

# Notes for Machine Learning – Brett Bernstein

## Recitation 2

### Intro Question

1. You have been given a data set  $(x_i, y_i)$  for  $i = 1, \dots, n$  where  $x_i \in \mathbb{R}^d$  and  $y_i \in \{-1, 1\}$ . Assume  $w \in \mathbb{R}^n$  and  $a \in \mathbb{R}$ .
  - (a) Suppose  $y_i(w^T x_i + a) > 0$  for all  $i$ . Use a picture to explain what this means when  $d = 2$ .
  - (b) Fix  $M > 0$ . Suppose  $y_i(w^T x_i + a) \geq M$  for all  $i$ . Use a picture to explain what this means when  $d = 2$ .

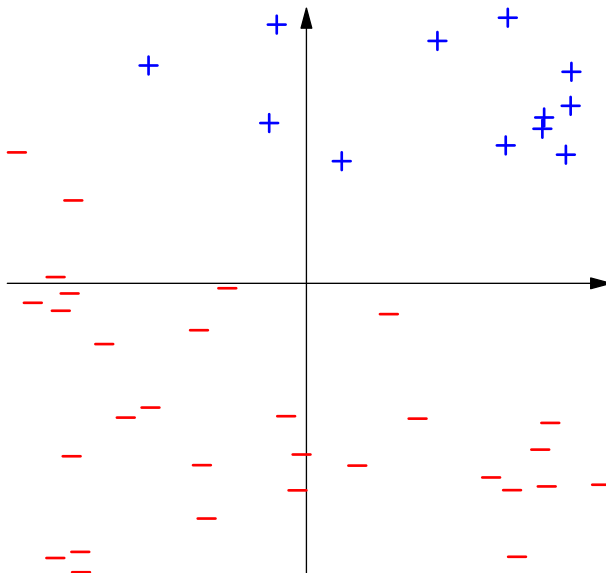


Figure 1: Data set with  $x_i \in \mathbb{R}^2$  and  $y_i \in \{+1, -1\}$

## Support Vector Machines

### Review of Geometry

If  $v, w \in \mathbb{R}^d$  then the component of  $v$  in the direction  $w$  is given by  $\frac{w^T v}{\|w\|_2}$ . This can also be thought of as the signed length of  $v$  when orthogonally projected onto the line through the vector  $w$ .

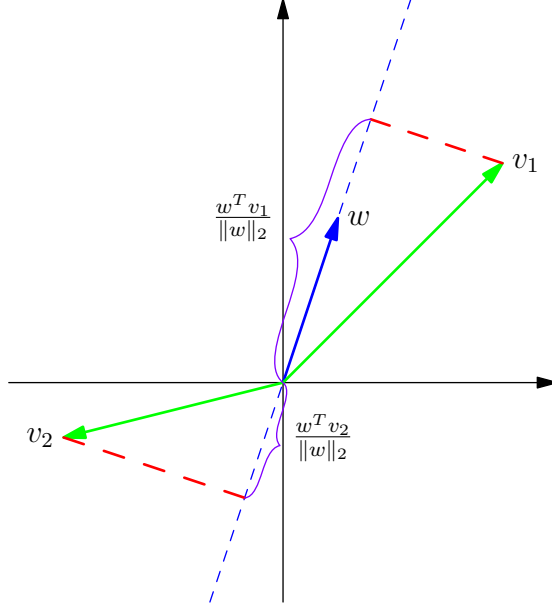


Figure 2: Component of  $v_1, v_2$  in the direction  $w$ .

Assuming  $w \neq 0$  we can use this to interpret the set

$$S = \{x \in \mathbb{R}^d \mid w^T x = b\}.$$

Noting that  $w^T x = b \iff \frac{w^T x}{\|w\|_2} = \frac{b}{\|w\|_2}$  we see that  $S$  contains all vectors whose component in the direction  $w$  is  $\frac{b}{\|w\|_2}$ . Using linear algebra we can see this is the hyperplane orthogonal to the vector  $w$  that passes through the point  $\frac{bw}{\|w\|_2^2}$ . Note also that there are infinitely many pairs  $(w, b)$  that give the same hyperplane. If  $c \neq 0$  then

$$\{x \in \mathbb{R}^d \mid w^T x = b\} \quad \text{and} \quad \{x \in \mathbb{R}^d \mid (cw)^T x = (cb)\}$$

result in the same hyperplanes.

