# Robotics
# Written Exercises 2: Control and Linear Analysis
# CS 603, Spring 2020

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# Contents

# 1 Chapter 3: Control

## 1.1 Problem 1: The Spring-Mass Damper

### 1.1.1 (a) Write the characteristic equation of the SMD

The homogeneous equation of motion for the 1 degree of freedom revolute motor unit is

$$\sum \tau = I\ddot{\theta} = -B\dot{\theta} - K\theta$$

$$I\ddot{\theta} + B\dot{\theta} + K\theta = 0$$

$$\ddot{\theta} + \frac{B}{I}\dot{\theta} + \frac{K}{I}\theta = 0 \qquad \text{characteristic equation of the SMD}$$

$$\ddot{\theta} + \frac{11}{1}\dot{\theta} + \frac{10}{1}\theta = 0$$

$$\ddot{\theta} + 11\dot{\theta} + 10\theta = 0$$

Hence, the characteristic equation of this SMD is

$$\boxed{\ddot{\theta} + 11\dot{\theta} + 10\theta = 0}$$

### 1.1.2 (b) Compute the natural frequency $\omega_n$ and the damping ratio $\zeta$

$$\begin{aligned} \zeta &= \frac{B}{2\sqrt{KI}} \\ &= \frac{11}{2\sqrt{10 \cdot 1}} \\ &\approx \boxed{1.74} \qquad \text{damping ratio} \end{aligned}$$

$$\begin{aligned} \omega_n &= \sqrt{\frac{K}{I}} \\ &= \sqrt{\frac{10}{1}} \\ &\approx \boxed{3.16}[rad/sec] \qquad \text{natural frequency} \end{aligned}$$

### 1.1.3 (c) What are the roots of the characteristic equation?

For equation

$$\ddot{\theta} + 11\dot{\theta} + 10\theta = 0$$

$a = 1$, $b = 11$ and $c = 10$.

$$\begin{aligned} s_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-11 \pm \sqrt{121 - 40}}{2} \\ &= \frac{-11 \pm \sqrt{71}}{2} \end{aligned}$$

$$s_1 = \frac{-11 + \sqrt{71}}{2} \quad \& \quad s_2 = \frac{-11 - \sqrt{71}}{2}$$

$s_1 = -1.287$ & $s_2 = -9.713$ are the two real distinct roots of the equation

### 1.1.4 (d) How much force must you apply to the mass to hold it rotated position?

$\theta = 0$ and $\theta_{ref} = \frac{\pi}{2}$

We want to hold the mass in that position, thus $\ddot{\theta} = 0$

$$\begin{aligned} \tau_m &= -B\ddot{\theta} - K(\theta - \theta_{ref}) \\ &= -11 \cdot 0 - 10 \cdot (0 - \frac{\pi}{2}) \\ &= 10 \cdot \frac{\pi}{2} \\ &= 5\pi \approx \boxed{15.7\ Nm} \end{aligned}$$

is the force we must apply to hold the mass in that rotated position

### 1.1.5 (e) At time zero, the SMD is released from rest $\dot{\theta} = 0$ at position $\theta = \pi/2$

**i. Solve for the time response $\theta(t)$**

$\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$, and $\theta(\infty) = 0$ are the boundary conditions.

Complete time-domain solution is determined by,

$$\theta(t) = \theta(\infty) + \frac{(\theta(0) - \theta(\infty))s_2 - \dot{\theta}(0)}{s_2 - s_1} e^{s_1 t} + \frac{(\theta(0) - \theta(\infty))s_1 - \dot{\theta}(0)}{s_1 - s_2} e^{s_2 t}$$

Plugging the $\theta$ and $s_{1,2}$ values we got from previous subexercises,

$$\begin{aligned}\theta(t) &= \frac{\frac{\pi}{2}s_2}{s_2 - s_1}e^{s_1 t} + \frac{\frac{\pi}{2}s_1}{s_1 - s_2}e^{s_2 t} \\ &= \frac{\frac{\pi}{2}(-9.713)}{-9.713 + 1.287}e^{-1.287t} + \frac{\frac{\pi}{2}(-1.287)}{-1.287 + 9.713}e^{-9.713t} \\ &= 1.81e^{-1.287t} - 0.24e^{-9.713t}\end{aligned}$$

Thus,

$$\boxed{\theta(t) = 1.81e^{-1.287t} - 0.24e^{-9.713t}}$$

is the time response equation.

