Lasso, Ridge, and Elastic Net

David Rosenberg

New York University

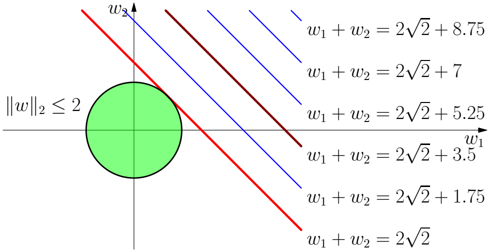
February 7, 2017

Suppose We Have 2 Equal Features

- Input features: $x_1, x_2 \in \mathbf{R}$.
- Outcome: $y \in \mathbf{R}$.
- Linear prediction functions $f(x) = w_1 x_2 + w_2 x_2$
- Suppose $x_1 = x_2$.
- Then all functions with $w_1 + w_2 = k$ are the same.
  - give same predictions and have same empirical risk

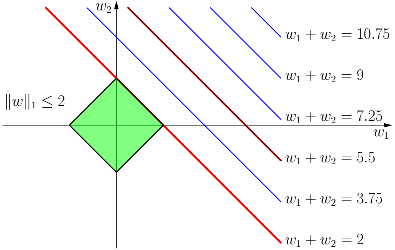
What function will we select if we do ERM with ℓ_1 or ℓ_2 constraint?

Equal Features, ℓ_2 Constraint



- Suppose the line $w_1 + w_2 = 2\sqrt{2} + 3.5$ corresponds to the empirical risk minimizers.
- Empirical risk increase as we move away from these parameter settings
- Intersection of $w_1 + w_2 = 2\sqrt{2}$ and the norm ball $\|w\|_2 \leqslant 2$ is ridge solution.
- Note that $w_1 = w_2$ at the solution

Equal Features, ℓ_1 Constraint



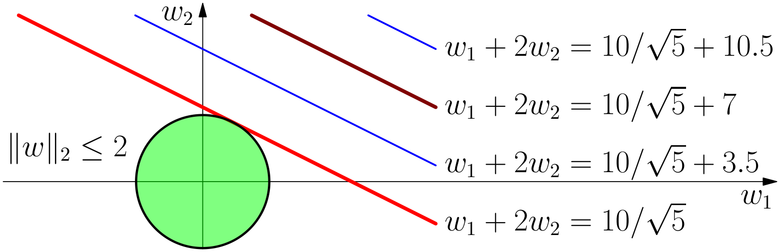
- Suppose the line $w_1 + w_2 = 5.5$ corresponds to the empirical risk minimizers.
- Intersection of $w_1 + w_2 = 2$ and the norm ball $\|w\|_1 \leqslant 2$ is lasso solution.
- Note that the solution set is $\{(w_1, w_2) : w_1 + w_2 = 2, w_1, w_2 \geqslant 0\}$.

Linearly Related Features

- Same setup, now suppose $x_1 = 2x_2$.
- Then all functions with $w_1 + 2w_2 = k$ are the same.
  - give same predictions and have same empirical risk

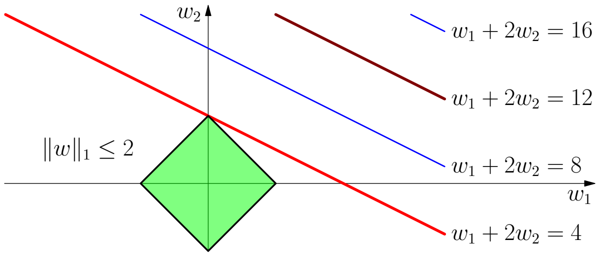
  What function will we select if we do ERM with ℓ_1 or ℓ_2 constraint?

Linearly Related Features, ℓ_2 Constraint



- $w_1 + 2w_2 = 10/\sqrt{5} + 7$ corresponds to the empirical risk minimizers.
- Intersection of $w_1 + 2w_2 = 10\sqrt{5}$ and the norm ball $\|w\|_2 \leqslant 2$ is ridge solution.
- At solution, $w_2 = 2w_1$.