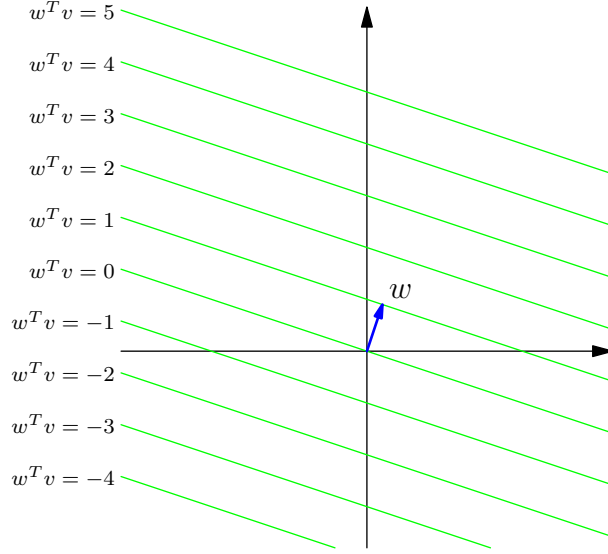


Figure 3: Level Surfaces of  $f(v) = w^T v$  with  $\|w\|_2 = 1$

Given a hyperplane  $\{v \mid w^T v = b\}$ , we can distinguish points  $x \in \mathbb{R}^d$  depending on whether  $w^T x - b$  is zero, positive, or negative, or in other words, whether  $x$  is on the hyperplane, on the side  $w$  is pointing at, or on the side  $-w$  is pointing at.

Figure 4: Sides of the Hyperplane  $w^T v = 15$

If we have a vector  $x \in \mathbb{R}^d$  and a hyperplane  $H = \{v \mid w^T v = b\}$  we can measure the distance from  $x$  to  $H$  by

$$d(x, H) = \left| \frac{w^T x - b}{\|w\|_2} \right|.$$

Without the absolute values we get the *signed distance*: a positive distance if  $w^T x > b$  and a negative distance if  $w^T x < b$ .

