Fourier Series

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1 Introduction

Since its introduction in the early 19th century, the Fourier series has been enshrined in the mathematical hall of fame as a vital tool of analysis. The motivation is as follows: suppose we are given a periodic function that we are trying to understand the properties of. In general this may be difficult, but there are 2 periodic functions in particular that mathematicians understand in superb detail, and have understood for several centuries: the sine and cosine functions.

We have certain tools of local approximation in mathematics; for example, supposing we want to approximate a special class of smooth functions known as *analytic* functions $f(x)$ at $x = 0$, we can use the Maclaurin series.

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

However, this approximation only converges *locally*. If we want to approximate a periodic function globally (i.e in the interval $[-\pi, \pi]$), that is when the **Fourier Series** comes in. In this paper we will be defining and exploring various types of convergence related to this series.

2 Definition

2.1 Key Terms

Before the main definition, it is important for us to define some key terms to have a basis for discussion.

- A function is **periodic** if it repeats over given intervals of the real axis. More formally, we say $f(x + 2\pi) = f(x)$ for all x. Note that if f has period T, we could rescale the function by taking $f((2\pi/T)x)$, and it would have period 2π and basically the same properties we'd be interested in analysing.

- We denote by $\mathcal{R}[a, a + 2\pi]$ the set of **Riemann integrable functions** on the interval $[a, a + 2\pi]$. The importance of the 2π is related to the fact that the period of the sine and cosine functions is 2π. By convention, we will use the interval $[-\pi, \pi]$.

- A **trigonometric polynomial** is a finite linear combination of $\sin(kx), \cos(mx)$, with k, m taking values in the natural numbers. We will see a lot of these pop up, since a Fourier series is the limit of a trigonometric polynomial as the number of terms tends to infinity.

2.2 Fourier Series

Definition. Take some $f \in \mathcal{R}[-\pi, \pi]$. Its **Fourier series** is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), \tag{1}$$

where:

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt, \quad n = 0, 1, 2...$$
$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt, \quad n = 1, 2, 3...$$

Note that we define it on the interval $[-\pi, \pi]$, but through simple linear transformations the series can be adapted to fit any period over any interval. Let us examine a simple example to see what's going on.

Example 2.2.1. *Consider the Fourier series of $f(x) = x, x \in [-\pi, \pi]$. We have:*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(nt)dt$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)dt$$

To avoid unnecessary computation, we use the fact that cosine is an even function and $f(x) = x$ is an odd function to deduce that $t\cos(nt)$ is an odd function. Thus, its integral evaluated from $-\pi$ to π will be 0, thus $a_n = 0$ for all n.
For b_n, using integral by parts we deduce that:

$$\begin{aligned}\int_{-\pi}^{\pi} t\sin(nt)dt &= \frac{1}{n}\Big[_{-\pi}^{\pi} - t\cos(nt) + \sin(nt)\Big] \\ &= \frac{1}{n}\Big(- \pi\cos(n\pi) + \sin(n\pi) - \pi\cos(-n\pi) - \sin(-n\pi)\Big) \\ &= \frac{1}{n}[-\pi(-1)^n - \pi(-1)^n] = \frac{2\pi(-1)^{n+1}}{n}\end{aligned}$$

Thus, we get $b_n = \frac{2(-1)^{n+1}}{n}$, and the Fourier series of $f(x) = x$ is:

$$\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$$

If we set $x = \pi/2$, and re-arrange, this gives us the famous expression:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ...$$

3 Convergence of Fourier Series

We proceed with a number of definitions and results related to pointwise convergence of the Fourier Series. We define the **partial** sums of the Fourier Series as:

$$S_N(x) := \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx))$$

We want to understand the behavior of $S_N(x)$ as $N \to \infty$.

