

EM Algorithm for Latent Variable Models

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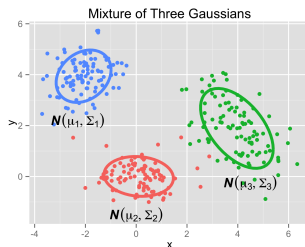
Gaussian Mixture Model: Review

Gaussian Mixture Model Parameters (k Components)

Cluster probabilities: $\pi = (\pi_1, \dots, \pi_k)$

Cluster means: $\mu = (\mu_1, \dots, \mu_k)$

Cluster covariance matrices: $\Sigma = (\Sigma_1, \dots, \Sigma_k)$



GMM: The Joint and the Marginal Likelihood

- **Generative model** description:

$z \sim \text{Categorical}(\pi_1, \dots, \pi_k)$ **Cluster assignment**

$x | z \sim \mathcal{N}(\mu_z, \Sigma_z)$ **Choose point from cluster distribution**

- **Joint distribution** (includes observed x and unobserved z):

$$\begin{aligned} p(x, z) &= p(x | z)p(z) \\ &= \mathcal{N}(x | \mu_z, \Sigma_z) \pi_z \end{aligned}$$

- **Marginal distribution** (just observed variable x):

$$p(x) = \sum_{z=1}^k p(x, z) = \sum_{z=1}^k \pi_z \mathcal{N}(x | \mu_z, \Sigma_z)$$

Maximum Likelihood for the Gaussian Mixture Model

- Find parameters that give **observed data** the **highest likelihood**.
- The model likelihood for $\mathcal{D} = \{x_1, \dots, x_n\}$ is

$$p(\mathcal{D}) = \prod_{i=1}^n p(x_i) = \prod_{i=1}^n \left[\sum_{z=1}^k \pi_z \mathcal{N}(x_i \mid \mu_z, \Sigma_z) \right].$$

- The log-likelihood objective function:

$$J(\pi, \mu, \Sigma) = \sum_{i=1}^n \log \left[\sum_{z=1}^k \pi_z \mathcal{N}(x_i \mid \mu_z, \Sigma_z) \right]$$

- MLE is $(\hat{\pi}, \hat{\mu}, \hat{\Sigma}) = \arg \max_{\pi, \mu, \Sigma} J(\pi, \mu, \Sigma)$. EM algorithm to find it...

The EM Algorithm for GMM

Cluster Probabilities and Expected Cluster Sizes

- Probability that observed value x_i comes from cluster c :

$$\gamma_i^c := \mathbb{P}(z_i = c \mid x_i).$$

- The vector $(\gamma_i^1, \dots, \gamma_i^k)$ gives the **soft cluster assignments** for x_i .
- Let n_c be the **expected number** of points in cluster c :

$$\begin{aligned} n_c &= \mathbb{E} \left[\sum_i 1(z_i = c) \mid x_1, \dots, x_n \right] \\ &= \sum_i \mathbb{P}(z_i = c \mid x_i) \\ &= \sum_{i=1}^n \gamma_i^c \end{aligned}$$

EM Algorithm for GMM

- 1 Initialize parameters μ, Σ, π .
- 2 “E step”. Evaluate the **responsibilities** using current parameters:

$$\gamma_i^j = \mathbb{P}(z_i = j | x_i) = \frac{\pi_j \mathcal{N}(x_i | \mu_j, \Sigma_j)}{\sum_{c=1}^k \pi_c \mathcal{N}(x_i | \mu_c, \Sigma_c)},$$

for $i = 1, \dots, n$ and $j = 1, \dots, k$.

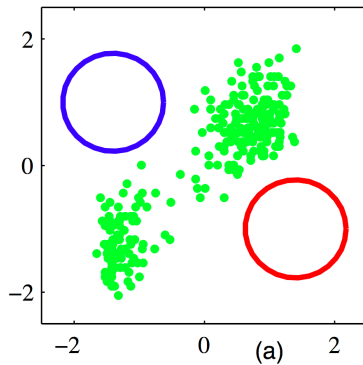
- 3 “M step”. Re-estimate the parameters using responsibilities:

$$\begin{aligned}\mu_c^{\text{new}} &= \frac{1}{n_c} \sum_{i=1}^n \gamma_i^c x_i \\ \Sigma_c^{\text{new}} &= \frac{1}{n_c} \sum_{i=1}^n \gamma_i^c (x_i - \mu_{\text{MLE}}) (x_i - \mu_{\text{MLE}})^T \\ \pi_c^{\text{new}} &= \frac{n_c}{n},\end{aligned}$$

- 4 Repeat from Step 2, until log-likelihood converges.

