

## Week 1 Lab: Concept Check Exercises

### Multivariable Calculus Exercises

1. If  $f'(x; u) < 0$  show that  $f(x + hu) < f(x)$  for sufficiently small  $h > 0$ .
2. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable, and assume that  $\nabla f(x) \neq 0$ . Prove

$$\arg \max_{\|u\|_2=1} f'(x; u) = \frac{\nabla f(x)}{\|\nabla f(x)\|_2} \quad \text{and} \quad \arg \min_{\|u\|_2=1} f'(x; u) = -\frac{\nabla f(x)}{\|\nabla f(x)\|_2}.$$

3. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given by  $f(x, y) = x^2 + 4xy + 3y^2$ . Compute the gradient  $\nabla f(x, y)$ .
4. Compute the gradient of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  where  $f(x) = x^T A x$  and  $A \in \mathbb{R}^{n \times n}$  is any matrix.
5. Compute the gradient of the quadratic function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  given by

$$f(x) = b + c^T x + x^T A x,$$

where  $b \in \mathbb{R}$ ,  $c \in \mathbb{R}^n$  and  $A \in \mathbb{R}^{n \times n}$ .

6. Fix  $s \in \mathbb{R}^n$  and consider  $f(x) = (x - s)^T A (x - s)$  where  $A \in \mathbb{R}^{n \times n}$ . Compute the gradient of  $f$ .
7. Consider the ridge regression objective function

$$f(w) = \|Aw - y\|_2^2 + \lambda \|w\|_2^2,$$

where  $w \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{m \times n}$ ,  $y \in \mathbb{R}^m$ , and  $\lambda \in \mathbb{R}_{\geq 0}$ .

- (a) Compute the gradient of  $f$ .
- (b) Express  $f$  in the form  $f(w) = \|Bw - z\|_2^2$  for some choice of  $B, z$ .
- (c) Using either of the parts above, compute

$$\arg \min_{w \in \mathbb{R}^n} f(w).$$

8. Compute the gradient of

$$f(\theta) = \lambda \|\theta\|_2^2 + \sum_{i=1}^n \log(1 + \exp(-y_i \theta^T x_i)),$$

where  $y_i \in \mathbb{R}$  and  $\theta \in \mathbb{R}^m$  and  $x_i \in \mathbb{R}^m$  for  $i = 1, \dots, n$ .

## Linear Algebra Exercises

1. When performing linear regression we obtain the *normal equations*  $A^T A x = A^T y$  where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $y \in \mathbb{R}^m$ .
  - (a) If  $\mathbf{rank}(A) = n$  then solve the normal equations for  $x$ .
  - (b) What if  $\mathbf{rank}(A) \neq n$ ?
2. Prove that  $A^T A + \lambda \mathbf{I}_{n \times n}$  is invertible if  $\lambda > 0$  and  $A \in \mathbb{R}^{n \times n}$ .
3. Describe the following set geometrically:

$$\left\{ v \in \mathbb{R}^2 \mid v^T \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} v = 4 \right\}.$$