| \, \mathrm{d}^* z$$

which gives us the desired result. □

proof of (b) in 5.1.5. Our functional equation in Lemma 5.1.6 tells us that:

$$\gamma(\chi, \psi, \, dx \,) := \frac{Z(\widehat{f}, \chi^\vee)}{Z(f, \chi)}$$

is independent of our choice of $f \in S(F)$. Note that:

$$\chi^l or = (\mu |\cdot|^s)^\vee = \mu^{-1} |\cdot|^{1-s}$$

By part (a) of the main theorem, we deduce that $Z(\widehat{f}, \chi^\vee)$ is absolutely convergent when $\sigma = \mathfrak{Re}\,(s) < 1$, and we have already seen that the denominator of γ is absolutely convergent for all positive σ, so the functional equation in terms of γ holds for all $\sigma \in (0, 1)$ as desired.

To show γ is meromorphic as a function of s, we turn to the proof of part (c) of the theorem. $\square$

proof of (c) in 5.1.5 in the Archimedean case. As shown in the proof of part (b) of the theorem, the value of the γ function is independent of our choice of $f \in S(F)$. Thus we may choose a special function, or family of functions, for each of the possible cases of F to make our computations convenient. We will select a standard measure dx for each case, which is self-dual for an appropriate choice of ψ, and then extend our results to arbitrary dx and ψ afterwords.

Case 1: $F = \mathbb{R}$

We take dx to be the usual Lebesgue measure, and set our additive character to be:

$$\psi(x) = e^{-2\pi i x}.$$

As we noted in our discussion of the local L-factors, any $\chi \in X(\mathbb{R}^\times)$ is of the form $|\cdot|^s$ or $\text{sgn}\,|\cdot|^s$.

Sub-Case (i): $\chi = |\cdot|^s$

Take $f(x) = e^{-\pi x^2}$, which is visibly in $S(\mathbb{R})$. Then recalling that we set the scaling constant in the measure transformation of Equation 7 to 1, we get:

$$Z(f, \chi) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \, d^* x \ = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} \, dx \, .$$

Using a substitution $u = \pi x^2$ (hence $du = 2\pi x \, dx$), the integral becomes:

$$\pi^{-s/2} \int_0^\infty e^{-u} u^{s/2-1} \, du \ = \pi^{-s/2} \Gamma(s/2).$$

and we see from Equation 6 that this is exactly our local L-factor, so in fact $Z(f, \chi) = L(\chi)$ for all characters of this form. Next we claim that f is self-dual via the Fourier transform, i.e.

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x y} \, dx \ = f(x).$$

We are omitting the details of the final equality but this is a well-known identity in classical functional analysis,

and provable via a simple countour integral. Thus, we have:

$$Z(\widehat{f}, \chi^\vee) = \int_{\mathbb{R}^\times} \widehat{f}(x)\chi^\vee(x)\,\mathrm{d}^*x \; = \; \int_{\mathbb{R}^\times} f(x)\chi^\vee(x)\,\mathrm{d}^*x \; = \; Z(f, \chi^\vee)$$

which is equal to $L(\chi^\vee)$ by our previous calculation in this proof. Thus if $\chi = |\cdot|^s$, we have:

$$\gamma(\chi) = \frac{L(\chi^\vee)}{L(\chi)}$$

so we may set $\varepsilon(\chi) = 1$ in this case.

Sub-Case (ii): $\chi = \mathrm{sgn}\,|\cdot|^s$

In this case we choose $f(x) = xe^{-\pi x^2}$, which is again immediately verifiable as rapidly decaying. Expanding the local zeta function we get:

$$\begin{aligned}
Z(f, \chi) &= \int_{\mathbb{R}^\times} xe^{-\pi x^2}\frac{x}{|x|}|x|^s\,\mathrm{d}^*x \\
&= \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^{s+1}\,\mathrm{d}^*x \\
&= \pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right). \\
&= \Gamma_{\mathbb{R}}(s+1)
\end{aligned}$$

which we again recall from Equation 6 as by definition our real local L-factor $L(\chi)$.