EM for GMM

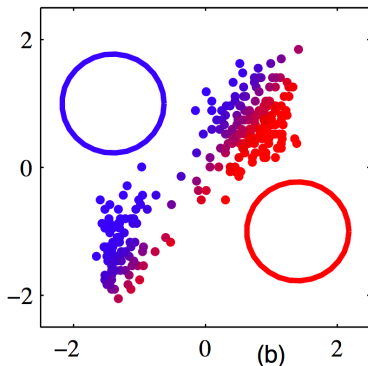
- Initialization



From Bishop's *Pattern recognition and machine learning*, Figure 9.8.

EM for GMM

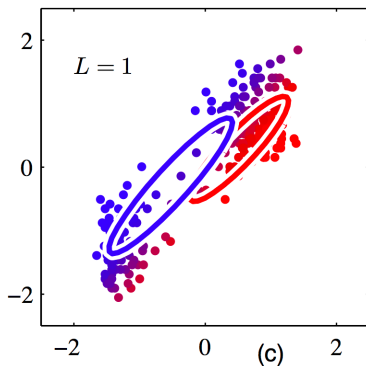
- First soft assignment:



From Bishop's *Pattern recognition and machine learning*, Figure 9.8.

EM for GMM

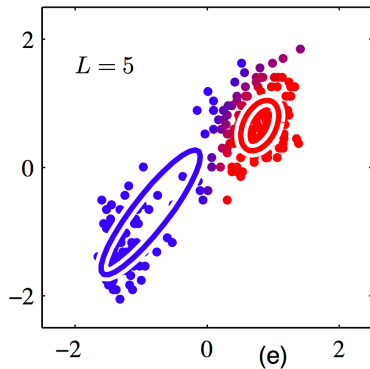
- First soft assignment:



From Bishop's *Pattern recognition and machine learning*, Figure 9.8.

EM for GMM

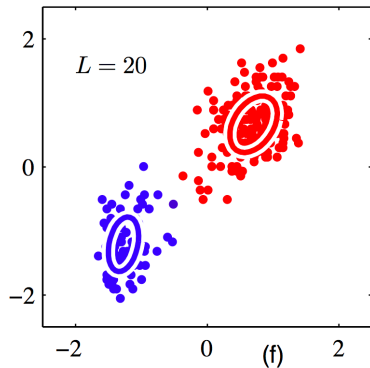
- After 5 rounds of EM:



From Bishop's *Pattern recognition and machine learning*, Figure 9.8.

EM for GMM

- After 20 rounds of EM:



From Bishop's *Pattern recognition and machine learning*, Figure 9.8.

Relation to K -Means

- EM for GMM seems a little like k -means.
- In fact, there is a precise correspondence.
- First, fix each cluster covariance matrix to be $\sigma^2 I$.
- As we take $\sigma^2 \rightarrow 0$, the update equations converge to doing k -means.
- If you do a quick experiment yourself, you'll find
 - Soft assignments converge to hard assignments.
 - Has to do with the tail behavior (exponential decay) of Gaussian.

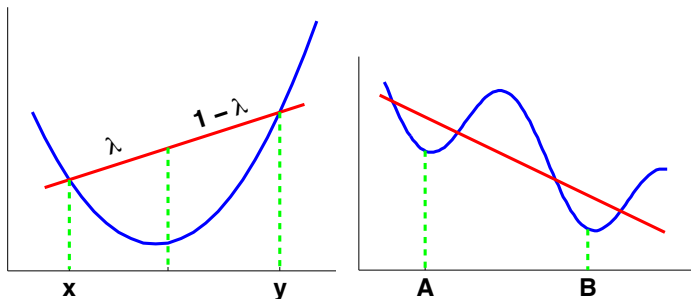
Math Prerequisites

Convex and Concave Functions

Definition

A function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is **convex** if for all $x, y \in \mathbf{R}^n$ and $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y).$$



Jensen's Inequality

Theorem (Jensen's Inequality)

If $f : \mathbf{R} \rightarrow \mathbf{R}$ is a **convex** function, and x is a random variable, then

$$\mathbb{E}f(x) \geq f(\mathbb{E}x).$$

Moreover, if f is **strictly convex**, then equality implies that $x = \mathbb{E}x$ with probability 1 (i.e. x is a constant).

