Back Propagation and the Chain Rule

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Learning with Back-Propagation

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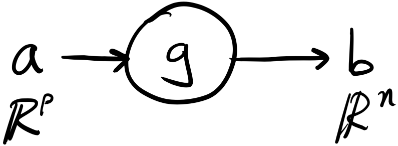
Learning with Back-Propagation

- Back-propagation is an **algorithm** for computing the gradient
- With lots of chain rule, you could also work out the gradient by hand.
- Back-propagation is
    - a clean way to organize the computation of the gradient
    - an efficient way to compute the gradient

Partial Derivatives

- Consider a function $g : \mathbf{R}^p \to \mathbf{R}^n$.
  - Typical computation graph:



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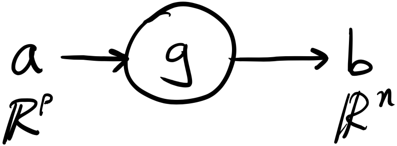
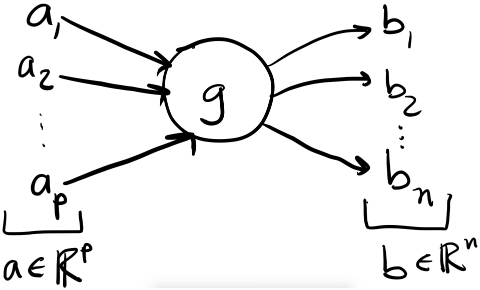


- Broken out into components:



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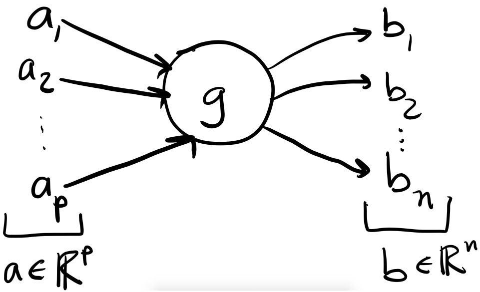

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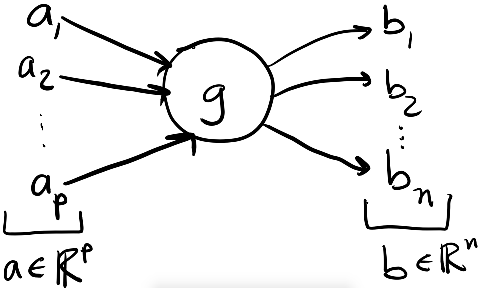
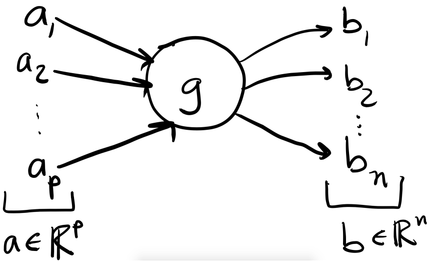


- Partial derivative $\frac{\partial b_i}{\partial a_j}$ is the instantaneous rate of change of b_i as we change a_j.
- If we change a_j slightly to $a_j + \delta$,
- Then (for small δ), b_i changes to approximately $b_i + \frac{\partial b_i}{\partial a_j}\delta$.

Partial Derivatives of an Affine Function

- Define the affine function $g(x) = Mx + c$, for $M \in \mathbf{R}^{n \times p}$ and $c \in \mathbf{R}$.



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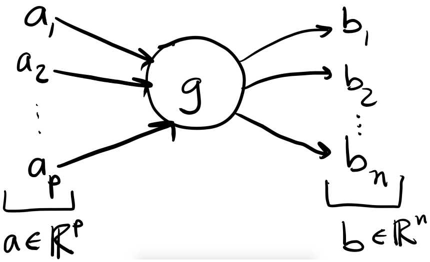


- If we let $b = g(a)$, then what is b_i?
- b_i depends on the ith row of M:

$$b_i = \sum_{k=1}^{p} M_{ik} a_k + c_i$$

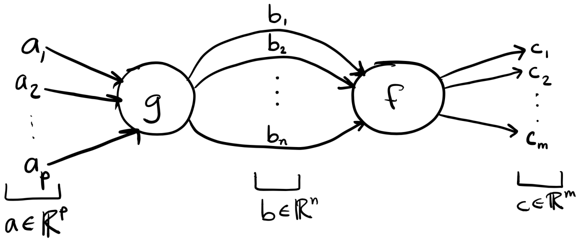
and

$$\frac{\partial b_i}{\partial a_j} = M_{ij}.$$

- So for an an affine mapping, entries of matrix M directly tell us the rates of change.

