

EM Algorithm for Latent Variable Models

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Latent Variable Models

General Latent Variable Model

- Two sets of random variables: z and x .
- z consists of unobserved **hidden variables**.
- x consists of **observed variables**.
- Joint probability model parameterized by $\theta \in \Theta$:

$$p(x, z \mid \theta)$$

Definition

A **latent variable model** is a probability model for which certain variables are never observed.

e.g. The Gaussian mixture model is a latent variable model.

Complete and Incomplete Data

- Suppose we observe some data (x_1, \dots, x_n) .
- To simplify notation, take x to represent the entire dataset

$$x = (x_1, \dots, x_n),$$

and z to represent the corresponding unobserved variables

$$z = (z_1, \dots, z_n).$$

- An observation of x is called an **incomplete data set**.
- An observation (x, z) is called a **complete data set**.

Our Objectives

- **Learning problem:** Given incomplete dataset $\mathcal{D} = x = (x_1, \dots, x_n)$, find MLE

$$\hat{\theta} = \arg \max_{\theta} p(\mathcal{D} \mid \theta).$$

- **Inference problem:** Given x , find conditional distribution over z :

$$p(z_i \mid x_i, \theta).$$

- For Gaussian mixture model, learning is hard, inference is easy.
- For more complicated models, inference can also be hard. (See DSGA-1005)

Log-Likelihood and Terminology

- Note that

$$\arg \max_{\theta} p(x \mid \theta) = \arg \max_{\theta} [\log p(x \mid \theta)] .$$

- Often easier to work with this “**log-likelihood**”.
- We often call $p(x)$ the **marginal likelihood**,
 - because it is $p(x, z)$ with z “marginalized out”:

$$p(x) = \sum_z p(x, z)$$

- We often call $p(x, z)$ the **joint**. (for “joint distribution”)
- Similarly, $\log p(x)$ is the **marginal log-likelihood**.

EM Algorithm (and Variational Methods) – The Big Picture

Big Picture Idea

- Want to find θ by maximizing the likelihood of the observed data x :

$$\hat{\theta} = \arg \max_{\theta \in \Theta} [\log p(x | \theta)]$$

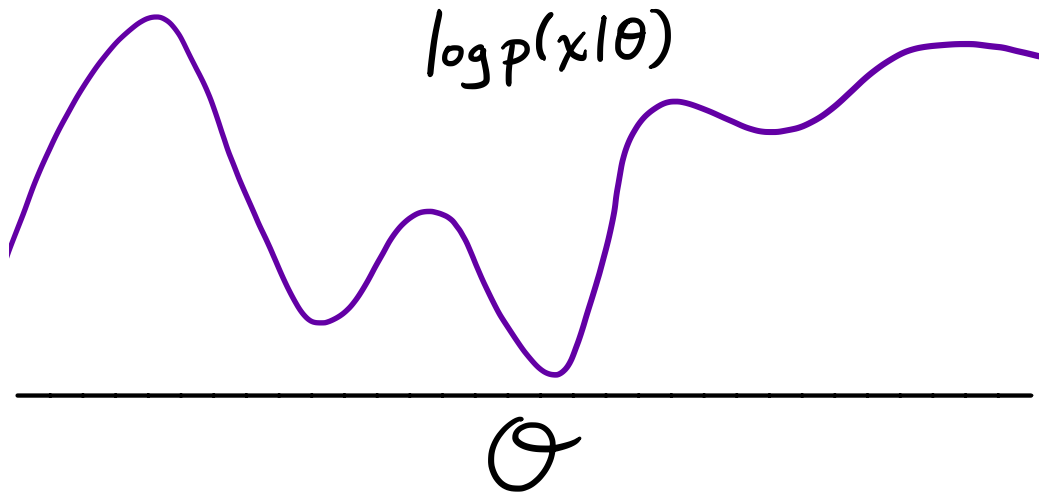
- Unfortunately this may be hard to do directly.
- Approach: Generate a **family of lower bounds** on $\theta \mapsto \log p(x | \theta)$.
- For every $q \in \mathcal{Q}$, we will have a lower bound:

$$\log p(x | \theta) \geq \mathcal{L}_q(\theta) \quad \forall \theta \in \Theta$$