- e.g. $f(x) = x^2$ is convex. So $\mathbb{E}x^2 \geq (\mathbb{E}x)^2$. Thus

$$\text{Var}x = \mathbb{E}x^2 - (\mathbb{E}x)^2 \geq 0.$$

Kullback-Leibler Divergence

- Let $p(x)$ and $q(x)$ be probability mass functions (PMFs) on \mathcal{X} .
- How can we measure how “different” p and q are?
- The **Kullback-Leibler** or “**KL**” **Divergence** is defined by

$$\text{KL}(p\|q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}.$$

(Assumes $q(x) = 0$ implies $p(x) = 0$.)

- Can also write this as

$$\text{KL}(p\|q) = \mathbb{E}_{x \sim p} \log \frac{p(x)}{q(x)}.$$

Gibbs Inequality ($KL(p||q) \geq 0$ and $KL(p||p) = 0$)

Theorem (Gibbs Inequality)

Let $p(x)$ and $q(x)$ be PMFs on \mathcal{X} . Then

$$KL(p||q) \geq 0,$$

with equality iff $p(x) = q(x)$ for all $x \in \mathcal{X}$.

- KL divergence measures the “distance” between distributions.
- Note:
 - KL divergence **not a metric**.
 - KL divergence is **not symmetric**.

Gibbs Inequality: Proof

$$\begin{aligned}
 \text{KL}(p\|q) &= \mathbb{E}_p \left[-\log \left(\frac{q(x)}{p(x)} \right) \right] \\
 &\geq -\log \left[\mathbb{E}_p \left(\frac{q(x)}{p(x)} \right) \right] \quad (\text{Jensen's}) \\
 &= -\log \left[\sum_{\{x|p(x)>0\}} p(x) \frac{q(x)}{p(x)} \right] \\
 &= -\log \left[\sum_{x \in \mathcal{X}} q(x) \right] \\
 &= -\log 1 = 0.
 \end{aligned}$$

- Since $-\log$ is strictly convex, we have strict equality iff $q(x)/p(x)$ is a constant, which implies $q = p$.

EM Algorithm for Latent Variable Models

General Latent Variable Model

- Two sets of random variables: z and x .
- z consists of unobserved **hidden variables**.
- x consists of **observed variables**.
- Joint probability model parameterized by $\theta \in \Theta$:

$$p(x, z \mid \theta)$$

Notation abuse

Notation $p(x, z \mid \theta)$ suggests a Bayesian setting, in which θ is a r.v. However we are **not** assuming a Bayesian setting. $p(x, z \mid \theta)$ is just easier to read than $p_{\theta}(x, z)$, once θ gets more complicated.

Complete and Incomplete Data

- An observation of x is called an **incomplete data set**.
- An observation (x, z) is called a **complete data set**.
 - We never have a complete data set for latent variable models.
 - But it's a useful construct.
- Suppose we have an incomplete data set $\mathcal{D} = (x_1, \dots, x_n)$.
- To simplify notation, take x to represent the entire dataset

$$x = (x_1, \dots, x_n),$$

and z to represent the corresponding unobserved variables

$$z = (z_1, \dots, z_n).$$

The EM Algorithm **Key Idea**

- Marginal log-likelihood is hard to optimize:

$$\max_{\theta} \log \left\{ \sum_z p(x, z \mid \theta) \right\}$$

- **Assume that** complete data log-likelihood would be easy to optimize:

$$\max_{\theta} \log p(x, z \mid \theta)$$

- What if we had a **distribution** $q(z)$ for the latent variables z ?
- Then maximize the **expected complete data log-likelihood**:

$$\max_{\theta} \sum_z q(z) \log p(x, z \mid \theta)$$

- EM **assumes** this maximization is feasible.