Via another contour integral that we handwave, it can be shown that the Fourier transform of this function is of the form:

$$\widehat{f}(y) = iye^{-\pi y^2} = if(y)$$

from which it follows

$$Z(\widehat{f}, \chi^\vee) = \int_{\mathbb{R}^\times} if(y)\chi^\vee\,\mathrm{d}^*x \; = \; iL(\chi^\vee),$$

and thus by a similar calculation to the last sub-case we deduce $\varepsilon(\chi) = i$ in this case.

Case 2: $F = \mathbb{C}$

Here we let our measure on $\mathbb{C}$ be $\mathrm{d}z\,\mathrm{d}\bar{z} \; = 2\,\mathrm{d}x\,\mathrm{d}y$, or double the standard Lebesgue measure. We choose this as it is self-dual with respect to the character

$$\psi(z) := e^{-2\pi i(z+\bar{z})}.$$

We also adjust the standard norm on $\mathbb{C}$ to agree with the module, i.e

$$|z| :=$$

[FINISH LATER] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Before we proceed complete the proof of the final part of the theorem for the non-Archimedean case, we need to build some more tools and lemmas to use.

Definition 5.1.7. Given a ring R with an additive (unitary) character $\psi : R \to S^1$ and a multiplicative (unitary) character $\chi : R^\times \to S^1$, we define their associated **Gauss sum** to be:

$$g(\chi, \psi) := \int_{R^\times} \chi(u) \psi(u) \, \mathrm{d}^* u \, .$$

Remark. Gauss sums turn out to be a hugely convenient notation that show up in a host of different applications, from theta-functions to cyclotomic fields. In fact, Weil made use of these Gauss-sums as a key step in his diagonal variety point-counting argument to motivate the famed Weil Conjectures in his original 1936 paper.

We now build the notion of a conductor of a character. As is standard we work with a local field F with ring of integers $\mathcal{O}_F$, prime ideal $\mathfrak{p}$, and uniformizer π.

Definition 5.1.8. Let ψ be an additive character on F. Then the **conductor** of ψ is the ideal $\mathfrak{p}^m$, where

$$m := \min \left\{ r \in \mathbb{Z} \, : \, \psi|_{\mathfrak{p}^r} = 1 \right\},$$

i.e the conductor is the largest fractional ideal of $\mathcal{O}_F$ on which ψ is trivial. In the case that ψ is trivial we set the conductor to be $\mathcal{O}_F$ itself.

Now suppose χ is a multiplicative character on $F^\times$. Then its **conductor** is $\mathfrak{p}^n$, where

$$n := \min \left\{ r \in \mathbb{N} \, : \, \chi|_{1+\mathfrak{p}^n} = 1 \right\},$$

i.e the conductor is the largest possible unit subgroup of $\mathcal{O}_F$ on which χ is trivial.

Sometimes the conductors are defined as the integers m and n themselves rather than the ideals, but we avoid this convention.

Warning (Overloaded Term). The conductor ideal is also something associated to ring extensions $A \subseteq B$, defined as $\mathrm{ann}_A(B/A)$. There is no meaningful relation between this conductor and the conductor defined above (as far as I'm aware).

Lemma 5.1.9. Let χ be a multiplicative (unitary) character on $\mathcal{O}_F^\times$ and ψ be an additive (unitary) character, and suppose they have conductors $\mathfrak{p}^n$ and $\mathfrak{p}^m$ respectively. Then:

(a) If $m < n$ then $g(\chi, \psi) = 0$.

(b) If $m = n$ then $|g(\chi, \psi)|^2 = c \operatorname{Vol}(\mathcal{O}_F, \mathrm{d}x) \operatorname{Vol}(1 + \mathfrak{p}^n, \mathrm{d}^*x)$.

(c) If $m > n$ then $|g(\chi, \psi)|^2 = c \operatorname{Vol}(\mathcal{O}_F, \mathrm{d}x) \left(\operatorname{Vol}(1 + \mathfrak{p}^n, \mathrm{d}^*x) - \operatorname{Vol}(1 + \mathfrak{p}^{m-1}, \mathrm{d}^*x) \right)$.

