ℓ_1 and ℓ_2 Regularization

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Tikhonov and Ivanov Regularization

Hypothesis Spaces

- We've spoken vaguely about "bigger" and "smaller" hypothesis spaces
- In practice, convenient to work with a **nested sequence** of spaces:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_n \cdots \subset \mathcal{F}$$

Polynomial Functions

- $\mathcal{F} = \{\text{all polynomial functions}\}$
- $\mathcal{F}_d = \{\text{all polynomials of degree } \leqslant d\}$

Complexity Measures for Decision Functions

- Number of variables / features
- Depth of a decision tree
- Degree of polynomial
- How about for **linear** decision functions, i.e. $x \mapsto w^T x = w_1 x_1 + \cdots + w_d x_d$?
    - ℓ_0 complexity: number of non-zero coefficients $\sum_{i=1}^{d} 1(w_i \neq 0)$.
    - ℓ_1 "lasso" complexity: $\sum_{i=1}^{d} |w_i|$, for coefficients $w_1, \ldots, w_d$
    - ℓ_2 "ridge" complexity: $\sum_{i=1}^{d} w_i^2$ for coefficients $w_1, \ldots, w_d$

Nested Hypothesis Spaces from Complexity Measure

- Hypothesis space: $\mathcal{F}$
- Complexity measure $\Omega : \mathcal{F} \to [0, \infty)$
- Consider all functions in $\mathcal{F}$ with complexity **at most** r**:**

$$\mathcal{F}_r = \{f \in \mathcal{F} \mid \Omega(f) \leqslant r\}$$

- Increasing complexities: $r = 0, 1.2, 2.6, 5.4, \ldots$ gives nested spaces:

$$\mathcal{F}_0 \subset \mathcal{F}_{1.2} \subset \mathcal{F}_{2.6} \subset \mathcal{F}_{5.4} \subset \cdots \subset \mathcal{F}$$

Constrained Empirical Risk Minimization

Constrained ERM (Ivanov regularization)

For complexity measure $\Omega : \mathcal{F} \to [0, \infty)$ and fixed $r \geqslant 0$,

$$\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \ell(f(x_i), y_i)$$
$$\text{s.t. } \Omega(f) \leqslant r$$

- Choose r using validation data or cross-validation.
- Each r corresponds to a different hypothesis spaces. Could also write:

$$\min_{f \in \mathcal{F}_r} \frac{1}{n} \sum_{i=1}^{n} \ell(f(x_i), y_i)$$

Penalized Empirical Risk Minimization

Penalized ERM (Tikhonov regularization)

For complexity measure $\Omega : \mathcal{F} \to [0, \infty)$ and fixed $\lambda \geqslant 0$,

$$\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \ell(f(x_i), y_i) + \lambda \Omega(f)$$

- Choose λ using validation data or cross-validation.
- (Ridge regression in homework is of this form.)

Ivanov vs Tikhonov Regularization

- Let $L : \mathcal{F} \to \mathbf{R}$ be any performance measure of f
  - e.g. $L(f)$ could be the empirical risk of f
- For many L and Ω, Ivanov and Tikhonov are "equivalent".
- What does this mean?
  - Any solution f^* you could get from Ivanov, can also get from Tikhonov.
  - Any solution f^* you could get from Tikhonov, can also get from Ivanov.
- In practice, both approaches are effective.
- Tikhonov convenient because it's *unconstrained* minimization.

Can get conditions for equivalence from Lagrangian duality theory – details in homework.

Ivanov vs Tikhonov Regularization (Details)

Ivanov and Tikhonov regularization are equivalent if:

1. For any choice of $r > 0$, any Ivanov solution

$$f_r^* \in \operatorname*{arg\,min}_{f \in \mathcal{F}} L(f) \text{ s.t. } \Omega(f) \leqslant r$$

is also a Tikhonov solution for some $\lambda > 0$. That is, $\exists \lambda > 0$ such that

$$f_r^* \in \operatorname*{arg\,min}_{f \in \mathcal{F}} L(f) + \lambda \Omega(f).$$

2. Conversely, for any choice of $\lambda > 0$, any Tikhonov solution:

$$f_\lambda^* \in \operatorname*{arg\,min}_{f \in \mathcal{F}} L(f) + \lambda \Omega(f)$$

is also an Ivanov solution for some $r > 0$. That is, $\exists r > 0$ such that

$$f_\lambda^* \in \operatorname*{arg\,min}_{f \in \mathcal{F}} L(f) \text{ s.t. } \Omega(f) \leqslant r$$