Lower Bound for Likelihood

- Let $q(z)$ be any PMF on \mathcal{Z} , the support of Z :

$$\begin{aligned}
 \log p(x | \theta) &= \log \left[\sum_z p(x, z | \theta) \right] \\
 &= \log \left[\sum_z q(z) \left(\frac{p(x, z | \theta)}{q(z)} \right) \right] \quad (\text{log of an expectation}) \\
 &\geq \sum_z q(z) \log \left(\frac{p(x, z | \theta)}{q(z)} \right) \quad (\text{expectation of log}) \\
 &=: \mathcal{L}(q, \theta).
 \end{aligned}$$

- The inequality is by Jensen's, by concavity of the log.

This is the **key step** for “variational methods”.

Lower Bound and Expected Complete Log-Likelihood

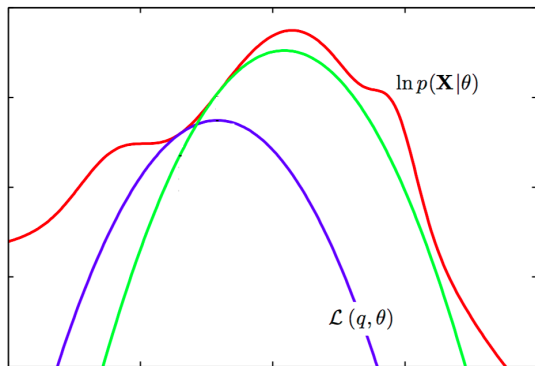
- Consider maximizing the lower bound $\mathcal{L}(q, \theta)$:

$$\begin{aligned}
 \mathcal{L}(q, \theta) &= \sum_z q(z) \log \left(\frac{p(x, z | \theta)}{q(z)} \right) \\
 &= \underbrace{\sum_z q(z) \log p(x, z | \theta)}_{\mathbb{E}[\text{complete data log-likelihood}]} - \underbrace{\sum_z q(z) \log q(z)}_{\text{no } \theta \text{ here}}
 \end{aligned}$$

- Maximizing $\mathcal{L}(q, \theta)$ equivalent to maximizing $\mathbb{E}[\text{complete data log-likelihood}]$.

A Family of Lower Bounds

- Each q gives a different lower bound: $\log p(\mathbf{x} | \theta) \geq \mathcal{L}(q, \theta)$
- Two lower bounds, as functions of θ :



From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

EM: Big Picture Idea

- The following inequality holds for all θ and q :

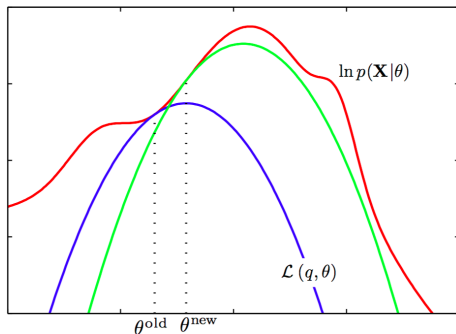
$$\log p(x | \theta) \geq \mathcal{L}(q, \theta).$$

- We want to find θ that maximizes $\log p(x | \theta)$.
- $\log p(x | \theta)$ is hard to maximize directly.
- Two step version of the EM algorithm:
 - 1 We vary q and θ , searching for the biggest $\mathcal{L}(q, \theta)$ we can find.
 - 2 Final result is $\hat{\theta}$ corresponding to the largest $\mathcal{L}(q, \theta)$ we found.
- Often this is a local maximum of the likelihood.
- One question left: How to choose the sequence of q 's and θ 's we try?

EM: Coordinate Ascent on Lower Bound

- Choose sequence of q 's and θ 's by “coordinate ascent”.
- EM Algorithm (high level):
 - 1 Choose initial θ^{old} .
 - 2 Let $q^* = \arg \max_q \mathcal{L}(q, \theta^{\text{old}})$
 - 3 Let $\theta^{\text{new}} = \arg \max_{\theta} \mathcal{L}(q^*, \theta^{\text{old}})$.
 - 4 Go to step 2, until converged.
- Will show: $p(x | \theta^{\text{new}}) \geq p(x | \theta^{\text{old}})$
- Get sequence of θ 's with monotonically increasing likelihood.