Linearly Related Features, ℓ_1 Constraint



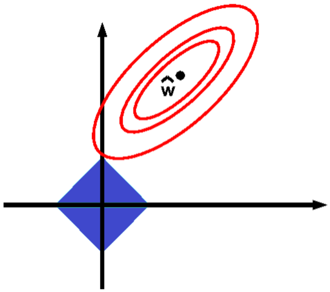
- Intersection of $w_1 + 2w_2 = 4$ and the norm ball $\|w\|_1 \leqslant 2$ is lasso solution.
- Solution is now a corner of the ℓ_1 ball, corresonding to a sparse solution.

Linearly Dependent Features: Take Away

- For identical features
  - ℓ_1 regularization spreads weight arbitrarily (all weights same sign)
  - ℓ_2 regularization spreads weight evenly
- Linearly related features
  - ℓ_1 regularization chooses variable with larger scale, 0 weight to others
  - ℓ_2 prefers variables with larger scale – spreads weight inversely proportional to scale

Empirical Risk for Square Loss and Linear Predictors

- Recall our discussion of linear predictors $f(x) = w^T x$ and square loss.
- Sets of w giving same empirical risk (i.e. level sets) formed ellipsoids around the ERM.



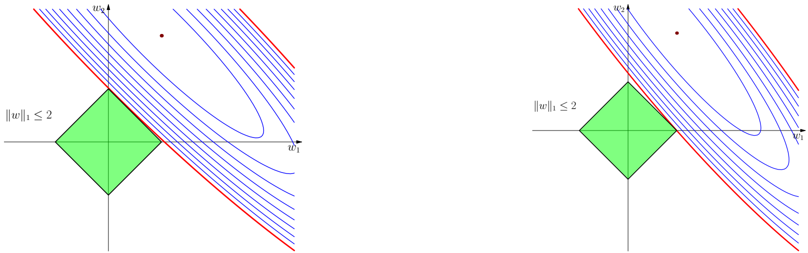
- With x_1 and x_2 linearly related, we get a degenerate ellipse.
- That's why level sets were lines (actually pairs of lines, one on each side of ERM).

KPM Fig. 13.3

Correlated Features – Same Scale

- Suppose x_1 and x_2 are highly correlated and the same scale.
- This is quite typical in real data, after normalizing data.
- Nothing degenerate here, so level sets are ellipsoids.
- But, the higher the correlation, the closer to degenerate we get.
- That is, ellipsoids keep stretching out, getting closer to two parallel lines.

Correlated Features, ℓ_1 Regularization



- Intersection could be anywhere on the top right edge.
- Minor perturbations can drastically change intersection point – very unstable solution.
- Makes division of weight among highly correlated features (of same scale) seem arbitrary.
  - If $x_1 \approx 2x_2$, ellipse changes orientation and we probably hit a corner.

Example with highly correlated features

- Model in words:
  - y is a linear combination of z_1 and z_2
  - But we don't observe z_1 and z_2 directly.
  - We get 3 noisy observations of z_1.
  - We get 3 noisy observations of z_2.
- We want to predict y from our noisy observations.

Example from Section 4.2 in Hastie et al's *Statistical Learning with Sparsity*.