ℓ_1 and ℓ_2 Regularization

Linear Least Squares Regression

- Consider linear models

$$\mathcal{F} = \left\{ f : \mathbf{R}^d \to \mathbf{R} \mid f(x) = w^T x \text{ for } w \in \mathbf{R}^d \right\}$$

- Loss: $\ell(\hat{y}, y) = (y - \hat{y})^2$
- Training data $\mathcal{D}_n = ((x_1, y_1), \ldots, (x_n, y_n))$
- Linear least squares regression is ERM for ℓ over $\mathcal{F}$:

$$\hat{w} = \underset{w \in \mathbf{R}^d}{\arg\min} \frac{1}{n} \sum_{i=1}^{n} \left\{ w^T x_i - y_i \right\}^2$$

- Can **overfit** when d is large compared to n.
- e.g.: $d \gg n$ very common in Natural Language Processing problems (e.g. a 1M features for 10K documents).

Ridge Regression (Tikhonov Form)

The ridge regression solution for regularization parameter $\lambda \geqslant 0$ is

$$\hat{w} = \underset{w \in \mathbf{R}^d}{\arg\min} \frac{1}{n} \sum_{i=1}^{n} \left\{ w^T x_i - y_i \right\}^2 + \lambda \|w\|_2^2,$$

where $\|w\|_2^2 = w_1^2 + \cdots + w_d^2$ is the square of the ℓ_2-norm.

Ridge Regression (Ivanov Form)

The ridge regression solution for complexity parameter $r \geqslant 0$ is

$$\hat{w} = \underset{\|w\|_2^2 \leqslant r^2}{\arg\min} \frac{1}{n} \sum_{i=1}^{n} \left\{ w^T x_i - y_i \right\}^2.$$

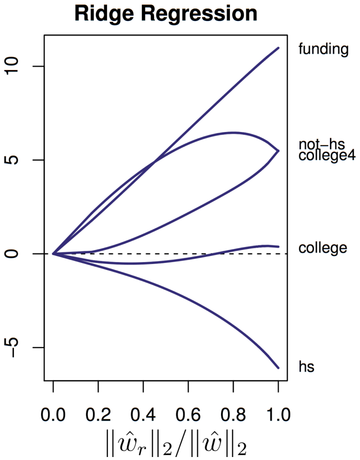
How does ℓ_2 regularization induce "regularity"?

- For $\hat{f}(x) = \hat{w}^T x$, $\hat{f}$ is **Lipschitz continuous** with Lipschitz constant $L = \|\hat{w}\|_2$.
- That is, when moving from x to $x+h$, $\hat{f}$ changes no more than $L\|h\|$.
- So ℓ_2 regularization controls the maximum rate of change of $\hat{f}$.
- Proof:

$$
\begin{aligned}
\left|\hat{f}(x+h) - \hat{f}(x)\right| &= |\hat{w}^T(x+h) - \hat{w}^T x| = \left|\hat{w}^T h\right| \\
&\leqslant \|\hat{w}\|_2 \|h\|_2 (\text{Cauchy-Schwarz inequality})
\end{aligned}
$$

- Since $\|\hat{w}\|_1 \geqslant \|\hat{w}\|_2$, an ℓ_1 constraint will also give a Lipschitz bound.

Ridge Regression: Regularization Path



$$\begin{aligned}\hat{w}_r &= \underset{\|w\|_2^2\le r^2}{\arg\min}\ \frac{1}{n}\sum_{i=1}^{n}\left(w^T x_i - y_i\right)^2\\ \hat{w} &= \hat{w}_\infty = \text{Unconstrained ERM}\end{aligned}$$

- For $r=0$, $\|\hat{w}_r\|_2/\|\hat{w}\|_2 = 0$.
- For $r=\infty$, $\|\hat{w}_r\|_2/\|\hat{w}\|_2 = 1$

Modified from Hastie, Tibshirani, and Wainwright's *Statistical Learning with Sparsity*, Fig 2.1. About predicting crime in 50 US cities.