EM: Coordinate Ascent on Lower Bound



- 1 Start at θ^{old} .
- 2 Find q giving best lower bound at $\theta^{\text{old}} \implies \mathcal{L}(q, \theta)$.
- 3 $\theta^{\text{new}} = \arg \max_{\theta} \mathcal{L}(q, \theta)$.

From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

The Lower Bound

- Let's investigate the lower bound:

$$\begin{aligned}
 \mathcal{L}(q, \theta) &= \sum_z q(z) \log \left(\frac{p(x, z | \theta)}{q(z)} \right) \\
 &= \sum_z q(z) \log \left(\frac{p(z | x, \theta) p(x | \theta)}{q(z)} \right) \\
 &= \sum_z q(z) \log \left(\frac{p(z | x, \theta)}{q(z)} \right) + \sum_z q(z) \log p(x | \theta) \\
 &= -\text{KL}[q(z), p(z | x, \theta)] + \log p(x | \theta)
 \end{aligned}$$

- Amazing! We get back an equality for the marginal likelihood:

$$\log p(x | \theta) = \mathcal{L}(q, \theta) + \text{KL}[q(z), p(z | x, \theta)]$$

The Best Lower Bound

- Find q maximizing

$$\mathcal{L}(q, \theta^{\text{old}}) = -\text{KL}[q(z), p(z | x, \theta^{\text{old}})] + \underbrace{\log p(x | \theta^{\text{old}})}_{\text{no } q \text{ here}}$$

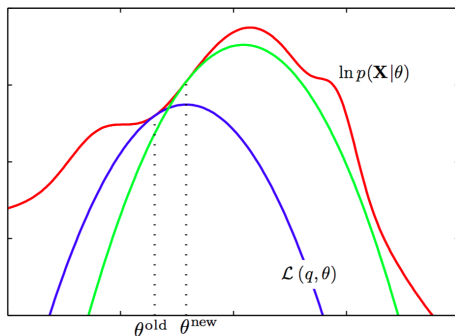
- Recall $\text{KL}(p||q) \geq 0$, and $\text{KL}(p||p) = 0$.
- Best q is $q^*(z) = p(z | x, \theta^{\text{old}})$. Proof:

$$\mathcal{L}(q^*, \theta^{\text{old}}) = -\underbrace{\text{KL}[p(z | x, \theta^{\text{old}}), p(z | x, \theta^{\text{old}})]}_{=0} + \log p(x | \theta^{\text{old}})$$

- Summary:

$$\begin{aligned} \log p(x | \theta^{\text{old}}) &= \mathcal{L}(q^*, \theta^{\text{old}}) \quad (\text{tangent at } \theta^{\text{old}}). \\ \log p(x | \theta) &\geq \mathcal{L}(q^*, \theta) \quad \forall \theta \end{aligned}$$

Tight lower bound for any chosen θ



Fix any θ' and take $q'(z) = p(z | x, \theta')$. Then

- ① $\log p(x | \theta) \geq \mathcal{L}(q', \theta) \forall \theta$. [Global lower bound].
- ② $\log p(x | \theta') = \mathcal{L}(q', \theta')$. [Lower bound is **tight** at θ' .]

From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

General EM Algorithm

1 Choose initial θ^{old} .

2 **Expectation Step**

- Let $q^*(z) = p(z | x, \theta^{\text{old}})$. [q^* gives best lower bound at θ^{old}]
- Let

$$J(\theta) := \mathcal{L}(q^*, \theta) = \underbrace{\sum_z q^*(z) \log \left(\frac{p(x, z | \theta)}{q^*(z)} \right)}_{\text{expectation w.r.t. } z \sim q^*(z)}$$

3 **Maximization Step**

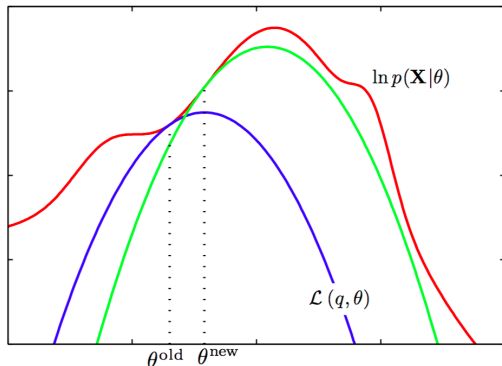
$$\theta^{\text{new}} = \arg \max_{\theta} J(\theta).$$

[Equivalent to maximizing expected complete log-likelihood.]

