

## Recitation 1: Gradients and Directional Derivatives

### Intro Question

1. We are given the data set  $(x_1, y_1), \dots, (x_n, y_n)$  where  $x_i \in \mathbb{R}^d$  and  $y_i \in \mathbb{R}$ . We want to fit a linear function to this data by performing empirical risk minimization. More precisely, we are using the hypothesis space  $\mathcal{F} = \{f(x) = w^T x \mid w \in \mathbb{R}^d\}$  and the loss function  $\ell(a, y) = (a - y)^2$ . Given an initial guess  $\tilde{w}$  for the empirical risk minimizing parameter vector, how could we improve our guess?

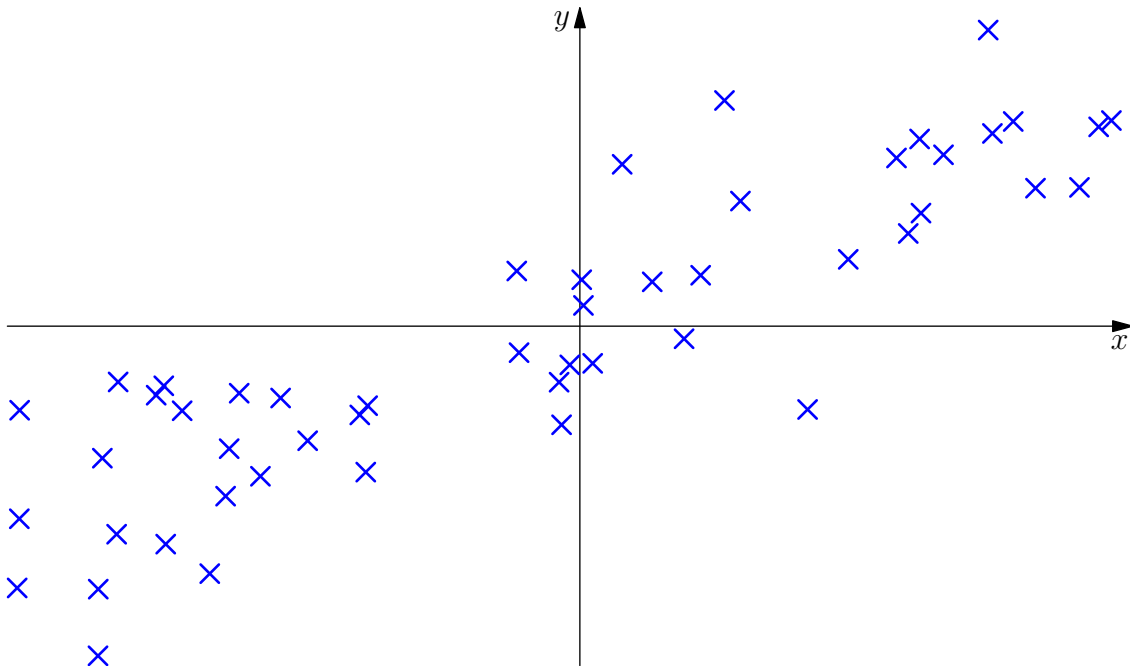


Figure 1: Data Set With  $d = 1$

### Multivariable Differentiation

Differential calculus allows us to convert non-linear problems into local linear problems, to which we can apply the well-developed techniques of linear algebra. Here we will review some of the important concepts needed in the rest of the course.

## Single Variable Differentiation

To gain intuition, we first recall the single variable case. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable. The derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

gives us a local linear approximation of  $f$  near  $x$ . This is seen more clearly in the following form:

$$f(x+h) = f(x) + hf'(x) + o(h) \quad \text{as } h \rightarrow 0,$$

where  $o(h)$  represents a function  $g(h)$  with  $g(h)/h \rightarrow 0$  as  $h \rightarrow 0$ . This can be used to show that if  $x$  is a local extremum of  $f$  then  $f'(x) = 0$ . Points with  $f'(x) = 0$  are called *critical points*.

