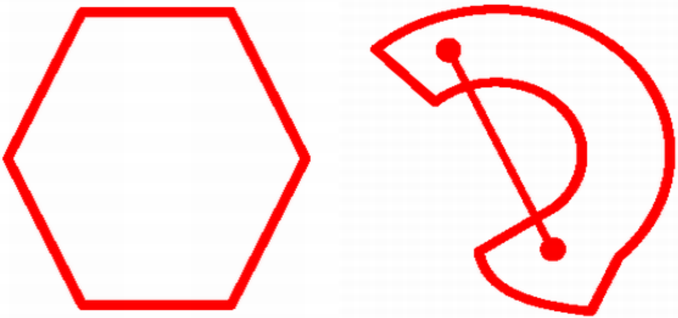
Subgradient Descent

February 17, 2016

Convex Sets

Definition

A set C is **convex** if the line segment between any two points in C lies in C.

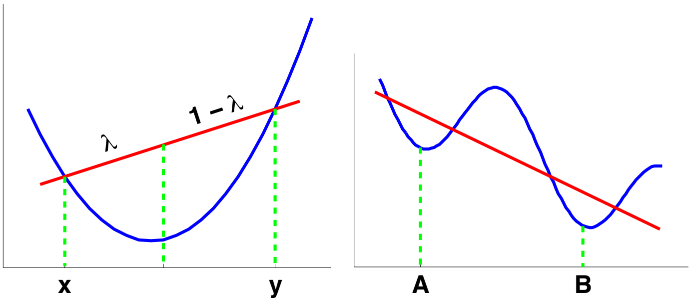


KPM Fig. 7.4

Convex and Concave Functions

Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is **convex** if the line segment connecting any two points on the graph of f lies above the graph. f is **concave** if $-f$ is convex.

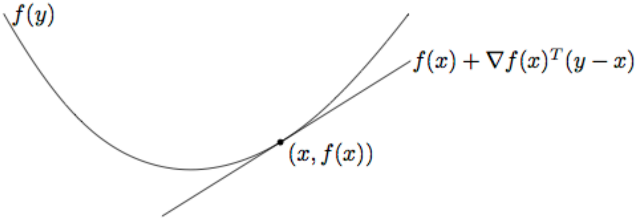


KPM Fig. 7.5

First-Order Approximation

- Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is **differentiable.**
- Predict $f(y)$ given $f(x)$ and $\nabla f(x)$?
- Linear (i.e. "**first order**") approximation:

$$f(y) \approx f(x) + \nabla f(x)^T (y - x)$$



Boyd & Vandenberghe Fig. 3.2

First-Order Condition for Convex, Differentiable Function

- Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is **convex** and **differentiable.**
- Then for any $x, y \in \mathbf{R}^n$

$$f(y) \geqslant f(x) + \nabla f(x)^T (y - x)$$

- The linear approximation to f at x is a **global underestimator** of f:

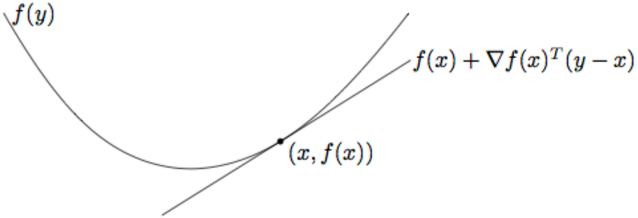


Figure from Boyd & Vandenberghe Fig. 3.2; Proof in Section 3.1.3

First-Order Condition for Convex, Differentiable Function

- Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is **convex** and **differentiable**
- Then for any $x, y \in \mathbf{R}^n$

$$f(y) \geqslant f(x) + \nabla f(x)^T (y - x)$$

Corollary

If $\nabla f(x) = 0$ then x is a global minimizer of f.

For convex functions, **local information gives global information.**

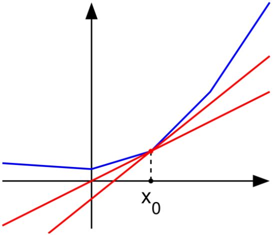
Subgradients

Definition

A vector $g \in \mathbf{R}^n$ is a **subgradient** of $f : \mathbf{R}^n \to \mathbf{R}$ at x if for all z,

$$f(z) \geqslant f(x) + g^T(z - x).$$

g is a subgradient iff $f(x) + g^T(z - x)$ is a global underestimator of f



Blue is a graph of $f(x)$.
Each red line is a lower bound: $x \mapsto f(x_0) + g^T(x - x_0)$

Subdifferential

Definitions

- f is **subdifferentiable** at x if $\exists$ at least one subgradient at x.
- The set of all subgradients at x is called the **subdifferential:** $\partial f(x)$

Basic Facts

- f is convex and differentiable $\implies \partial f(x) = \{\nabla f(x)\}$.
- Any point x, there can be 0, 1, or infinitely many subgradients.
- $\partial f(x) = \emptyset \implies f$ is not convex.