4 Go to step 2, until converged.

EM Monotonically Increases Likelihood

EM Gives Monotonically Increasing Likelihood: By Picture



From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

EM Gives Monotonically Increasing Likelihood: By Math

- 1 Start at θ^{old} .
- 2 Choose $q^*(z) = \arg \max_q \mathcal{L}(q, \theta^{\text{old}})$. We've shown

$$\log p(x | \theta^{\text{old}}) = \mathcal{L}(q^*, \theta^{\text{old}})$$

- 3 Choose $\theta^{\text{new}} = \arg \max_{\theta} \mathcal{L}(q^*, \theta^{\text{old}})$. So

$$\mathcal{L}(q^*, \theta^{\text{new}}) \geq \mathcal{L}(q^*, \theta^{\text{old}}).$$

Putting it together, we get

$$\begin{aligned}
 \log p(x | \theta^{\text{new}}) &\geq \mathcal{L}(q^*, \theta^{\text{new}}) && \mathcal{L} \text{ is a lower bound} \\
 &\geq \mathcal{L}(q^*, \theta^{\text{old}}) && \text{By definition of } \theta^{\text{new}} \\
 &= \log p(x | \theta^{\text{old}}) && \text{Bound is tight at } \theta^{\text{old}}.
 \end{aligned}$$

Suppose We Maximize the Lower Bound...

- Suppose we have found a **global maximum** of $\mathcal{L}(q, \theta)$:

$$L(q^*, \theta^*) \geq L(q, \theta) \quad \forall q, \theta,$$

where of course

$$q^*(z) = p(z | x, \theta^*).$$

- Claim: θ^* is a global maximum of $\log p(x | \theta^*)$.
- Proof: For any θ' , we showed that for $q'(z) = p(z | x, \theta')$ we have

$$\begin{aligned} \log p(x | \theta') &= \mathcal{L}(q', \theta') + \text{KL}[q', p(z | x, \theta')] \\ &= \mathcal{L}(q', \theta') \\ &\leq \mathcal{L}(q^*, \theta^*) \\ &= \log p(x | \theta^*) \end{aligned}$$

Convergence of EM

- Let θ_n be value of EM algorithm after n steps.
- Define “transition function” $M(\cdot)$ such that $\theta_{n+1} = M(\theta_n)$.
- Suppose log-likelihood function $\ell(\theta) = \log p(x | \theta)$ is differentiable.
- Let S be the set of stationary points of $\ell(\theta)$. (i.e. $\nabla_{\theta} \ell(\theta) = 0$)

Theorem

Under mild regularity conditions^a, for any starting point θ_0 ,

- *$\lim_{n \rightarrow \infty} \theta_n = \theta^*$ for some stationary point $\theta^* \in S$ and*
- *θ^* is a fixed point of the EM algorithm, i.e. $M(\theta^*) = \theta^*$. Moreover,*
- *$\ell(\theta_n)$ strictly increases to $\ell(\theta^*)$ as $n \rightarrow \infty$, unless $\theta_n \equiv \theta^*$.*

^aFor details, see “Parameter Convergence for EM and MM Algorithms” by Florin Vaida in *Statistica Sinica* (2005).

<http://www3.stat.sinica.edu.tw/statistica/oldpdf/a15n316.pdf>

Variations on EM

EM Gives Us Two New Problems

- The “E” Step: Computing

$$J(\theta) := \mathcal{L}(q^*, \theta) = \sum_z q^*(z) \log \left(\frac{p(x, z | \theta)}{q^*(z)} \right)$$

- The “M” Step: Computing

$$\theta^{\text{new}} = \arg \max_{\theta} J(\theta).$$

- Either of these can be too hard to do in practice.

Generalized EM (GEM)

- Addresses the problem of a difficult “M” step.
- Rather than finding

$$\theta^{\text{new}} = \arg \max_{\theta} J(\theta),$$

find **any** θ^{new} for which

$$J(\theta^{\text{new}}) > J(\theta^{\text{old}}).$$

- Can use a standard nonlinear optimization strategy
 - e.g. take a gradient step on J .
- We still get monotonically increasing likelihood.