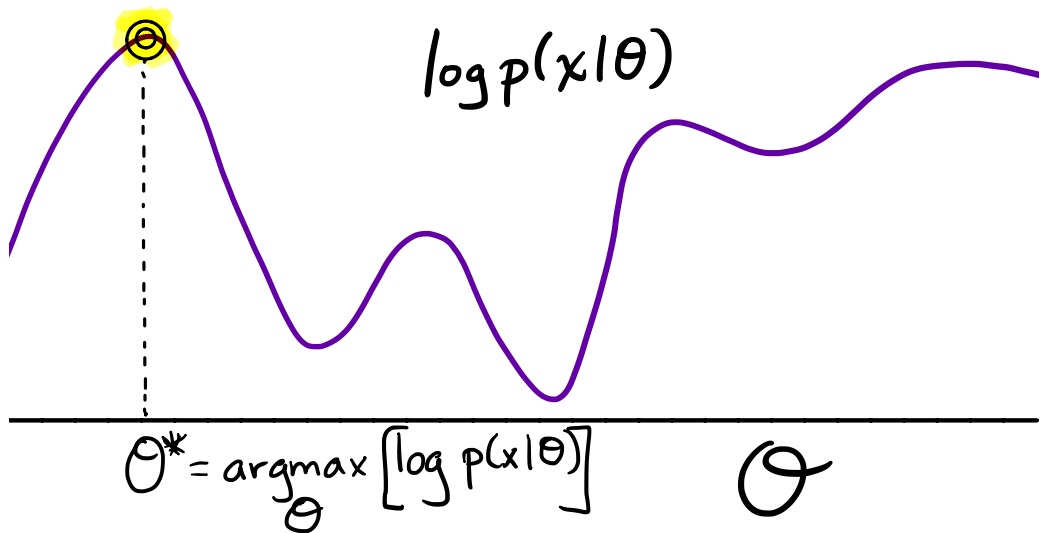
- We will try to find the maximum over all lower bounds:

$$\hat{\theta} = \arg \max_{\theta \in \Theta} \left[\sup_{q \in \mathcal{Q}} \mathcal{L}_q(\theta) \right]$$