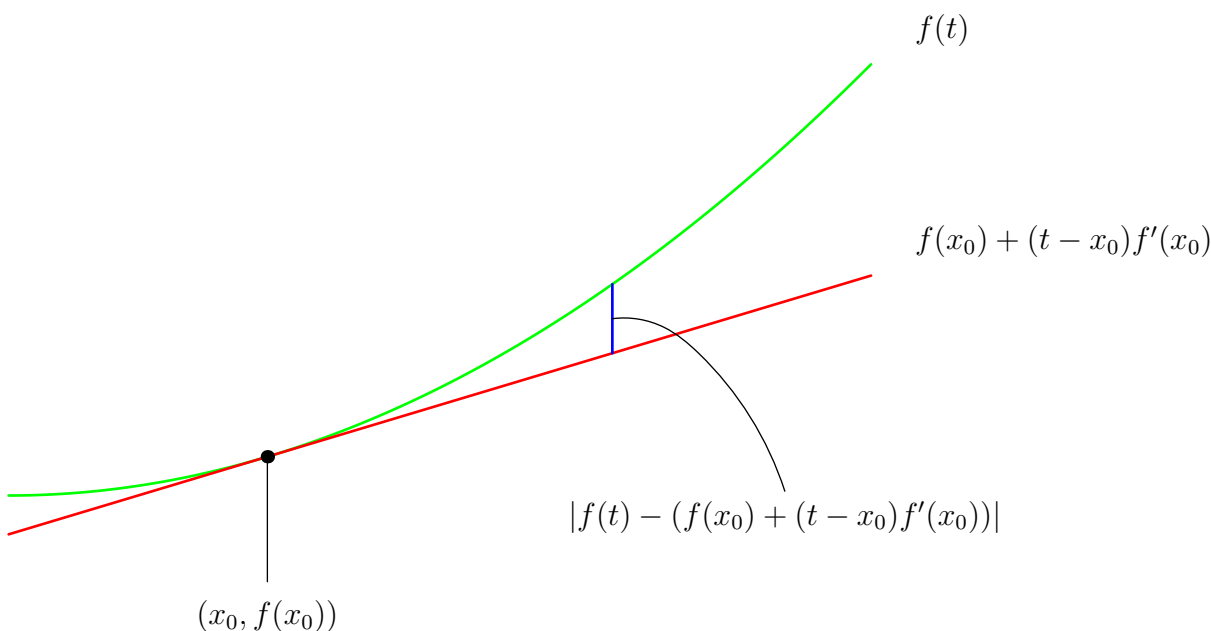


Figure 2: 1D Linear Approximation By Derivative

## Multivariate Differentiation

More generally, we will look at functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . In the single-variable case, the derivative was just a number that signified how much the function increased when we moved in the positive  $x$ -direction. In the multivariable case, we have many possible directions we can move along from a given point  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ .

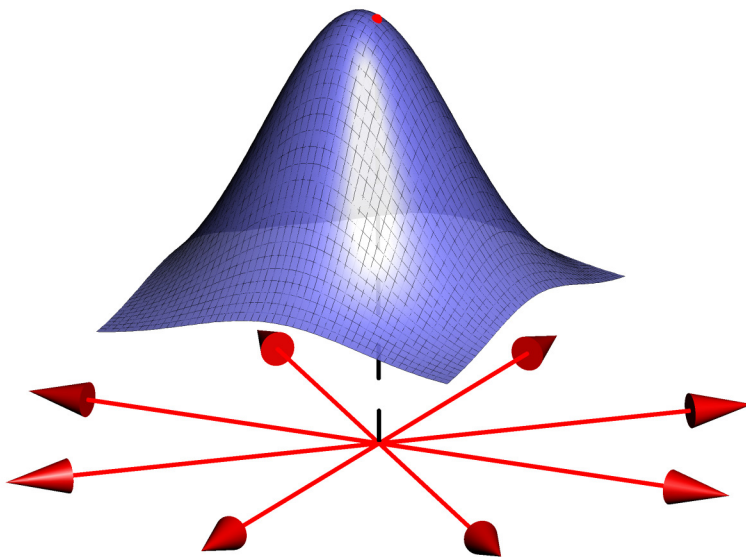


Figure 3: Multiple Possible Directions for  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

If we fix a direction  $u$  we can compute the directional derivative  $f'(x; u)$  as

$$f'(x; u) = \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h}.$$

This allows us to turn our multidimensional problem into a 1-dimensional computation. For instance,

$$f(x + hu) = f(x) + hf'(x; u) + o(h),$$

mimicking our earlier 1-d formula. This says that nearby  $x$  we can get a good approximation of  $f(x+hu)$  using the linear approximation  $f(x)+hf'(x; u)$ . In particular, if  $f'(x; u) < 0$  (such a  $u$  is called a *descent direction*) then for sufficiently small  $h > 0$  we have  $f(x + hu) < f(x)$ .