Chain Rule (in terms of partial derivatives)

- $g : \mathbf{R}^p \to \mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}^m$. Let $b = g(a)$. Let $c = f(b)$.



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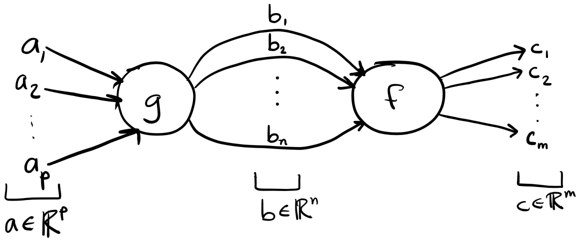

- **Chain rule** says that

$$\frac{\partial c_i}{\partial a_j} = \sum_{k=1}^{n} \frac{\partial c_i}{\partial b_k} \frac{\partial b_k}{\partial a_j}.$$

- Change in a_j may change each of $b_1, \ldots, b_n$.
- Changes in $b_1, \ldots, b_n$ may each effect c_i.
- Chain rule tells us that, to first order, the net change in c_i is
  - the sum of the changes induced along each path from a_j to c_i.

Example: Least Squares Regression

Review: Linear least squares

- Hypothesis space $\left\{ f(x) = w^T x + b \mid w \in \mathbf{R}^d, b \in \mathbf{R} \right\}$.

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- Data set $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbf{R}^d \times \mathbf{R}$.
- Define

$$\ell_i(w, b) = \left[\left(w^T x_i + b\right) - y_i\right]^2.$$

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- Define
$$\ell_i(w, b) = \left[\left(w^T x_i + b\right) - y_i\right]^2.$$
- In SGD, in each round we'd choose a random index $i \in 1, \ldots, n$ and take a gradient step
$$\begin{aligned} w_j &\leftarrow w_j - \eta \frac{\partial \ell_i(w, b)}{\partial w_j}, \text{ for } j = 1, \ldots, d \\ b &\leftarrow b - \eta \frac{\partial \ell_i(w, b)}{\partial b}, \end{aligned}$$
for some step size $\eta > 0$.
- Let's revisit how to calculate these partial derivatives...

Computation Graph and Intermediate Variables

- For a generic training point (x,y), denote the loss by

$$\ell(w,b) = \left[\left(w^T x + b\right) - y\right]^2.$$

- Let's break this down into some intermediate computations:

$$(\text{prediction})\ \hat{y} \quad = \quad \sum_{j=1}^{d} w_j x_j + b$$

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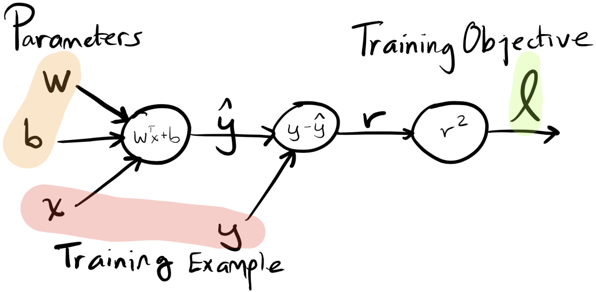
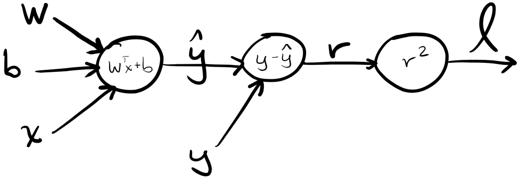


- We'll work our way from graph output ℓ back to the parameters w and b:



