Basic Hilbert Space Theory

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Inner Product Spaces (or "Pre-Hilbert" Spaces)

Definitions

A **[real] inner product space** is $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is

- a [real] vector space $\mathcal{V}$ and
- an inner product $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$.
- Inner product induces a **norm**:

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Example

$\mathbf{R}^d$ with standard Euclidean inner product is an inner product space:

$$\langle x, y \rangle := x^T y \qquad \forall x, y \in \mathbf{R}^d.$$

What norms can we get from an inner product?

Theorem

Parallelogram LawA norm $\|v\|$ can be generated by an inner product on $\mathcal{V}$ if and only if $\forall x, y \in \mathcal{V}$

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2.$$

If it can be, then the [unique] inner product is given by the polarization identity

$$\langle x, y \rangle = \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2}.$$

Example

ℓ_1 norm on $\mathbf{R}^d$ is NOT generated by an inner product. [Exercise]

Is ℓ_2 norm on $\mathbf{R}^d$ generated by an inner product?

Pythagorean Theroem

Definition

Two vectors are **orthogonal** if $\langle x, y \rangle = 0$. We denote this by $x \perp y$.

Definition

Vector x is orthogonal to a set S ($x \perp S$) if $x \perp s$ for all $x \in S$.

Theorem

If $x \perp y$, *then* $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Proof.

We have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle \\ &= \|x\|^2 + \|y\|^2. \end{aligned}$$

Projection onto a Plane (Rough Definition)

- $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ an inner product space.
- M is a subspace of $\mathcal{V}$.
- $x \in \mathcal{V}$
- Then $m_0 \in M$ is the **projection of** x **onto** M
  - if m_0 is the closest point to x in M.
- In math: For all $m \in M$,

$$\|x - m_0\| \leqslant \|x - m\|.$$

- To Do: projections exist and characterized by $x - m_0 \perp M$.

Hilbert Space

- Projections exist for all finite-dimensional inner product spaces.
- We want to allow infinite-dimensional spaces.
- Need an extra condition called **completeness**.
- (Recall: A space is **complete** if all Cauchy sequences in the space converge.)

Definition

A **Hilbert space** is a complete inner product space.

Example

Any finite dimensional inner product space is a Hilbert space.

Hilbert Space

Theorem (Classical Projection Theorem)

- *$\mathcal{H}$ a Hilbert space*
- *M a closed subspace of $\mathcal{H}$*
- *For any $x \in \mathcal{H}$, there **exists a unique** $m_0 \in M$ for which*

$$\|x - m_0\| \leqslant \|x - m\| \; \forall m \in M.$$

- *This m_0 is called the **[orthogonal] projection of** x **onto** M.*
- *Furthermore, $m_0 \in M$ is the projection of x onto M iff*

$$x - m_0 \perp M.$$

Projection Reduces Norm

Theorem

Let M be a closed subspace of $\mathcal{H}$. For any $x \in \mathcal{H}$, let $m_0 = Proj_M x$ be the projection of x onto M. Then

$$\|m_0\| \leqslant \|x\|.$$

Proof.

$$\begin{aligned}
\|x\|^2 &= \|m_0 + (x - m_0)\|^2 \text{ (note: } x - m_0 \perp m_0) \\
&= \|m_0\|^2 + \|x - m_0\|^2 \text{ by Pythagorean theorem} \\
\|m_0\|^2 &= \|x\|^2 - \|x - m_0\|^2 \\
\|m_0\|^2 &\leqslant \|x\|^2
\end{aligned}$$

□

Orthogonal Complements

Definition

Consider $S \subset \mathcal{V}$, for an inner product space $\mathcal{V}$. The set

$$S^{\perp} = \{v \in \mathcal{V} \mid v \perp S\}$$

is called the **orthogonal complement** of S [in $\mathcal{V}$].

Theorem

$S^{\perp}$ is a closed subspace of $\mathcal{V}$.

Theorem

If M is a closed subspace of a Hilbert space $\mathcal{H}$, then every $x \in \mathcal{H}$ has a unique representation of the form

$$x = m + m^{\perp},$$