SVM and Complementary Slackness

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SVM Review: Primal and Dual Formulations

Support Vector Machine

- Hypothesis space $\mathcal{F} = \left\{ f(x) = w^T x + b \mid w \in \mathbf{R}^d, b \in \mathbf{R} \right\}$.
- ℓ_2 regularization (Tikhonov style)
- Loss $\ell(m) = \max\{1-m, 0\} = (1-m)_+$
- The SVM prediction function is the solution to

$$\min_{w \in \mathbf{R}^d, b \in \mathbf{R}} \frac{1}{2}||w||^2 + \frac{c}{n}\sum_{i=1}^{n} \max\left(0, 1 - y_i\left[w^T x_i + b\right]\right).$$

SVM as a Quadratic Program

- The SVM optimization problem is equivalent to

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}\|w\|^2 + \frac{c}{n}\sum_{i=1}^{n}\xi_i \\
\text{subject to} & -\xi_i \leqslant 0 \text{ for } i = 1,\ldots,n \\
 & \left(1 - y_i\left[w^T x_i + b\right]\right) - \xi_i \leqslant 0 \text{ for } i = 1,\ldots,n
\end{array}
$$

- Differentiable objective function
- $2n$ affine constraints.
- A quadratic program that can be solved by any off-the-shelf QP solver.
- Let's learn more by examining the dual.

SVM Lagrange Multipliers

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}\|w\|^2 + \frac{c}{n}\sum_{i=1}^{n}\xi_i \\
\text{subject to} & -\xi_i \leqslant 0 \text{ for } i = 1,\ldots,n \\
 & \left(1 - y_i\left[w^T x_i + b\right]\right) - \xi_i \leqslant 0 \text{ for } i = 1,\ldots,n
\end{array}
$$

Lagrange Multiplier	Constraint
λ_i	-$\xi_i \leqslant 0$
α_i	$\left(1 - y_i\left[w^T x_i + b\right]\right) - \xi_i \leqslant 0$

$$
L(w,b,\xi,\alpha,\lambda) = \frac{1}{2}\|w\|^2 + \frac{c}{n}\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - y_i\left[w^T x_i + b\right] - \xi_i\right) + \sum_{i=1}^{n}\lambda_i\left(-\xi_i\right)
$$

SVM Primal and Dual

- Lagrangian:

$$L(w,b,\xi,\alpha,\lambda) \;=\; \frac{1}{2}\|w\|^2 + \frac{c}{n}\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - y_i\left[w^T x_i + b\right] - \xi_i\right) + \sum_{i=1}^{n}\lambda_i\left(-\xi_i\right)$$

- Primal and dual formulations:

$$p^* = \inf_{w,\xi,b} \underbrace{\sup_{\alpha,\lambda\succeq 0} L(w,b,\xi,\alpha,\lambda)}_{\text{primal objective}} \geqslant \sup_{\alpha,\lambda\succeq 0} \underbrace{\inf_{w,b,\xi} L(w,b,\xi,\alpha,\lambda)}_{\text{dual objective } g(\alpha,\lambda)} = d^*$$

- Constraints are satisfied by $w=b=0$ and $\xi_i=1$ for $i=1,\ldots,n$.
- So we have strong duality by Slater's conditions. That is: $p^*=d^*$.

First Order Conditions (KKT)

- Suppose (w^*, b^*) are primal optimal and $(\xi^*, \alpha^*, \lambda^*)$ are dual optimal.
- By strong duality, $p^* = d^*$, and by "sandwich proof",

$$p^* = d^* = L(w^*, b^*, \xi^*, \alpha^*, \lambda^*).$$

- Since L is differentiable, we must have first order conditions:

$$\begin{aligned} \nabla_w L(w^*, b^*, \xi^*, \alpha^*, \lambda^*) &= 0 \\ \text{(shown last week)} \iff w^* &= \sum_{i=1}^{n} \alpha_i^* y_i x_i \end{aligned}$$

- Conclude that w^* is "in the span of the data" – i.e. a linear combination of $x_1, \ldots, x_n$.

The SVM Dual Problem

- We found the SVM dual problem can be written as:

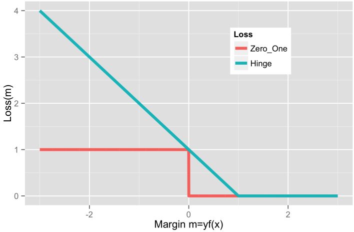
$$
\begin{aligned}
\sup_{\alpha} \quad & \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j y_i y_j x_j^T x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \alpha_i y_i = 0 \\
& \alpha_i \in \left[0, \frac{c}{n}\right] \quad i = 1, \ldots, n.
\end{aligned}
$$

- Given solution α^* to dual, primal solution is $w^* = \sum_{i=1}^{n} \alpha_i^* y_i x_i$.
- Note $\alpha_i^* \in [0, \frac{c}{n}]$. So c controls max weight on each example. (**Robustness**!)

Insights From Complementary Slackness: Margin and Support Vectors

The Margin and Some Terminology

- For notational convenience, define $f^*(x) = x^T w^* + b^*$.
- Margin $yf^*(x)$



- Incorrect classification: $yf^*(x) \leqslant 0$.
- Margin error: $yf^*(x) < 1$.
- "On the margin": $yf^*(x) = 1$.
- "Good side of the margin": $yf^*(x) > 1$.