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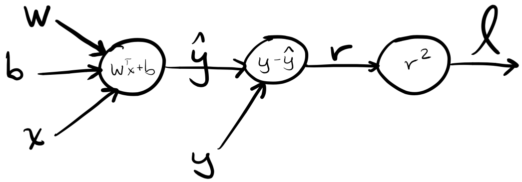

$$\frac{\partial \ell}{\partial r} =$$

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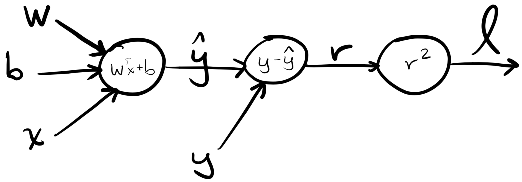

$$
\begin{aligned}
\frac{\partial \ell}{\partial r} &= 2r \\
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\end{aligned}
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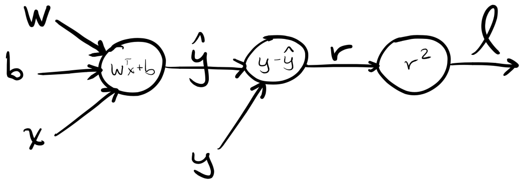

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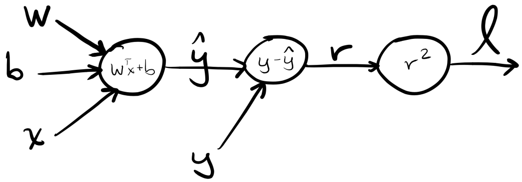

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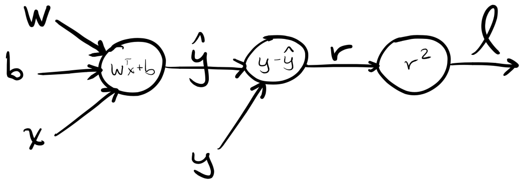

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Example: Ridge Regression

Ridge Regression: Computation Graph and Intermediate Variables

- For training point (x,y), the ℓ_2-regularized objective function is

$$J(w,b) = \left[\left(w^T x + b\right) - y\right]^2 + \lambda w^T w.$$

- Let's break this down into some intermediate computations:

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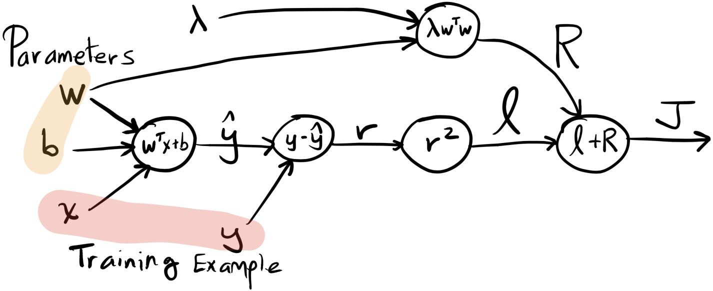

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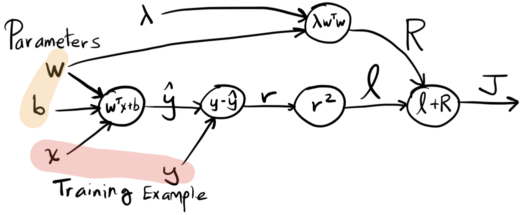

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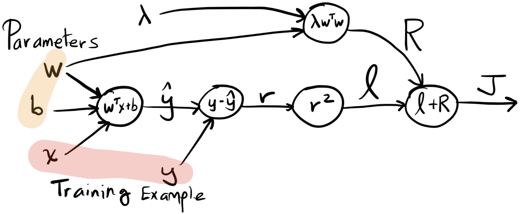

$$\frac{\partial J}{\partial \ell} = \qquad \frac{\partial J}{\partial R} =$$

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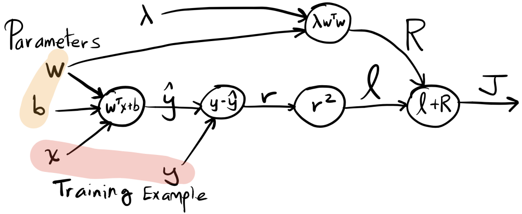

