

# Lagrangian Duality and Convex Optimization

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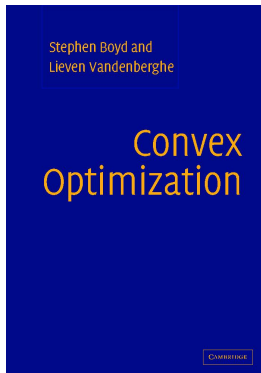
# Introduction

# Why Convex Optimization?

- Historically:
  - **Linear programs** (linear objectives & constraints) were the focus
  - **Nonlinear programs**: some easy, some hard
- By early 2000s:
  - Main distinction is between **convex** and **non-convex** problems
  - Convex problems are the ones we know how to solve efficiently
  - Mostly batch methods until... around 2010? (earlier if you were into neural nets)
- By 2010 +/- few years, most people understood the
  - optimization / estimation / approximation error tradeoffs
  - accepted that **stochastic methods** were often faster to get good results
    - (especially on big data sets)
  - now nobody's scared to try convex optimization machinery on non-convex problems

# Your Reference for Convex Optimization

- Boyd and Vandenberghe (2004)
  - Very clearly written, but has a ton of detail for a first pass.
  - See the [Extreme Abridgement of Boyd and Vandenberghe](#).



# Notation from Boyd and Vandenberghe

- $f : \mathbf{R}^p \rightarrow \mathbf{R}^q$  to mean that  $f$  maps from some *subset* of  $\mathbf{R}^p$ 
  - namely  $\mathbf{dom} f \subset \mathbf{R}^p$ , where  $\mathbf{dom} f$  is the domain of  $f$

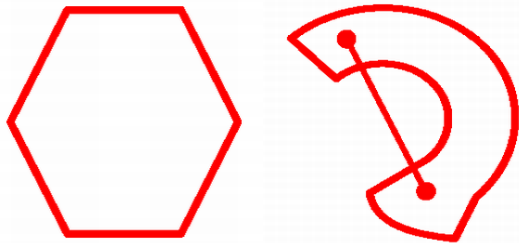
# Convex Sets and Functions

# Convex Sets

## Definition

A set  $C$  is **convex** if for any  $x_1, x_2 \in C$  and any  $\theta$  with  $0 \leq \theta \leq 1$  we have

$$\theta x_1 + (1 - \theta)x_2 \in C.$$



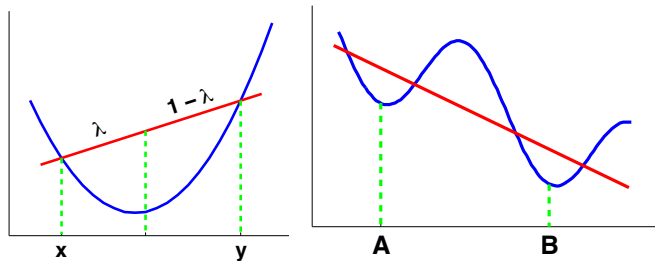
KPM Fig. 7.4

# Convex and Concave Functions

## Definition

A function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is **convex** if  $\mathbf{dom} f$  is a convex set and if for all  $x, y \in \mathbf{dom} f$ , and  $0 \leq \theta \leq 1$ , we have

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y).$$



KPM Fig. 7.5



# Examples of Convex Functions on $\mathbf{R}$

## Examples

- $x \mapsto ax + b$  is both convex and concave on  $\mathbf{R}$  for all  $a, b \in \mathbf{R}$ .
- $x \mapsto |x|^p$  for  $p \geq 1$  is convex on  $\mathbf{R}$
- $x \mapsto e^{ax}$  is convex on  $\mathbf{R}$  for all  $a \in \mathbf{R}$
- Every norm on  $\mathbf{R}^n$  is convex (e.g.  $\|x\|_1$  and  $\|x\|_2$ )
- Max:  $(x_1, \dots, x_n) \mapsto \max\{x_1, \dots, x_n\}$  is convex on  $\mathbf{R}^n$

# Convex Functions and Optimization

## Definition

A function  $f$  is **strictly convex** if the line segment connecting any two points on the graph of  $f$  lies **strictly** above the graph (excluding the endpoints).

Consequences for optimization:

- **convex**: if there is a local minimum, then it is a **global** minimum
- **strictly convex**: if there is a local minimum, then it is the **unique global** minimum

# The General Optimization Problem

# General Optimization Problem: Standard Form

## General Optimization Problem: Standard Form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p, \end{array}$$

where  $x \in \mathbf{R}^n$  are the **optimization variables** and  $f_0$  is the **objective function**.

Assume **domain**  $\mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$  is nonempty.

# General Optimization Problem: More Terminology

- The set of points satisfying the constraints is called the **feasible set**.
- A point  $x$  in the feasible set is called a **feasible point**.
- If  $x$  is feasible and  $f_i(x) = 0$ ,
  - then we say the inequality constraint  $f_i(x) \leq 0$  is **active** at  $x$ .
- The **optimal value**  $p^*$  of the problem is defined as

$$p^* = \inf \{f_0(x) \mid x \text{ satisfies all constraints}\}.$$

- $x^*$  is an **optimal point** (or a solution to the problem) if  $x^*$  is feasible and  $f(x^*) = p^*$ .

