Directional Derivatives and Optimality

David Rosenberg

New York University

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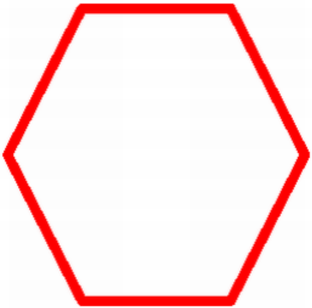
Convex Sets and Functions

Convex Sets

Definition

A set C is **convex** if for any $x_1, x_2 \in C$ and any θ with $0 \leqslant \theta \leqslant 1$ we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$



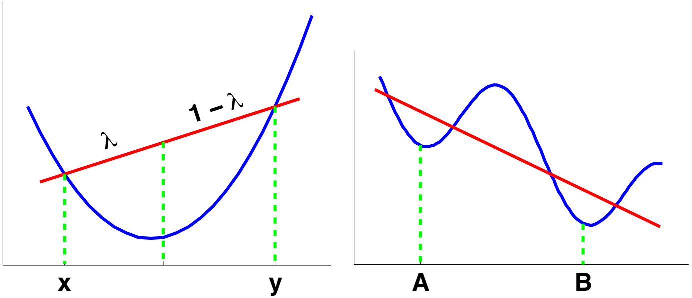
KPM Fig. 7.4

Convex and Concave Functions

Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is **convex** if $\mathbf{dom}\, f$ is a convex set and if for all $x, y \in \mathbf{dom}\, f$, and $0 \leqslant \theta \leqslant 1$, we have

$$f(\theta x + (1-\theta)y) \leqslant \theta f(x) + (1-\theta) f(y).$$



KPM Fig. 7.5

Directional Derivatives and Minima

Directional Derivatives

Definition

A **[one-sided] directional derivative** of f at x in the direction v is

$$f'(x;v) = \lim_{h \downarrow 0} \frac{f(x+hv) - f(x)}{h},$$

and it can be $\pm\infty$ (e.g. for discontinuous functions).

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- If f is convex and finite near x, then $f'(x;v)$ exists.
- f is **differentiable** at x iff for some $g(= \nabla f(x))$ and all v,

$$f'(x;v) = g^T v.$$

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Theorem

If f is convex and finite near x, then either

- *x minimizes f, or*
- *there is a descent direction for f at x.*

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- Let's see what that means in terms of our directional derivative characterization.

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$$J_\lambda'(0; v) = \lim_{h \downarrow 0} \frac{J(hv) - J(0)}{h}.$$
- For $w = 0$ to be a minimizer, need to have $J_\lambda'(0; v) \geqslant 0$ for every direction v.
- Can find $\lambda_{\max}$ by finding conditions on λ for this to be the case.