**ii. Identify the term in the solution that dominates the behavior of the system for large $t$**

The real root ($s_1$ or $s_2$) with the smallest absolute value approaches its asymptote more slowly and will, therefore, dominate the system's asymptotic behavior. It produces a relatively sluggish, non-oscillatory response that reflects the excessively dissipative influence of the damper.

Thus, root $s_2 = -9.713$ and thus, the term

$$\frac{\frac{\pi}{2}s_1}{s_1 - s_2}e^{s_2 t} = -0.24e^{-9.713t}$$

dominates the behavior of the system for large $t$.

As a result, overdamped control configurations take comparatively longer to converge to equilibrium setpoints.

**iii. Plot the phase portrait for this system, i.e. $\theta$ vs. $\dot{\theta}$ for $t = 0, \infty$**

See Figure 1

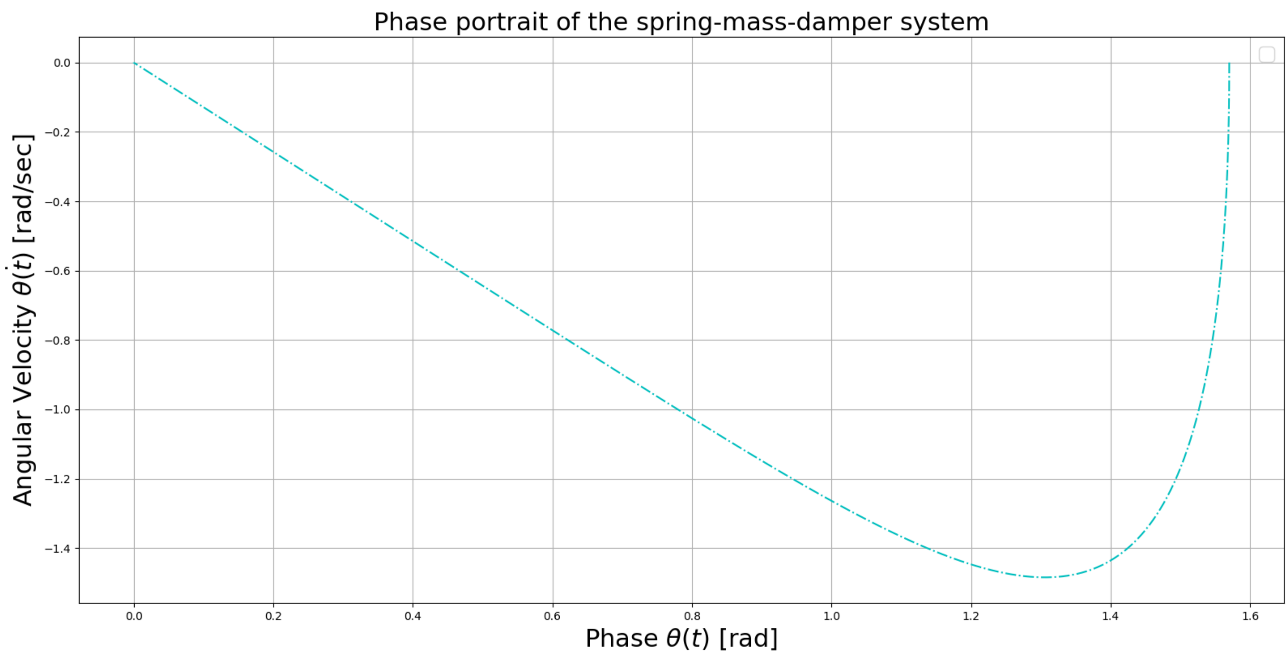


Figure 1: Phase portrait for this system, i.e. $\theta$ vs. $\dot{\theta}$ for $t = 0, \infty$

## 1.2 Problem 4: Critically Damping -

### 1.2.1 (a) Find the damping coefficient $B$

The second order system is $\ddot{x} + B\dot{x} + 16x = 0$.