3.1 The Dirichlet Kernel

Let us write out $S_N(x)$ in a fuller form and see what we get:

$$\begin{aligned}
S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{N}\Big[\cos(kx)\int_{-\pi}^{\pi} f(t)\cos(kt)dt + \sin(kx)\int_{-\pi}^{\pi} f(t)\sin(kt)dt\Big] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{N}\int_{-\pi}^{\pi} f(t)[\cos(kx)\cos(kt) + \sin(kx)\sin(kt)]dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{N}\int_{-\pi}^{\pi} f(t)\cos(k(x-t))dt
\end{aligned}$$

We now make the change of variable $t = x - t$ (note this t is a 'new' t). This gives us:

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)dt + \frac{1}{\pi}\sum_{k=1}^{N}\int_{-\pi}^{\pi} f(x-t)\cos(kt)dt \tag{2}$$

We now have the motivation to define the Dirichlet Kernel:

Definition. The **Dirichlet Kernel** is:

$$D_N(t) := \frac{1}{2} + \sum_{k=1}^{N}\cos(kt)$$

Defining this object gives us an elegant representation of the partial sums:

Lemma 3.1.1.

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt$$

Proof. This follows immediately from the definition of $D_N(t)$, with a simple substitution into Equation 2. □

We now prove a couple more results about the Dirichlet kernel.

Lemma 3.1.2. *The Dirichlet kernel has the following 2 properties:*

1.
$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(t)dt = 1$$

2.
$$D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{2\sin(t/2)}, \text{ for } \sin(t/2) \neq 0$$

Proof. For part 1. we note that the anti-derivative of cosine is sine. By the Fundamental Theorem of Calculus, we get that $\int_b^a \cos(x) = \sin(a) - \sin(b)$.

$$\begin{aligned}\int_{-\pi}^{\pi} D_N(t)dt &= \int_{-\pi}^{\pi} \frac{1}{2} + \int_{-\pi}^{\pi} [\cos(t) + \cos(2t) + ... \cos(Nt)]dt \\ &= \pi + \sum_{k=1}^{N} [\sin(k\pi) - \sin(-k\pi)]\end{aligned}$$

We now use the fact that $\sin(x)$ evaluates to 0 at integers multiple of π to deduce that that everything after the first term is 0, and thus the integral evaluates to π, and multiplying by the reciprocal of π gives us the identity in part 1.

For part 2, we use the following identity:

$$\sin(a+b) - \sin(a-b) = 2\sin(b)\cos(a)$$

By letting $a = kt$, $b = t/2$, this allows us to deduce that:

$$\begin{aligned}\sin\left[(k+\frac{1}{2})t\right] - \sin\left[(k-\frac{1}{2})t\right] &= \sin\left[(k+\frac{1}{2})t\right] - \sin\left[((k-1)+\frac{1}{2})t\right] \\ &= 2\sin(t/2)\cos(kt)\end{aligned}$$

We now take a telescoping sum:

$$\begin{aligned}\sum_{k=1}^{N} 2\sin(t/2)\cos(kt) &= \sum_{k=1}^{N}\left(\sin\left[(k+\frac{1}{2})t\right] - \sin\left[((k-1)+\frac{1}{2})t\right]\right) \\ &= \sin\left[(N+\frac{1}{2})t\right] - \sin(t/2) \\ \implies \sum_{k=1}^{N}\cos(kt) &= \frac{\sin((N+\frac{1}{2})t)}{2\sin(t/2)} - \frac{1}{2},\end{aligned}$$

and moving the 1/2 from the right side to the left side of the equation gives us the desired inequality. ◻

3.2 A Convergence Theorem

We want to prove the following about convergence of the Fourier Series:

Theorem 3.2.1 (Pointwise Convergence)**.** *Let $f(x)$ be a continuously differentiable periodic function on $[-\pi, \pi]$. Then, $\lim_{N\to\infty} S_N(x) = f(x)$ for all $x \in [-\pi, \pi]$.*

Essentially what this theorem tells us is as long as $f(x)$ is differentiable, the Fourier series of f converges pointwise to f itself. Intuitively what this means is that the Fourier series is a 'good global approximation' of the function f.

To prove this, we will need a lemma due to Riemann and Lebesgue whose proof will be beyond the scope of this text.