Lasso Regression (Tikhonov Form)

The lasso regression solution for regularization parameter $\lambda \geqslant 0$ is

$$\hat{w} = \operatorname*{arg\,min}_{w \in \mathbf{R}^d} \frac{1}{n} \sum_{i=1}^{n} \left\{ w^T x_i - y_i \right\}^2 + \lambda \|w\|_1,$$

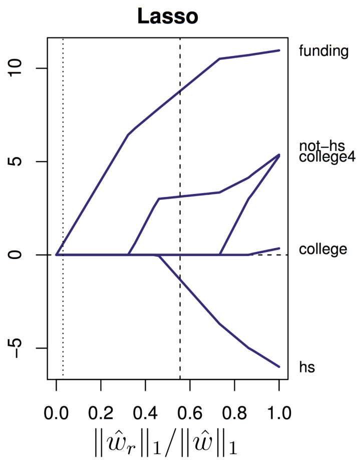
where $\|w\|_1 = |w_1| + \cdots + |w_d|$ is the ℓ_1-norm.

Lasso Regression (Ivanov Form)

The lasso regression solution for complexity parameter $r \geqslant 0$ is

$$\hat{w} = \operatorname*{arg\,min}_{\|w\|_1 \leqslant r} \frac{1}{n} \sum_{i=1}^{n} \left\{ w^T x_i - y_i \right\}^2.$$

Lasso Regression: Regularization Path



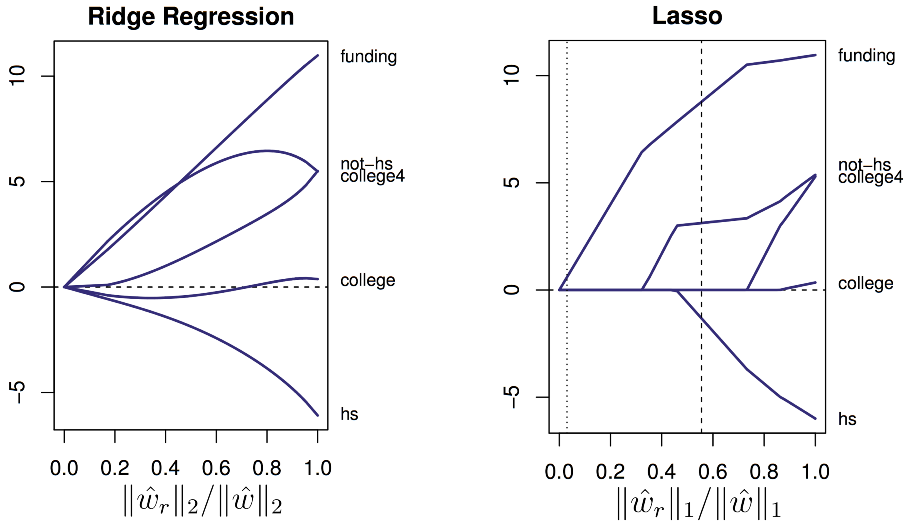
$$\hat{w}_r = \underset{\|w\|_1 \le r}{\arg\min} \frac{1}{n}\sum_{i=1}^{n}\left(w^T x_i - y_i\right)^2$$

$$\hat{w} = \hat{w}_\infty = \text{Unconstrained ERM}$$

- For $r = 0$, $\|\hat{w}_r\|_1/\|\hat{w}\|_1 = 0$.
- For $r = \infty$, $\|\hat{w}_r\|_1/\|\hat{w}\|_1 = 1$

Modified from Hastie, Tibshirani, and Wainwright's *Statistical Learning with Sparsity*, Fig 2.1. About predicting crime in 50 US cities.

Ridge vs. Lasso: Regularization Paths



Modified from Hastie, Tibshirani, and Wainwright's *Statistical Learning with Sparsity*, Fig 2.1. About predicting crime in 50 US cities.

Lasso Gives Feature Sparsity: So What?

Coefficient are 0 $\implies$ don't need those features. What's the gain?

- Time/expense to compute/buy features
- Memory to store features (e.g. real-time deployment)
- Identifies the important features
- Better prediction? sometimes
- As a feature-selection step for training a slower non-linear model

Ivanov and Tikhonov Equivalent?

- For ridge regression and lasso regression (and much more)
  - the Ivanov and Tikhonov formulations are equivalent
  - [Optional homework problem, upcoming.]
- We will use whichever form is most convenient.

Why does Lasso regression give sparse solutions?