Thus, here $K = 16$ and $m = 1$ and the system is *critically damped*, so $\zeta = 1$.

$$\zeta = \frac{B}{2\sqrt{Km}} = \frac{B}{2\sqrt{16}} = 1$$

Hence,

$$B = 2\sqrt{16} = 2 \cdot 4 \qquad \boxed{\therefore B = 8\ [N\ m\ sec/rad]}$$

### 1.2.2 (b) Satisfy the original differential equation

Show that when $\zeta = 1$ and the roots of the characteristic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ are $s_1, s_2 = -\omega_n$, that terms in the time domain solution like $Ae^{-\omega_n t}$ and $Ate^{-\omega_n t}$ both satisfy the original differential equation $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

The roots of the equation are,

$$\frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n}}{2}$$

For $\zeta = 1$ and $s_1, s_2 = -\omega_n$, the discriminant will be $0$.

For the term $Ae^{-\omega_n t}$, let's show that the original differential equation is satisfied:

$$\begin{aligned}
\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x &= \frac{d^2(Ae^{-\omega_n t})}{dt^2} + 2 \cdot 1 \cdot \omega_n \frac{d(Ae^{-\omega_n t})}{dt} + \omega_n^2 Ae^{-\omega_n t} \\
&= -A\omega_n \frac{d(e^{-\omega_n t})}{dt} - 2A\omega_n^2 e^{-\omega_n t} + \omega_n^2 Ae^{-\omega_n t} \\
&= A\omega_n^2 e^{-\omega_n t} - 2A\omega_n^2 e^{-\omega_n t} + A\omega_n^2 e^{-\omega_n t} \\
&= 2A\omega_n^2 e^{-\omega_n t} - 2A\omega_n^2 e^{-\omega_n t} = \boxed{0}
\end{aligned}$$

For the term $Ate^{-\omega_n t}$, let's show that the original differential equation is satisfied:

$$\begin{aligned}
\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x &= \frac{d^2(Ate^{-\omega_n t})}{dt^2} + 2\cdot 1\cdot\omega_n\frac{d(Ate^{-\omega_n t})}{dt} + \omega_n^2 Ate^{-\omega_n t} \\
&= \frac{A\cdot d(e^{-\omega_n t} - \omega_n te^{-\omega_n t})}{dt} + 2A\omega_n[e^{-\omega_n t} - \omega_n te^{-\omega_n t}] + A\omega_n^2 te^{-\omega_n t} \\
&= -A\omega_n e^{-\omega_n t} - A\omega_n[e^{-\omega_n t} - \omega_n te^{-\omega_n t}] + 2A\omega_n[e^{-\omega_n t} - \omega_n te^{-\omega_n t}] + A\omega_n^2 te^{-\omega_n t} \\
&= -A\omega_n e^{-\omega_n t} + A\omega_n[e^{-\omega_n t} - \omega_n te^{-\omega_n t}] + A\omega_n^2 te^{-\omega_n t} \\
&= -A\omega_n^2 te^{-\omega_n t} + A\omega_n^2 te^{-\omega_n t} = \boxed{0}
\end{aligned}$$

Hence, both terms $Ae^{-\omega_n t}$ and $Ate^{-\omega_n t}$ satisfy the original differential equation $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

## 1.3 Problem 5: Natural Frequency

Design a second order control law, $m\ddot{x} + B\dot{x} + Kx$ with natural frequency, $\omega_n = 50\ [rad/sec]$

### 1.3.1 (a) For $m = 1[kg]$, find $K$ and $B$ for critical damping

For a second order control law, $m\ddot{x}+B\dot{x}+Kx$, the characteristic equation will be

$$s^2 + Bs + K = 0$$
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Thus, $\zeta = \frac{B}{2\sqrt{K}}$ and $\omega_n = \sqrt{K}$ for $m = 1[kg]$.

$$\begin{aligned} \omega_n &= \sqrt{K} \\ \therefore \sqrt{K} &= 50 \\ &\boxed{K = 2500\ [N\ m/rad]} \end{aligned}$$

$$\begin{aligned} \zeta &= \frac{B}{2\sqrt{K}} \\ 1 &= \frac{B}{2\sqrt{K}} && \zeta = 1 \text{ for critical damping} \\ &\boxed{B = 2 \times 50 = 100\ [N\ m\ sec/rad]} \end{aligned}$$

### 1.3.2 (b) Comment on why it might be useful to design the natural frequency of a controlled system

The natural frequency is the frequency at which a system would oscillate if there was no damping effect. It describes the frequency at which the undamped spring and mass exchange potential and kinetic energy if the system is set in motion.