$$\frac{\partial J}{\partial \ell} = \frac{\partial J}{\partial R} = 1$$

$$\frac{\partial J}{\partial \hat{y}} =$$

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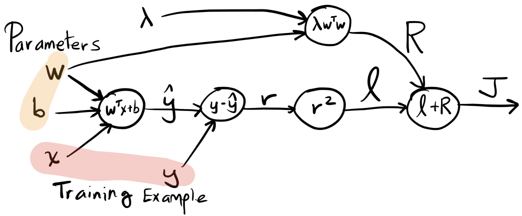

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$$\frac{\partial J}{\partial \hat{y}} = \frac{\partial J}{\partial \ell}\frac{\partial \ell}{\partial r}\frac{\partial r}{\partial \hat{y}} = (1)(2r)(-1) = -2r$$

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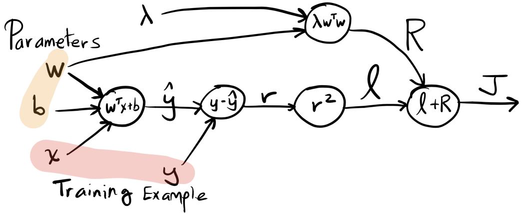

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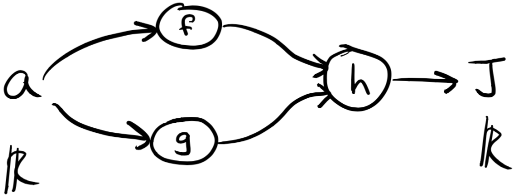
$$\frac{\partial J}{\partial b} = \frac{\partial J}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial b} = (-2r)(1) = -2r$$

$$\frac{\partial J}{\partial w_j} = ?$$

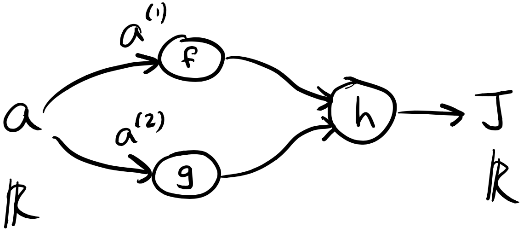
Handling Nodes with Multiple Children

- Consider $a \mapsto J = h(f(a), g(a))$.



- It's helpful to think about having two independent copies of a, call them $a^{(1)}$ and $a^{(2)}$...

Handling Nodes with Multiple Children



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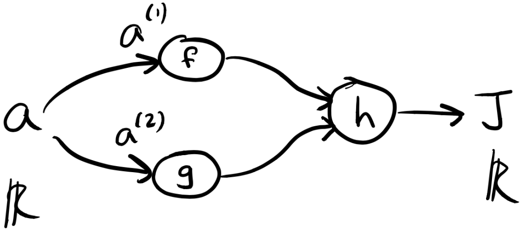

$$\frac{\partial J}{\partial a} =$$

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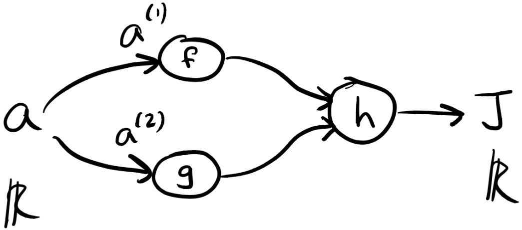

$$
\begin{aligned}
\frac{\partial J}{\partial a} &= \frac{\partial J}{\partial a^{(1)}} \frac{\partial a^{(1)}}{\partial a} + \frac{\partial J}{\partial a^{(2)}} \frac{\partial a^{(2)}}{\partial a} \\
&=
\end{aligned}
$$

