

Bayesian Networks

David Rosenberg

New York University

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Introduction

Probabilistic Reasoning

- **Represent** system of interest by a set of random variables

$$(X_1, \dots, X_d).$$

- Suppose by research or ML, we get a joint probability distribution

$$p(x_1, \dots, x_d).$$

- We'd like to be able to do “**inference**” on this model – essentially, answer queries:
 - ① What is the most likely of value X_1 ?
 - ② What is the most likely of value X_1 , given we've observed $X_2 = 1$?
 - ③ Distribution of (X_1, X_2) given observation of $(X_3 = x_3, \dots, X_d = x_d)$?

Example: Medical Diagnosis

- Variables for each **symptom**
 - fever, cough, fast breathing, shaking, nausea, vomiting
- Variables for each **disease**
 - pneumonia, flu, common cold, bronchitis, tuberculosis
- Diagnosis is performed by **inference** in the model:

$$p(\text{pneumonia} = 1 \mid \text{cough} = 1, \text{fever} = 1, \text{vomiting} = 0)$$

- The QMR-DT (Quick Medical Reference - Decision Theoretic) has
 - 600 diseases
 - 4000 symptoms

Discrete Probability Distribution Review

Some Notation

- This lecture we'll only be considering **discrete** random variables.
- Capital letters X_1, \dots, X_d, Y , etc. denote **random variables**.
- Lower case letters x_1, \dots, x_n, y denote the values taken.
- Probability that $X_1 = x_1$ and $X_2 = x_2$ will be denoted

$$\mathbb{P}(X_1 = x_1, X_2 = x_2).$$

- We'll generally write things in terms of the probability mass function:

$$p(x_1, x_2, \dots, x_d) := \mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_d = x_d)$$

Representing Probability Distributions

- Let's consider the case of discrete random variables.
- Conceptually, everything can be represented with probability tables.
- Variables
 - Temperature $T \in \{\text{hot}, \text{cold}\}$
 - Weather $W \in \{\text{sun}, \text{rain}\}$

t	$p(t)$
hot	0.5
cold	0.5

w	$p(w)$
sun	0.6
rain	0.4

- These are the **marginal** probability distributions.
- To do reasoning, we need the **joint probability distribution**.

Joint Probability Distributions

- A joint probability distribution for T and W is given by

t	w	$p(t, w)$
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- With the joint we can answer question such as
 - $p(\text{sun} \mid \text{hot})=?$
 - $p(\text{rain} \mid \text{cold})=?$

Representing Joint Distributions

- Consider random variables $X_1, \dots, X_d \in \{0, 1\}$.
- How many parameters do we need to represent the joint distribution?
- Joint probability table has 2^d rows.
- For QMR-DT, that's $2^{4600} > 10^{1000}$ rows.
- That's not going to happen.
- Only $\sim 10^{80}$ atoms in the universe.
- Having exponentially many parameters is a problem for
 - storage
 - computation (inference is summing over exponentially many rows)
 - statistical estimation / learning

How to Restrict the Complexity?

- Restrict the space of probability distributions
- We will make various **independence** assumptions.
- Extreme assumption: X_1, \dots, X_d are **mutually independent**.

Definition

Discrete random variables X_1, \dots, X_d are **mutually independent** if their joint probability mass function (PMF) factorizes as

$$p(x_1, x_2, \dots, x_d) = p(x_1)p(x_2) \cdots p(x_d).$$

- Note: We usually just write **independent** for “mutually independent”.
- How many parameters to represent the joint distribution, assuming independence?

Assume Full Independence

- How many parameters to represent the joint distribution?
- Say $p(X_i = 1) = \theta_i$, for $i = 1, \dots, d$.
- **Clever representation:** Since $x_i \in \{0, 1\}$, we can write

$$\mathbb{P}(X_i = x_i) = \theta_i^{x_i} (1 - \theta_i)^{1-x_i}.$$

- Then by independence,

$$p(x_1, \dots, x_d) = \prod_{i=1}^d \theta_i^{x_i} (1 - \theta_i)^{1-x_i}$$

- How many parameters?
- d parameters needed to represent the joint.

Conditional Interpretation of Independence

- Suppose X and Y are independent, then

$$p(x | y) = p(x).$$

- Proof:

$$\begin{aligned} p(x | y) &= \frac{p(x, y)}{p(y)} \\ &= \frac{p(x)p(y)}{p(y)} = p(x). \end{aligned}$$

- With full independence, we have no relationships among variables.
- Information about one variable says nothing about any other variable.
 - Would mean diseases don't have symptoms.