Lemma 3.2.2 (Riemann-Lebesgue Lemma)**.** *Let $f : [-\pi, \pi] \to \mathbb{R}$ be continuous with a period of 2π, and let a_n, b_n be the coefficients in its Fourier Series. Then:*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

That is to say, the coefficients of the Fourier series diminish to zero as n grows large. This will be useful since in our proof to show convergence, we will express the difference of $S_N(x)$ and $f(x)$ as a trigonometric polynomial whose limit becomes a sum of Fourier series. We are now equipped to prove Theorem 3.2.1.

proof of Theorem 3.2.1. We want to show that $\lim_{N\to\infty} S_N(x) = f(x)$. By 3.1.1, it suffices to show:

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt = f(x)$$

By part 1 of 3.1.2, we note that $f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) D_N(t) dt$, and so essentially we want to show:

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x-t) - f(x)] D_N(t) dt = 0 \tag{3}$$

By part 2 of 3.1.2, we make the substitution:

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x-t) - f(x)}{2\sin(t/2)} \sin\Big[(N + 1/2)t\Big] dt$$

We want to express this as the coefficients of a Fourier series, which will let us employ the Riemann-Lebesgue lemma to conclude that the limit is 0. Let us now set $g(t) := \frac{f(x-t)-f(x)}{2\sin(t/2)}$. We note that g is continuous, as it is the composition of continuous functions. However, it is not defined at 0, but we resolve this by setting $g(0) = f'(0) = \lim_{t\to 0} \frac{f(x-t)-f(x)}{t} = \lim_{t\to 0} g(t)$, which makes g continuous at 0. Note this is only possible when f is differentiable.
We also make use of the trigonometric identity $\sin[(N+1/2)t] = \sin(Nt)\cos(t/2) + \sin(t/2)\cos(Nt)$. This allows us to transform the limit to:

$$\lim_{N\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)[\sin(Nt)\cos(t/2) + \sin(t/2)\cos(Nt)]dt$$
$$= \lim_{N\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} [g(t)\cos(t/2)]\sin(Nt)dt + \frac{1}{\pi}\int^{\pi} [g(t)\sin(t/2)]\cos(Nt)]dt$$

Setting $g_1(t) := g(t)\cos(t/2)$ and $g_2(t) := g(t)\sin(t/2)$, we get:

$$\lim_{N\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} g_1(t)\sin(Nt)dt + \frac{1}{\pi}\int^{\pi} g_2(t)\cos(Nt)]dt$$

The first integral is the coefficient b_N of the Fourier series of $g_1(t)$, and the second integral is the coefficient a_N of the Fourier series of $g_2(t)$.
By 3.2.2, the limit of both of those integrals is 0 as $N \to \infty$, and therefore their sum is also 0. Thus, we have shown Equation 3 is true, and thus the Fourier series is pointwise convergent for differentiable f.

□

We also want to understand the behavior of the Fourier series for non-differentiable f. Is differentiability a necessary condition for convergence?

3.3 Fejér Kernel

In general, it is hard to talk about convergence of the Fourier Series for even continuous functions f. However, we *can* talk about convergence of a related series. We introduce a new object known as the Fejér Kernel, which is a Cesáro sum of Dirichlet kernels.

Definition. The **Fejér Kernel** is:

$$K_N(t) := \frac{1}{N+1}\sum_{n=0}^{N} D_n(t)$$

We now prove some properties of this object

Lemma 3.3.1. *The Fejér Kernel has the following properties:*

1. $$\frac{1}{\pi}\int_{-\pi}^{\pi} K_N(t)dt = 1$$

2. $$K_N(t) = \frac{1}{2(N+1)}\left(\frac{\sin[(N+1)t/2]}{\sin(t/2)}\right)^2, \textit{ for } \sin(t/2) \neq 0$$

Proof. For part 1, we do a simple calculation using property 1 of 3.1.2.

$$\int_{-\pi}^{\pi} K_N(t)dt = \frac{1}{N+1}\int_{-\pi}^{\pi}\sum_{n=0}^{N} D_n(t)dt$$
$$= \frac{1}{N+1}\sum_{n=0}^{N}\pi = \pi$$

Multiplying by the reciprocal of π, we get the desired identity in part 1.

