Recitation 4

Subgradients

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Intro Question

Question

When stating a convex optimization problem in standard form we write

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad \text{for all } i = 1, \ldots, n. \end{array}$$

where $f_0, f_1, \ldots, f_n$ are convex. Why don't we use $\geq$ or $=$ instead of $\leq$?

Review of Convexity

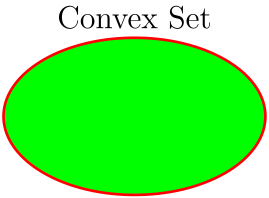
Definition (Convex Set)

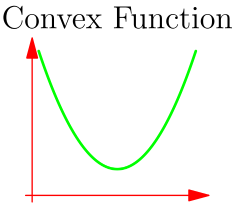
A set $S \subseteq \mathbb{R}^d$ is convex if for any $x, y \in S$ and $\theta \in (0,1)$ we have $(1-\theta)x + \theta y \in S$.

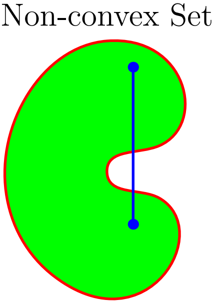
Definition (Convex Function)

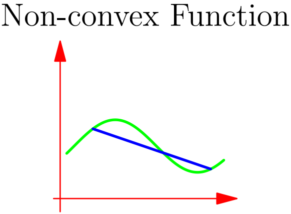
A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if for any $x, y \in \mathbb{R}^d$ and $\theta \in (0,1)$ we have $f((1-\theta)x + \theta y) \leq (1-\theta)f(x) + \theta f(y)$.

Review of Convexity









(Sub-)Level Sets of Convex Functions

Definition ((Sub-)Level Sets)

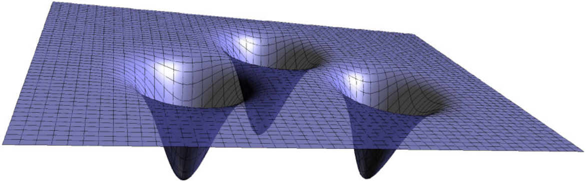
For a function $f : \mathbb{R}^d \to \mathbb{R}$, a *level set* (or contour line) corresponding to the value c is given by the set of all points $x \in \mathbb{R}^d$ where $f(x) = c$:

$$f^{-1}\{c\} = \{x \in \mathbb{R}^d \mid f(x) = c\}.$$

Analogously, the *sublevel set* for the value c is the set of all points $x \in \mathbb{R}^d$ where $f(x) \leq c$:

$$f^{-1}(-\infty, c] = \{x \in \mathbb{R}^d \mid f(x) \leq c\}.$$

3D Plot and Contour Plot With Sublevel Set



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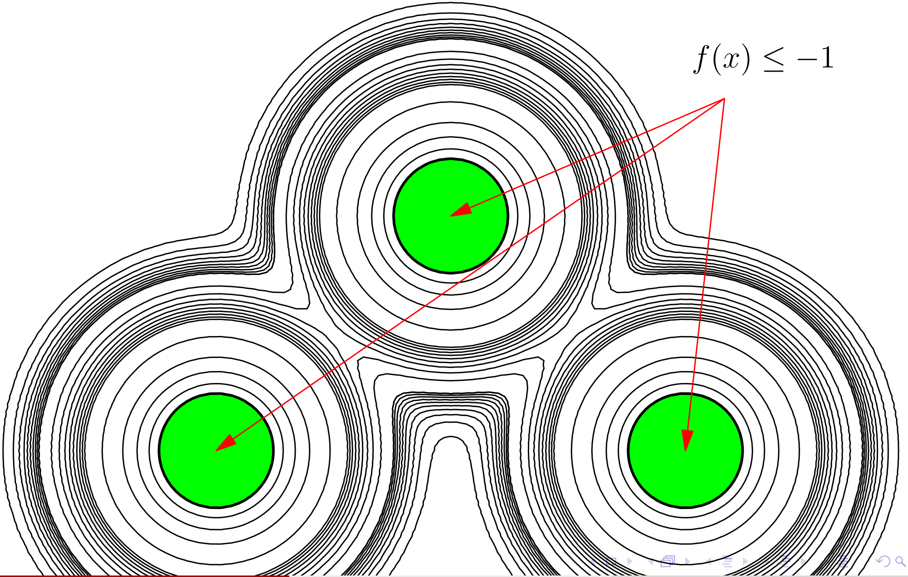

Sublevel Sets of Convex Functions

Theorem

If $f : \mathbb{R}^d \to \mathbb{R}$ is convex then the sublevel sets are convex.