Conditional Independence

- Consider 3 events:
 - 1 $W = \{\text{The grass is wet}\}$
 - 2 $S = \{\text{The road is slippery}\}$
 - 3 $R = \{\text{It's raining}\}$
- These events are certainly **not** independent.
 - Raining (R) \implies Grass is wet AND The road is slippery ($W \cap S$)
 - Grass is wet (W) \implies More likely that the road is slippery (S)
- Suppose we know that **it's raining**.
 - Then, we learn that **the grass is wet**.
 - Does this tell us anything new about whether **the road is slippery**?
- Once we know R , then W and S become independent.
- This is called **conditional independence**, and we'll denote it as

$$W \perp S \mid R.$$

Conditional Independence

Definition

We say W and S are **conditionally independent** given R , denoted

$$W \perp S \mid R,$$

if the conditional joint factorizes as

$$p(w, s \mid r) = p(w \mid r)p(s \mid r).$$

Also holds when W , S , and R represent **sets of random variables**.

- Can have conditional independence without independence.
- Can have independence without conditional independence.

Example: Rainy, Slippery, Wet

- Consider 3 events:
 - 1 $W = \{\text{The grass is wet}\}$
 - 2 $S = \{\text{The road is slippery}\}$
 - 3 $R = \{\text{It's raining}\}$
- Represent joint distribution as

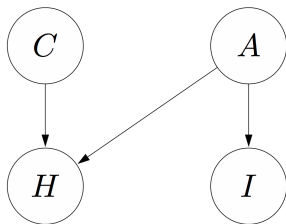
$$\begin{aligned}
 p(w, s, r) &= p(w, s | r)p(r) && \text{(no assumptions so far)} \\
 &= p(w | r)p(s | r)p(r) && \text{(assuming } W \perp S | R)
 \end{aligned}$$

- How many parameters to specify the joint?
 - $p(w | r)$ requires two parameters: one for $r = 1$ and one for $r = 0$.
 - $p(s | r)$ requires two.
 - $p(r)$ requires one parameter,
- Full joint: 7 parameters. Conditional independence: 5 parameters.
Full independence: 3 parameters.

Bayesian Networks

Bayesian Networks: Introduction

- Bayesian Networks are
 - used to specify joint probability distributions that
 - have a particular factorization.



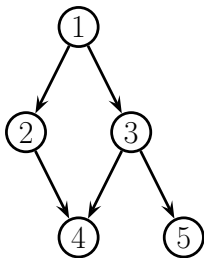
$$p(c, h, a, i) = p(c)p(a) \times p(h | c, a)p(i | a)$$

- With practice, one can read conditional independence relationships directly from the graph.

Directed Graphs

A **directed graph** is a pair $G = (\mathcal{V}, \mathcal{E})$, where

- $\mathcal{V} = \{1, \dots, d\}$ is a set of **nodes** and
- $\mathcal{E} = \{(s, t) \mid s, t \in \mathcal{V}\}$ is a set of **directed edges**.



$$\text{Parents}(5) = \{3\}$$

$$\text{Parents}(4) = \{2, 3\}$$

$$\text{Children}(3) = \{4, 5\}$$

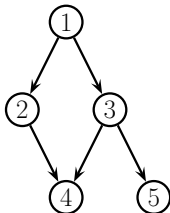
$$\text{Descendants}(1) = \{2, 3, 4, 5\}$$

$$\text{NonDescendants}(3) = \{1, 2\}$$

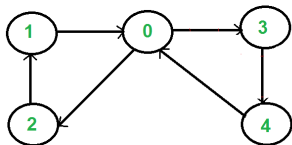
Directed Acyclic Graphs (DAGs)

A **DAG** is a directed graph with **no directed cycles**.

DAG



Not a DAG



Every DAG has a **topological ordering**, in which parents have lower numbers than their children.

<http://www.geeksforgeeks.org/wp-content/uploads/SCC1.png> and KPM Figure 10.2(a).

Bayesian Networks

Definition

A **Bayesian network** is a

- DAG $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, d\}$, and
- a corresponding set of random variables $X = \{X_1, \dots, X_d\}$

where

- the joint probability distribution over X factorizes as

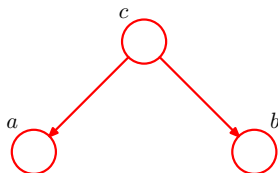
$$p(x_1, \dots, x_d) = \prod_{i=1}^d p(x_i \mid x_{\text{Parents}(i)}).$$

Bayesian networks are also known as

- **directed graphical models**, and
- **belief networks**.

