# Lagrangian Duality and Convex Optimization

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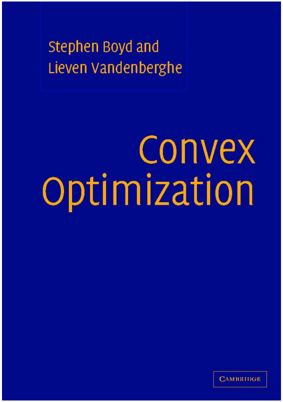
# Why Convex Optimization?

- Historically:
  - **Linear programs** (linear objectives & constraints) were the focus
  - **Nonlinear programs**: some easy, some hard
- Today:
  - Main distinction is between **convex** and **non-convex** problems
  - Convex problems are the ones we know how to solve efficiently
- Many techniques that are well understood for convex problems are applied to non-convex problems
  - e.g. SGD is routinely applied to neural networks

# Your Reference for Convex Optimization

- Boyd and Vandenberghe (2004)
  - Very clearly written, but has a ton of detail for a first pass.
  - See my "Extreme Abridgement of Boyd and Vandenberghe".



# Notation from Boyd and Vandenberghe

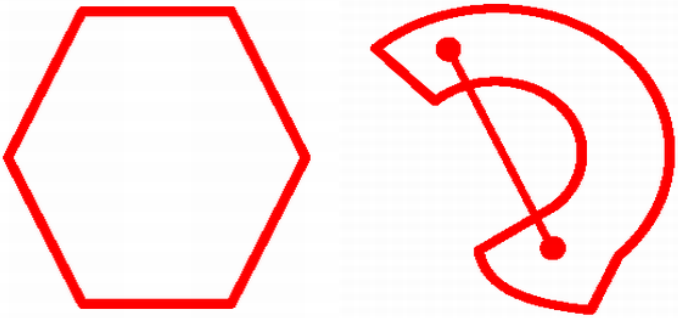
- $f : \mathbf{R}^p \to \mathbf{R}^q$ to mean that $f$ maps from some *subset* of $\mathbf{R}^p$
    - namely $\mathbf{dom}\, f \subset \mathbf{R}^p$, where $\mathbf{dom}\, f$ is the domain of $f$

# Convex Sets

### Definition

A set $C$ is **convex** if for any $x_1, x_2 \in C$ and any $\theta$ with $0 \leqslant \theta \leqslant 1$ we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$



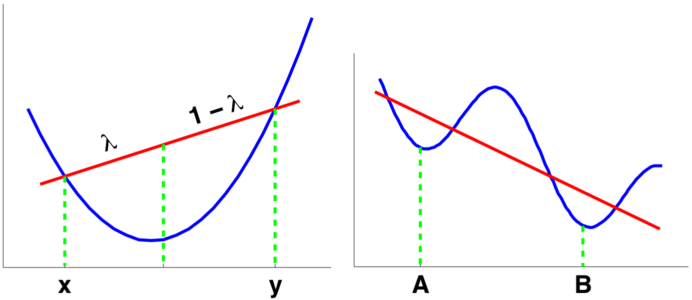
KPM Fig. 7.4

# Convex and Concave Functions

## Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is **convex** if $\mathbf{dom}\ f$ is a convex set and if for all $x, y \in \mathbf{dom}\ f$, and $0 \leqslant \theta \leqslant 1$, we have

$$f(\theta x + (1-\theta)y) \leqslant \theta f(x) + (1-\theta) f(y).$$




KPM Fig. 7.5

# Examples of Convex Functions on $\mathbf{R}$

## Examples

- $x \mapsto ax + b$ is both convex and concave on $\mathbf{R}$for all $a, b \in \mathbf{R}$.
- $x \mapsto |x|^p$ for $p \geqslant 1$ is convex on $\mathbf{R}$
- $x \mapsto e^{ax}$ is convex on $\mathbf{R}$ for all $a \in \mathbf{R}$

# Maximum of Convex Functions is Convex

**Theorem**

*If $f_1, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$ are convex, then their pointwise maximum*

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

*is also convex with domain* $\textbf{dom } f = \textbf{dom } f_1 \cap \cdots \cap \textbf{dom } f_m$.