If the system is excited at or near this frequency, large amplitude oscillations can occur, leading to resonance. It is thus important to model the natural frequency to keep it separate from the excitation frequency.

## 1.4 Problem 8: Stability - Complex Frequency Domain

### 1.4.1 (c) Closed-Loop Transfer Function

**i.** $\frac{G}{1+GH}$ **=** $\frac{s^2}{s^2+s-2}$

The denominator of the transfer function is the characteristic equation, and its roots are the poles of the transfer function.

For $s^2 + s - 2$, roots will be

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \qquad \therefore s_1 = 1 \; \& \; s_2 = -2$$

The roots are not both negative, the system **cannot be stable**.

**ii.** $\frac{G}{1+GH}$ **=** $\frac{s^2}{s^2-s-6}$

The denominator of the transfer function is the characteristic equation, and its roots are the poles of the transfer function.

For $s^2 - s - 6$, roots will be

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} \qquad \therefore s_1 = -2 \; \& \; s_2 = 3$$

The roots are not both negative, the system **cannot be stable**.

**iii.** In the circuit, $G = Bs^2$ and $H = 1/s$, because the $s$ is going as a positive feedback, we need a negative feedback to make the closed loop transfer function, we can inverse the $H' = s$ to become $H = 1/s$. The closed loop function will be,

$$\frac{G}{1+GH} = \frac{Bs^2}{1 + Bs^2 \cdot 1/s} = \frac{Bs^2}{1+Bs}$$

If we solve the equation in denominator, we get $s = -\frac{1}{B}$. The damping coefficient $B$ will be always positive, that means solution of that equality will render $s$ as negative.

This will generate a decaying component in the homogeneous response, the **system is stable**.

**iv.** In the circuit, $G = Bs + 3$ and $H = \frac{s^2+3s+1}{Bs+3}$, the closed loop function will be,

$$\frac{G}{1+GH} = \frac{Bs+3}{1+s^2+3s+1} = \frac{Bs+3}{s^2+3s+2}$$

The denominator of the transfer function is the characteristic equation, and its roots are the poles of the transfer function.

For $s^2 + 3s + 2$, roots will be

$$s_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2} \qquad \therefore s_1 = -2 \ \& \ s_2 = -1$$

The roots are both negative, real and distinct. The system **is stable**.

## 1.5 Problem 9: Stability - Lyapunov

### 1.5.1 (b) Skew Symmetric Term

For the first Lyapunov's condition; where $x = 0$, $V(0,t) = (1/2) \cdot m\dot{x}^2 = 0$ and if $x = 0$, then $\dot{x} = 0$, so $V(0,t) = 0$. This satisfies the first condition.

For the second condition, where $x \neq 0$, $V(x,t) = (1/2) \cdot m\dot{x}^2 + (1/2) \cdot Kx^2 + \epsilon m\dot{x}x > 0$ because all terms are positive, we just need to take care when $\dot{x} = 0$ can happen.

If $\boxed{\frac{1}{2}(m+K) > \epsilon}$, then even if $\dot{x} < 0$, the whole $V(x,t)$ will stay positive definite. $\therefore V(x,t) > 0$.

Let's take the derivative of the modified function,

$$V(\mathbf{x},t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 + \epsilon m\dot{x}x$$

We also have the result $\ddot{x} = -(B/m)\dot{x} - (K/m)x$ from our SMD system.

$$\begin{aligned}
\frac{d}{dt}V(\mathbf{x},t) &= \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2 + \epsilon m\dot{x}x\right] \\
&= m\dot{x}\ddot{x} + Kx\dot{x} + \epsilon m(\dot{x}^2 + x\ddot{x}) \\
&= m\dot{x}\left[-\left(\frac{B}{m}\right)\dot{x} - \left(\frac{K}{m}\right)x\right] + Kx\dot{x} + \epsilon m\dot{x}^2 + \epsilon mx\left[-\left(\frac{B}{m}\right)\dot{x} - \left(\frac{K}{m}\right)x\right] \\
&= -B\dot{x}^2 - Kx\dot{x} + Kx\dot{x} + \epsilon m\dot{x}^2 - \epsilon Bx\dot{x} - \epsilon Kx^2 \\
&= -B\dot{x}^2 + \epsilon m\dot{x}^2 - \epsilon Bx\dot{x} - \epsilon Kx^2 < 0
\end{aligned}$$

The subterms $x^2$, $\dot{x}^2$, $K$ and $m$ will be non-zero positives because of the square and inherent property of mass and spring co-efficient.