For part 2, we again use the parallel part 2 of 3.1.2 and the identity $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$.

$$\begin{aligned}
K_N(t) &= \frac{1}{N+1}\sum_{n=0}^{N} D_n(t) \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin((n+1/2)t)}{2\sin(t/2)} \\
&= \frac{1}{2(N+1)\sin^2(t/2)}\sum_{n=0}^{N}\sin((n+1/2)t)\sin(t/2) \\
&= \frac{1}{2(N+1)\sin^2(t/2)}\frac{1}{2}\sum_{n=0}^{N}\cos(nt)-\cos((n+1)t) \\
&= \frac{1}{2(N+1)\sin^2(t/2)}\frac{\cos(0)-\cos((N+1)t)}{2} \\
&= \frac{1}{2(N+1)\sin^2(t/2)}\sin^2[(N+1)t/2] \\
&= \frac{1}{2(N+1)}\left(\frac{\sin[(N+1)t/2]}{\sin(t/2)}\right)^2
\end{aligned}$$

□

Corollary 3.3.1.1. *If* $\sin(t/2)$ *cannot be arbitrarily close to 0, i.e there is some* $\delta > 0$ *such that* $\delta < |t| < 2\pi - \delta$*, then* $K_N(t)$ *grows arbitrarily small as* N *becomes large, since the value inside the big bracket is bounded by* $(1/\sin(\delta))^2$*. Moreover,* $K_N(t) > 0$ *for all* $x \in [-\pi, \pi]$*.*

3.4 Another Convergence Theorem

We now define the Cesàro sums of the partial sums of the Fourier Series.

Definition. The N**th Cesàro sum** of the partial sum of the Fourier series is given by:

$$\sigma_N(x) := \frac{1}{N+1}\sum_{n=0}^{N} S_n(x)$$

We now prove a theorem about the convergence of these Cesàro Sums.

Theorem 3.4.1 (Uniform Convergence). *If $f(x)$ is continuous and periodic on $[-\pi, \pi]$, then $\sigma_N(x)$ converges uniformly to $f(x)$ on $[-\pi, \pi]$*

To prove this, we need the following lemma about the Fejér Kernel.

Lemma 3.4.2. *Let $0 < |t| \leq \pi$. For all $\epsilon > 0$, there exists N such that $n > N$ implies $K_n(t) < \epsilon$*

Proof. For a fixed t, pick N such that $1/(N+1) < 2\epsilon \sin^2(t/2)$, which is always possible by the Archimedean Property. Then, we have:

$$\begin{aligned} K_N(t) &= \frac{1}{2(N+1)} \left(\frac{\sin[(N+1)t/2]}{\sin(t/2)} \right)^2 \\ &\leq \frac{1}{2(N+1)\sin^2(t/2)} \\ &< \frac{2\epsilon \sin^2(t/2)}{2\sin^2(t/2)} = \epsilon \end{aligned}$$

□

Corollary 3.4.2.1. *The function $K_N(t)$ is even, or in other words, $K_N(t) = K_N(-t)$.*

Now we have the tools to prove 3.4.1, the Uniform Convergence result.

proof of 3.4.1. Substituting the integral form of $S_N(x)$ from Lemma 3.1.1, we get:

$$\begin{aligned} \sigma_N(x) &= \frac{1}{N+1} \sum_{n=0}^{N} S_n(x) \\ &= \frac{1}{N+1} \sum_{n=0}^{N} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{1}{N+1} \sum_{n=0}^{N} D_n(t) dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt \end{aligned}$$

To show convergence, we now consider the difference between this Cesàro sum and $f(x)$ and try and make it arbitrarily small. In other words, for any $\epsilon > 0$, we want to show that there is some N such that $n > N$ implies $\sigma_n(x) - f(x) < \epsilon$ for all $x \in [-\pi, \pi]$.

$$\sigma_N(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+t) - f(x)]K_N(t)dt$$

Note that we replaced the variable t with $-t$, which leaves $K_N(t)$ unperturbed since it is an even function.