# Do We Need Equality Constraints?

- Note that

$$h(x) = 0 \iff (h(x) \geq 0 \text{ AND } h(x) \leq 0)$$

- Consider an equality-constrained problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & h(x) = 0 \end{array}$$

- Can be rewritten as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & h(x) \leq 0 \\ & -h(x) \leq 0. \end{array}$$

- For simplicity, we'll drop equality constraints from this presentation.

## Lagrangian Duality: Convexity not required

# The Lagrangian

The general [inequality-constrained] optimization problem is:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \end{array}$$

## Definition

The **Lagrangian** for this optimization problem is

$$L(x, \lambda) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x).$$

- $\lambda_i$ 's are called **Lagrange multipliers** (also called the **dual variables**).



# The Lagrangian Encodes the Objective and Constraints

- Supremum over Lagrangian gives back encoding of objective and constraints:

$$\begin{aligned}\sup_{\lambda \succeq 0} L(x, \lambda) &= \sup_{\lambda \succeq 0} \left( f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \right) \\ &= \begin{cases} f_0(x) & \text{when } f_i(x) \leq 0 \text{ all } i \\ \infty & \text{otherwise.} \end{cases}\end{aligned}$$

- Equivalent **primal form** of optimization problem:

$$p^* = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$$

# The Primal and the Dual

- Original optimization problem in **primal form**:

$$p^* = \inf_x \sup_{\lambda \succeq 0} L(x, \lambda)$$

- Get the **Lagrangian dual problem** by “swapping the inf and the sup”:

$$d^* = \sup_{\lambda \succeq 0} \inf_x L(x, \lambda)$$

- We will show **weak duality**:  $p^* \geq d^*$  for any optimization problem

# Weak Max-Min Inequality

## Theorem

For *any*  $f : W \times Z \rightarrow \mathbf{R}$ , we have

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) \leq \inf_{w \in W} \sup_{z \in Z} f(w, z).$$

## Proof.

For any  $w_0 \in W$  and  $z_0 \in Z$ , we clearly have

$$\inf_{w \in W} f(w, z_0) \leq f(w_0, z_0) \leq \sup_{z \in Z} f(w_0, z).$$

Since  $\inf_{w \in W} f(w, z_0) \leq \sup_{z \in Z} f(w_0, z)$  for all  $w_0$  and  $z_0$ , we must also have

$$\sup_{z_0 \in Z} \inf_{w \in W} f(w, z_0) \leq \inf_{w_0 \in W} \sup_{z \in Z} f(w_0, z).$$

# Weak Duality

- For any optimization problem (**not just convex**), weak max-min inequality implies **weak duality**:

$$\begin{aligned}
 p^* &= \inf_x \sup_{\lambda \succeq 0} \left[ f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \right] \\
 &\geq \sup_{\lambda \succeq 0} \inf_x \left[ f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \right] = d^*
 \end{aligned}$$

- The difference  $p^* - d^*$  is called the **duality gap**.
- For *convex* problems, we often have **strong duality**:  $p^* = d^*$ .

# The Lagrange Dual Function

- The **Lagrangian dual problem**:

$$d^* = \sup_{\lambda \succeq 0} \inf_x L(x, \lambda)$$

## Definition

The **Lagrange dual function** (or just **dual function**) is

$$g(\lambda) = \inf_x L(x, \lambda) = \inf_x \left( f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \right).$$

- The dual function may take on the value  $-\infty$  (e.g.  $f_0(x) = x$ ).
- The dual function is always **concave**
  - since pointwise min of affine functions

# The Lagrange Dual Problem: Search for Best Lower Bound

- In terms of Lagrange dual function, we can write weak duality as

$$p^* \geq \sup_{\lambda \geq 0} g(\lambda) = d^*$$

- So for any  $\lambda$  with  $\lambda \geq 0$ , **Lagrange dual function gives a lower bound on optimal solution:**

$$p^* \geq g(\lambda) \text{ for all } \lambda \geq 0$$

# The Lagrange Dual Problem: Search for Best Lower Bound

- The **Lagrange dual problem** is a search for best lower bound on  $p^*$ :

$$\begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

- $\lambda$  **dual feasible** if  $\lambda \succeq 0$  and  $g(\lambda) > -\infty$ .
- $\lambda^*$  **dual optimal** or **optimal Lagrange multipliers** if they are optimal for the Lagrange dual problem.
- Lagrange dual problem often easier to solve (simpler constraints).
- $d^*$  can be used as stopping criterion for primal optimization.
- Dual can reveal hidden structure in the solution.

# Convex Optimization



# Convex Optimization Problem: Standard Form

## Convex Optimization Problem: Standard Form

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m\end{array}$$

where  $f_0, \dots, f_m$  are convex functions.