Sublevel Sets of Convex Functions

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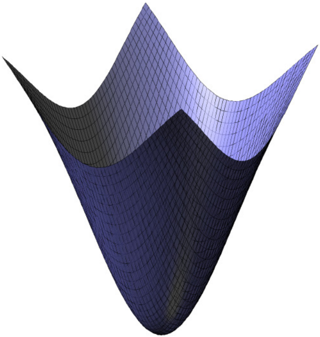
Proof.

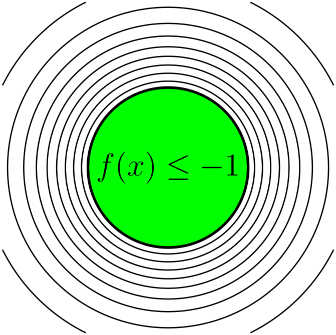
Fix a sublevel set $S = \{x \in \mathbb{R}^d \mid f(x) \leq c\}$ for some fixed $c \in \mathbb{R}$. If $x, y \in S$ and $\theta \in (0, 1)$ then we have

$$f((1-\theta)x + \theta y) \leq (1-\theta)f(x) + \theta f(y) \leq (1-\theta)c + \theta c = c.$$

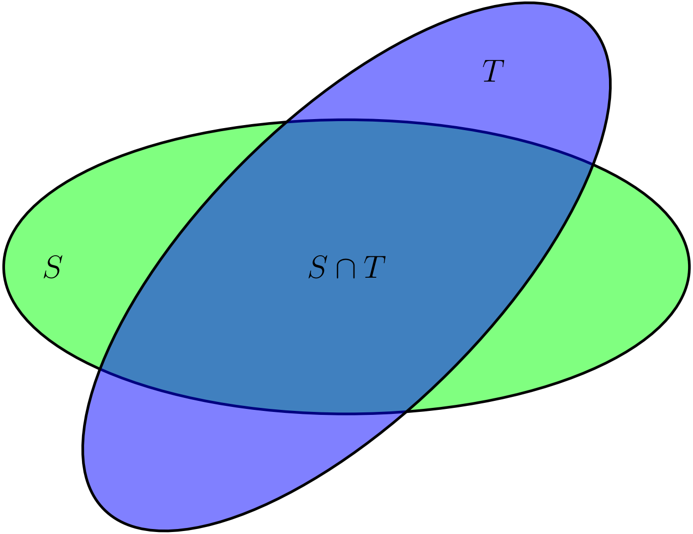
$\square$

Plots of Convex Function With Sublevel Set

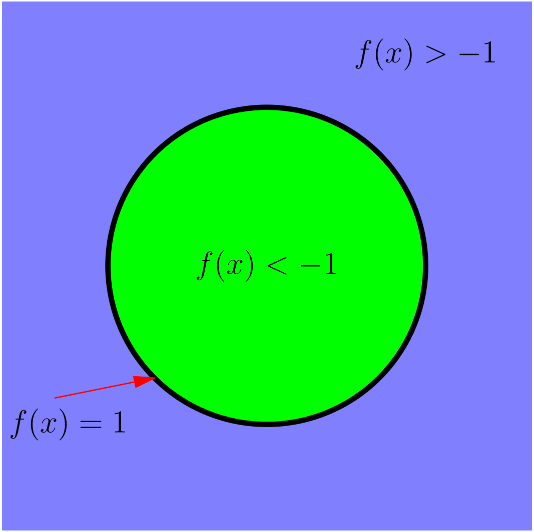




Intersection of Convex Sets is Convex



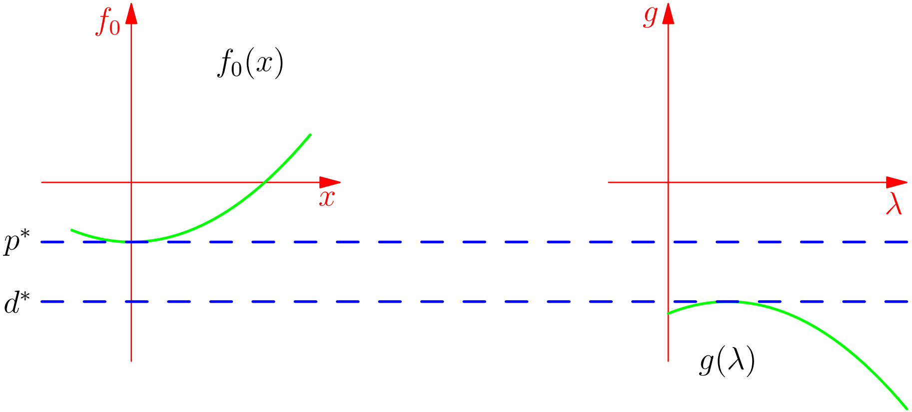
Level Sets and Superlevel Sets Not Convex