**This result extends to sup over arbitrary [infinite] sets of functions.**

**Proof.**

(For $m = 2$.) Fix an $0 \leqslant \theta \leqslant 1$ and $x, y \in \textbf{dom } f$. Then

$$\begin{aligned}
f(\theta x + (1-\theta)y) &= \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\
&\leqslant \max\{\theta f_1(x) + (1-\theta)f_1(y), \theta f_2(x) + (1-\theta)f_2(y)\} \\
&\leqslant \max\{\theta f_1(x), \theta f_2(x)\} + \max\{(1-\theta)f_1(y), (1-\theta)f_2(y)\} \\
&= \theta f(x) + (1-\theta)f(y)
\end{aligned}$$

# Convex Functions and Optimization

### Definition

A function $f$ is **strictly convex** if the line segment connecting any two points on the graph of $f$ lies **strictly** above the graph (excluding the endpoints).

Consequences for optimization:

- **convex:** if there is a local minimum, then it is a **global** minimum
- **strictly convex:** if there is a local minimum, then it is the **unique global** minumum

# General Optimization Problem: Standard Form

## General Optimization Problem: Standard Form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \;\; i = 1, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, \ldots p, \end{array}$$

where $x \in \mathbf{R}^n$ are the **optimization variables** and $f_0$ is the **objective function.**

Assume **domain** $\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathbf{dom}\, h_i$ is nonempty.

# General Optimization Problem: More Terminology

- The set of points satisfying the constraints is called the **feasible set**.
- A point $x$ in the feasible set is called a **feasible point.**
- If $x$ is feasible and $f_i(x) = 0$,
  - then we say the inequality constraint $f_i(x) \leqslant 0$ is **active** at $x$.
- The **optimal value** $p^*$ of the problem is defined as

$$p^* = \inf\{f_0(x) \mid f_i(x) \leqslant 0, i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}.$$

- $x^*$ is an **optimal point** (or a solution to the problem) if $x^*$ is feasible and $f(x^*) = p^*$.

# The Lagrangian

Recall the general optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \;\; i = 1, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, \ldots p, \end{array}$$

### Definition

The **Lagrangian** for the general optimization problem is

$$L(x, \lambda, \nu) = f_0(x) + \sum_{l=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x),$$

- $\lambda_i$'s and $\nu$'s are called **Lagrange multipliers**
- $\lambda$ and $\nu$ also called the **dual variables** .

# The Lagrangian Encodes the Objective and Constraints

- Supremum over Lagrangian gives back objective and constraints:

$$\begin{aligned} \sup_{\lambda \succeq 0, \nu} L(x, \lambda, \nu) &= \sup_{\lambda \succeq 0, \nu} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x), \right) \\ &= \begin{cases} f_0(x) & f_i(x) \leqslant 0 \text{ and } h_i(x) = 0, \text{ all } i \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

- Equivalent **primal form** of optimization problem:

$$p^* = \inf_x \sup_{\lambda \succeq 0, \nu} L(x, \lambda, \nu)$$

# The Primal and the Dual

- Original optimization problem in **primal form:**

$$p^* = \inf_x \sup_{\lambda \succeq 0, \nu} L(x, \lambda, \nu)$$

- The **Lagrangian dual problem**:

$$d^* = \sup_{\lambda \succeq 0, \nu} \inf_x L(x, \lambda, \nu)$$

- We will show **weak duality**: $p^* \geqslant d^*$ for any optimization problem

# Weak Max-Min Inequality

### Theorem

*For **any** $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, $W \subseteq \mathbf{R}^n$, or $Z \subseteq \mathbf{R}^m$, we have*

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) \leqslant \inf_{w \in W} \sup_{z \in Z} f(w, z).$$

### Proof.

For any $w_0 \in W$ and $z_0 \in Z$, we clearly have

$$\inf_{w \in W} f(w, z_0) \leqslant f(w_0, z_0) \leqslant \sup_{z \in Z} f(w_0, z).$$

Since this is true for all $w_0$ and $z_0$, we must also have

$$\sup_{z_0 \in Z} \inf_{w \in W} f(w, z_0) \leqslant \inf_{w_0 \in W} \sup_{z \in Z} f(w_0, z).$$

□

# Weak Duality

- For any optimization problem (**not just convex**), weak max-min inequality implies **weak duality**:

$$
\begin{aligned}
p^* = & \inf_x \sup_{\lambda \succcurlyeq 0, \nu} \left[ f_0(x) + \sum_{l=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right] \\
\geqslant & \sup_{\lambda \succcurlyeq 0, \nu} \inf_x \left[ f_0(x) + \sum_{l=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right] = d^*
\end{aligned}
$$

- The difference $p^* - d^*$ is called the **duality gap**.
- For *convex* problems, we often have **strong duality**: $p^* = d^*$.