Example with highly correlated features

- Suppose (x, y) generated as follows:

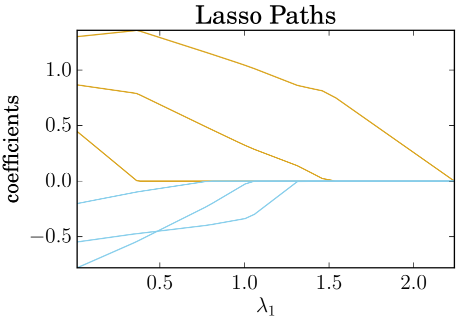
$$
\begin{aligned}
z_1, z_2 &\sim \mathcal{N}(0,1) \text{ (independent)} \\
\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_6 &\sim \mathcal{N}(0,1) \text{ (independent)} \\
y &= 3z_1 - 1.5z_2 + 2\varepsilon_0 \\
x_j &= \begin{cases} z_1 + \varepsilon_j/5 & \text{for } j = 1,2,3 \\ z_2 + \varepsilon_j/5 & \text{for } j = 4,5,6 \end{cases}
\end{aligned}
$$

- Generated a sample of (x, y) pairs of size 100.
- Correlations within the groups of x's were around 0.97.

Example from Section 4.2 in Hastie et al's *Statistical Learning with Sparsity*.

Example with highly correlated features

- Lasso regularization paths:



- Lines with the same color correspond to features with essentially the same information
- Distribution of weight among them seems almost arbitrary

Hedge Bets When Variables Highly Correlated

- When variables are highly correlated (and same scale, after normalization),
  - we want to give them roughly the same weight.
- Why?
  - Let their error cancel out
- How can we get the weight spread more evenly?

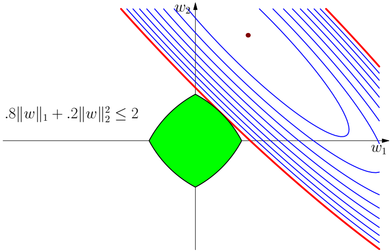
Elastic Net

- The **elastic net** combines lasso and ridge penalties:

$$\hat{w} = \operatorname*{arg\,min}_{w \in \mathbf{R}^d} \frac{1}{n} \sum_{i=1}^{n} \left\{ w^T x_i - y_i \right\}^2 + \lambda_1 \|w\|_1 + \lambda_2 \|w\|_2^2$$

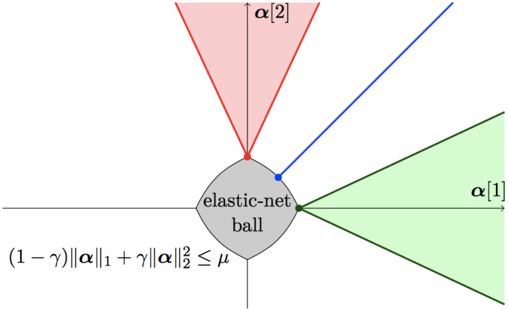
- We expect correlated random variables to have similar coefficients.

Highly Correlated Features, Elastic Net Constraint



- Elastic net solution is closer to $w_2 = w_1$ line, despite high correlation.

Elastic Net - "Sparse Regions"



- Suppose design matrix X is orthogonal, so $X^T X = I$, and contours are circles (and features uncorrelated)
- Then OLS solution in green or red regions implies elastic-net constrained solution will be at corner

Fig from Mairal et al.'s Sparse Modeling for Image and Vision Processing Fig 1.9

Elastic Net – A Theorem for Correlated Variables

Theorem

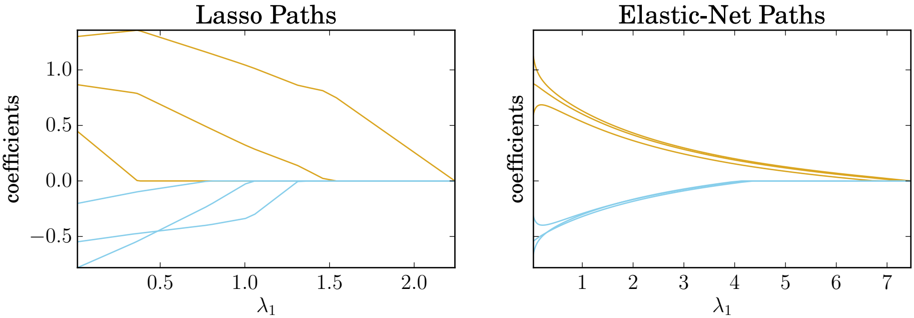
[a] *Let* $\rho_{ij} = \widehat{corr}(x_i, x_j)$. *Suppose* $\hat{w}_i$ *and* $\hat{w}_j$ *are selected by elastic net, with* y *centered and predictors* $x_1, \ldots, x_d$ *standardized. If* $\hat{w}_i \hat{w}_j > 0$, *then*