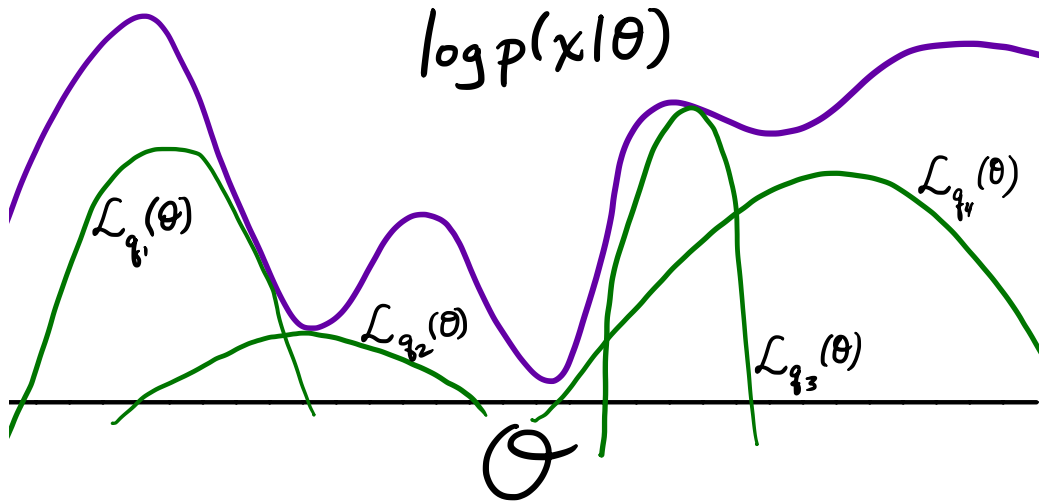
The Marginal Log-Likelihood Function



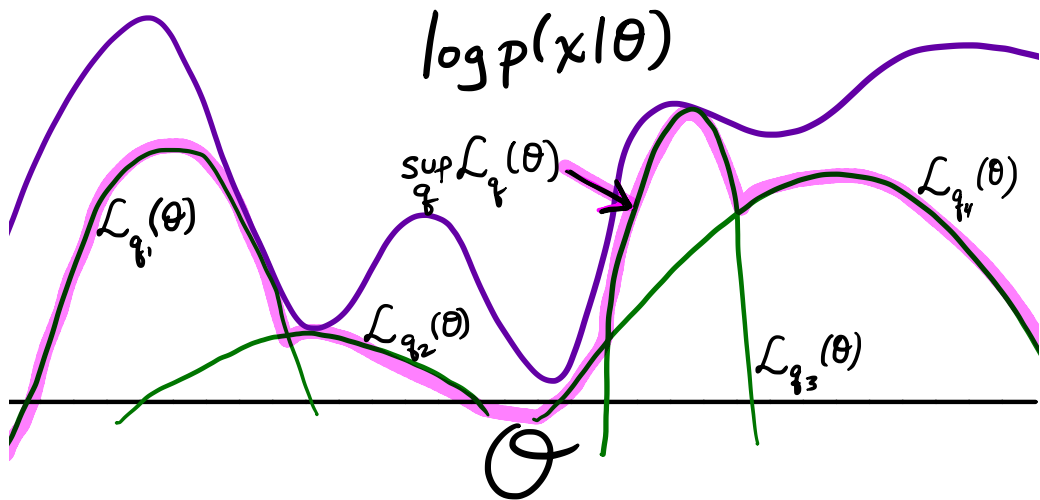
The Maximum Likelihood Estimator



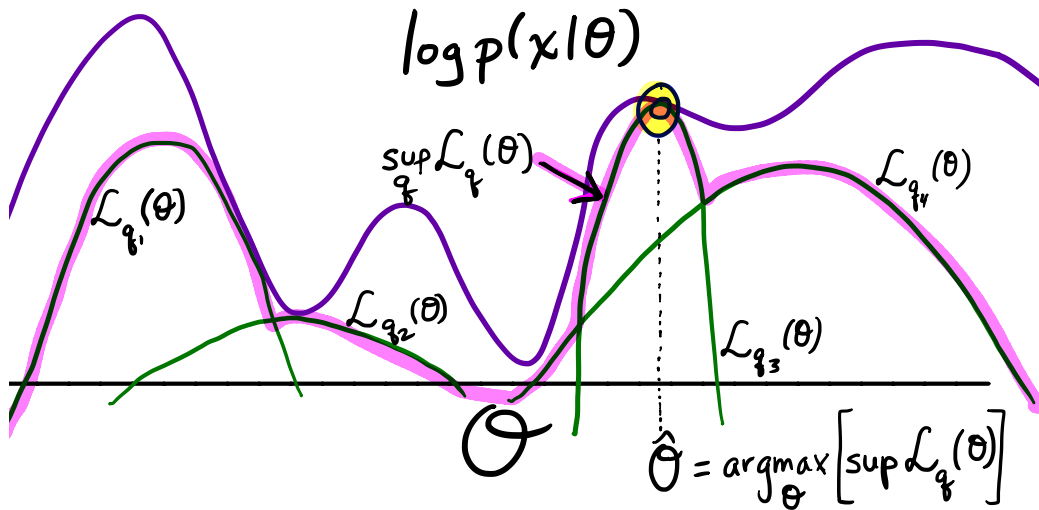
Lower Bounds on Marginal Log-Likelihood



Supremum over Lower Bounds is a Lower Bound



Parameter Estimate: Max over all lower bounds



The Expected Complete Data Log-Likelihood

- Marginal log-likelihood is hard to optimize:

$$\max_{\theta} \log p(x \mid \theta)$$

- **Typically** the complete data log-likelihood is easy to optimize:

$$\max_{\theta} \log p(x, z \mid \theta)$$

- What if we had a **distribution** $q(z)$ for the latent variables z ?

The Expected Complete Data Log-Likelihood

- Suppose we have a distribution $q(z)$ on latent variable z .
- Then maximize the **expected complete data log-likelihood**:

$$\max_{\theta} \sum_z q(z) \log p(x, z \mid \theta)$$

- If q puts lots of weight on actual z , this could be a good approximation to MLE
- EM **assumes this maximization is relatively easy**.
- (This is true for GMM.)

Math Prerequisites

Jensen's Inequality

Theorem (Jensen's Inequality)

If $f : \mathbf{R} \rightarrow \mathbf{R}$ is a **convex** function, and x is a random variable, then

$$\mathbb{E}f(x) \geq f(\mathbb{E}x).$$

Moreover, if f is **strictly convex**, then equality implies that $x = \mathbb{E}x$ with probability 1 (i.e. x is a constant).