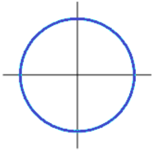
Parameter Space

- Illustrate affine prediction functions in parameter space.

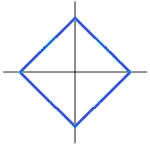
The ℓ_1 and ℓ_2 Norm Constraints

- For visualization, restrict to 2-dimensional input space
- $\mathcal{F} = \{f(x) = w_1x_1 + w_2x_2\}$ (linear hypothesis space)
- Represent $\mathcal{F}$ by $\left\{(w_1, w_2) \in \mathbf{R}^2\right\}$.

- ℓ_2 contour:
  $w_1^2 + w_2^2 = r$



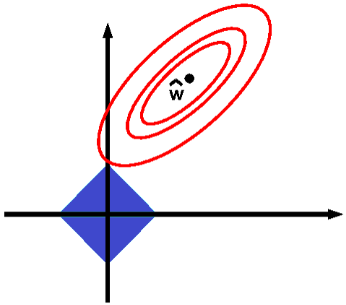
- ℓ_1 contour:
  $|w_1| + |w_2| = r$



Where are the "sparse" solutions?

The Famous Picture for ℓ_1 Regularization

- $f_r^* = \arg\min_{w \in \mathbf{R}^2} \frac{1}{n} \sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2$ subject to $|w_1| + |w_2| \leqslant r$



- Blue region: Area satisfying complexity constraint: $|w_1| + |w_2| \leqslant r$
- Red lines: contours of $\hat{R}_n(w) = \sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2$.

KPM Fig. 13.3

The Empirical Risk for Square Loss

- Denote the empirical risk of $f(x) = w^T x$ by

$$\hat{R}_n(w) = \frac{1}{n}\|Xw - y\|^2,$$

  where X is the **design matrix**.
- $\hat{R}_n$ is minimized by $\hat{w} = \left(X^T X\right)^{-1} X^T y$, the OLS solution.
- What does $\hat{R}_n$ look like around $\hat{w}$?

The Empirical Risk for Square Loss

- By "completing the square", we can show for any $w \in \mathbf{R}^d$:

$$\hat{R}_n(w) = \frac{1}{n}(w - \hat{w})^T X^T X (w - \hat{w}) + \hat{R}_n(\hat{w})$$

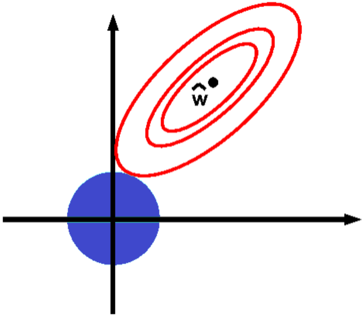
- Set of w with $\hat{R}_n(w)$ exceeding $\hat{R}_n(\hat{w})$ by $c > 0$ is

$$\left\{ w \mid \hat{R}_n(w) = c + \hat{R}_n(\hat{w}) \right\} = \left\{ w \mid (w - \hat{w})^T X^T X (w - \hat{w}) = nc \right\},$$

  which is an **ellipsoid centered at** $\hat{w}$.
- We'll derive this in homework.

The Famous Picture for ℓ_2 Regularization

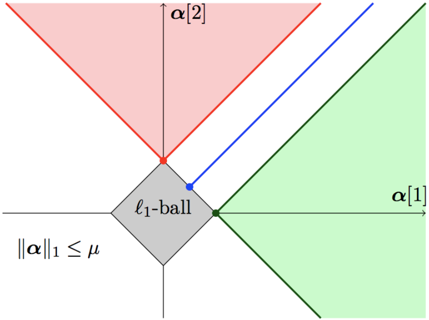
- $f_r^* = \arg\min_{w \in \mathbf{R}^2} \sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2$ subject to $w_1^2 + w_2^2 \leqslant r$



- Blue region: Area satisfying complexity constraint: $w_1^2 + w_2^2 \leqslant r$
- Red lines: contours of $\hat{R}_n(w) = \sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2$.

KPM Fig. 13.3

Why are Lasso Solutions Often Sparse?

