Statistical Learning Theory: Recap and Example

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October 29, 2016

Statistical Learning Theory Framework

The Spaces

- $\mathcal{X}$: input space
- $\mathcal{Y}$: output space
- $\mathcal{A}$: action space

Decision Function

A **decision function** produces an action $a \in \mathcal{A}$ for any input $x \in \mathcal{X}$:

$$\begin{aligned} f: \quad \mathcal{X} &\to \mathcal{A} \\ x &\mapsto f(x) \end{aligned}$$

Loss Function

A **loss function** evaluates an action in the context of the output y.

$$\begin{aligned} \ell: \quad \mathcal{A} \times \mathcal{Y} &\to \mathbf{R}^{\geqslant 0} \\ (a, y) &\mapsto \ell(a, y) \end{aligned}$$

The Gold Standard: Bayes Decision Function

Definition

The **expected loss** or **"risk"** of a decision function $f : \mathcal{X} \to \mathcal{A}$ is

$$R(f) = \mathbb{E}\ell(f(X), Y),$$

where the expectation taken is over $(X, Y) \sim P_{\mathcal{X}\times\mathcal{Y}}$.

Definition

A **Bayes decision function** $f^* : \mathcal{X} \to \mathcal{A}$ is a function that achieves the *minimal risk* among all possible functions:

$$R(f^*) = \inf_f \mathbb{E}\ell(f(X), Y).$$

- But Risk function cannot be computed because we don't know $P_{\mathcal{X}\times\mathcal{Y}}$!

Empirical Risk Minimization

- Let $\mathcal{D}_n = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ be drawn i.i.d. from $\mathcal{P}_{\mathcal{X}\times\mathcal{Y}}$.

Definition

The **empirical risk** of $f : \mathcal{X} \to \mathcal{A}$ with respect to $\mathcal{D}_n$ is

$$\hat{R}_n(f) = \frac{1}{n}\sum_{i=1}^{n} \ell(f(X_i), Y_i).$$

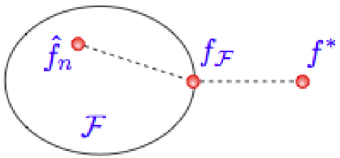
- Minimizing empirical risk is a good idea, but overfits!

Constrained Empirical Risk Minimization

- Hypothesis space $\mathcal{F} \subset \mathcal{A}^{\mathcal{X}}$, a set of functions mapping $\mathcal{X} \to \mathcal{A}$
- **Empirical risk minimizer** (ERM) **in** $\mathcal{F}$ is $\hat{f} \in \mathcal{F}$, where

$$\hat{R}(\hat{f}) = \inf_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \ell(f(X_i), Y_i).$$

Error Decomposition



$$f^* = \arg\min_f \mathbb{E}\ell(f(X), Y)$$

$$f_{\mathcal{F}} = \arg\min_{f \in \mathcal{F}} \mathbb{E}\ell(f(X), Y))$$

$$\hat{f}_n = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \ell(f(x_i), y_i)$$

- **Approximation Error** (of $\mathcal{F}$) $= R(f_{\mathcal{F}}) - R(f^*)$
- **Estimation error** (of $\hat{f}_n$ in $\mathcal{F}$) $= R(\hat{f}_n) - R(f_{\mathcal{F}})$

Optimization Error

- There's still the *algorithmic* problem of *finding* ERM $\hat{f}_n \in \mathcal{F}$.
- **Optimization error:** If $\tilde{f}_n$ is the function our optimization method returns, and $\hat{f}_n$ is the empirical risk minimizer, then

$$\text{Optimization Error } = R(\tilde{f}_n) - R(\hat{f}_n).$$

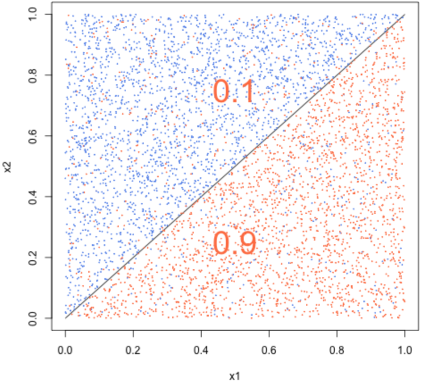
Error Decomposition

Definition

The **excess risk** of f is the amount by which the risk of f exceeds the Bayes risk.