$\epsilon$ must be a positive constant, so we set lower non-inclusive limit as $0$. Suppose $B$ is positive here, then only term we need to worry about is $\epsilon m\dot{x}^2$, with compared to $-B\dot{x}^2$.

$$\begin{aligned}
\epsilon m\dot{x}^2 &< B\dot{x}^2 \\
\epsilon &< \frac{B}{m}
\end{aligned}$$

This makes sure that the positive term isn't greater than absolute value of the negative term.

But still, we must make sure that the case where $B = 0$ is taken care of. If the limit is $\boxed{\epsilon < \frac{BK}{m}}$, the term will have greater absolute value for its negative term.

$$\frac{d}{dt}V(\mathbf{x}, t) < -B\dot{x}^2 + \frac{BK}{m}m\dot{x}^2 - \frac{B^2K}{m}x\dot{x} - \frac{BK^2}{m}x^2 < 0$$

Combining the two conditions $\boxed{\frac{1}{2}(m + K) > \epsilon}$ and $\boxed{\epsilon < \frac{BK}{m}}$;

**The acceptable values of $\epsilon$ will be in range** $\left(0, \frac{B(K+m)}{m}\right)$

### 1.5.2 (d) Pendulum

**i. Use the total energy function $KE$ + $PE$ as a candidate Lyapunov function. Is the system stable? Is it asymptotically stable?**

Total energy function is,

$$E = KE + PE = \frac{1}{2}I\dot{\theta}^2 + mgl(1 - \cos(\theta))$$

First, we check for the system's stability.

The origin of the state space is at the middle of the pendulum's swinging path, where the pendulum is, when it is in static state, under influence of no external force.

After giving it any kind of external force, the pendulum will swing but because of the dissipation of energy through environmental resistance, it will come back to the origin and stay there until next external push.

Thus, *the origin of the state space is stable, because there exists a region $S(q)$, containing the origin, such that system trajectories that begin at states within $S(q)$ remain within $S(q)$.*

Note that, for an ideal virtual lab environment, a pendulum will always keep swinging because of the exchange between kinetic and potential energies without any dissipation or damping. But practically, there is always dissipation and thus, the system always returns to its origin, making is a **stable system**.

Now let's check for asymptomatically stable system. The system's equation is,

$$E = KE + PE = \frac{1}{2}I\dot{\theta}^2 + mgl(1 - \cos(\theta))$$

At $t = 0$, $\theta$ will be $0$ and so $\dot{\theta}$ will be $0$. Thus,

$$E = \frac{1}{2}I \cdot 0 + mgl(1 - \cos(0)) = 0 + mgl(1 - 1) = 0$$

*The first condition of Lyapunov's Direct Method has been satisfied.*

We know that $I$, $m$, $g$ and $l$ are all positive terms and maximum value $\cos(\theta)$ can get is $1$, making the term $mgl(1 - \cos(\theta))$ $0$. Thus the entire equation stay positive.

$$E > 0, \text{for non-origin starting states}$$

*The second condition of Lyapunov's Direct Method has been satisfied.*

$$\begin{aligned}\frac{d}{dt}E &= \frac{1}{2}I(2\theta\dot{\theta}) + mgl(\sin(\theta))\dot{\theta} \\ &= I\dot{\theta}\ddot{\theta} + \dot{\theta}mgl\sin\theta \\ &= I\dot{\theta}\ddot{\theta} - I\dot{\theta}\ddot{\theta} && \text{because } I\ddot{\theta} = -mgl\sin\theta \\ &= 0\end{aligned}$$

*The third condition of Lyapunov's Direct Method has **not** been satisfied.*

We can conclude that **Pendulum is not asymptotically stable**.

**ii. Suppose that the pendulum is released from rest at** $theta - \pi/4$
I have assumed $m = 1\ [kg]$ and $l = 1\ [m]$ to calculate the values for the phase space plot.