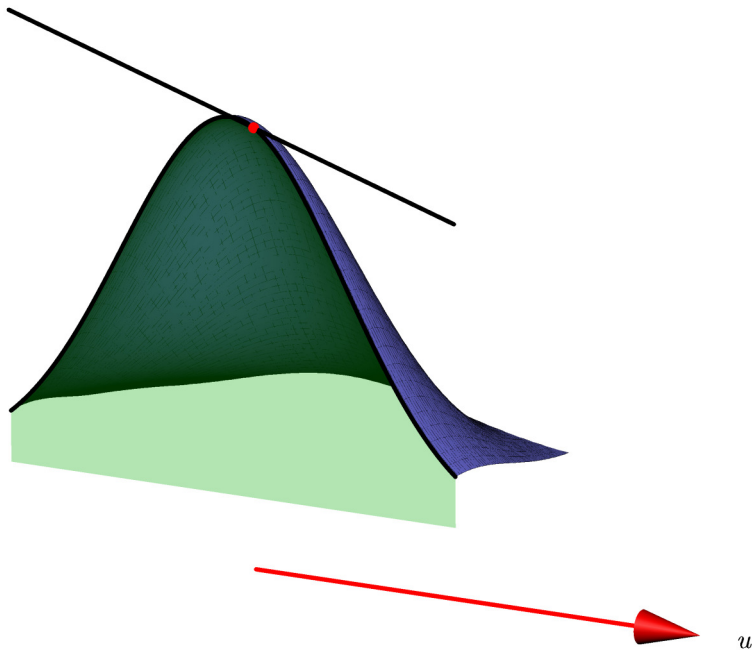


Figure 4: Directional Derivative as a Slope of a Slice

Let  $e_i = (\overbrace{0, 0, \dots, 0}^{i-1}, 1, 0, \dots, 0)$  be the  $i$ th standard basis vector. The directional derivative in the direction  $e_i$  is called the  $i$ th partial derivative and can be written in several ways:

$$\frac{\partial}{\partial x_i} f(x) = \partial_{x_i} f(x) = \partial_i f(x) = f'(x; e_i).$$

We say that  $f$  is differentiable at  $x$  if

$$\lim_{v \rightarrow 0} \frac{f(x + v) - f(x) - g^T v}{\|v\|_2} = 0,$$

for some  $g \in \mathbb{R}^n$  (note that the limit for  $v$  is taken in  $\mathbb{R}^n$ ). This  $g$  is uniquely determined, and is called the gradient of  $f$  at  $x$  denoted by  $\nabla f(x)$ . It is easy to show that the gradient is the vector of partial derivatives:

$$\nabla f(x) = \begin{pmatrix} \partial_{x_1} f(x) \\ \vdots \\ \partial_{x_n} f(x) \end{pmatrix}.$$

The  $k$ th entry of the gradient (i.e., the  $k$ th partial derivative) is the approximate change in  $f$  due to a small positive change in  $x_k$ . Sometimes we will split the variables of  $f$  into two parts. For instance, we could write  $f(x, w)$  with  $x \in \mathbb{R}^p$  and  $w \in \mathbb{R}^q$ . It is often useful to take the gradient with respect to some of the variables. Here we would write  $\nabla_x$  or  $\nabla_w$  to specify which part:

$$\nabla_x f(x, w) := \begin{pmatrix} \partial_{x_1} f(x, w) \\ \vdots \\ \partial_{x_p} f(x, w) \end{pmatrix} \quad \text{and} \quad \nabla_w f(x, w) := \begin{pmatrix} \partial_{w_1} f(x, w) \\ \vdots \\ \partial_{w_q} f(x, w) \end{pmatrix}.$$

Analogous to the univariate case, can express the condition for differentiability in terms of a gradient approximation:

$$f(x + v) = f(x) + \nabla f(x)^T v + o(\|v\|_2).$$

The approximation  $f(x + v) \approx f(x) + \nabla f(x)^T v$  gives a tangent plane at the point  $x$  as we let  $v$  vary.