Proof. Case 1: $m < n$
We decompose $\mathcal{O}_F^\times$ into the disjoint union of (finitely many!) cosets of the unit subgroup $U_m := 1 + \mathfrak{p}^m$. Note for

any $a, b \in \mathcal{O}_F$ we have

$$\psi(a(1 + \pi^m b)) = \psi(a)\psi(a\pi^m b) = \psi(a),$$

by definition of the conductor $\mathfrak{p}^m = \langle \pi^m \rangle$. The Gauss sum thus becomes:

$$g(\chi, \psi) = \sum_{\mathcal{O}_F^\times / U_m} \left(\psi(a)\chi(a) \int_{U_m} \chi(u) \, \mathrm{d}^* u \right).$$

However since $m < n$, the subgroup U_m is a strict superset of U_n, and so, by definition of the conductor, the character χ is non-trivial on U_m. Recall the integral of a character over a compact subgroup is zero, which in turn makes the entire Gauss sum zero, as desired. This proves (i).

Case 2: $m \geq n$

Expanding out the expression and making use of Fubini, we get:

$$|g(\chi, \psi)|^2 = g(\chi, \psi)\overline{g(\chi, \psi)}$$
$$= \int_{\mathcal{O}_F^\times} \chi(x)\psi(x) \, \mathrm{d}^* x \cdot \overline{\int_{\mathcal{O}_F^\times} \chi(y)\psi(y) \, \mathrm{d}^* y}$$
$$= \iint_{\mathcal{O}_F^\times} \chi(xy^{-1})\psi(x - y) \, \mathrm{d}^* x \ \mathrm{d}^* y .$$

Making the substitution $xy^{-1} \mapsto z$ the above integral becomes

$$\int_{\mathcal{O}_F^\times} \chi(z)h(z) \, \mathrm{d}^* z ,$$

where

$$h(z) := \int_{\mathcal{O}_F^\times} \psi(yz - y) \, \mathrm{d}^* y \ = c \int_{\mathcal{O}_F^\times} \psi(y(z - 1)) \, \mathrm{d} y .$$

Note we can switch the Haar measure in the second equality using Equation 7 and the fact that $|y| = 1$ for all $y \in \mathcal{O}_F^\times$. Making use of the fact that $\mathcal{O}_F$ is a local ring, we get:

$$h(z) = c \left(\int_{\mathcal{O}_F} \psi(y(z - 1)) \, \mathrm{d} y \ - \int_{\mathfrak{p}} \psi(y(z - 1)) \, \mathrm{d} y \right).$$

Note that if $v_{\mathfrak{p}}(z - 1) < m - 1$ then in the above expression ψ takes values outside of its conductor in both of the integrals, and thus is a non-trivial character over $\mathcal{O}_F$. If $v_{\mathfrak{p}}(z - 1) = m - 1$ then ψ is trivial only over $\mathfrak{p}$, not $\mathcal{O}_F$, and otherwise it is trivial over the entire ring. These observations give us:

$$h(z) = \begin{cases} 0, & v_{\mathfrak{p}}(z - 1) < m - 1 \\ -c\frac{\mathrm{Vol}(\mathcal{O}_F, \, \mathrm{d} x \ )}{q}, & v_{\mathfrak{p}}(z - 1) = m - 1 \\ c \, \mathrm{Vol}(\mathcal{O}_F, \, \mathrm{d} x \ ) \left(1 - \frac{1}{q} \right), & v_{\mathfrak{p}}(z - 1) \geq m. \end{cases}$$

All that remains is to substitute these cases back into the integral $\int \chi(z)h(z) \, \mathrm{d} \ast^z$ to show the desired equalities in part (ii) and part (iii), which we leave as an exercise. $\qquad \square$

> **EXERCISE 5.1.B.** Finish the last part of the proof of 5.1.9.

proof of (c) in 5.1.5 in the non-Archimedean char 0 case. Here F is a finite extension of $\mathbb{Q}_p$ for p a prime number. We first define our "canonical" additive character ψ_p.

finish this

$\square$

§5.2   The Root Number

§5.3   Riemann-Roch Theorem

A standard result of classical Fourier analysis is the Poisson summation formula, and its nontrivial extension the Riemann-Roch theorem. We explore both these results in an adelic setting. We begin with another slew of notation. Let K be a global field.

> **DEFINITION 5.3.1.** An **adelic Schwartz-Bruhat function** is a "restricted tensor product" of Schwartz-Bruhat functions on the completions of K, i.e
>
> $$f = \bigotimes f_v \ : \ \forall v, \ f_v \in S(K_v) \text{ and } f_v \mid_{\mathcal{O}_v} = 1 \text{ for almost all } v.$$
>
> The space of such functions is denoted $S(\mathbb{A}_K)$.