$$\begin{aligned}\textbf{Excess Risk}(\tilde{f}_n) &= R(\tilde{f}_n) - R(f^*) \\ &= \underbrace{R(\tilde{f}_n) - R(\hat{f}_n)}_{\text{optimization error}} + \underbrace{R(\hat{f}_n) - R(f_{\mathcal{F}}^*)}_{\text{estimation error}} + \underbrace{R(f_{\mathcal{F}}^*) - R(f^*)}_{\text{approximation error}}\end{aligned}$$

Excess Risk Decomposition, Nested Space, and Trees

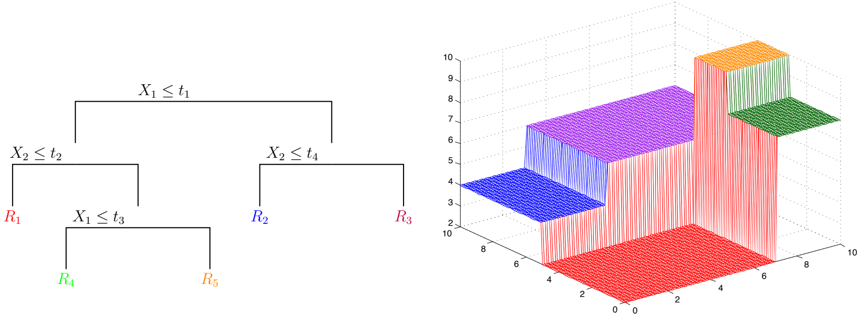


$$\mathcal{Y} = \{\text{blue}, \text{orange}\}$$
$$P_{\mathcal{X}} = \text{Uniform}([0,1]^2)$$
$$\mathbb{P}(\text{orange} \mid x_1 > x_2) = .9$$
$$\mathbb{P}(\text{orange} \mid x_1 < x_2) = .1$$

Bayes Error Rate $= 0.1$

Regression Trees

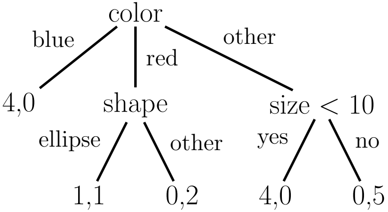
- Partition space on one variable at a time



KPM Figure 16.1

Classification Trees

- Classification Tree
- $4,0$ in the leaf node means 4 successes, 0 failures



- Depth of the tree is one measure of complexity

KPM Figure 16.2

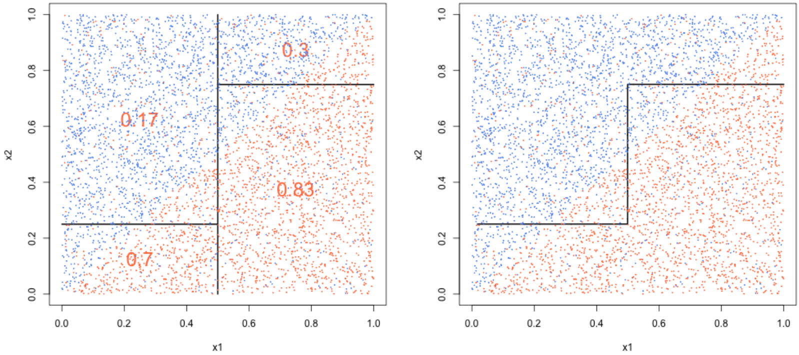
Hypothesis Space: Decision Tree

- $\mathcal{F} = \left\{ \text{all decision tree classifiers on } [0,1]^2 \right\}$
- $\mathcal{F}_d = \left\{ \text{all decision tree classifiers on } [0,1]^2 \text{ with DEPTH} \leqslant d \right\}$
- We'll consider

$$\mathcal{F}_2 \subset \mathcal{F}_3 \subset \mathcal{F}_4 \cdots \subset \mathcal{F}_{15}$$