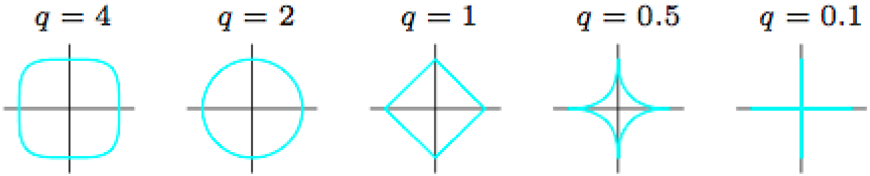


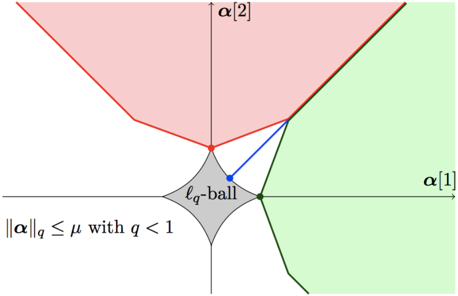
- Suppose design matrix X is orthogonal, so $X^T X = I$, and contours are circles.
- Then OLS solution in green or red regions implies ℓ_1 constrained solution will be at corner

Fig from Mairal et al.'s Sparse Modeling for Image and Vision Processing Fig 1.6

The $(\ell_q)^q$ Constraint

- Generalize to ℓ_q : $(\|w\|_q)^q = |w_1|^q + |w_2|^q$.
- Note: $\|w\|_q$ is a norm if $q \geqslant 1$, but not for $q \in (0,1)$
- $\mathcal{F} = \{f(x) = w_1x_1 + w_2x_2\}$.
- Contours of $\|w\|_q^q = |w_1|^q + |w_2|^q$:





(b) ℓ_q-ball with $q < 1$.

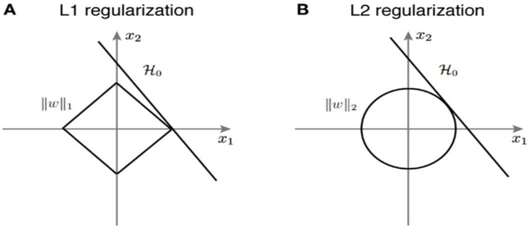
- Suppose design matrix X is orthogonal, so $X^T X = I$, and contours are circles.
- Then OLS solution in green or red regions implies ℓ_q constrained solution will be at corner

ℓ_q-ball constraint is not convex, so more difficult to optimize.

Fig from Mairal et al.'s Sparse Modeling for Image and Vision Processing Fig 1.9

The Quora Picture

- From Quora: "Why is L1 regularization supposed to lead to sparsity than L2? [sic]" (google it)



- Does this picture have any interpretation that makes sense? (Aren't those lines supposed to be ellipses?)
- Yes... we can revisit.

Figure from https://www.quora.com/Why-is-L1-regularization-supposed-to-lead-to-sparsity-than-L2.

Finding the Lasso Solution: Lasso as Quadratic Program

How to find the Lasso solution?

- How to solve the Lasso?
$$\min_{w \in \mathbf{R}^d} \sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2 + \lambda \|w\|_1$$
- $\|w\|_1 = |w_1| + |w_2|$ is not differentiable!

Splitting a Number into Positive and Negative Parts

- Consider any number $a \in \mathbf{R}$.
- Let the **positive part** of a be
$$a^+ = a1(a \geqslant 0).$$
- Let the **negative part** of a be
$$a^- = -a1(a \leqslant 0).$$
- Do you see why $a^+ \geqslant 0$ and $a^- \geqslant 0$?
- How do you write a in terms of a^+ and a^-?
- How do you write $|a|$ in terms of a^+ and a^-?

How to find the Lasso solution?

- The Lasso problem

$$\min_{w \in \mathbf{R}^d} \sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2 + \lambda \|w\|_1$$

- Replace each w_i by $w_i^+ - w_i^-$.
- Write $w^+ = \left(w_1^+, \ldots, w_d^+\right)$ and $w^- = \left(w_1^-, \ldots, w_d^-\right)$.

The Lasso as a Quadratic Program

We **will show:** substituting $w = w^+ - w^-$ and $|w| = w^+ + w^-$ gives an **equivalent** problem:

$$
\begin{array}{ll}
\min_{w^+, w^-} & \sum_{i=1}^{n} \left(\left(w^+ - w^- \right)^T x_i - y_i \right)^2 + \lambda 1^T \left(w^+ + w^- \right) \\
\text{subject to} & w_i^+ \geqslant 0 \text{ for all } i \qquad w_i^- \geqslant 0 \text{ for all } i,
\end{array}
$$

- Objective is **differentiable** (in fact, **convex and quadratic**)
- $2d$ variables vs d variables and $2d$ constraints vs no constraints
- A "**quadratic program**": a convex quadratic objective with linear constraints.
  - Could plug this into a generic QP solver.