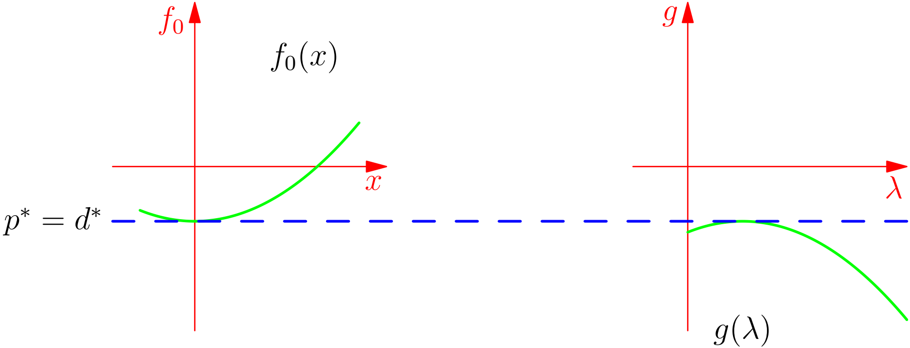
Lagrange Duality

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0 \end{array} \longrightarrow \boxed{\text{Lagrange Dual}} \longrightarrow \begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \ge 0 \end{array}$$

Weak Duality



Strong Duality



Gradient Characterization of Convexity

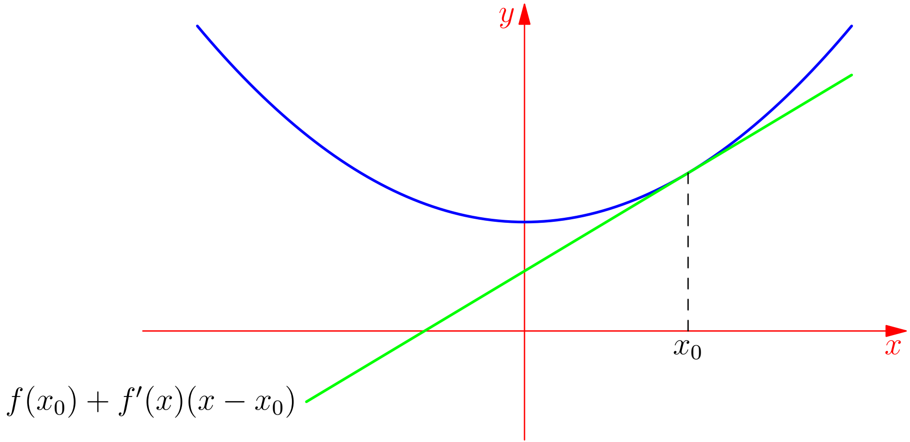
Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable. Then f is convex iff

$$f(x+v) \geq f(x) + \nabla f(x)^T v$$

hold for all $x, v \in \mathbb{R}^d$.

Gradient Approximation Gives Global Underestimator



Subgradients

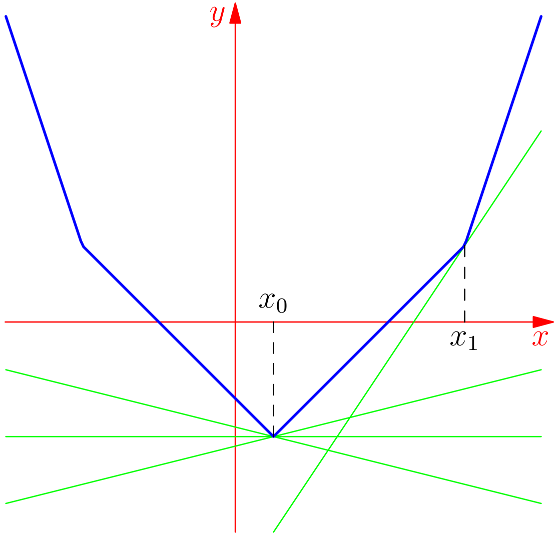
Definition (Subgradient, Subdifferential, Subdifferentiable)