Conditional Independencies

Bayesian Networks: “A Common Cause”



$$p(a, b, c) = p(c)p(a | c)p(b | c)$$

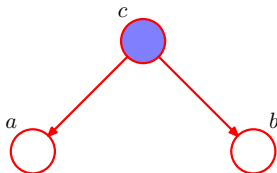
Are a and b independent? (c =Rain, a =Slippery, b =Wet?)

$$p(a, b) = \sum_c p(c)p(a | c)p(b | c),$$

which in general will not be equal to $p(a)p(b)$.

From Bishop's *Pattern recognition and machine learning*, Figure 8.15.

Bayesian Networks: “A Common Cause”



$$p(a, b, c) = p(c)p(a | c)p(b | c)$$

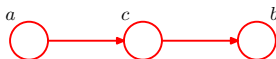
Are a and b independent, conditioned on observing c ? (c =Rain, a =Slippery, b =Wet?)

$$\begin{aligned} p(a, b | c) &= p(a, b, c) / p(c) \\ &= p(a | c)p(b | c) \end{aligned}$$

So $a \perp b | c$.

From Bishop's *Pattern recognition and machine learning*, Figure 8.16.

Bayesian Networks: “An Indirect Effect”



$$p(a, b, c) = p(a)p(c | a)p(b | c)$$

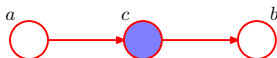
Are a and b independent? (Note: This is a **Markov chain**)
(e.g. a =raining, c =wet ground, b =mud on shoes)

$$\begin{aligned} p(a, b) &= \sum_c p(a, b, c) \\ &= p(a) \sum_c p(c | a)p(b | c) \end{aligned}$$

So doesn't factorize, thus not independent, in general.

From Bishop's *Pattern recognition and machine learning*, Figure 8.17.

Bayesian Networks: “An Indirect Effect”



$$p(a, b, c) = p(a)p(c | a)p(b | c)$$

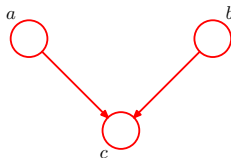
Are a and b independent after observing c ?
 (e.g. a =raining, c =wet ground, b =mud on shoes)

$$\begin{aligned}
 p(a, b | c) &= p(a, b, c) / p(c) \\
 &= p(a)p(c | a)p(b | c) / p(c) \\
 &= p(a | c)p(b | c)
 \end{aligned}$$

So $a \perp b | c$.

From Bishop's *Pattern recognition and machine learning*, Figure 8.18.

Bayesian Networks: “A Common Effect”



$$p(a, b, c) = p(a)p(b)p(c | a, b)$$

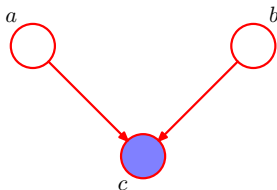
Are a and b independent? (a =course difficulty, b =knowledge, c = grade)

$$\begin{aligned}
 p(a, b) &= \sum_c p(a)p(b)p(c | a, b) \\
 &= p(a)p(b) \sum_c p(c | a, b) \\
 &= p(a)p(b)
 \end{aligned}$$

So $a \perp b$.

From Bishop's *Pattern recognition and machine learning*, Figure 8.19.

Bayesian Networks: “A Common Effect” or “V-Structure”



$$p(a, b, c) = p(a)p(b)p(c | a, b)$$

Are a and b independent, given observation of c ? (a =course difficulty, b =knowledge, c = grade)

$$p(a, b | c) = p(a)p(b)p(c | a, b)/p(c)$$

which does not factorize into $p(a | c)p(b | c)$, in general.

From Bishop's *Pattern recognition and machine learning*, Figure 8.20.

Conditional Independence from Graph Structure

- In general, given 3 sets of nodes A , B , and C
- How can we determine whether

$$A \perp B \mid C?$$

- There is a purely graph-theoretic notion of “**d-separation**” that is equivalent to conditional independence.
- Suppose we have observed C and we want to do inference on A .
- We could ignore any evidence collected about B , where $A \perp B \mid C$.
- See KPM Section 10.5.1 for details.