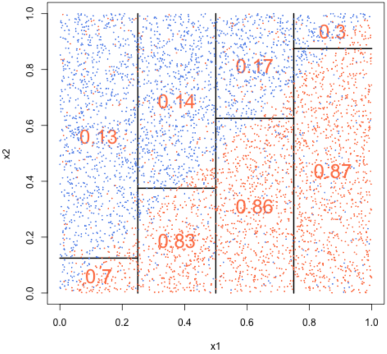
- Bayes error rate $= 0.1$

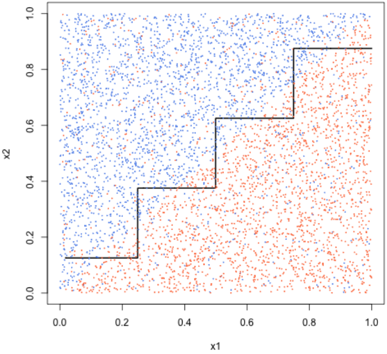
Theoretical Best in $\mathcal{F}_2$



- Risk Minimizer (e.g. assuming **infinite training data**)
- Risk = P(error) = 0.2
- Approximation Error = 0.2 - 0.1 = 0.1

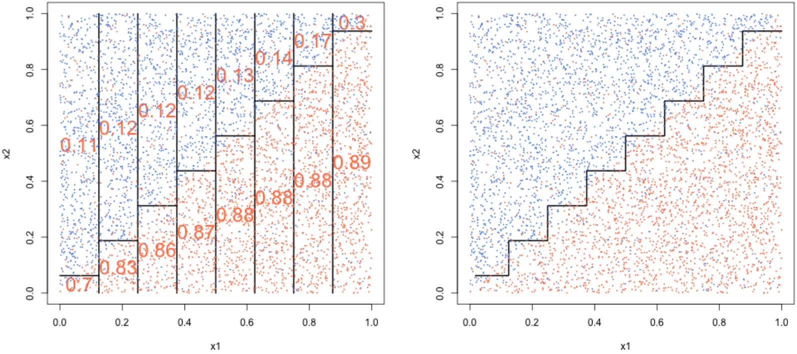
Theoretical Best in $\mathcal{F}_3$





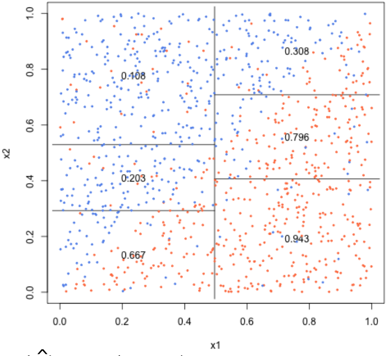
- Risk Minimizer (e.g. assuming **infinite training data**)
- Risk = P(error) = 0.15
- Approximation Error = 0.15 - 0.1 = 0.05

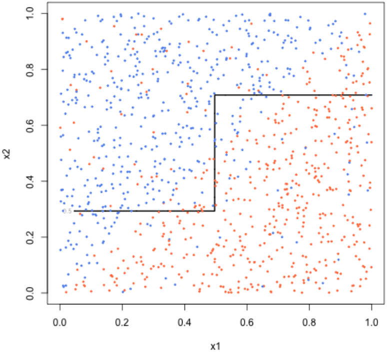
Theoretical Best in $\mathcal{F}_4$



- Risk Minimizer (e.g. assuming **infinite training data**)
- Risk = P(error) = 0.125
- Approximation Error = 0.125 - 0.1 = 0.025

Decision Tree in $\mathcal{F}_3$ Estimated From Sample ($n = 1024$)