Let $f : \mathbb{R}^d \to \mathbb{R}$. We say that $g \in \mathbb{R}^d$ is a *subgradient* of f at $x \in \mathbb{R}^d$ if

$$f(x+v) \geq f(x) + g^T v$$

for all $v \in \mathbb{R}^d$. The *subdifferential* $\partial f(x)$ is the set of all subgradients of f at x. We say that f is *subdifferentiable* at x if $\partial f(x) \neq \emptyset$ (i.e., if there is at least one subgradient).

Subgradients at x_0 and x_1



Facts About Subgradients

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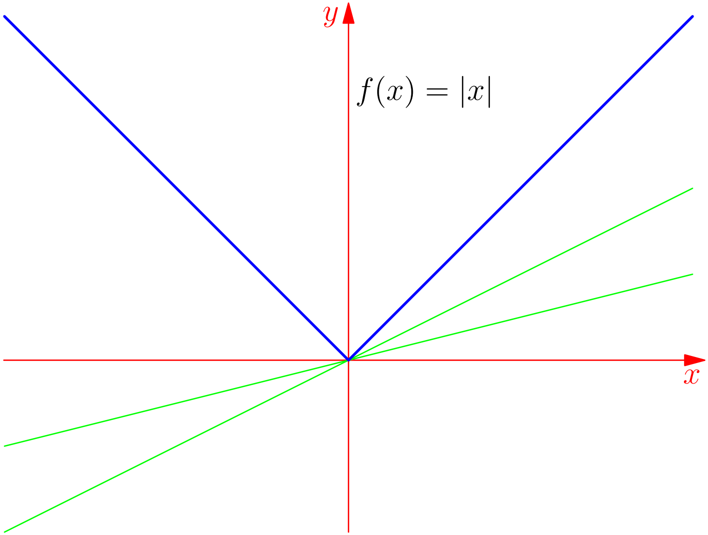
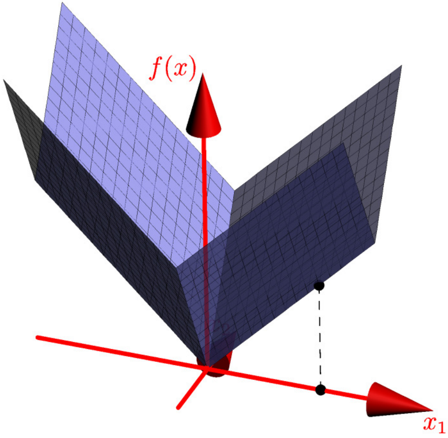
4. If the zero vector is a subgradient of f at x, then x is a global minimum.
5. If g is a subgradient of f at x, then $(g, -1)$ is orthogonal to the underestimating hyperplane $\{(x + v, f(x) + g^T v) \mid v \in \mathbb{R}^d\}$ at $(x, f(x))$.

Compute the Subdifferentials of $f(x) = |x|$



Compute $\partial f(3,0)$ For $f(x_1,x_2) = |x_1| + 2|x_2|$



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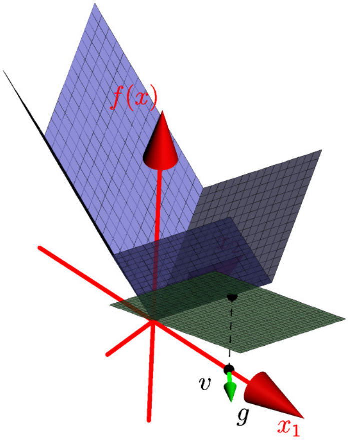
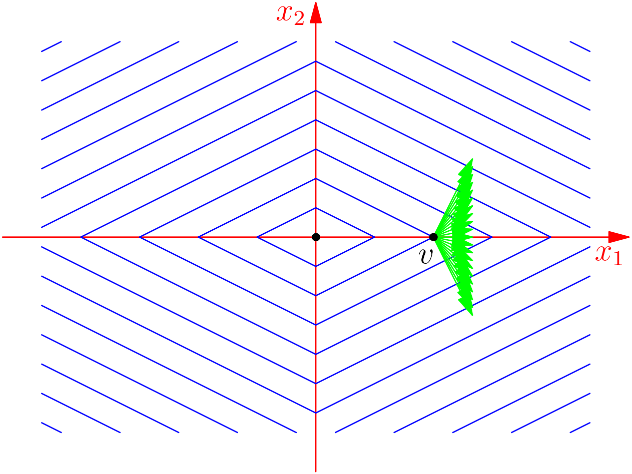