**A. Label this starting state on the phase plane and sketch the trajectory that this system should execute after it is released**
See Figure 2

**B. What is the magnitude of the velocity when** $\theta = 0$?
With assumption stated above ($m = 1\ [kg]$ and $l = 1\ [m]$), the magnitude of angular velocity $\dot{\theta}$, at $\theta = 0$ is $\underline{\dot{\theta} = 2.458\ [rad/sec]}$

**iii. What term was missing in the dynamic model of this system?**
An undamped pendulum can be realized only virtually as here in the Figure 2. In reality dissipation of energy leading to damping is unavoidable. Usually dissipation is included in the equation of motion by adding a viscous damping term which is a damping constant times the velocity. Thus, the equation of the damped system becomes

$$f_d = I\ddot{\theta} + \gamma\dot{\theta} + K\theta$$

The equation of motion of the damped pendulum reads

$$\ddot{\theta} + \gamma\dot{\theta} + \frac{g}{l}sin(\theta) = 0$$

where $\gamma$ is the damping constant.

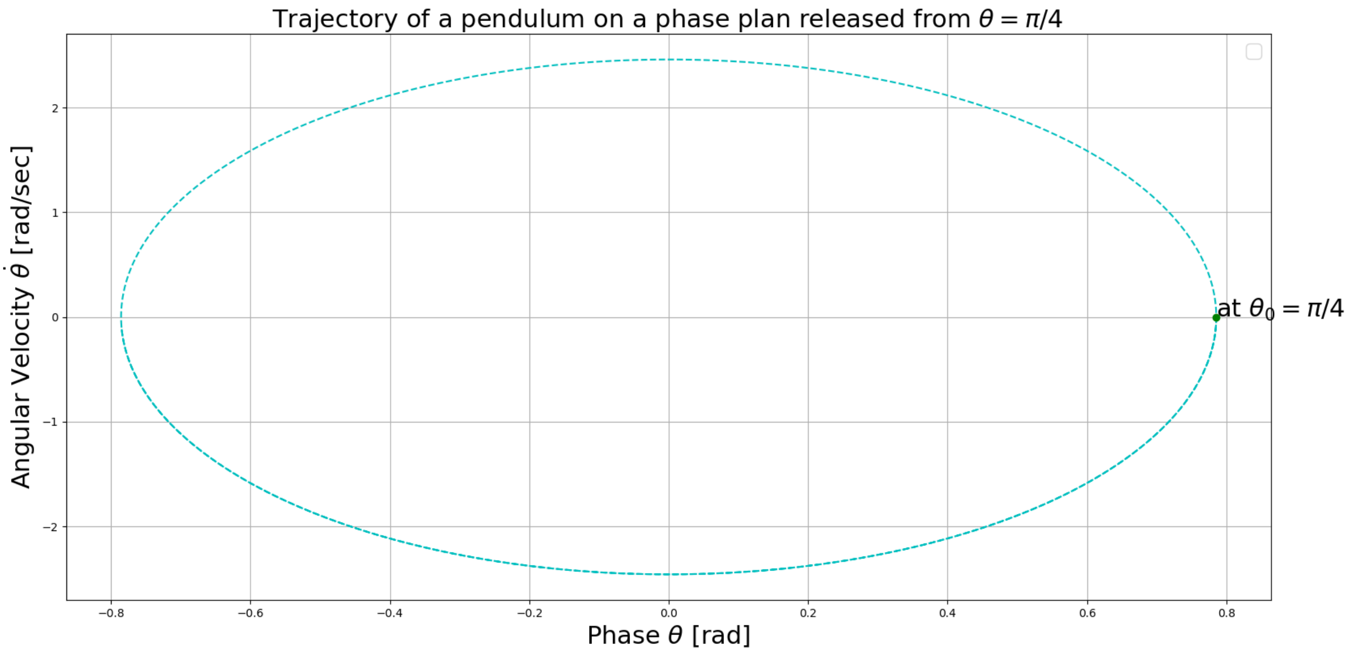


Figure 2: Qualitative phase space trajectory of the undamped pendulum

## References

[1] Rod Grupen, *Control* http://www-robotics.cs.umass.edu/~grupen/book2020/3-Control.pdf

[2] Rod Grupen, *Appendix 1* http://www-robotics.cs.umass.edu/~grupen/book2020/A-Linear-Analysis.pdf