Possible point of confusion

Equivalent to lasso problem:

$$\begin{array}{ll}\min_{w^+, w^-} & \sum_{i=1}^{n} \left(\left(w^+ - w^-\right)^T x_i - y_i \right)^2 + \lambda 1^T \left(w^+ + w^-\right) \\ \text{subject to} & w_i^+ \geqslant 0 \text{ for all } i \qquad w_i^- \geqslant 0 \text{ for all } i,\end{array}$$

- When we plug this optimization problem into a QP solver,
  - it just sees $2d$ variables and $2d$ constraints.
  - Doesn't know we want w_i^+ and w_i^- to be positive and negative parts of w_i.
- Turns out – they will come out that way as a result of the optimization!
- But to eliminate confusion, let's start by calling them a_i and b_i and prove our claim...

The Lasso as a Quadratic Program

Lasso problem is trivially equivalent to the following:

$$
\begin{aligned}
\min_{w} \min_{a,b} \quad & \sum_{i=1}^{n} \left((a-b)^T x_i - y_i \right)^2 + \lambda 1^T (a+b) \\
\text{subject to} \quad & a_i \geqslant 0 \text{ for all } i \qquad b_i \geqslant 0 \text{ for all } i, \\
& a - b = w \\
& a + b = |w|
\end{aligned}
$$

- Claim: Don't need constraint $a+b=|w|$.
- $a' \leftarrow a - \min(a,b)$ and $b' \leftarrow b - \min(a,b)$ at least as good
- So if a and b are minimizers, at least one is 0.
- Since $a-b=w$, we must have $a=w^+$ and $b=w^-$. So also $a+b=|w|$.

The Lasso as a Quadratic Program

$$
\begin{aligned}
\min_{w} \min_{a,b} \quad & \sum_{i=1}^{n} \left((a-b)^T x_i - y_i \right)^2 + \lambda 1^T (a+b) \\
\text{subject to} \quad & a_i \geqslant 0 \text{ for all } i \qquad b_i \geqslant 0 \text{ for all } i, \\
& a - b = w
\end{aligned}
$$

- Claim: Can remove $\min_w$ and the constraint $a-b=w$.
- One way to see this is by switching the order of minimization...

The Lasso as a Quadratic Program

$$\begin{array}{rl} \min_{a,b} \min_{w} & \sum_{i=1}^{n} \left((a-b)^T x_i - y_i \right)^2 + \lambda 1^T (a+b) \\ \text{subject to} & a_i \geqslant 0 \text{ for all } i \qquad b_i \geqslant 0 \text{ for all } i, \\ & a - b = w \end{array}$$

- For any $a \geqslant 0, b \geqslant 0$, there's always a single w that satisfies the constraints.
- So the inner minimum is always attained at $w = a - b$.
- Since w doesn't show up in the objective function,
  - nothing changes if we drop $\min_w$ and the constraint.

Projected SGD

$$\min_{w^+, w^- \in \mathbf{R}^d} \sum_{i=1}^{n} \left(\left(w^+ - w^- \right)^T x_i - y_i \right)^2 + \lambda 1^T \left(w^+ + w^- \right)$$

$$\text{subject to } w_i^+ \geqslant 0 \text{ for all } i$$

$$w_i^- \geqslant 0 \text{ for all } i$$

- Just like SGD, but after each step
  - Project w^+ and w^- into the constraint set.
  - In other words, if any component of w^+ or w^- becomes negative, set it back to 0.

Finding the Lasso Solution: Coordinate Descent (Shooting Method)

Coordinate Descent Method

- **Goal:** Minimize $L(w) = L(w_1, \ldots, w_d)$ over $w = (w_1, \ldots, w_d) \in \mathbf{R}^d$.
- In gradient descent or SGD,
  - each step potentially changes all entries of w.
- In each step of **coordinate descent**,
  - we adjust only a single w_i.
- In each step, solve

$$w_i^{\text{new}} = \underset{w_i}{\arg\min}\, L(w_1, \ldots, w_{i-1}, \mathbf{w_i}, w_{i+1}, \ldots, w_d)$$

- Solving this argmin may itself be an iterative process.
- Coordinate descent is great when
  - it's easy or easier to minimize w.r.t. one coordinate at a time

Coordinate Descent Method

Goal: Minimize $L(w) = L(w_1, \ldots w_d)$ over $w = (w_1, \ldots, w_d) \in \mathbf{R}^d$.