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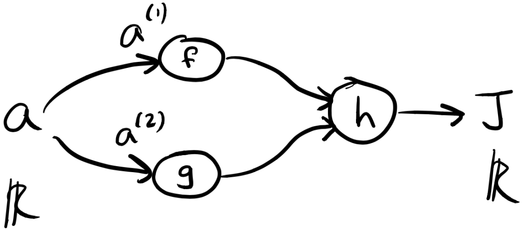
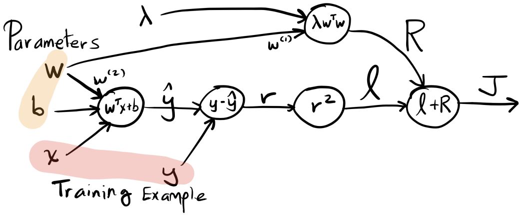


$$\begin{aligned}\frac{\partial J}{\partial a} &= \frac{\partial J}{\partial a^{(1)}}\frac{\partial a^{(1)}}{\partial a} + \frac{\partial J}{\partial a^{(2)}}\frac{\partial a^{(2)}}{\partial a} \\ &= \frac{\partial J}{\partial a^{(1)}} + \frac{\partial J}{\partial a^{(2)}}\end{aligned}$$

- Derivative w.r.t. a is the sum of derivatives w.r.t. each copy of a.

- We'll work our way from graph output ℓ back to the parameters w and b:



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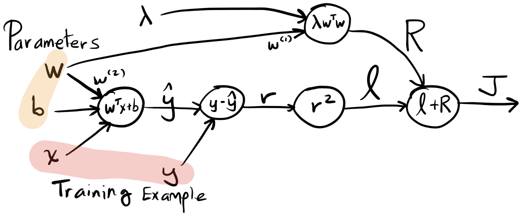

$$\frac{\partial J}{\partial \hat{y}} = \frac{\partial J}{\partial \ell}\frac{\partial \ell}{\partial r}\frac{\partial r}{\partial \hat{y}} = (1)(2r)(-1) = -2r$$

$$\frac{\partial J}{\partial w_j^{(2)}} =$$

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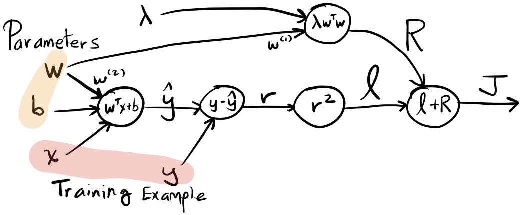

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\frac{\partial J}{\partial w_j^{(1)}} &=
\end{aligned}
$$

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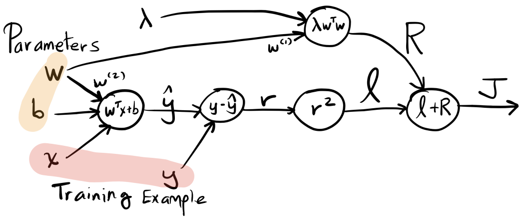

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\frac{\partial J}{\partial w_j^{(1)}} &= \frac{\partial J}{\partial R}\frac{\partial R}{\partial w_j^{(1)}} = (1)\left(2\lambda w_j^{(1)}\right) \\
\frac{\partial J}{\partial w_j} &=
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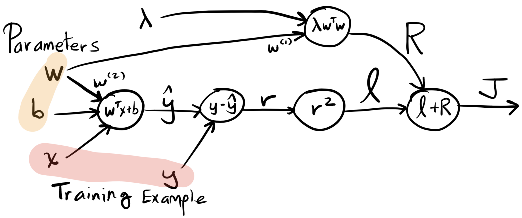
$$

- We'll work our way from graph output ℓ back to the parameters w and b:



$$
\begin{aligned}
\frac{\partial J}{\partial \hat{y}} &= \frac{\partial J}{\partial \ell}\frac{\partial \ell}{\partial r}\frac{\partial r}{\partial \hat{y}} = (1)(2r)(-1) = -2r \\
\frac{\partial J}{\partial w_j^{(2)}} &= \frac{\partial J}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial w_j^{(2)}} = \frac{\partial J}{\partial \hat{y}} x_j \\
\frac{\partial J}{\partial w_j^{(1)}} &= \frac{\partial J}{\partial R}\frac{\partial R}{\partial w_j^{(1)}} = (1)\left(2\lambda w_j^{(1)}\right) \\
\frac{\partial J}{\partial w_j} &= \frac{\partial J}{\partial w_j^{(1)}} + \frac{\partial J}{\partial w_j^{(2)}}
\end{aligned}
$$

General Backpropagation

Backpropagation: Overview

- Backpropagation is a specific way to evaluate the partial derivatives of a computation graph output J w.r.t. the inputs and outputs of all nodes.

