Generating Functions

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1 Introduction

Given a sequence $\{a_n\} = \{a_0, a_1, a_2 ...\}$, we want some nice way of packaging all the terms in something we can work with in order to detect patterns in the sequence. This is the point of generating functions.

Definition. The **generating function** $A(x)$ of a sequence $\{a_n\}$ is the *power series* defined as:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i$$

That is, the coefficient of x^i is the i-th term of the sequence a_i.

This power series encodes all of the data of the sequence in it. If we can find a closed form expression for $A(x)$, then we can manipulate it to unpack information about the sequence we may not have been able to deduce beforehand. In particular, they are a powerful tools for solving *recurrence relations.*

Remark. When working with Generating Functions, we 'forget' about issues of convergence

Example 1.0.1. *To give a basic example, consider the number of ways we can pay \$10 using \$1 and \$2 bills. If we take the polynomial:*

$$(1 + x + x^2 + ...x^{10})(1 + x^2 + x^4 + ...x^{10})$$

Observe that the coefficient of x^{10} will enumerate the number of ways we can add 1s and 2s to make 10.

We now present a gallery of examples.

2 Sicherman Dice

Here's a problem that bends nicely to the power of generating functions. Suppose we have two fair 6-sided dice, each labeled from 1 to 6. If we roll both dice, t is not hard to compute the probability distribution of the sum of the faces of the two dice. For instance, the probability that the sum of the two faces is 8 is $\frac{5}{36}$. The question that we pose is to find two fair 6-sided dice with positive integer face values such that if we roll the two dice, the probability distribution of the sum of the faces is the same as the original distribution. For example, for the two new dice, the probability that the sum of the two faces is 8 is still $\frac{5}{36}$.

We can express each die as a generating function. The fair 6-sided die with faces from 1 to 6 can be expressed as $h(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$. Here, the exponents represent the face values, and the coefficients represent the number of faces of that value. All the coefficients in h are 1 because each face value is unique.

Now consider $h(x)^2 = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$. Note that the probability distribution is laid out clear in the expansion of h^2: the coefficient of the x^8 term is 5, as there are 5 ways for the two dice to sum to 8. We wish, now, to construct two polynomials f, g such that $f \cdot g = h^2$.

We need a bit more conditions on f, g, though, before we begin factoring. The two conditions on our new dice are that each die has six faces with positive integer values. The face values correspond to the exponents in the generating function and the coefficients encode the number of faces, so we require $f(1) = g(1) = 6$ and $x \mid f, g$, i.e. the term with smallest degree is non-constant.

Now let's factor h^2. We have

$$\begin{aligned}
h(x)^2 &= x^2(1 + x + x^2 + x^3 + x^4 + x^5)^2 \\
&= x^2 \left(\frac{1 - x^6}{1 - x} \right)^2 \\
&= x^2 \left(\frac{(1 - x^3)(1 + x^3)}{1 - x} \right)^2 \\
&= x^2(1 + x + x^2)^2(1 + x^3)^2 \\
&= x^2(1 + x + x^2)^2(1 + x)^2(1 - x + x^2)^2.
\end{aligned}$$

One can check that all these terms are irreducible in $\mathbb{Z}[x]$. First, note that $x \mid f, g$. Define $h_1(x) = 1 + x + x^2$, $h_2(x) = 1 + x$, and $h_3(x) = 1 - x + x^2$. Observe that $h_1(1) = 3$, $h_2(1) = 2$, and $h_3(x) = 1$. Thus, each of f, g are divisible by both h_1, h_2 to force $f(1) = g(1) = 6$.

If $h_3 \mid f, g$, then $f = g$, in which case $f = g = h$, i.e. f, g represent the fair 6-sided dice with faces from 1 to 6. Thus, WLOG we will force $h_3^2 \mid f$ and $h_3 \nmid g$. Explicitly,

$$\begin{aligned}
f(x) &= x(1 + x + x^2)(1 + x)(1 - x + x^2)^2 \\
g(x) &= x(1 + x + x^2)(1 + x).
\end{aligned}$$

Expanding, we have

$$\begin{aligned}
f(x) &= x + x^3 + x^4 + x^5 + x^6 + x^8 \\
g(x) &= x + 2x^2 + 2x^3 + x^4,
\end{aligned}$$

so the two dice have faces $\{1, 3, 4, 5, 6, 8\}$ and $\{1, 2, 2, 3, 3, 4\}$, respectively.