Since f is continuous on a compact interval, it is uniform continuous over that interval. We thus pick $\delta > 0$ such that for any pair $x, y \in [-\pi, \pi]$, it holds that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon/2\pi$. We now separate $\sigma_N(x) - f(x)$ into 2 integrals as follows:

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+t) - f(x)]K_N(t)dt &= \int_{|t|\in[\delta,\pi]}[f(x+t) - f(x)]K_N(t)dt \\ &\quad + \int_{-\delta}^{\delta}[f(x+t) - f(x)]K_N(t)dt\end{aligned}$$

If we show both integrals are less than $\epsilon/2$, then we will have shown convergence. We use the fact that a continuous f attains its maximum and minimum over a compact interval to set M such that $|f(x)| < M$ for all x. This also means that $|f(x+t) - f(x)|$ is bounded by $2M$ for all t. Next, we use Corollary 3.3.1.1 to pick N such that for all $n > N$, for $\delta \leq |t| \leq \pi$, we get $0 < K_n(t) < \epsilon/(8\pi M)$. This gives us:

$$\begin{aligned}\int_{|t|\in[\delta,\pi]}[f(x+t) - f(x)]K_N(t)dt &\leq \left|\int_{|t|\in[\delta,\pi]}[f(x+t) - f(x)]K_N(t)dt\right| \\ &\leq \int_{|t|\in[\delta,\pi]}|f(x+t) - f(x)|K_N(t)dt \\ &< 2\pi \times 2M \times \frac{\epsilon}{8\pi M} \\ &= \epsilon/2\end{aligned}$$

For the second integral, we use the uniform continuity of f to get:

$$\begin{aligned}\int_{-\delta}^{\delta}[f(x+t) - f(x)]K_N(t)dt &\leq \left|\int_{-\delta}^{\delta}[f(x+t) - f(x)]K_N(t)dt\right| \\ &\leq \int_{-\delta}^{\delta}|f(x+t) - f(x)|K_N(t)dt \\ &< \int_{-\delta}^{\delta}\frac{\epsilon}{2\pi}K_N(t)dt \\ &< \epsilon/2\pi \times \pi = \epsilon/2\end{aligned}$$

Note the last line follows from part 1 of 3.3.1. Thus, putting it all together, since each integral is less than $\epsilon < 2$ for any choice of $\epsilon > 0$, it follows that there exists some N such that $n > N$ implies $|\sigma_n(x) - f(x)| < \epsilon/2$ for all $x \in [-\pi, \pi]$, and we have uniform convergence.

$\square$

Note that this proof was possible due to the behavior of the Fejér kernel getting arbitrarily close to 0 as N grew arbitrarily large; a property that is not shared by the Dirichlet Kernel. This is why we needed it to show uniform convergence.

4 Conclusion and Acknowledgements

In this text we have explored the Fourier series, looking at its definition and some motivating examples. We have also shown important results about when and how the Fourier series (or related Cesàro sums of the Fourier series) converge of f globally converge to f pointwise and uniformly, respectively. The importance and use of this series is difficult to understate: Fourier Series pop up in signal processing, control theory to solve PDEs, DNA sequence alignment, and a host of other applications. All thanks to a few sums of sines and cosines.

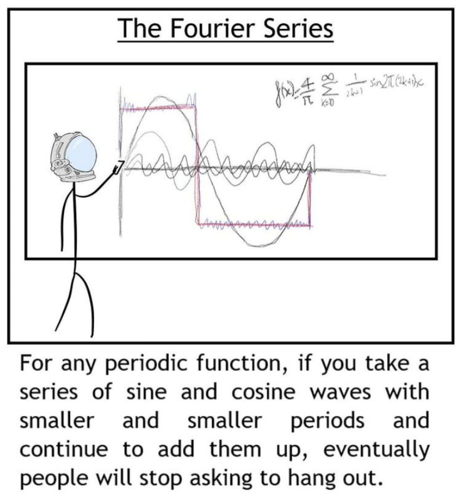

Figure 1: The sad truth of real analysis

The concepts, background and ideas in this text were taken primarily from the Johnsonbaugh and Pfaffenberger's *Foundations of Mathematical Analysis*, along with class material taught by Joonhyun La.