$$|\hat{w}_i - \hat{w}_j| \leqslant \frac{\|y\|\sqrt{2}}{\lambda_2}\sqrt{1-\rho_{ij}}.$$

---

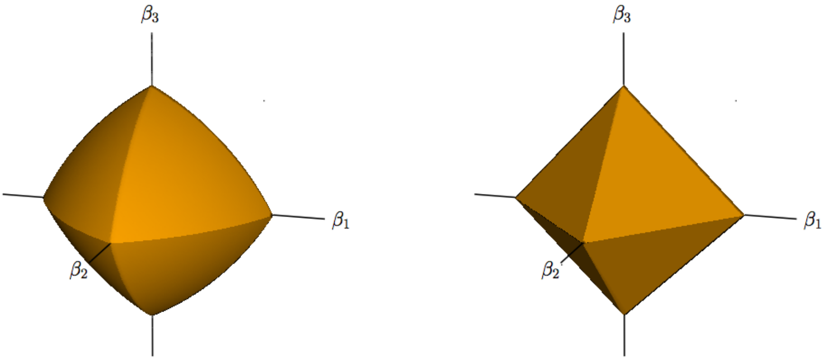
[a] Theorem 1 in "Regularization and variable selection via the elastic net": https://web.stanford.edu/~hastie/Papers/B67.2%20(2005)%20301-320%20Zou%20&%20Hastie.pdf

Elastic Net Results on Model



- Lasso on left; Elastic net on right.
- Ratio of ℓ_2 to ℓ_1 regularization roughly $2:1$.

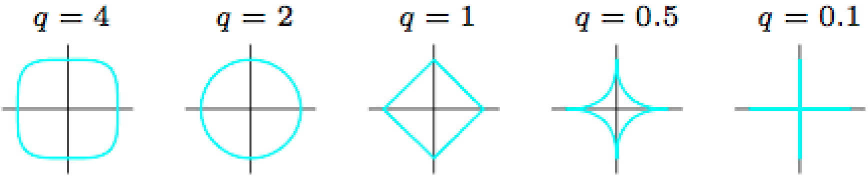
Elastic Net vs Lasso Norm Ball



From Figure 4.2 of Hastie et al's *Statistical Learning with Sparsity*.

The $(\ell_q)^q$ Norm Constraint

- Generalize to ℓ_q : $(\|w\|_q)^q = |w_1|^q + |w_2|^q$.
- Note: $\|w\|_q$ is a norm if $q \geqslant 1$, but not for $q \in (0,1)$
- $\mathcal{F} = \{f(x) = w_1x_1 + w_2x_2\}$.
- Contours of $\|w\|_q^q = |w_1|^q + |w_2|^q$:



$\ell_{1.2}$ vs Elastic Net

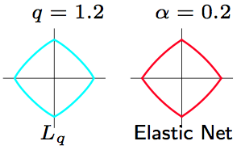


FIGURE 3.13. *Contours of constant value of* $\sum_j |\beta_j|^q$ *for* $q = 1.2$ *(left plot), and the elastic-net penalty* $\sum_j (\alpha\beta_j^2 + (1-\alpha)|\beta_j|)$ *for* $\alpha = 0.2$ *(right plot). Although visually very similar, the elastic-net has sharp (non-differentiable) corners, while the* $q = 1.2$ *penalty does not.*

From Hastie et al's *Elements of Statistical Learning*.