

Notes for Machine Learning – Brett Bernstein

Recitation 2

Intro Question

1. You have been given a data set (x_i, y_i) for $i = 1, \dots, n$ where $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, 1\}$. Assume $w \in \mathbb{R}^n$ and $a \in \mathbb{R}$.
 - (a) Suppose $y_i(w^T x_i + a) > 0$ for all i . Use a picture to explain what this means when $d = 2$.
 - (b) Fix $M > 0$. Suppose $y_i(w^T x_i + a) \geq M$ for all i . Use a picture to explain what this means when $d = 2$.

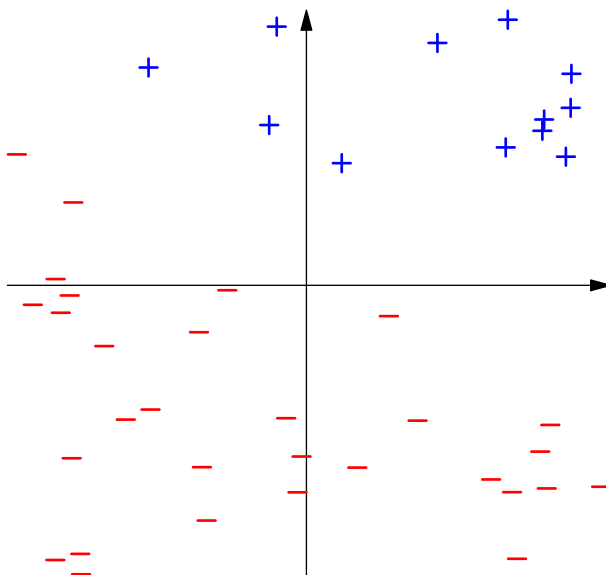


Figure 1: Data set with $x_i \in \mathbb{R}^2$ and $y_i \in \{+1, -1\}$

Support Vector Machines

Review of Geometry

If $v, w \in \mathbb{R}^d$ then the component of v in the direction w is given by $\frac{w^T v}{\|w\|_2}$. This can also be thought of as the signed length of v when orthogonally projected onto the line through the vector w .

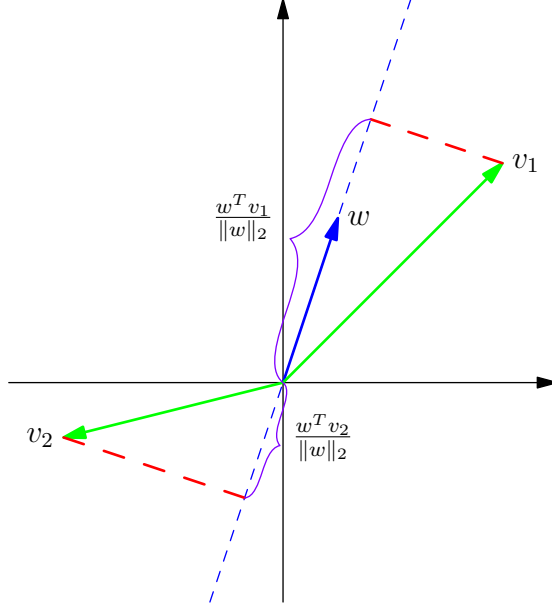


Figure 2: Component of v_1, v_2 in the direction w .

Assuming $w \neq 0$ we can use this to interpret the set

$$S = \{x \in \mathbb{R}^d \mid w^T x = b\}.$$

Noting that $w^T x = b \iff \frac{w^T x}{\|w\|_2} = \frac{b}{\|w\|_2}$ we see that S contains all vectors whose component in the direction w is $\frac{b}{\|w\|_2}$. Using linear algebra we can see this is the hyperplane orthogonal to the vector w that passes through the point $\frac{bw}{\|w\|_2^2}$. Note also that there are infinitely many pairs (w, b) that give the same hyperplane. If $c \neq 0$ then

$$\{x \in \mathbb{R}^d \mid w^T x = b\} \quad \text{and} \quad \{x \in \mathbb{R}^d \mid (cw)^T x = (cb)\}$$

result in the same hyperplanes.

