

# Loss Functions for Regression and Classification

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# Regression Loss Functions

# Regression Notation

- Regression spaces:
  - Input space  $\mathcal{X} = \mathbf{R}^d$
  - Action space  $\mathcal{A} = \mathbf{R}$
  - Outcome space  $\mathcal{Y} = \mathbf{R}$ .
- Since  $\mathcal{A} = \mathcal{Y}$ , we can use more traditional notation:
  - $\hat{y}$  is the predicted value (the action)
  - $y$  is the actual observed value (the outcome)

# Loss Functions for Regression

- In general, loss function may take the form

$$(\hat{y}, y) \mapsto \ell(\hat{y}, y) \in \mathbf{R}$$

- Regression losses usually only depend on the **residual**  $r = y - \hat{y}$ .
  - what you have to add to your prediction to get the right answer
- Loss  $\ell(\hat{y}, y)$  is called **distance-based** if it
  - ① only depends on the residual:

$$\ell(\hat{y}, y) = \psi(y - \hat{y}) \quad \text{for some } \psi: \mathbf{R} \rightarrow \mathbf{R}$$

- ② loss is zero when residual is 0:

$$\psi(0) = 0$$

# Distance-Based Losses are Translation Invariant

- Distance-based losses are translation-invariant. That is,

$$\ell(\hat{y} + b, y + b) = \ell(\hat{y}, y) \quad \forall b \in \mathbf{R}.$$

- When might you not want to use a translation-invariant loss?
- Sometimes relative error  $\frac{\hat{y}-y}{y}$  is a more natural loss (but not translation-invariant)
- Often you can transform response  $y$  so it's translation-invariant (e.g. log transform)

## Some Losses for Regression

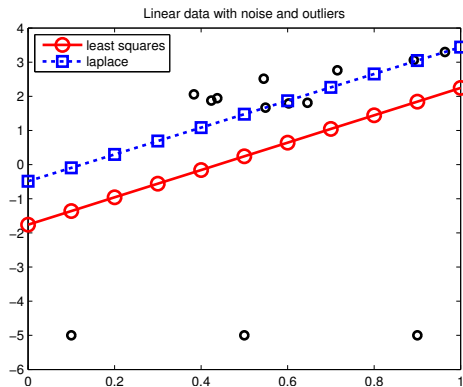
- **Residual:**  $r = y - \hat{y}$
- **Square** or  $\ell_2$  Loss:  $\ell(r) = r^2$
- **Absolute** or **Laplace** or  $\ell_1$  Loss:  $\ell(r) = |r|$

$y$	$\hat{y}$	$ r  =  y - \hat{y} $	$r^2 = (y - \hat{y})^2$
1	0	1	1
5	0	5	25
10	0	10	100
50	0	50	2500

- Outliers typically have large residuals.
- Square loss much more affected by outliers than absolute loss.

# Loss Function Robustness

- **Robustness** refers to how affected a learning algorithm is by outliers.

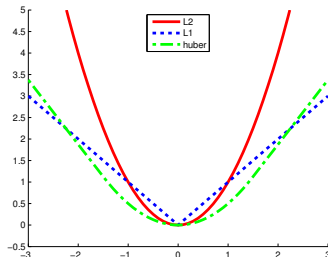


KPM Figure 7.6



## Some Losses for Regression

- **Square** or  $\ell_2$  Loss:  $\ell(r) = r^2$  (*not robust*)
- **Absolute** or **Laplace** Loss:  $\ell(r) = |r|$  (*not differentiable*)
  - gives **median regression**
- **Huber** Loss: Quadratic for  $|r| \leq \delta$  and linear for  $|r| > \delta$  (*robust and differentiable*)



- x-axis is the residual  $y - \hat{y}$ .

## Classification Loss Functions

# The Classification Problem

- Outcome space  $\mathcal{Y} = \{-1, 1\}$
- Action space  $\mathcal{A} = \{-1, 1\}$
- **0-1 loss** for  $f : \mathcal{X} \rightarrow \{-1, 1\}$ :

$$\ell(f(x), y) = 1(f(x) \neq y)$$

- But let's allow **real-valued predictions**  $f : \mathcal{X} \rightarrow \mathbf{R}$ :

$$f(x) > 0 \implies \text{Predict } 1$$

$$f(x) < 0 \implies \text{Predict } -1$$

# The Score Function

- Action space  $\mathcal{A} = \mathbf{R}$       Output space  $\mathcal{Y} = \{-1, 1\}$
- **Real-valued prediction function**  $f : \mathcal{X} \rightarrow \mathbf{R}$

## Definition

The value  $f(x)$  is called the **score** for the input  $x$ .

- In this context,  $f$  may be called a **score function**.
- Intuitively, magnitude of the score represents the **confidence of our prediction**.

# The Margin

## Definition

The **margin** (or **functional margin**) for predicted score  $\hat{y}$  and true class  $y \in \{-1, 1\}$  is  $y\hat{y}$ .

- The margin often looks like  $yf(x)$ , where  $f(x)$  is our score function.
- The margin is a measure of how **correct** we are.
  - If  $y$  and  $\hat{y}$  are the same sign, prediction is **correct** and margin is **positive**.
  - If  $y$  and  $\hat{y}$  have different sign, prediction is **incorrect** and margin is **negative**.
- We want to **maximize the margin**.

# Margin-Based Losses

- Most classification losses depend only on the margin.
- Such a loss is called a **margin-based loss**.
- There is a related concept, the **geometric margin**, in the notes on hard-margin SVM.)