3 Fibonacci Numbers

We know the Fibonacci Numbers to be defined as $F_0 = 0, F_1 = 0$, and for $n \geq 2$, they are determined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. However, can we find a *closed form* for them? In other words, if I give you some n, can you compute the n-th Fibonacci number without necessarily computing the previous $n-1$ ones? As it turns out, you can!

Theorem 3.0.1. *The Fibonacci numbers are given by the formula*

$$F_n = \frac{1}{\sqrt{5}}\Big((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\Big)$$

Let us consider the generating function defined by:

$$\mathcal{F}(x) = \sum_{k=0}^{\infty} F_k x^k$$

Using the recurrence relation, we get:

$$\begin{aligned}
\mathcal{F}(x) &= 0 + x + \sum_{k=2}^{\infty} F_k x^k \\
&= x + \sum^{\big(F_{k-2}+F_{k-1}\big)x^k} \\
&= x + \sum_{k=2}^{\infty} F_{k-2}x^k + \sum_{k=2}^{\infty} F_{k-1}x^k
\end{aligned}$$

We now do a little bit of *re-indexing*. Let $i = k-2$ and $j = k-1$. Now, we re-write the last two terms of the sum as follows:

$$\begin{aligned}
\mathcal{F}(x) &= x + \sum_{i=0}^{\infty} F_i x^{i+2} + \sum_{j=1}^{\infty} F_j x^{j+1} \\
&= x + x^2 \sum_{i=2}^{\infty} F_i x^i + x \sum_{j=0}^{\infty} F_j x^j \\
&= x + x^2\mathcal{F}(x) + x\mathcal{F}(x)
\end{aligned}$$

Note that in going from Line 1 to Line 2 in the above expression, we went from starting at $j = 1$ in the last term to $j = 0$, and we can do that since $F_0 = 0$, so it doesn't affect anything.

If we rearrange to solve for $\mathcal{F}(x)$, we get:

$$\mathcal{F}(x) = \frac{x}{1 - x^2 - x} \tag{1}$$

Note that the roots of the denominator are $\frac{1 \pm \sqrt{5}}{2}$, and we denote the plus case by ϕ_1, and the minus case by ϕ_2. Note that $\phi_2 = -\frac{1}{\phi_1}$. Thus, we can decompose Equation 1 into:

$$
\begin{aligned}
\mathcal{F}(x) &= \frac{A}{x-\phi_1} + \frac{B}{x-\phi_2} \\
&= \frac{B'}{1-\phi_2 x} + \frac{A'}{1-\phi_1 x}
\end{aligned}
$$

Note that we went from Line 1 to Line 2 by multiplying the numerator and denominator by The reason for this decomposition will become apparent soon. Solving for A', B', we get that $A' = \frac{1}{\sqrt{5}}$ and $B' = \frac{-1}{\sqrt{5}}$. Thus, we end up with:

$$
\mathcal{F}(x) = \frac{1/\sqrt{5}}{1-\phi_1 x} - \frac{1/\sqrt{5}}{1-\phi_2 x}
$$

Now we use the well-known result of infinite geometric series:

Fact.

$$
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
$$

Technically, this convergence holds only if $|x| < 1$, but remember since we are dealing with formal variables when working with generating functions, we don't worry about issues of convergence. We use this to rewrite the generating function as:

$$
\begin{aligned}
\mathcal{F}(x) &= \frac{1}{\sqrt{5}}\Big(\frac{1}{1-\phi_1 x} - \frac{1}{1-\phi_2 x}\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\sum_{n=0}^{\infty}(\phi_1 x)^n - \sum_{n=0}^{\infty}(\phi_2 x)^n\Big) \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(\phi_1^n - \phi_2^n)x^n
\end{aligned}
$$