# The Lagrange Dual Function

- The **Lagrangian dual problem**:

$$d^* = \sup_{\lambda \succeq 0, \nu} \underbrace{\inf_x L(x, \lambda, \nu)}_{\text{Lagrange dual function}}$$

### Definition

The **Lagrange dual function** (or just **dual function**) is

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right).$$

- The dual function may take on the value $-\infty$ (e.g. $f_0(x) = x$).

# The Lagrange Dual Problem

- In terms of Lagrange dual function, we can write weak duality as

$$p^* \geqslant \sup_{\lambda \geqslant 0, \nu} g(\lambda, \nu) = d^*$$

- So for any $(\lambda, \nu)$ with $\lambda \geqslant 0$, **Lagrange dual function gives a lower bound on optimal solution**:

$$g(\lambda, \nu) \leqslant p^*$$

# The Lagrange Dual Problem

- The **Lagrange dual problem** is a search for best lower bound:

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0. \end{array}$$

  - $(\lambda, \nu)$ **dual feasible** if $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$.
  - $(\lambda^*, \nu^*)$ are **dual optimal** or **optimal Lagrange multipliers** if they are optimal for the Lagrange dual problem.
- Lagrange dual problem often easier to solve (simpler constraints).
- $d^*$ can be used as stopping criterion for primal optimization.
- Dual can reveal hidden structure in the solution.

# Convex Optimization Problem: Standard Form

## Convex Optimization Problem: Standard Form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \;\; i = 1, \ldots, m \\ & a_i^T x = b_i, \;\; i = 1, \ldots p \end{array}$$

where $f_0, \ldots, f_m$ are convex functions.
Note: Equality constraints are now linear. Why?

# Strong Duality for Convex Problems

- For a convex optimization problems, we **usually** have strong duality, but not always.
    - For example:

$$
\begin{array}{ll}
\text{minimize} & e^{-x} \\
\text{subject to} & x^2/y \leqslant 0 \\
 & y > 0
\end{array}
$$

- The additional conditions needed are called **constraint qualifications**.

Example from Laurent El Ghaoui's EE 227A: Lecture 8 Notes, Feb 9, 2012

# Slater's Constraint Qualifications for Strong Duality

- Sufficient conditions for strong duality in a **convex** problem.
- Roughly: the problem must be **strictly** feasible.
- Qualifications when problem domain $\mathcal{D} \subset \mathbf{R}^n$ is an open set:
  - $\exists x$ such that $Ax = b$ and $f_i(x) < 0$ for $i = 1, \ldots, m$
  - For any affine inequality constraints, $f_i(x) \leqslant 0$ is sufficient
- Otherwise, $x$ must be in the "relative interior" of $\mathcal{D}$
  - See notes, or BV Section 5.2.3, p. 226.

# Complementary Slackness

- Consider a general optimization problem (i.e. not necessarily convex).
- If we have **strong duality**, we get an interesting relationship between
  - the optimal Lagrange multiplier $\lambda_i$ and
  - the $i$th constraint at the optimum: $f_i(x^*)$
- Relationship is called "**complementary slackness**":

$$\lambda_i^* f_i(x^*) = 0$$

- Lagrange multiplier is zero unless the constraint is active at the optimum.

# Complementary Slackness Proof

- Assume strong duality: $p^* = d^*$ in a general optimization problem
- Let $x^*$ be primal optimal and $(\lambda^*, \nu^*)$ be dual optimal. Then:

$$
\begin{aligned}
f_0(x^*) &= g(\lambda^*, \nu^*) \\
&= \inf_x \left( f_0(x) + \sum_{l=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x) \right) \\
&\leqslant f_0(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^* f_i(x^*)}_{\leqslant 0} + \sum_{i=1}^{p} \underbrace{\nu_i^* h_i(x^*)}_{=0} \\
&\leqslant f_0(x^*).
\end{aligned}
$$

Each term in sum $\sum_{i=1} \lambda_i^* f_i(x^*)$ must actually be 0. That is

$$\lambda_i^* f_i(x^*) = 0, \quad i = 1, \ldots, m.$$

This condition is known as **complementary slackness**.