## Classification Losses: 0 – 1 Loss

- Empirical risk for 0 – 1 loss:

$$\hat{R}_n(f) = \frac{1}{n} \sum_{i=1}^n 1(y_i f(x_i) \leq 0)$$

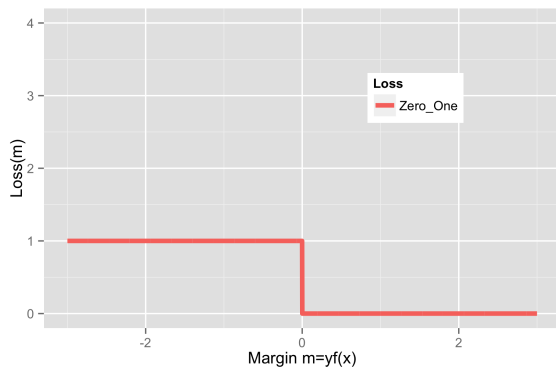
Minimizing empirical 0 – 1 risk not computationally feasible

$\hat{R}_n(f)$  is non-convex, not differentiable (in fact, discontinuous!).

Optimization is **NP-Hard**.

# Classification Losses

Zero-One loss:  $\ell_{0-1} = 1(m \leq 0)$

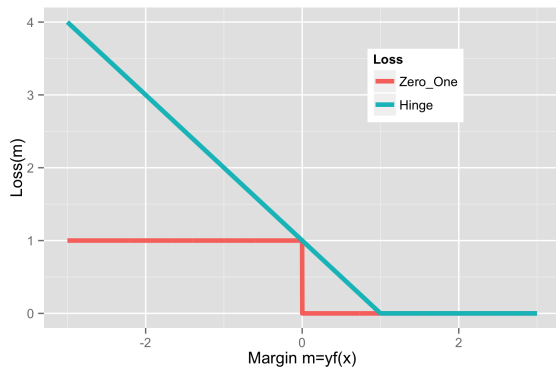


- x-axis is **margin**:  $m > 0 \iff$  correct classification



# Classification Losses

SVM/Hinge loss:  $\ell_{\text{Hinge}} = \max(1 - m, 0)$



Hinge is a **convex, upper bound** on 0–1 loss. Not differentiable at  $m = 1$ .  
We have a “margin error” when  $m < 1$ .

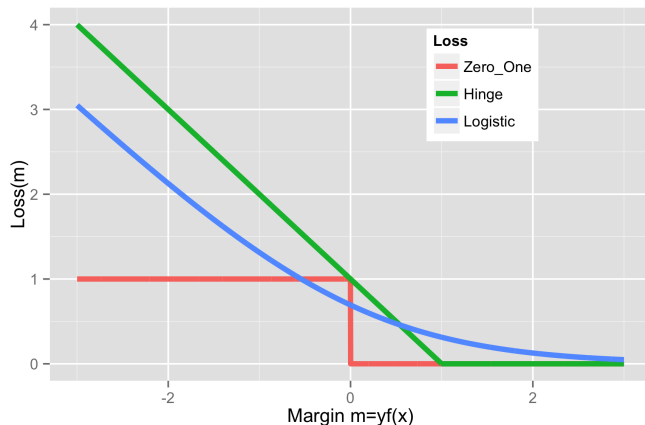
# (Soft Margin) Linear Support Vector Machine

- Hypothesis space:  $\mathcal{F} = \{f_w(x) = w^T x \mid w \in \mathbf{R}^d\}$ .
- Loss:  $\ell(m) = \max(1 - m, 0)$  [**Hinge loss** – sometimes called **SVM loss**]
- Regularization:  $\ell_2$

$$\min_{w \in \mathbf{R}^d} \frac{1}{n} \sum_{i=1}^n \max(1 - y_i f_w(x_i), 0) + \lambda \|w\|_2^2$$

# Classification Losses

Logistic/Log loss:  $\ell_{\text{Logistic}} = \log(1 + e^{-m})$



Logistic loss is differentiable. Logistic loss always wants more margin (loss never 0).

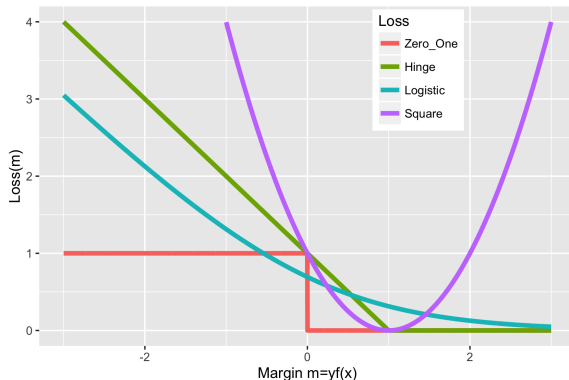
# What About Square Loss for Classification?

- Action space  $\mathcal{A} = \mathbf{R}$       Output space  $\mathcal{Y} = \{-1, 1\}$
- Loss  $\ell(f(x), y) = (f(x) - y)^2$ .
- Turns out, can write this in terms of margin  $m = f(x)y$ :

$$\ell(f(x), y) = (f(x) - y)^2 = (1 - f(x)y)^2 = (1 - m)^2$$

- Prove using fact that  $y^2 = 1$ , since  $y \in \{-1, 1\}$ .

# What About Square Loss for Classification?



Heavily penalizes outliers (e.g. mislabeled examples).

May have higher sample complexity (i.e. needs more data) than hinge & logistic<sup>1</sup>.

<sup>1</sup>Rosasco et al's "Are Loss Functions All the Same?" <http://web.mit.edu/lrosasco/www/publications/loss.pdf>