- e.g. $f(x) = x^2$ is convex. So $\mathbb{E}x^2 \geq (\mathbb{E}x)^2$. Thus

$$\text{Var}(x) = \mathbb{E}x^2 - (\mathbb{E}x)^2 \geq 0.$$

Kullback-Leibler Divergence

- Let $p(x)$ and $q(x)$ be probability mass functions (PMFs) on \mathcal{X} .
- How can we measure how “different” p and q are?
- The **Kullback-Leibler** or “**KL**” **Divergence** is defined by

$$\text{KL}(p\|q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}.$$

(Assumes $q(x) = 0$ implies $p(x) = 0$.)

- Can also write this as

$$\text{KL}(p\|q) = \mathbb{E}_{x \sim p} \log \frac{p(x)}{q(x)}.$$

Gibbs Inequality ($\text{KL}(p\|q) \geq 0$ and $\text{KL}(p\|p) = 0$)

Theorem (Gibbs Inequality)

Let $p(x)$ and $q(x)$ be PMFs on \mathcal{X} . Then

$$\text{KL}(p\|q) \geq 0,$$

with equality iff $p(x) = q(x)$ for all $x \in \mathcal{X}$.

- KL divergence measures the “distance” between distributions.
- Note:
 - KL divergence **not a metric**.
 - KL divergence is **not symmetric**.

Gibbs Inequality: Proof

$$\begin{aligned}\text{KL}(p\|q) &= \mathbb{E}_p \left[-\log \left(\frac{q(x)}{p(x)} \right) \right] \\ &\geq -\log \left[\mathbb{E}_p \left(\frac{q(x)}{p(x)} \right) \right] \quad (\text{Jensen's}) \\ &= -\log \left[\sum_{\{x|p(x)>0\}} p(x) \frac{q(x)}{p(x)} \right] \\ &= -\log \left[\sum_{x \in \mathcal{X}} q(x) \right] \\ &= -\log 1 = 0.\end{aligned}$$

- Since $-\log$ is strictly convex, we have strict equality iff $q(x)/p(x)$ is a constant, which implies $q = p$.

The ELBO: Family of Lower Bounds on $\log p(x | \theta)$

Lower Bound for Marginal Log-Likelihood

- Let $q(z)$ be any PMF on \mathcal{Z} , the support of z :

$$\begin{aligned}\log p(x | \theta) &= \log \left[\sum_z p(x, z | \theta) \right] \\ &= \log \left[\sum_z q(z) \left(\frac{p(x, z | \theta)}{q(z)} \right) \right] \quad (\text{log of an expectation}) \\ &\geq \underbrace{\sum_z q(z) \log \left(\frac{p(x, z | \theta)}{q(z)} \right)}_{\mathcal{L}(q, \theta)} \quad (\text{expectation of log})\end{aligned}$$

- Inequality is by Jensen's, by concavity of the log.

This inequality is the basis for “**variational methods**”, of which EM is a basic example.

The ELBO

- For any PMF $q(z)$, we have a lower bound on the marginal log-likelihood

$$\log p(x | \theta) \geq \underbrace{\sum_z q(z) \log \left(\frac{p(x, z | \theta)}{q(z)} \right)}_{\mathcal{L}(q, \theta)}$$

- Marginal log likelihood $\log p(x | \theta)$ also called the **evidence**.
- $\mathcal{L}(q, \theta)$ is the **evidence lower bound**, or “**ELBO**”.

In EM algorithm (and variational methods more generally), we maximize $\mathcal{L}(q, \theta)$ over q and θ .

MLE, EM, and the ELBO

- For any PMF $q(z)$, we have a lower bound on the marginal log-likelihood

$$\log p(x | \theta) \geq \mathcal{L}(q, \theta).$$

- The MLE is defined as a maximum over θ :

$$\hat{\theta}_{\text{MLE}} = \arg \max_{\theta} [\log p(x | \theta)].$$

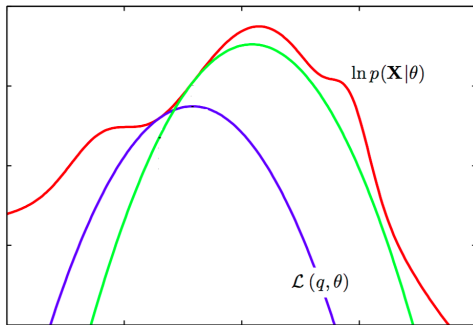
- In EM algorithm, we maximize the lower bound (ELBO) over θ and q :

$$\hat{\theta}_{\text{EM}} \approx \arg \max_{\theta} \left[\max_q \mathcal{L}(q, \theta) \right]$$

- In EM algorithm, q ranges over all distributions on z .