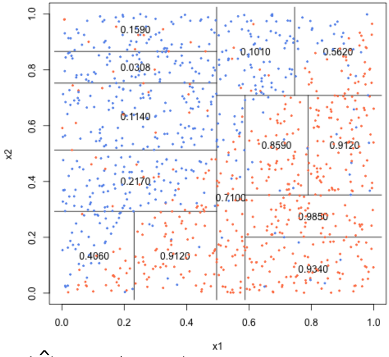


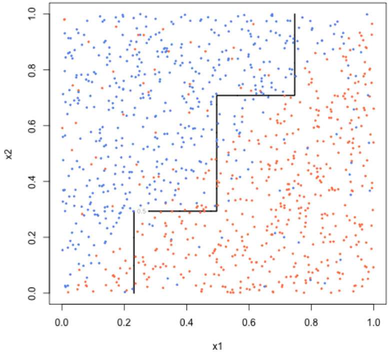
$$R(\hat{f}) = \mathbb{P}(\text{error}) = 0.176 \pm .004$$

$$\text{Estimation Error+Optimization Error} = \underbrace{0.176 \pm .004}_{R(\hat{f})} - \underbrace{0.150}_{\min_{f \in \mathcal{F}_3} R(f)}$$

$$= .026 \pm .004$$

Decision Tree in $\mathcal{F}_4$ Estimated From Sample ($n = 1024$)



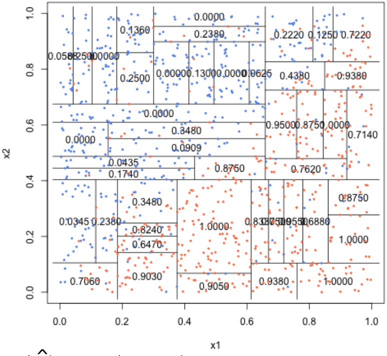


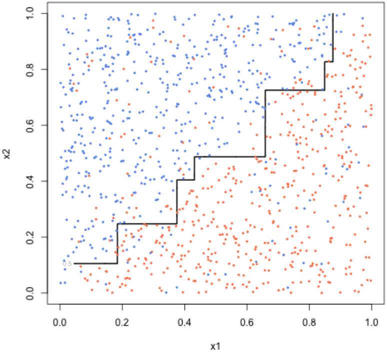
$$R(\hat{f}) = \mathbb{P}(\text{error}) = 0.144 \pm .005$$

$$\text{Estimation Error+Optimization Error} = \underbrace{0.144 \pm .005}_{R(\hat{f})} - \underbrace{0.125}_{\min_{f \in \mathcal{F}_4} R(f)}$$

$$= .019 \pm .005$$

Decision Tree in $\mathcal{F}_6$ Estimated From Sample ($n = 1024$)



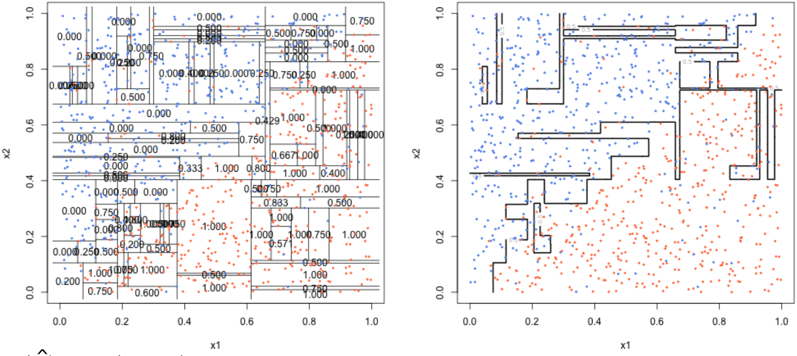


$$R(\hat{f}) = \mathbb{P}(\text{error}) = 0.148 \pm .007$$

$$\text{Estimation Error+Optimization Error} = \underbrace{0.148 \pm .007}_{R(\hat{f})} - \underbrace{0.106}_{\min_{f \in \mathcal{F}_6} R(f)}$$

$$= .042 \pm .008$$

Decision Tree in $\mathcal{F}_8$ Estimated From Sample ($n = 1024$)