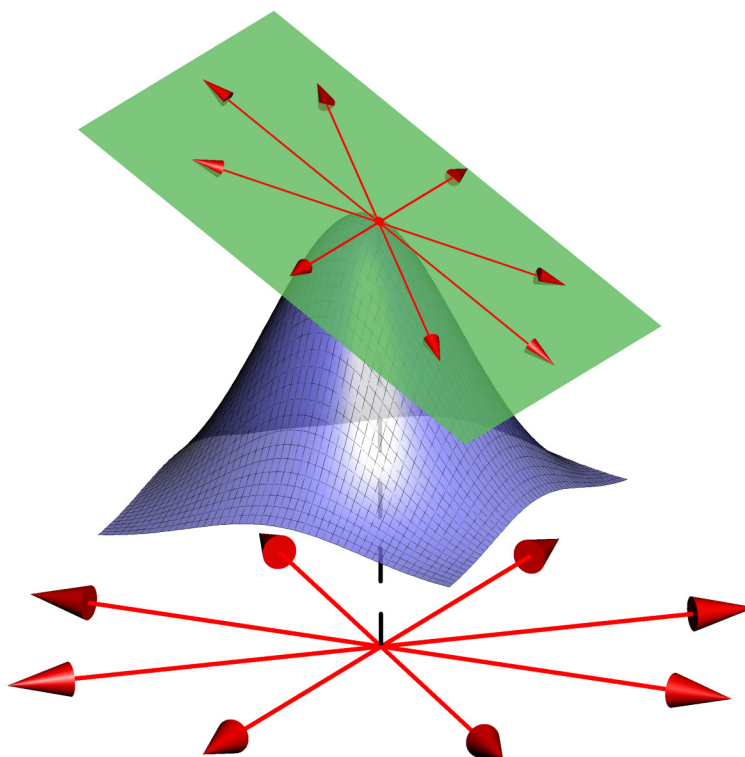


Figure 5: Tangent Plane for  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

If  $f$  is differentiable, we can use the gradient to compute an arbitrary directional derivative:

$$f'(x; u) = \nabla f(x)^T u.$$

From this expression we can quickly see that (assuming  $\nabla f(x) \neq 0$ )

$$\arg \max_{\|u\|_2=1} f'(x; u) = \frac{\nabla f(x)}{\|\nabla f(x)\|_2} \quad \text{and} \quad \arg \min_{\|u\|_2=1} f'(x; u) = -\frac{\nabla f(x)}{\|\nabla f(x)\|_2}.$$

In words, we say that the gradient points in the direction of steepest ascent, and the negative gradient points in the direction of steepest descent.

As in the 1-dimensional case, if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable and  $x$  is a local extremum of  $f$  then we must have  $\nabla f(x) = 0$ . Points  $x$  with  $\nabla f(x) = 0$  are called *critical points*. As we will see later in the course, if a function is differentiable and convex, then a point is critical if and only if it is a global minimum.

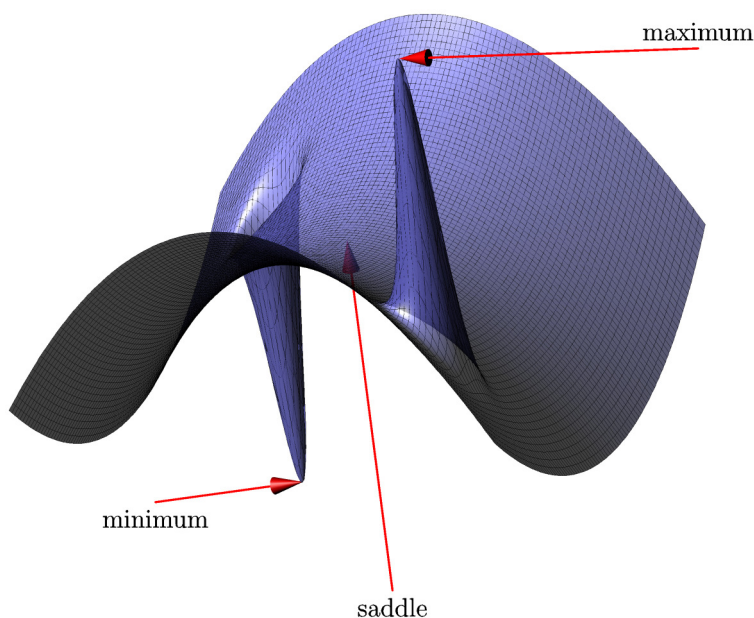


Figure 6: Critical Points of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