A Family of Lower Bounds

- For each q , we get a lower bound function: $\log p(x | \theta) \geq \mathcal{L}(q, \theta) \forall \theta$.
- Two lower bounds (blue and green curves), **as functions of θ** :



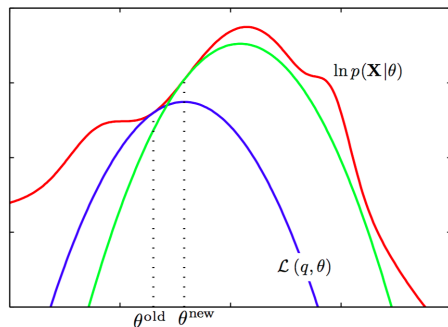
- Ideally, we'd find the maximum of the red curve. Maximum of green is close.

From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

EM: Coordinate Ascent on Lower Bound

- Choose sequence of q 's and θ 's by “coordinate ascent”.
- EM Algorithm (high level):
 - 1 Choose initial θ^{old} .
 - 2 Let $q^* = \arg \max_q \mathcal{L}(q, \theta^{\text{old}})$
 - 3 Let $\theta^{\text{new}} = \arg \max_{\theta} \mathcal{L}(q^*, \theta^{\text{old}})$.
 - 4 Go to step 2, until converged.
- Will show: $p(x | \theta^{\text{new}}) \geq p(x | \theta^{\text{old}})$
- Get sequence of θ 's with monotonically increasing likelihood.

EM: Coordinate Ascent on Lower Bound



- 1 Start at θ^{old} .
- 2 Find q giving best lower bound at $\theta^{\text{old}} \Rightarrow \mathcal{L}(q, \theta)$.
- 3 $\theta^{\text{new}} = \arg \max_{\theta} \mathcal{L}(q, \theta)$.

From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

EM: Next Steps

- In EM algorithm, we need to repeatedly solve the following steps:
 - $\arg \max_q \mathcal{L}(q, \theta)$, for a given θ , and
 - $\arg \max_{\theta} \mathcal{L}(q, \theta)$, for a given q .
- We now give two re-expressions of ELBO $\mathcal{L}(q, \theta)$ that make these easy to compute...

ELBO in Terms of KL Divergence and Entropy

- Let's investigate the lower bound:

$$\begin{aligned}\mathcal{L}(q, \theta) &= \sum_z q(z) \log \left(\frac{p(x, z | \theta)}{q(z)} \right) \\ &= \sum_z q(z) \log \left(\frac{p(z | x, \theta) p(x | \theta)}{q(z)} \right) \\ &= \sum_z q(z) \log \left(\frac{p(z | x, \theta)}{q(z)} \right) + \sum_z q(z) \log p(x | \theta) \\ &= -\text{KL}[q(z), p(z | x, \theta)] + \log p(x | \theta)\end{aligned}$$

- Amazing! We get back an equality for the marginal likelihood:

$$\log p(x | \theta) = \mathcal{L}(q, \theta) + \text{KL}[q(z), p(z | x, \theta)]$$

Maximizing over q for fixed θ .

- Find q maximizing

$$\mathcal{L}(q, \theta) = -\text{KL}[q(z), p(z | x, \theta)] + \underbrace{\log p(x | \theta)}_{\text{no } q \text{ here}}$$

- Recall $\text{KL}(p||q) \geq 0$, and $\text{KL}(p||p) = 0$.
- Best q is $q^*(z) = p(z | x, \theta)$ and