Globla Optimality Condition

Definition

A vector $g \in \mathbf{R}^n$ is a **subgradient** of $f : \mathbf{R}^n \to \mathbf{R}$ at x if for all z,

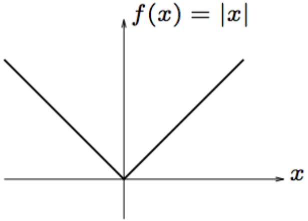
$$f(z) \geqslant f(x) + g^T(z - x).$$

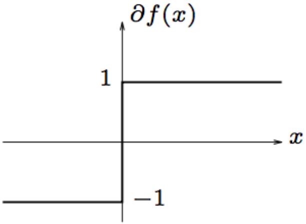
Corollary

*If $0 \in \partial f(x)$, then x is a **global minimizer** of f.*

Subdifferential of Absolute Value

- Consider $f(x) = |x|$





- Plot on right shows $\{(x, g) \mid x \in \mathbf{R},\ g \in \partial f(x)\}$

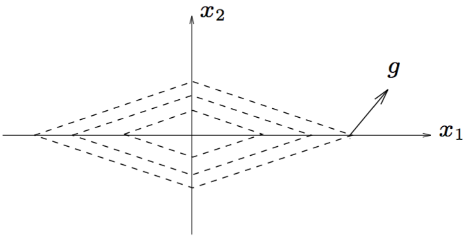
Boyd EE364b: Subgradients Slides

Descent Directions

- For differentiable f, $-\nabla f(x)$ is a descent direction.
- What can we do for non-differentiable f?
- Can we use $-g$ as a step, for some $g \in \partial f(x)$?
- Is $-g$ a descent direction?

Subgradient Not a Descent Direction

$$f(x) = |x_1| + 2|x_2|$$



- Diamonds are level sets of $f(x)$. (f minimized at origin)
- g is a subgradient at the point it's drawn.
- Moving in $-g$ direction increases the function.

Figure from Boyd EE364b: Subgradients Slides, `http://web.stanford.edu/class/ee364b/lectures/subgradients_slides.pdf`, slide 28.

Subgradient Descent

- Suppose f is convex, and we start optimizing at x_0.
- Repeat
    - Step in a negative subgradient direction:

      $$x = x_0 - tg,$$

      where $t > 0$ is the step size and $g \in \partial f(x_0)$.

$-g$ not a descent direction – can this work?

Subgradient Gets Us Closer To Minimizer

Theorem

Suppose f is convex.

- *Let $x = x_0 - tg$, for $g \in \partial f(x_0)$.*
- *Let z be any point for which $f(z) < f(x_0)$.*
- *Then for small enough $t > 0$,*

$$\|x - z\|_2 < \|x_0 - z\|_2.$$

- Apply this with $z = x^* \in \arg\min_x f(x)$.

$\Longrightarrow$ **Negative subgradient step gets us closer to minimizer.**

Subgradient Gets Us Closer To Minimizer (Proof)

- Let $x = x_0 - tg$, for $g \in \partial f(x_0)$ and $t > 0$.
- Let z be any point for which $f(z) < f(x_0)$.
- Then

$$\begin{aligned}
\|x - z\|_2^2 &= \|x_0 - tg - z\|_2^2 \\
&= \|x_0 - z\|_2^2 - 2tg^T(x_0 - z) + t^2\|g\|_2^2 \\
&\leqslant \|x_0 - z\|_2^2 - 2t[f(x_0) - f(z)] + t^2\|g\|_2^2
\end{aligned}$$

- Consider $-2t[f(x_0) - f(z)] + t^2\|g\|_2^2$.
  - It's a convex quadratic (facing upwards).
  - Has zeros at $t = 0$ and $t = 2(f(x_0) - f(z))/\|g\|_2^2 > 0$.
  - Therefore, it's negative for any

$$t \in \left(0, \frac{2(f(x_0) - f(z))}{\|g\|_2^2}\right).$$

Based on Boyd EE364b: Subgradients Slides

Convergence Theorem for Fixed Step Size

Assume $f : \mathbf{R}^n \to \mathbf{R}$ is convex and

- f is Lipschitz continuous with constant $G > 0$:

$$|f(x) - f(y)| \leqslant G\|x - y\| \text{ for all } x, y$$

Theorem

For fixed step size t, subgradient method satisfies:

$$\lim_{k\to\infty} f(x_{best}^{(k)}) \leqslant f(x^*) + G^2 t/2$$

Based on `https://www.cs.cmu.edu/~ggordon/10725-F12/slides/06-sg-method.pdf`

Convergence Theorems for Decreasing Step Sizes

Assume $f : \mathbf{R}^n \to \mathbf{R}$ is convex and

- f is Lipschitz continuous with constant $G > 0$:

$$|f(x) - f(y)| \leqslant G\|x - y\| \text{ for all } x, y$$

Theorem

For step size respecting Robbins-Monro conditions,

$$\lim_{k\to\infty} f(x_{best}^{(k)}) \leqslant f(x^*)$$

Based on `https://www.cs.cmu.edu/~ggordon/10725-F12/slides/06-sg-method.pdf`