SANITY CHECK. Verify for any $x = (x_v) \in \mathbb{A}_K$ and $f \in S(\mathbb{A}_K)$ that it makes sense to write $f(x) = \prod f_v(x_v)$ (i.e. this is a well-defined product), and that the integral $\int_{\mathbb{A}_K} f(x) \, dx$ is finite, for dx a Haar measure on $\mathbb{A}_K$.

> **EXERCISE 5.3.A.** Let dx be a Haar measure on $\mathbb{A}_K$, and define the space $L^2(\mathbb{A}_K)$ using this measure. Show $S(\mathbb{A}_K)$ is dense in $L^2(\mathbb{A}_K)$.

Fix a nontrivial, continuous unitary character ψ on $\mathbb{A}_K$ such that $\psi \mid_K = 1$, and normalize a Haar measure dx to be self dual for ψ. We define the adelic Fourier transform on any $f \in S(\mathbb{A}_K)$ by:

$$\widehat{f}(y) := \int_{\mathbb{A}_K} f(x)\psi(xy) \, dx \ .$$

It is a non-obvious fact that this transform $f \mapsto \widehat{f}$ induces an automorphism of $S(\mathbb{A}_K)$. [Put in appendix or smth].

We want to explore functions on the adele group that are invariant under translations by K (recall we can diagonally embed K into $\mathbb{A}_K$). One such example is the aforementioned ψ. Another strategy we can do is to take an "average" over all of K. Specifically, for any $\phi \in S(\mathbb{A}_K)$ we can take the following sum:

$$\widetilde{\phi}(x) := \sum_{y \in K} \phi(x + y).$$

When this sum is convergent then for any $z \in K$ we use the fact that $y \mapsto y + z$ is an automorphism of K to see $\widetilde{\phi}(x + z) = \widetilde{x}$ for all $x \in \mathbb{A}_K$.

> **Definition 5.3.2.** For any complex-valued function f on $\mathbb{A}_K$, define $\widetilde{f}$ as before. We say f is **admissible** if both $\widetilde{f}$ and $\widetilde{\widehat{f}}$ are absolutely and uniformly convergent on a compact subset.

> **Lemma 5.3.3.** All $f \in S(\mathbb{A}_K)$ are admissible.

Finish later. □

> **Theorem 5.3.4 (Poisson Summation Formula).** For a global field K, any adelic Schwartz-Bruhat function f on $\mathbb{A}_K$ satisfies
> $$\widetilde{f} = \widetilde{\widehat{f}}.$$

FINISH LATER. □

Number theorists are interested [Why??] in the "average"

$$\sum_{\alpha \in K} f(x\alpha)$$

for $x \in \mathbb{I}_K$.

> **Theorem 5.3.5 ((Adelic) Riemann-Roch).** Let x be an idele of a global field K and let f be an element of $S(\mathbb{A}_K)$. Then
> $$\sum_{\alpha \in K} f(x\alpha) = \frac{1}{|x|} \sum_{\alpha \in K} \widehat{f}(x^{-1}\alpha)$$

Proof. Fix some $x \in \mathbb{A}_K$ and for some arbitrary $y \in \mathbb{A}_K$ define

$$h(y) := f(xy).$$

It is evident that $h \in S(\mathbb{A}_K)$ as well, so by the Poisson summation formula (5.3.4) we have:

$$\sum_{\alpha \in K} h(\alpha) = \sum_{\alpha \in K} \widehat{h}(\alpha).$$

Expanding and manipulating the Fourier transform, we get:

$$\begin{aligned}
\widehat{h}(y) &= \int_{\mathbb{A}_K} f(xy)\psi(\alpha y)\,\mathrm{d}\alpha \\
&= \frac{1}{|x|} \int_{\mathbb{A}_K} f(y)\psi(\alpha y x^{-1})\,\mathrm{d}\alpha \\
&= \frac{1}{|x|} \widehat{f}(\alpha x^{-1}),
\end{aligned}$$

whence the theorem follows. □

We now explore the Riemann-Roch in a more geometric setting.

DEFINITION 5.3.6. A **divisor** on a