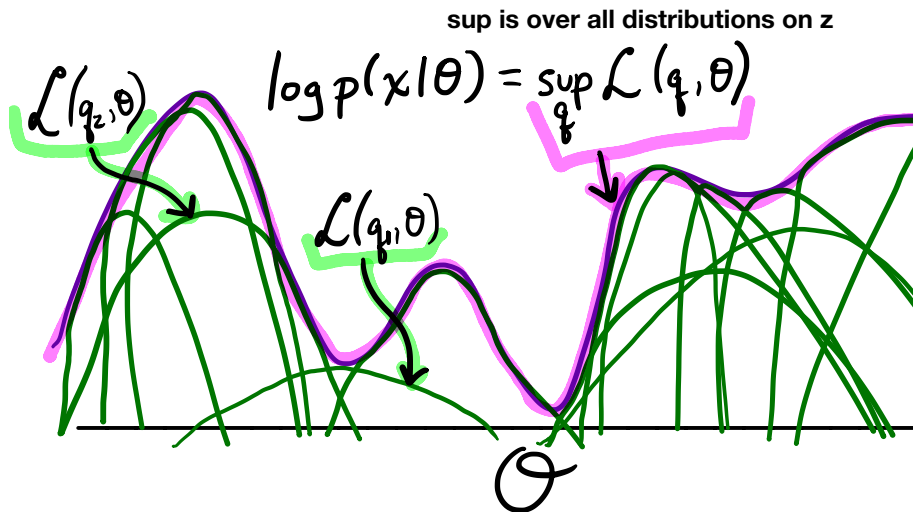
$$\mathcal{L}(q^*, \theta) = -\underbrace{\text{KL}[p(z | x, \theta), p(z | x, \theta)]}_{=0} + \log p(x | \theta)$$

- Summary:

$$\log p(x | \theta) = \sup_q \mathcal{L}(q, \theta) \quad \forall \theta$$

- For any θ , **sup is attained** at $q(z) = p(z | x, \theta)$.

Marginal Log-Likelihood **IS** the Supremum over Lower Bounds



Maximum of ELBO is MLE

- Suppose we find a **maximum** of $\mathcal{L}(q, \theta)$ over **all distributions** q on z and all $\theta \in \Theta$:

$$\mathcal{L}(q^*, \theta^*) = \sup_{\theta} \sup_q \mathcal{L}(q, \theta).$$

(where of course $q^*(z) = p(z | x, \theta^*)$.)

- Claim: θ^* is a maximizes $\log p(x | \theta)$.
- Proof: Trivial, since $\log p(x | \theta) = \sup_q \mathcal{L}(q, \theta)$.

Summary: Maximizing over q for fixed $\theta = \theta^{\text{old}}$.

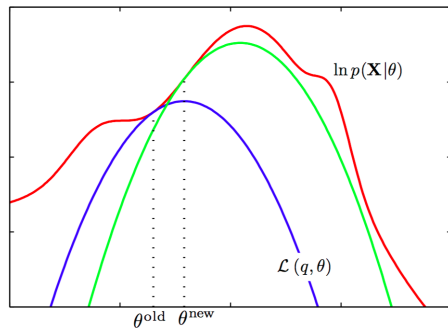
- At given $\theta = \theta^{\text{old}}$, want to find q giving best lower bound.
- Answer is $q^* = p(z \mid x, \theta^{\text{old}})$.
- This gives lower bound $\mathcal{L}(q^*, \theta)$ that is tight (equality) at θ^{old}

$$\log p(x \mid \theta^{\text{old}}) = \mathcal{L}(q^*, \theta^{\text{old}}) \quad (\text{tangent at } \theta^{\text{old}}).$$

- And elsewhere, of course, $\mathcal{L}(q^*, \theta)$ is just a lower bound:

$$\log p(x \mid \theta) \geq \mathcal{L}(q^*, \theta) \quad \forall \theta$$

Tight lower bound for any chosen θ



For θ^{old} , take $q(z) = p(z | x, \theta^{\text{old}})$. Then

- 1 $\log p(x | \theta^{\text{old}}) = \mathcal{L}(q, \theta^{\text{old}})$. [Lower bound is **tight** at θ^{old} .]
- 2 $\log p(x | \theta) \geq \mathcal{L}(q, \theta) \forall \theta$. [Global lower bound].

From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

Maximizing over θ for fixed q

- Consider maximizing the lower bound $\mathcal{L}(q, \theta)$:

$$\begin{aligned}\mathcal{L}(q, \theta) &= \sum_z q(z) \log \left(\frac{p(x, z | \theta)}{q(z)} \right) \\ &= \underbrace{\sum_z q(z) \log p(x, z | \theta)}_{\mathbb{E}[\text{complete data log-likelihood}]} - \underbrace{\sum_z q(z) \log q(z)}_{\text{no } \theta \text{ here}}\end{aligned}$$

- Maximizing $\mathcal{L}(q, \theta)$ equivalent to maximizing $\mathbb{E}[\text{complete data log-likelihood}]$ (for fixed q).

General EM Algorithm

① Choose initial θ^{old} .