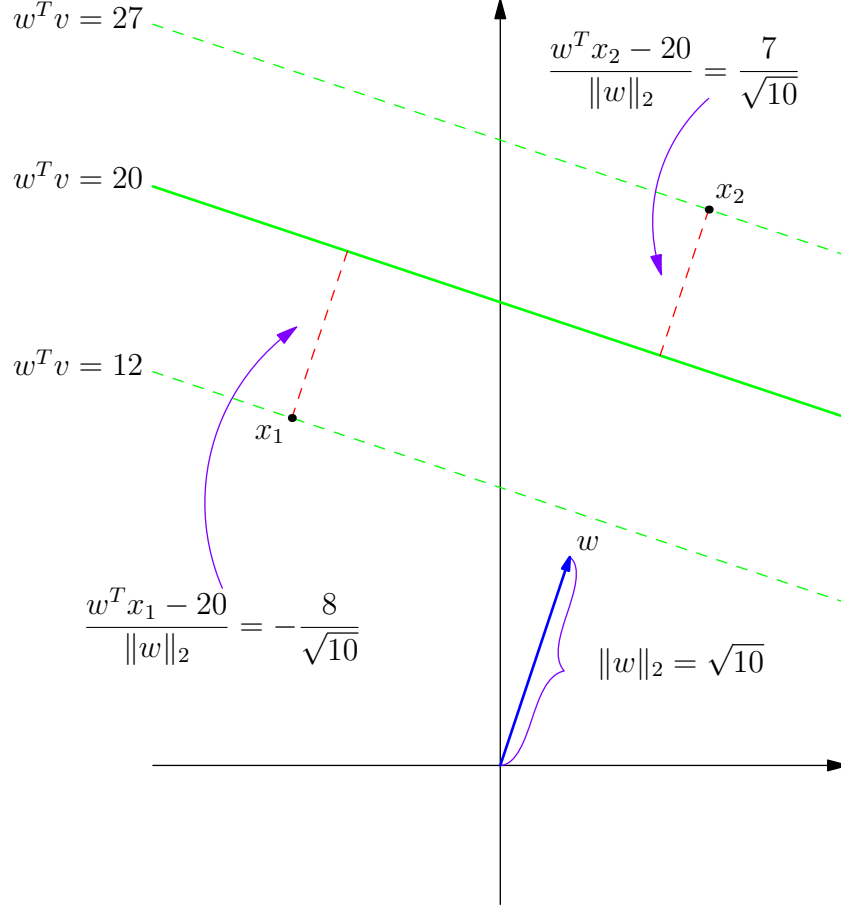


Figure 5: Signed Distance from  $x_1, x_2$  to Hyperplane  $w^T v = 20$

## Hard Margin SVM

Returning to the initial question, suppose we have the data set  $(x_i, y_i)$  for  $i = 1, \dots, n$  where  $x_i \in \mathbb{R}^d$  and  $y_i \in \{-1, 1\}$ .

**Definition 1** (Linearly Separable). We say  $(x_i, y_i)$  for  $i = 1, \dots, n$  is *linearly separable* if there is a  $w \in \mathbb{R}^d$  and  $a \in \mathbb{R}$  such that  $y_i(w^T x_i + a) > 0$  for all  $i$ . The set  $\{v \in \mathbb{R}^d \mid w^T v + a = 0\}$  is called a *separating hyperplane*.

Let's examine what this definition says. If  $y_i = +1$  then we require that  $w^T x_i > -a$  and if  $y_i = -1$  we require that  $w^T x_i < -a$ . Thus linearly separable means that we can separate all of the  $+1$ 's from the  $-1$ 's using the hyperplane  $\{v \mid w^T v = -a\}$ . For the rest of this section, we assume our data is linearly separable. [Linearly separable data] If we can find the  $w, a$  corresponding to a hyperplane that separates the data, we then have a decision function for classifying elements of  $\mathcal{X}$ :  $f(x) = \text{sgn}(w^T x + a)$ . Before we look for such a hyperplane, we must address another issue. If the data is linearly separable, then there are infinitely

many choices of separating hyperplanes. [Show many separating planes.] We will choose the hyperplane that maximizes a quantity called the *margin*.

**Definition 2** (Margin). Let  $H$  be a hyperplane that separates the data  $(x_i, y_i)$  for  $i = 1, \dots, n$ . The margin of this hyperplane is

$$\min_i d(x_i, H),$$

the distance from the hyperplane to the closest data point.

Fix  $w \in \mathbb{R}^d$  and  $a \in \mathbb{R}$  with  $y_i(w^T x_i + a) > 0$  for all  $i$ . Then we saw earlier that

$$d(x_i, H) = \left| \frac{w^T x_i + a}{\|w\|_2} \right| = \frac{y_i(w^T x_i + a)}{\|w\|_2}.$$

This gives us the following optimization problem:

$$\text{maximize}_{w,a} \quad \min_i \frac{y_i(w^T x_i + a)}{\|w\|_2}.$$

We can rewrite this in a more standard form:

$$\begin{aligned} &\text{maximize}_{w,a,M} && M \\ &\text{subject to} && \frac{y_i(w^T x_i + a)}{\|w\|_2} \geq M \quad \text{for all } i. \end{aligned}$$

[Image of maximum margin hyperplane.] The expression  $y_i(w^T x_i + a)/\|w\|_2$  allows us to choose any positive value for  $\|w\|_2$  by changing  $a$  accordingly. Thus we can fix  $\|w\|_2 = 1/M$  and obtain

$$\begin{aligned} &\text{maximize}_{w,a} && 1/\|w\|_2 \\ &\text{subject to} && y_i(w^T x_i + a) \geq 1 \quad \text{for all } i. \end{aligned}$$

To find the optimal  $w, a$  we can solve the corresponding minimization problem

$$\begin{aligned} &\text{minimize}_{w,a} && \|w\|_2^2 \\ &\text{subject to} && y_i(w^T x_i + a) \geq 1 \quad \text{for all } i. \end{aligned}$$

This is a quadratic program that can be solved quickly on fairly large datasets.

## Soft Margin SVM