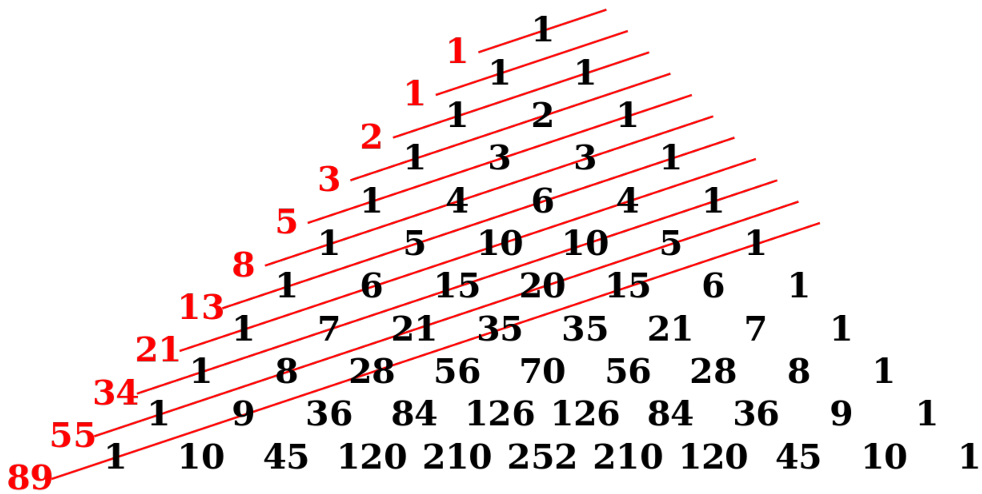
By using this manipulation, we have now rewritten the power series in explicit form. But recall that the coefficient of x^n in $\mathcal{F}(x)$ was precisely the n-th term in the Fibonacci sequence! Thus, we are left to conclude that:

$$
F_n = \frac{1}{\sqrt{5}}\Big(\big(\frac{1+\sqrt{5}}{2}\big)^n - \big(\frac{1-\sqrt{5}}{2}\big)^n\Big),
$$

which is exactly what we wanted to show.

4 Relating Fibonacci to Pascal's

Consider the diagonals on Pascal's triangle below. Taking the sum along these diagonals gives us, almost magically, the same Fibonacci numbers we dealt with in the last section! Fitting, because generating functions are truly magical.



Let's consider the seventh diagonal. Expressing this sum in terms of binomial coefficients, we can write this as $\binom{3}{3} + \binom{4}{2} + \binom{5}{1} + \binom{6}{0}$. In general, every sum along these diagonals can be expressed as

$$\sum_{k \geq 0} \binom{k}{n-k} = \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor} + \cdots + \binom{n}{0}$$

for $n \geq 0$.

As we want to compute this sum, we will force our way to make this sum as the coefficients of a generating function. Consider the series

$$\sum_{n \geq 0} \sum_{k \geq 0} \binom{k}{n-k} x^n.$$

We will reverse the order of our summation and split x^n as $x^k \cdot x^{n-k}$ as follows:

$$\begin{aligned}
\sum_{n \geq 0} \sum_{k \geq 0} \binom{k}{n-k} x^n &= \sum_{k \geq 0} x^k \sum_{n \geq 0} \binom{k}{n-k} x^{n-k} \\
&= \sum_{k \geq 0} x^k (1+x)^k \\
&= \frac{1}{1 - x(1+x)} = \frac{1}{1 - x - x^2},
\end{aligned}$$

which is exactly the generating function of the Fibonacci sequence, as derived in the previous section, but if the sequence starts with 1 instead of 0.

5 Final Notes

We hope that this minicourse has convinced you that generating functions are powerful machinery and allows us to play with sequences with great flexibility. The type of generating functions we

dealt with here are called *ordinary generating functions*, and indeed there are some extraordinary types of generating functions which are equally, if not more, powerful. A good place to start future exploration is *exponential generating functions*.
Herbert Wilf's book *Generatingfunctionology* (yes, that is actually the book title) is perhaps the seminal text for generating functions, and it explains many topics, including things outside of the ordinary generating function realm, with great clarity and intuitiveness. We suggest you check it out!
Here are some interesting problems related to generating functions that you might like to experiment with or search up:

- Let p_k be the number of ways you can pay \$$k$ using \$1 and \$2 bills. Can you find a generating function for $\{p_n\}$?
- The *Catalan Numbers* satisfy $C_0 = 1$, and $C_n = \sum_{i=0}^{n} C_i C_{n-i}$. See if you can find the generating function of the Catalan numbers, and use that to find an explicit form.
- Let S_n denote the set of all triples (a, b, c) such that $a+b+c = n$. Using generating functions, see if you can compute $\sum_{(a,b,c)\in s_{16}} abc$.

6 References

Laplace, Euler, Ramanujan, Pentland.