② **Expectation Step**

- Let $q^*(z) = p(z \mid x, \theta^{\text{old}})$. [q^* gives best lower bound at θ^{old}]
- Let

$$J(\theta) := \mathcal{L}(q^*, \theta) = \underbrace{\sum_z q^*(z) \log \left(\frac{p(x, z \mid \theta)}{q^*(z)} \right)}_{\text{expectation w.r.t. } z \sim q^*(z)}$$

③ **Maximization Step**

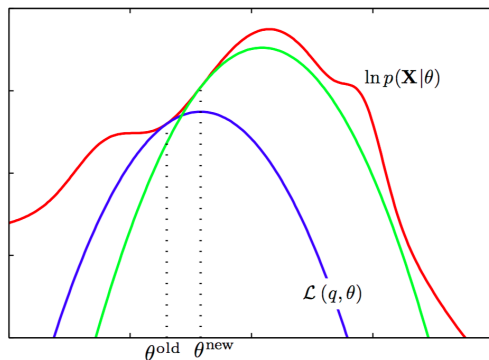
$$\theta^{\text{new}} = \arg \max_{\theta} J(\theta).$$

[Equivalent to maximizing expected complete log-likelihood.]

④ Go to step 2, until converged.

Does EM Work?

EM Gives Monotonically Increasing Likelihood: By Picture



From Bishop's *Pattern recognition and machine learning*, Figure 9.14.

EM Gives Monotonically Increasing Likelihood: By Math

- 1 Start at θ^{old} .
- 2 Choose $q^*(z) = \arg \max_q \mathcal{L}(q, \theta^{\text{old}})$. We've shown

$$\log p(x | \theta^{\text{old}}) = \mathcal{L}(q^*, \theta^{\text{old}})$$

- 3 Choose $\theta^{\text{new}} = \arg \max_{\theta} \mathcal{L}(q^*, \theta^{\text{old}})$. So

$$\mathcal{L}(q^*, \theta^{\text{new}}) \geq \mathcal{L}(q^*, \theta^{\text{old}}).$$

Putting it together, we get

$$\begin{aligned} \log p(x | \theta^{\text{new}}) &\geq \mathcal{L}(q^*, \theta^{\text{new}}) && \mathcal{L} \text{ is a lower bound} \\ &\geq \mathcal{L}(q^*, \theta^{\text{old}}) && \text{By definition of } \theta^{\text{new}} \\ &= \log p(x | \theta^{\text{old}}) && \text{Bound is tight at } \theta^{\text{old}}. \end{aligned}$$

Convergence of EM

- Let θ_n be value of EM algorithm after n steps.
- Define “transition function” $M(\cdot)$ such that $\theta_{n+1} = M(\theta_n)$.
- Suppose log-likelihood function $\ell(\theta) = \log p(x | \theta)$ is differentiable.
- Let S be the set of stationary points of $\ell(\theta)$. (i.e. $\nabla_{\theta} \ell(\theta) = 0$)

Theorem

Under mild regularity conditions^a, for any starting point θ_0 ,

- $\lim_{n \rightarrow \infty} \theta_n = \theta^*$ for some stationary point $\theta^* \in S$ and
- θ^* is a fixed point of the EM algorithm, i.e. $M(\theta^*) = \theta^*$. Moreover,
- $\ell(\theta_n)$ strictly increases to $\ell(\theta^*)$ as $n \rightarrow \infty$, unless $\theta_n \equiv \theta^*$.

^aFor details, see “Parameter Convergence for EM and MM Algorithms” by Florin Vaida in *Statistica Sinica* (2005). <http://www3.stat.sinica.edu.tw/statistica/oldpdf/a15n316.pdf>

Variations on EM

EM Gives Us Two New Problems

- The “E” Step: Computing

$$J(\theta) := \mathcal{L}(q^*, \theta) = \sum_z q^*(z) \log \left(\frac{p(x, z | \theta)}{q^*(z)} \right)$$

- The “M” Step: Computing

$$\theta^{\text{new}} = \arg \max_{\theta} J(\theta).$$

- Either of these can be too hard to do in practice.

Generalized EM (GEM)

- Addresses the problem of a difficult “M” step.
- Rather than finding

$$\theta^{\text{new}} = \arg \max_{\theta} J(\theta),$$

find **any** θ^{new} for which

$$J(\theta^{\text{new}}) > J(\theta^{\text{old}}).$$

- Can use a standard nonlinear optimization strategy
 - e.g. take a gradient step on J .
- We still get monotonically increasing likelihood.