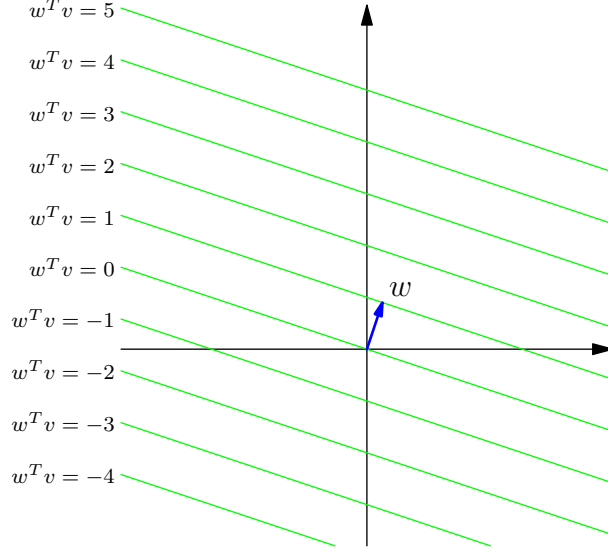


Figure 3: Level Surfaces of $f(v) = w^T v$ with $\|w\|_2 = 1$

Given a hyperplane $\{v \mid w^T v = b\}$, we can distinguish points $x \in \mathbb{R}^d$ depending on whether $w^T x - b$ is zero, positive, or negative, or in other words, whether x is on the hyperplane, on the side w is pointing at, or on the side $-w$ is pointing at.

Figure 4: Sides of the Hyperplane $w^T v = 15$

If we have a vector $x \in \mathbb{R}^d$ and a hyperplane $H = \{v \mid w^T v = b\}$ we can measure the distance from x to H by

$$d(x, H) = \left| \frac{w^T x - b}{\|w\|_2} \right|.$$

Without the absolute values we get the *signed distance*: a positive distance if $w^T x > b$ and a negative distance if $w^T x < b$.

Figure 5: Signed Distance from x_1, x_2 to Hyperplane $w^T v = 20$

Hard Margin SVM

Returning to the initial question, suppose we have the data set (x_i, y_i) for $i = 1, \dots, n$ where $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, 1\}$.

Definition 1 (Linearly Separable). We say (x_i, y_i) for $i = 1, \dots, n$ is *linearly separable* if there is a $w \in \mathbb{R}^d$ and $a \in \mathbb{R}$ such that $y_i(w^T x_i + a) > 0$ for all i . The set $\{v \in \mathbb{R}^d \mid w^T v + a = 0\}$ is called a *separating hyperplane*.

Let's examine what this definition says. If $y_i = +1$ then we require that $w^T x_i > -a$ and if $y_i = -1$ we require that $w^T x_i < -a$. Thus linearly separable means that we can separate all of the $+1$'s from the -1 's using the hyperplane $\{v \mid w^T v = -a\}$. For the rest of this section, we assume our data is linearly separable. [Linearly separable data] If we can find the w, a corresponding to a hyperplane that separates the data, we then have a decision function for classifying elements of \mathcal{X} : $f(x) = \text{sgn}(w^T x + a)$. Before we look for such a hyperplane, we must address another issue. If the data is linearly separable, then there are infinitely many choices of separating hyperplanes. [Show many separating planes.] We will choose the hyperplane that maximizes a quantity called the *margin*.

Definition 2 (Margin). Let H be a hyperplane that separates the data (x_i, y_i) for $i = 1, \dots, n$. The margin of this hyperplane is

$$\min_i d(x_i, H),$$

the distance from the hyperplane to the closest data point.

Fix $w \in \mathbb{R}^d$ and $a \in \mathbb{R}$ with $y_i(w^T x_i + a) > 0$ for all i . Then we saw earlier that

$$d(x_i, H) = \left| \frac{w^T x_i + a}{\|w\|_2} \right| = \frac{y_i(w^T x_i + a)}{\|w\|_2}.$$

This gives us the following optimization problem:

$$\text{maximize}_{w,a} \quad \min_i \frac{y_i(w^T x_i + a)}{\|w\|_2}.$$

We can rewrite this in a more standard form:

$$\begin{aligned} & \text{maximize}_{w,a,M} && M \\ & \text{subject to} && \frac{y_i(w^T x_i + a)}{\|w\|_2} \geq M \quad \text{for all } i. \end{aligned}$$

[Image of maximum margin hyperplane.] The expression $y_i(w^T x_i + a)/\|w\|_2$ allows us to choose any positive value for $\|w\|_2$ by changing a accordingly. Thus we can fix $\|w\|_2 = 1/M$ and obtain

$$\begin{aligned} & \text{maximize}_{w,a} && 1/\|w\|_2 \\ & \text{subject to} && y_i(w^T x_i + a) \geq 1 \quad \text{for all } i. \end{aligned}$$

To find the optimal w, a we can solve the corresponding minimization problem

$$\begin{aligned} & \text{minimize}_{w,a} && \|w\|_2^2 \\ & \text{subject to} && y_i(w^T x_i + a) \geq 1 \quad \text{for all } i. \end{aligned}$$

This is a quadratic program that can be solved quickly on fairly large datasets.

Soft Margin SVM