# Strong Duality for Convex Problems

- For a convex optimization problems, we **usually** have strong duality, but not always.
  - For example:

$$\begin{array}{ll}\text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq 0 \\ & y > 0\end{array}$$

- The additional conditions needed are called **constraint qualifications**.

# Slater's Constraint Qualifications for Strong Duality

- Sufficient conditions for strong duality in a **convex** problem.
- Roughly: the problem must be **strictly** feasible.
- Qualifications when problem domain<sup>1</sup>  $\mathcal{D} \subset \mathbf{R}^n$  is an open set:
  - **Strict feasibility is sufficient.** ( $\exists x \ f_i(x) < 0$  for  $i = 1, \dots, m$ )
  - For any affine inequality constraints,  $f_i(x) \leq 0$  is sufficient.
- Otherwise, see notes or BV Section 5.2.3, p. 226.

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<sup>1</sup> $\mathcal{D}$  is the set where all functions are defined, NOT the feasible set.

## Complementary Slackness

# Complementary Slackness

- Consider a general optimization problem (i.e. not necessarily convex).
- If we have **strong duality**, we get an interesting relationship between
  - the optimal Lagrange multiplier  $\lambda_i$  and
  - the  $i$ th constraint at the optimum:  $f_i(x^*)$
- Relationship is called “**complementary slackness**”:

$$\lambda_i^* f_i(x^*) = 0$$

- Lagrange multiplier is zero unless constraint is active at optimum.

# Complementary Slackness Proof

- Assume strong duality:  $p^* = d^*$  in a general optimization problem
- Let  $x^*$  be primal optimal and  $\lambda^*$  be dual optimal. Then:

$$\begin{aligned}
 f_0(x^*) &= g(\lambda^*) = \inf_x L(x, \lambda^*) \quad (\text{strong duality and definition}) \\
 &\leq L(x^*, \lambda^*) \\
 &= f_0(x^*) + \underbrace{\sum_{i=1}^m \lambda_i^* f_i(x^*)}_{\leq 0} \\
 &\leq f_0(x^*).
 \end{aligned}$$

Each term in sum  $\sum_{i=1}^m \lambda_i^* f_i(x^*)$  must actually be 0. That is

$$\boxed{\lambda_i^* f_i(x^*) = 0, \quad i = 1, \dots, m.}$$

This condition is known as **complementary slackness**.

# Reiterating Consequences of Strong Duality

- Let  $x^*$  be primal optimal and  $\lambda^*$  be dual optimal.
- If we have strong duality, then

$$p^* = d^* = f_0(x^*) = g(\lambda^*) = L(x^*, \lambda^*)$$

and

$$\lambda_i^* f_i(x^*) = 0, \quad i = 1, \dots, m.$$

- What more we can say if we have differentiable objective and constraint functions?

# Karush-Kuhn-Tucker (KKT) Conditions

- Suppose we have strong duality:  $p^* = d^* = f_0(x^*) = g(\lambda^*) = L(x^*, \lambda^*)$ ,
- and  $f_0, \dots, f_m$  are differentiable, but ***not necessarily convex***.
- Since  $x^*$  minimizes  $L(x, \lambda^*)$  over  $x$ , we must have  $\nabla_x L(x, \lambda^*) = 0$ .
- Thus  $x^*, \lambda^*$  satisfy the following **Karush-Kuhn-Tucker (KKT)** conditions:
  - ① Primal and dual feasibility:  $f_i(x^*) \leq 0, \lambda_i^* \geq 0$  for all  $i$ .
  - ② Complementary slackness:  $\lambda_i^* f_i(x^*) = 0$  for all  $i$ .
  - ③ First order conditions:  $\nabla_x L(x, \lambda^*) = \nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) = 0$ .
- Only complementary slackness is not completely obvious.



# KKT Conditions for Convex, Differentiable Problems

Suppose

- $f_0, \dots, f_m$  are differentiable and convex
- $\tilde{x}$  and  $\tilde{\lambda}$  satisfy the KKT conditions

Then we have strong duality and  $(\tilde{x}, \tilde{\lambda})$  are primal and dual optimal, respectively.

Proof.

Convexity and first order conditions implies  $\tilde{x} \in \arg \min_x L(x, \tilde{\lambda})$ . So

$$g(\tilde{\lambda}) = \inf_x L(x, \tilde{\lambda}) = L(\tilde{x}, \tilde{\lambda}) = f_0(\tilde{x}) + \sum_{i=1}^m \tilde{\lambda}_i f_i(\tilde{x}) = f_0(\tilde{x}) \quad \text{by complementary slackness.}$$

But  $g(\tilde{\lambda}) \leq \sup_{\lambda \succeq 0} g(\lambda) \leq \inf_x f_0(x) \leq f_0(\tilde{x})$  (middle inequality by weak duality).

So  $g(\tilde{\lambda}) = \sup_{\lambda \succeq 0} g(\lambda) = \inf_x f_0(x) = f_0(\tilde{x})$