Markov Blanket

- Suppose we have a very large Bayesian network.
- We're interested in a single variable A , which we cannot observe.
- To get maximal information about A , do we have to observe all other variables?
- No! We only need to observe the **Markov blanket** of A :

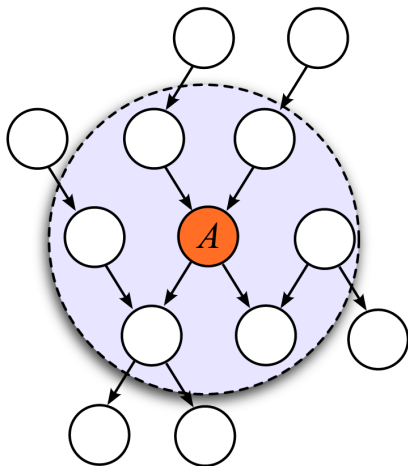
$$p(A \mid \text{all other nodes}) = p(A \mid \text{MarkovBlanket}(A)).$$

- In a Bayesian network, the Markov blanket of A consists of
 - the parents of A
 - the children of A
 - the “co-parents” of A , i.e. the parents of the children of A

(See KPM Sec. 10.5.3 for details.)

Markov Blanket

Markov Blanket of A in a Bayesian Network:



From http://en.wikipedia.org/wiki/Markov_blanket: "Diagram of a Markov blanket" by Laughsinthestocks - Licensed under CC0 via Wikimedia Commons

When to use Bayesian Networks?

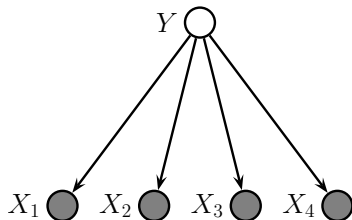
Bayesian Networks

- Bayesian Networks are great when
 - you know something about the relationships between your variables, or
 - you will routinely need to make inferences with incomplete data.
- Challenges:
 - The naive approach to inference doesn't work beyond small scale.
 - Need more sophisticated algorithm:
 - exact inference
 - approximate inference

Naive Bayes

Binary Naive Bayes: A Generative Model for Classification

- $\mathcal{X} = \left\{ (X_1, X_2, X_3, X_4) \in \{0, 1\}^4 \right\}$ $\mathcal{Y} = \{0, 1\}$ be a class label.
- Consider the Bayesian network depicted below:



- BN structure implies joint distribution factors as:

$$p(x_1, x_2, x_3, x_4, y) = p(y)p(x_1 | y)p(x_2 | y)p(x_3 | y)p(x_4 | y)$$

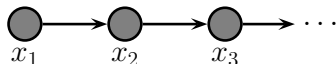
- Features X_1, \dots, X_4 are independent given the class label Y .

KPM Figure 10.2(a).

Markov Models

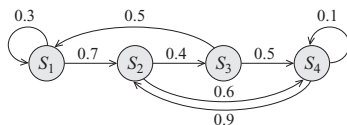
Markov Chain Model

- A Markov chain model has structure:



$$p(x_1, x_2, x_3, \dots) = p(x_1)p(x_2 | x_1)p(x_3 | x_2)\dots$$

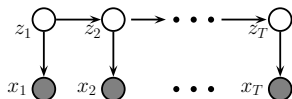
- Conditional distributions $p(x_i | x_{i-1})$ is called the **transition model**.
- When conditional distribution independent of i , called **time-homogeneous**.
- 4-state transition model for $X_i \in \{S_1, S_2, S_3, S_4\}$:



KPM Figure 10.3(a) and Koller and Friedman's *Probabilistic Graphical Models* Figure 6.04.

Hidden Markov Model

- A hidden Markov model (HMM) has structure:



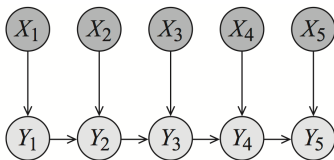
$$p(x_1, z_1, x_2, z_2, x_3, z_3, \dots) = p(z_1) \underbrace{\prod_{t=2}^T p(z_t | z_{t-1})}_{\text{Transition Model}} \underbrace{\prod_{t=1}^T p(x_t | z_t)}_{\text{Observation Model}}$$

- At deployment time, we typically only observe X_1, \dots, X_T .
- Want to infer Z_1, \dots, Z_T .
- e.g. Want to most likely sequence (Z_1, \dots, Z_T) . (Use **Viterbi algorithm**.)

KPM Figure 10.4

Maximum Entropy Markov Model

- A maximum entropy Markov model (MEMM) has structure:



$$p(y_1 \dots, y_5 | x) = \underbrace{p(y_0) \prod_{t=1}^5 p(y_t | y_{t-1}, x)}_{\text{Conditional Transition Model}}$$

- At deployment time, we only observe X_1, \dots, X_T .
- This is a **conditional model**. (And not a generative model).