- **Initialize** $w^{(0)} = 0$
- **while** not converged:
  - Choose a coordinate $j \in \{1, \ldots, d\}$
  - $w_j^{\text{new}} \leftarrow \arg\min_{w_j} L(w_1^{(t)}, \ldots, w_{j-1}^{(t)}, \mathbf{w_j}, w_{j+1}^{(t)}, \ldots, w_d^{(t)})$
  - $w_j^{(t+1)} \leftarrow w_j^{\text{new}}$ and $w^{(t+1)} \leftarrow w^{(t)}$
  - $t \leftarrow t+1$

- Random coordinate choice $\implies$ **stochastic coordinate descent**
- Cyclic coordinate choice $\implies$ **cyclic coordinate descent**

In general, **we will adjust each coordinate several times.**

Coordinate Descent Method for Lasso

- Why mention coordinate descent for Lasso?
- In Lasso, the coordinate minimization has a **closed form solution**!

Closed Form Coordinate Minimization for Lasso

$$\hat{w}_j = \arg\min_{w_j \in \mathbf{R}} \sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2 + \lambda |w|_1$$

Then

$$\hat{w}_j = \begin{cases} (c_j + \lambda)/a_j & \text{if } c_j < -\lambda \\ 0 & \text{if } c_j \in [-\lambda, \lambda] \\ (c_j - \lambda)/a_j & \text{if } c_j > \lambda \end{cases}$$

$$a_j = 2\sum_{i=1}^{n} x_{i,j}^2 \qquad\qquad c_j = 2\sum_{i=1}^{n} x_{i,j}(y_i - w_{-j}^T x_{i,-j})$$

where w_{-j} is w without component j and similarly for $x_{i,-j}$.

Coordinate Descent: When does it work?

- Suppose we're minimizing $f : \mathbf{R}^d \to \mathbf{R}$.
- Sufficient conditions:
  1. f is continuously differentiable and
  2. f is strictly convex in each coordinate
- But lasso objective

$$\sum_{i=1}^{n} \left(w^T x_i - y_i\right)^2 + \lambda \|w\|_1$$

  is not differentiable...
- Luckily there are weaker conditions...

Coordinate Descent: The Separability Condition

Theorem

[a] *If the objective f has the following structure*

$$f(w_1,\ldots,w_d) = g(w_1,\ldots,w_d) + \sum_{j=1}^{d} h_j(x_j),$$

where

- $g : \mathbf{R}^d \to \mathbf{R}$ *is differentiable and convex, and*
- *each* $h_j : \mathbf{R} \to \mathbf{R}$ *is convex (but not necessarily differentiable)*

then the coordinate descent algorithm converges to the global minimum.

[a] Tseng 1988: "Coordinate ascent for maximizing nondifferentiable concave functions", Technical Report LIDS-P

Coordinate Descent Method – Variation

- Suppose there's no closed form? (e.g. logistic regression)
- Do we really need to fully solve each inner minimization problem?
- A single projected gradient step is enough for ℓ_1 regularization!
    - Shalev-Shwartz & Tewari's "Stochastic Methods..." (2011)

- Let $\tilde{w} = (w^+, w^-) \in \mathbf{R}^{2d}$ and

$$L(\tilde{w}) = \sum_{i=1}^{n} \left(\left(w^+ - w^- \right)^T x_i - y_i \right)^2 + \lambda \left(w^+ + w^- \right)$$

Stochastic Coordinate Descent for Lasso - Variation

Goal: Minimize $L(\tilde{w})$ s.t. $w_i^+, w_i^- \geqslant 0$ for all i.

- **Initialize** $\tilde{w}^{(0)} = 0$
  - **while** not converged:
    - Randomly choose a coordinate $j \in \{1, \ldots, 2d\}$
    - $\tilde{w}_j \leftarrow \tilde{w}_j + \max\left\{-\tilde{w}_j, -\nabla_j L(\tilde{w})\right\}$