## Computing Gradients

A simple method to compute the gradient of a function is to compute each partial derivative separately. For example, if  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is given by

$$f(x_1, x_2, x_3) = \log(1 + e^{x_1 + 2x_2 + 3x_3})$$

then we can directly compute

$$\partial_{x_1} f(x_1, x_2, x_3) = \frac{e^{x_1+2x_2+3x_3}}{1 + e^{x_1+2x_2+3x_3}}, \quad \partial_{x_2} f(x_1, x_2, x_3) = \frac{2e^{x_1+2x_2+3x_3}}{1 + e^{x_1+2x_2+3x_3}}, \quad \partial_{x_3} f(x_1, x_2, x_3) = \frac{3e^{x_1+2x_2+3x_3}}{1 + e^{x_1+2x_2+3x_3}}$$

and obtain

$$\nabla f(x_1, x_2, x_3) = \begin{pmatrix} \frac{e^{x_1+2x_2+3x_3}}{1 + e^{x_1+2x_2+3x_3}} \\ \frac{2e^{x_1+2x_2+3x_3}}{1 + e^{x_1+2x_2+3x_3}} \\ \frac{3e^{x_1+2x_2+3x_3}}{1 + e^{x_1+2x_2+3x_3}} \end{pmatrix}.$$

Alternatively, we could let  $w = (1, 2, 3)^T$  and write

$$f(x) = \log(1 + e^{w^T x}).$$

Then we can apply the chain rule which says that if  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  are differentiable then

$$\nabla g(h(x)) = g'(h(x)) \nabla h(x).$$

Applying the chain rule twice (for log and exp) we obtain

$$\nabla f(x) = \frac{1}{1 + e^{w^T x}} e^{w^T x} w,$$

where we use the fact that  $\nabla_x(w^T x) = w$ . This last expression is more concise, and is more amenable to vectorized computation in many languages.

Another useful technique is to compute a general directional derivative and then infer the gradient from the computation. For example, let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be given by

$$f(x) = \|Ax - y\|_2^2 = (Ax - y)^T (Ax - y) = x^T A^T A x - 2y^T A x + y^T y,$$

for some  $A \in \mathbb{R}^{m \times n}$  and  $y \in \mathbb{R}^m$ . Assuming  $f$  is differentiable (so that  $f'(x, v) = \nabla f(x)^T v$ ) we have

$$\begin{aligned} f(x + tv) &= (x + tv)^T A^T A (x + tv) - 2y^T A (x + tv) + y^T y \\ &= x^T A^T A x + t^2 v^T A^T A v + 2tx^T A^T A v - 2y^T A x - 2ty^T A v + y^T y \\ &= f(x) + t(2x^T A^T A - 2y^T A)v + t^2 v^T A^T A v. \end{aligned}$$

Thus we have

$$\frac{f(x + tv) - f(x)}{t} = (2x^T A^T A - 2y^T A)v + tv^T A^T A v.$$

Taking the limit as  $t \rightarrow 0$  shows

$$f'(x, v) = (2x^T A^T A - 2y^T A)v \implies \nabla f(x) = (2x^T A^T A - 2y^T A)^T = 2A^T A x - 2A^T y.$$

Using this we can determine the critical points of  $f$ . Let's assume here that  $A$  has full column rank. Then  $A^T A$  is invertible, and the unique critical point is  $x = (A^T A)^{-1} A^T y$ . As we will see later in the course, this is a global minimum since  $f$  is convex (the *Hessian* of  $f$  satisfies  $\nabla^2 f(x) = 2A^T A \succ 0$ ).

### (★) Proving Differentiability

With a little extra work we can make the previous technique give a proof of differentiability. Using the computation above, we can rewrite  $f(x + v)$  as  $f(x)$  plus terms depending on  $v$ :

$$f(x + v) = f(x) + (2x^T A^T A - 2y^T A)v + v^T A^T A v.$$

Note that

$$\frac{v^T A^T A v}{\|v\|_2} = \frac{\|Av\|_2^2}{\|v\|_2} \leq \frac{\|A\|_2^2 \|v\|_2^2}{\|v\|_2} = \|A\|_2^2 \|v\|_2 \rightarrow 0,$$

as  $\|v\|_2 \rightarrow 0$ . (This section is starred since we used the matrix norm  $\|A\|_2$  here.) This shows  $f(x + v)$  above has the form

$$f(x + v) = f(x) + \nabla f(x)^T v + o(\|v\|_2).$$

This proves that  $f$  is differentiable and that

$$\nabla f(x) = 2A^T A x - 2A^T y.$$

Another method we could have used to establish differentiability is to observe that the partial derivatives are all continuous. This relies on the following theorem.

**Theorem 1.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and suppose  $\partial_{x_i} f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous for all  $x \in \mathbb{R}^n$  and all  $i = 1, \dots, n$ . Then  $f$  is differentiable.*