Compute $\partial f(3,0)$ For $f(x_1,x_2) = |x_1| + 2|x_2|$

$$\partial f(3,0) = \{(1,b)^T \mid b \in [-2,2]\}$$

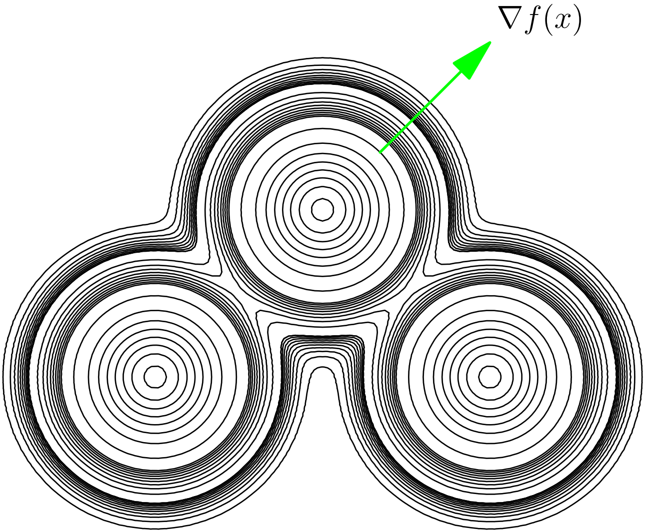


Gradient Lies Normal To Contours

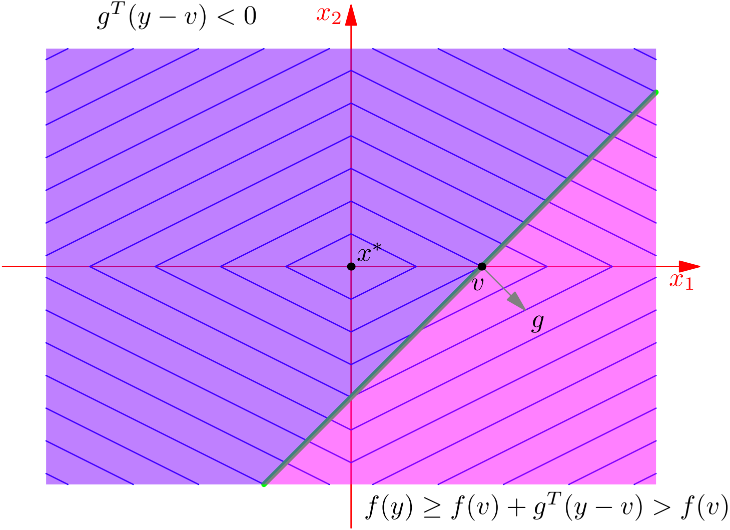
Theorem

If $f : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable and $x_0 \in \mathbb{R}^d$ with $\nabla f(x_0) \neq 0$ then $\nabla f(x_0)$ is normal to the level set $S = \{x \in \mathbb{R}^d \mid f(x) = f(x_0)\}$.

Gradient Lies Normal To Contours



Normal Plane to Subgradient Splits Space



Subgradient Descent

1. Let $x^{(0)}$ denote the initial point.
2. For $k = 1, 2, \ldots$
   1. Assign $x^{(k)} = x^{(k-1)} - \alpha_k g$, where $g \in \partial f(x^{(k-1)})$ and α_k is the step size.
   2. Set $f_{\text{best}}^{(k)} = \min_{i=1,\ldots,k} f(x^{(i)})$. (Used since this isn't a descent method.)

Convergence of Subgradient Descent

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and Lipschitz with constant G, and let x^* be a minimizer. For a fixed step size t, the subgradient method satisfies:

$$\lim_{k\to\infty} f(x_{best}^{(k)}) \leq f(x^*) + G^2 t/2.$$

For step sizes respecting the Robbins-Monro conditions,

$$\lim_{k\to\infty} f(x_{best}^{(k)}) = f(x^*).$$