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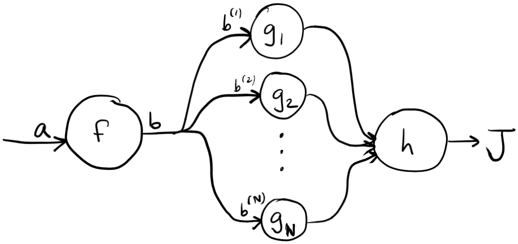
- Backpropagation works node-by-node.
- To run a "backward" step at a node f, we assume
  - we've already run "backward" for all of f's children.
- **Backward** at node $f : a \mapsto b$ returns
  - Partial of objective value J w.r.t. f's output: $\frac{\partial J}{\partial b}$
  - Partial of objective value J w.r.t f's input: $\frac{\partial J}{\partial a}$

Backpropagation: Simple Case

- Simple case: all nodes take a single scalar as input and have a single scalar output.



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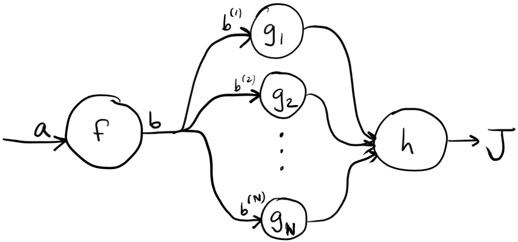
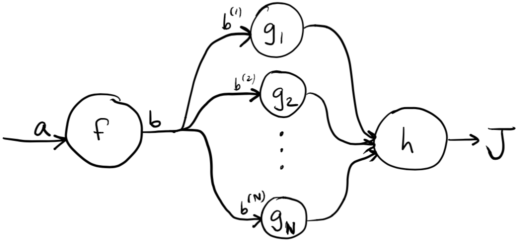


- Backprop for node f:
- **Input**: $\frac{\partial J}{\partial b^{(1)}}, \ldots, \frac{\partial J}{\partial b^{(N)}}$
  (Partials w.r.t. inputs to all children)
- **Output**:

$$\begin{aligned}\frac{\partial J}{\partial b} &= \sum_{k=1}^{N} \frac{\partial J}{\partial b^{(k)}} \\ \frac{\partial J}{\partial a} &= \frac{\partial J}{\partial b}\frac{\partial b}{\partial a}\end{aligned}$$

Backpropagation (General case)

- More generally, consider $f : \mathbf{R}^d \to \mathbf{R}^n$.



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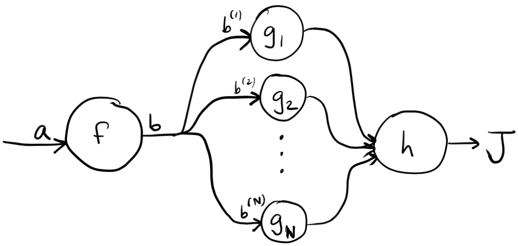

- **Input**: $\frac{\partial J}{\partial b_j^{(i)}}$, $i = 1, \ldots, N$, $j = 1, \ldots, n$
- **Output**:

$$\frac{\partial J}{\partial b_j} = \sum_{k=1}^{N} \frac{\partial J}{\partial b_j^{(k)}}$$

$$\frac{\partial J}{\partial a_i} = \sum_{j=1}^{n} \frac{\partial J}{\partial b_j} \frac{\partial b_j}{\partial a_i}$$

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  - is an ordering which every node appears before its children.
- So we'll evaluate backward on nodes in a **reverse topological ordering**.