EM and More General Variational Methods

- Suppose “E” step is difficult:
 - Hard to take expectation w.r.t. $q^*(z) = p(z | x, \theta^{\text{old}})$.
- Solution: Restrict to distributions \mathcal{Q} that are easy to work with.
- Lower bound now looser:

$$q^* = \arg \min_{q \in \mathcal{Q}} \text{KL}[q(z), p(z | x, \theta^{\text{old}})]$$

EM in Bayesian Setting

- Suppose we have a prior $p(\theta)$.
- Want to find MAP estimate: $\hat{\theta}_{\text{MAP}} = \arg \max_{\theta} p(\theta | x)$:

$$\begin{aligned} p(\theta | x) &= p(x | \theta)p(\theta)/p(x) \\ \log p(\theta | x) &= \log p(x | \theta) + \log p(\theta) - \log p(x) \end{aligned}$$

- Still can use our lower bound on $\log p(x, \theta)$.

$$J(\theta) := \mathcal{L}(q^*, \theta) = \sum_z q^*(z) \log \left(\frac{p(x, z | \theta)}{q^*(z)} \right)$$

- Maximization step becomes

$$\theta^{\text{new}} = \arg \max_{\theta} [J(\theta) + \log p(\theta)]$$

- Homework: Convince yourself our lower bound is still tight at θ .

Summer Homework: Gaussian Mixture Model (Hints)

Homework: Derive EM for GMM from General EM Algorithm

- Subsequent slides may help set things up.
- Key skills:
 - MLE for multivariate Gaussian distributions.
 - Lagrange multipliers

Gaussian Mixture Model (k Components)

- GMM Parameters

Cluster probabilities: $\pi = (\pi_1, \dots, \pi_k)$

Cluster means: $\mu = (\mu_1, \dots, \mu_k)$

Cluster covariance matrices: $\Sigma = (\Sigma_1, \dots, \Sigma_k)$

- Let $\theta = (\pi, \mu, \Sigma)$.

- Marginal log-likelihood

$$\log p(x | \theta) = \log \left\{ \sum_{z=1}^k \pi_z \mathcal{N}(x | \mu_z, \Sigma_z) \right\}$$

$q^*(z)$ are “Soft Assignments”

- Suppose we observe n points: $X = (x_1, \dots, x_n) \in \mathbf{R}^{n \times d}$.
- Let $z_1, \dots, z_n \in \{1, \dots, k\}$ be corresponding hidden variables.
- Optimal distribution q^* is:

$$q^*(z) = p(z | x, \theta).$$

- Convenient to define the conditional distribution for z_i given x_i as

$$\begin{aligned} \gamma_i^j &:= p(z = j | x_i) \\ &= \frac{\pi_j \mathcal{N}(x_i | \mu_j, \Sigma_j)}{\sum_{c=1}^k \pi_c \mathcal{N}(x_i | \mu_c, \Sigma_c)} \end{aligned}$$

Expectation Step

- The complete log-likelihood is

$$\begin{aligned}\log p(x, z \mid \theta) &= \sum_{i=1}^n \log [\pi_z \mathcal{N}(x_i \mid \mu_z, \Sigma_z)] \\ &= \sum_{i=1}^n \left(\log \pi_z + \underbrace{\log \mathcal{N}(x_i \mid \mu_z, \Sigma_z)}_{\text{simplifies nicely}} \right)\end{aligned}$$

- Take the expected complete log-likelihood w.r.t. q^* :

$$\begin{aligned}J(\theta) &= \sum_z q^*(z) \log p(x, z \mid \theta) \\ &= \sum_{i=1}^n \sum_{j=1}^k \gamma_i^j [\log \pi_j + \log \mathcal{N}(x_i \mid \mu_j, \Sigma_j)]\end{aligned}$$

Maximization Step

- Find θ^* maximizing $J(\theta)$:

$$\mu_c^{\text{new}} = \frac{1}{n_c} \sum_{i=1}^n \gamma_i^c x_i$$

$$\Sigma_c^{\text{new}} = \frac{1}{n_c} \sum_{i=1}^n \gamma_i^c (x_i - \mu_{\text{MLE}}) (x_i - \mu_{\text{MLE}})^T$$

$$\pi_c^{\text{new}} = \frac{n_c}{n},$$

for each $c = 1, \dots, k$.