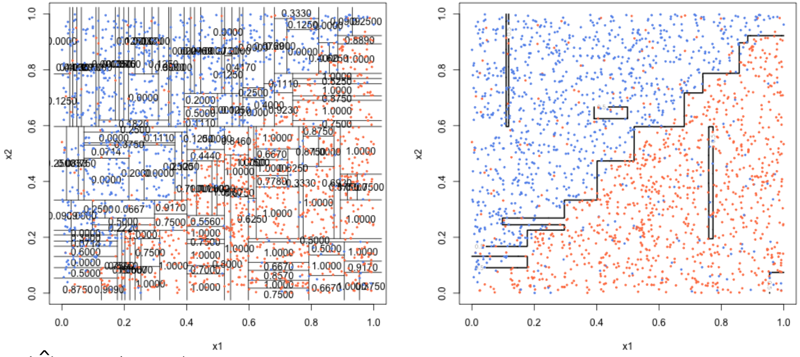


$$R(\hat{f}) = \mathbb{P}(\text{error}) = 0.162 \pm .009$$

$$\text{Estimation Error+Optimization Error} = \underbrace{0.162 \pm .009}_{R(\hat{f})} - \underbrace{0.102}_{\min_{f \in \mathcal{F}_8} R(f)}$$

$$= .061 \pm .009$$

Decision Tree in $\mathcal{F}_8$ Estimated From Sample ($n = 2048$)

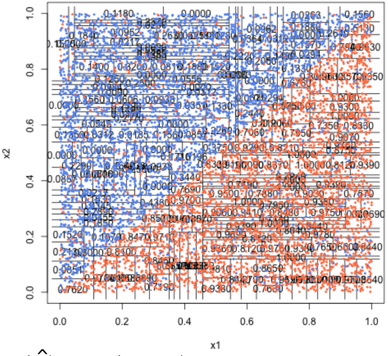


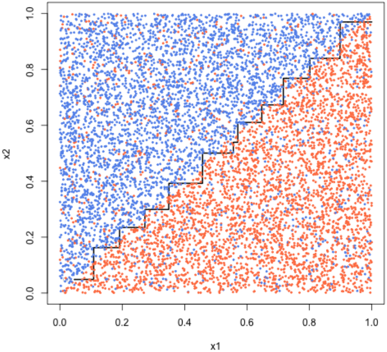
$$R(\hat{f}) = \mathbb{P}(\text{error}) = 0.146 \pm .006$$

$$\text{Estimation Error+Optimization Error} = \underbrace{0.146 \pm .006}_{R(\hat{f})} - \underbrace{0.102}_{\min_{f \in \mathcal{F}_3} R(f)}$$

$$= .045 \pm .006$$

Decision Tree in $\mathcal{F}_8$ Estimated From Sample ($n = 8192$)



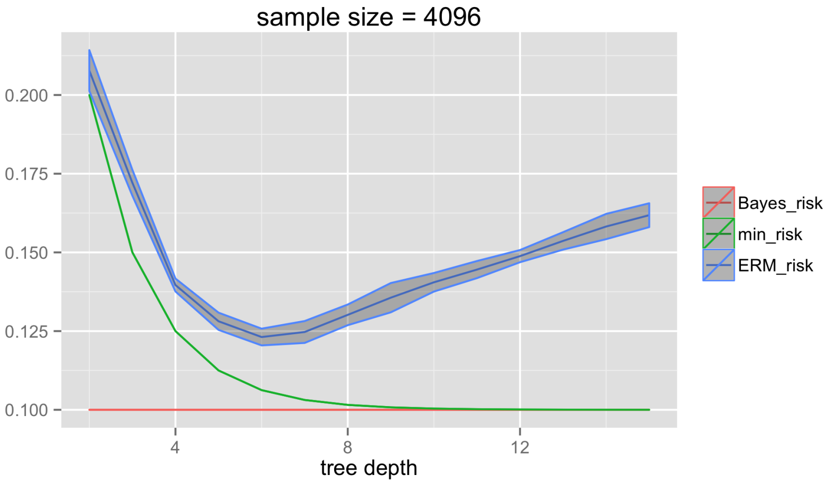


$R(\hat{f}) = \mathbb{P}(\text{error}) = 0.121 \pm .002$

$$\text{Estimation Error+Optimization Error} = \underbrace{0.121 \pm .002}_{R(\hat{f})} - \underbrace{0.102}_{\min_{f \in \mathcal{F}_3} R(f)}$$

$$= .019 \pm .002$$

Risk Summary



Excess Risk Decomposition