EM and More General Variational Methods

- Suppose “E” step is difficult:
 - Hard to take expectation w.r.t. $q^*(z) = p(z \mid x, \theta^{\text{old}})$.
- Solution: Restrict to distributions \mathcal{Q} that are easy to work with.
- Lower bound now looser:

$$q^* = \arg \min_{q \in \mathcal{Q}} \text{KL}[q(z), p(z \mid x, \theta^{\text{old}})]$$

EM in Bayesian Setting

- Suppose we have a prior $p(\theta)$.
- Want to find MAP estimate: $\hat{\theta}_{\text{MAP}} = \arg \max_{\theta} p(\theta | x)$:

$$\begin{aligned} p(\theta | x) &= p(x | \theta)p(\theta)/p(x) \\ \log p(\theta | x) &= \log p(x | \theta) + \log p(\theta) - \log p(x) \end{aligned}$$

- Still can use our lower bound on $\log p(x, \theta)$.

$$J(\theta) := \mathcal{L}(q^*, \theta) = \sum_z q^*(z) \log \left(\frac{p(x, z | \theta)}{q^*(z)} \right)$$

- Maximization step becomes

$$\theta^{\text{new}} = \arg \max_{\theta} [J(\theta) + \log p(\theta)]$$

- Homework: Convince yourself our lower bound is still tight at θ .

Homework: Gaussian Mixture Model (Hints)

Homework: Derive EM for GMM from General EM Algorithm

- Subsequent slides may help set things up.
- Key skills:
 - MLE for multivariate Gaussian distributions.
 - Lagrange multipliers

Gaussian Mixture Model (k Components)

- GMM Parameters

Cluster probabilities: $\pi = (\pi_1, \dots, \pi_k)$

Cluster means: $\mu = (\mu_1, \dots, \mu_k)$

Cluster covariance matrices: $\Sigma = (\Sigma_1, \dots, \Sigma_k)$

- Let $\theta = (\pi, \mu, \Sigma)$.

- Marginal log-likelihood

$$\log p(x | \theta) = \log \left\{ \sum_{z=1}^k \pi_z \mathcal{N}(x | \mu_z, \Sigma_z) \right\}$$

$q^*(z)$ are “Soft Assignments”

- Suppose we observe n points: $X = (x_1, \dots, x_n) \in \mathbf{R}^{n \times d}$.
- Let $z_1, \dots, z_n \in \{1, \dots, k\}$ be corresponding hidden variables.
- Optimal distribution q^* is:

$$q^*(z) = p(z | x, \theta).$$

- Convenient to define the conditional distribution for z_i given x_i as

$$\begin{aligned} \gamma_i^j &:= p(z = j | x_i) \\ &= \frac{\pi_j \mathcal{N}(x_i | \mu_j, \Sigma_j)}{\sum_{c=1}^k \pi_c \mathcal{N}(x_i | \mu_c, \Sigma_c)} \end{aligned}$$

Expectation Step

- The complete log-likelihood is

$$\begin{aligned}\log p(x, z | \theta) &= \sum_{i=1}^n \log [\pi_z \mathcal{N}(x_i | \mu_z, \Sigma_z)] \\ &= \sum_{i=1}^n \left(\log \pi_z + \underbrace{\log \mathcal{N}(x_i | \mu_z, \Sigma_z)}_{\text{simplifies nicely}} \right)\end{aligned}$$

- Take the expected complete log-likelihood w.r.t. q^* :

$$\begin{aligned}J(\theta) &= \sum_z q^*(z) \log p(x, z | \theta) \\ &= \sum_{i=1}^n \sum_{j=1}^k \gamma_i^j [\log \pi_j + \log \mathcal{N}(x_i | \mu_j, \Sigma_j)]\end{aligned}$$

Maximization Step

- Find θ^* maximizing $J(\theta)$:

$$\mu_c^{\text{new}} = \frac{1}{n_c} \sum_{i=1}^n \gamma_i^c x_i$$

$$\Sigma_c^{\text{new}} = \frac{1}{n_c} \sum_{i=1}^n \gamma_i^c (x_i - \mu_{\text{MLE}}) (x_i - \mu_{\text{MLE}})^T$$

$$\pi_c^{\text{new}} = \frac{n_c}{n},$$

for each $c = 1, \dots, k$.