Support Vectors and The Margin

- Recall "**slack variable**" $\xi_i^* = \max(0, 1 - y_i f^*(x_i))$ is the hinge loss on (x_i, y_i).
- Suppose $\xi_i^* = 0$.
- Then $y_i f^*(x_i) \geqslant 1$
  - "on the margin" $(= 1)$, or
  - "on the good side" (> 1)

Complementary Slackness Conditions

- Recall our primal constraints and Lagrange multipliers:

Lagrange Multiplier	Constraint
λ_i	$-\xi_i \leqslant 0$
α_i	$(1-y_i f(x_i)) - \xi_i \leqslant 0$

- Recall first order condition $\nabla_{\xi_i} L = 0$ gave us $\lambda_i^* = \frac{c}{n} - \alpha_i^*$.
- By strong duality, we must have **complementary slackness**:

$$\alpha_i^* \left(1 - y_i f^*(x_i) - \xi_i^*\right) = 0$$

$$\lambda_i^* \xi_i^* = \left(\frac{c}{n} - \alpha_i^*\right) \xi_i^* = 0$$

Consequences of Complementary Slackness

- By strong duality, we must have **complementary slackness**:

$$\alpha_i^*\left(1-y_i f^*(x_i)-\xi_i^*\right)=0$$
$$\left(\frac{c}{n}-\alpha_i^*\right)\xi_i^*=0$$

- If $y_i f^*(x)>1$ then the margin loss is $\xi_i^*=0$, and we get $\alpha_i^*=0$.

- If $y_i f^*(x_i)<1$ then the margin loss is $\xi_i^*>0$, so $\alpha_i^*=\frac{c}{n}$.
- If $\alpha_i^*=0$, then $\xi_i^*=0$, which implies no loss, so $y_i f^*(x)\geqslant 1$.
- If $\alpha_i^*\in\left(0,\frac{c}{n}\right)$, then $\xi_i^*=0$, which implies $1-y_i f^*(x_i)=0$.

Complementary Slackness Results: Summary

$$
\begin{aligned}
\alpha_i^* = 0 \quad &\implies \quad y_i f^*(x_i) \geqslant 1 \\
\alpha_i^* \in \left(0, \frac{c}{n}\right) \quad &\implies \quad y_i f^*(x_i) = 1 \\
\alpha_i^* = \frac{c}{n} \quad &\implies \quad y_i f^*(x_i) \leqslant 1
\end{aligned}
$$

$$
\begin{aligned}
y_i f^*(x_i) < 1 \quad &\implies \quad \alpha_i^* = \frac{c}{n} \\
y_i f^*(x_i) = 1 \quad &\implies \quad \alpha_i^* \in \left[0, \frac{c}{n}\right] \\
y_i f^*(x_i) > 1 \quad &\implies \quad \alpha_i^* = 0
\end{aligned}
$$

Support Vectors

- If α^* is a solution to the dual problem, then primal solution is

$$w^* = \sum_{i=1}^{n} \alpha_i^* y_i x_i$$

  with $\alpha_i^* \in [0, \frac{c}{n}]$.
- The x_i's corresponding to $\alpha_i^* > 0$ are called **support vectors.**
- Few margin errors or "on the margin" examples $\Longrightarrow$ **sparsity in input examples**.

Complementary Slackness To Get b^*

The Bias Term: b

- For our SVM primal, the complementary slackness conditions are:

$$\alpha_i^* \left(1 - y_i \left[x_i^T w^* + b\right] - \xi_i^*\right) = 0 \tag{1}$$

$$\lambda_i^* \xi_i^* = \left(\frac{c}{n} - \alpha_i^*\right) \xi_i^* = 0 \tag{2}$$

- Suppose there's an i such that $\alpha_i^* \in \left(0, \frac{c}{n}\right)$.
- (2) implies $\xi_i^* = 0$.
- (1) implies

$$\begin{aligned} & y_i \left[x_i^T w^* + b^*\right] = 1 \\ \iff \quad & x_i^T w^* + b^* = y_i \text{ (use } y_i \in \{-1, 1\}) \\ \iff \quad & \boxed{b^* = y_i - x_i^T w^*} \end{aligned}$$

The Bias Term: b

- The optimal b is

$$b^* = y_i - x_i^T w^*$$

- We get the same b^* for any choice of i with $\alpha_i^* \in \left(0, \frac{c}{n}\right)$
  - **With exact calculations!**
- With numerical error, more robust to average over all eligible i's:

$$b^* = \text{mean}\left\{ y_i - x_i^T w^* \mid \alpha_i^* \in \left(0, \frac{c}{n}\right) \right\}.$$

- If there are no $\alpha_i^* \in \left(0, \frac{c}{n}\right)$?
  - Then we have a **degenerate SVM training problem**[1] ($w^* = 0$).

[1] See Rifkin et al.'s "A Note on Support Vector Machine Degeneracy", an MIT AI Lab Technical Report.

Teaser for Kernelization

Dual Problem: Dependence on x through inner products

- SVM Dual Problem:

$$
\begin{aligned}
\sup_{\alpha} \quad & \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j y_i y_j x_j^T x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \alpha_i y_i = 0 \\
& \alpha_i \in \left[0, \frac{c}{n}\right] \quad i = 1, \ldots, n.
\end{aligned}
$$

- Note that all dependence on inputs x_i and x_j is through their inner product: $\langle x_j, x_i \rangle = x_j^T x_i$.
- We can replace $x_j^T x_i$ by any other product...
- This is a "kernelized" objective function.