1 Defining average and instantaneous rates of change at a point

1.1 Derivative as a concept:

Derivative is defined as the instantaneous rate of change at a point, or the slope of a tangent line at that point.
which is defined as the $\lim_{x\to 0} \frac{dy}{dx} = f'(x)$.

1.2 Secant lines and average rate of change:

the average rate of change between two points in an interval $[a, b]$ of a curve is defined as the slope of the secant line that connects these points.

example:

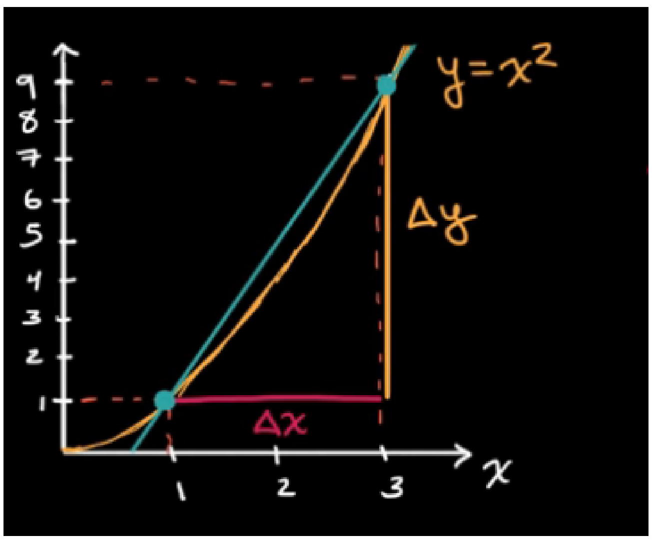

Figure 1: secant line

2 Defining the derivative of a function and using derivative notation

2.1 Formal definition of the derivative as a limit:

h represent some distance more than x. the derivative of a point x_0 formally defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

2.2 Alternate form of the derivative:

the derivative of a point x_0 in alternate form is written as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

3 Connecting differentiability and continuity

3.1 Differentiability and continuity

- if f (x) is differentiable at $x = c$, then f (x) is continuous at $x = c$
- if f (x) is not continuous at $x = c$, then f (x) is not differentiable at $x = c$
- if f (x) is not differentiable at a point, then f (x) may or may not be continuous at that point.

4 Power rule

let the function $f(x) = x^n, n \neq 0$, the derivative of $f(x)$ is given according to the power rule as

$$f'(x) = nx^{n-1}$$

5 Derivative rules: constant, sum, difference and constant multiple

5.1 Basic rules

- let A be a constant then $\frac{d}{dx}[A] = 0$.
- let $f(x)$ be a defined function then:

$$\frac{d}{dx}[Af(x)] = A\frac{d}{dx}[f(x)] = Af'(x)$$

- let g (x) be a defined function then:
$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = f'(x) + g'(x)$$

6 Derivatives of $\cos(x)$, $\sin(x)$, e^x and $\ln(x)$

6.1 Derivatives of $\sin(x)$ and $\cos(x)$

- $\frac{d}{dx}[\sin(x)] = \cos(x)$
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$

6.2 Derivative of e^x

$\frac{d}{dx}[e^x] = e^x$

6.3 Derivative of $\ln(x)$

$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

7 The product rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

8 The Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{g(x)^2}$$

9 Derivatives of $\tan(x)$ and $\cot(x)$

tan (x):

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cdot\cos(x) + \sin(x)\cdot\sin(x)}{\cos x^2} = \frac{1}{\cos x^2} = \sec x^2$$

cot (x):

$$\frac{d}{dx}[\cot(x)] = \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] = \frac{-\sin(x)\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\sin x^2} = -\frac{1}{\sin x^2} = -\csc x^2$$

10 Chain rule

$$\frac{d}{dx}[f\Big(g(x)\Big)] = f'\Big(g(x)\Big) \cdot g'(x)$$

11 The chain rule: further practice

11.1 Derivative of a^x(for any positive base a)

let $a = e^{\ln a}$ then:

$$\frac{d}{dx}[a^x] = \frac{d}{dx}\Big[\Big(e^{\ln a}\Big)^x\Big] = e^{(\ln a)\cdot x} \cdot \ln a = (\ln a) \cdot a^x$$

11.2 Derivative of $\log_a x$ (for any positive base $a \neq 1$)

let $\frac{d}{dx}[\ln x] = \frac{1}{x}$ and $\log_a b = \frac{\log_c b}{\log_c a}$ then:

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\Big[\frac{1}{\ln a} \cdot \ln x\Big] = \frac{1}{\ln a} \cdot \frac{d}{dx}[\ln x] = \frac{1}{(\ln a)x}$$

11.3 Proving the chain rule

- $u(x)$ continuous at $x = c$ implies that $\Delta u \to 0$ as $\Delta x \to 0$
- $u(x)$ is continuous $\iff$ $\lim_{x\to c} u(x) = u(c) \equiv \lim_{x\to c}(u(x) - u(c)) = 0$

we have:

- $\Delta u = u(x) - u(c)$
- $\Delta x = x - c$
- $\lim_{\Delta x\to 0} \Delta u = 0$

chain rule prove:
assume y, u differentiable at x.

$$\frac{d}{dx}[y(u(x))] = \frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{dy}{dx}$$

$$\frac{dx}{dy} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \Big(\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta u}\Big) \cdot \Big(\lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x}\Big)$$

$$= \Big(\lim_{\Delta u\to 0} \frac{\Delta y}{\Delta u}\Big) \cdot \frac{du}{dx} = \frac{dy}{dy} \cdot \frac{du}{dx}$$

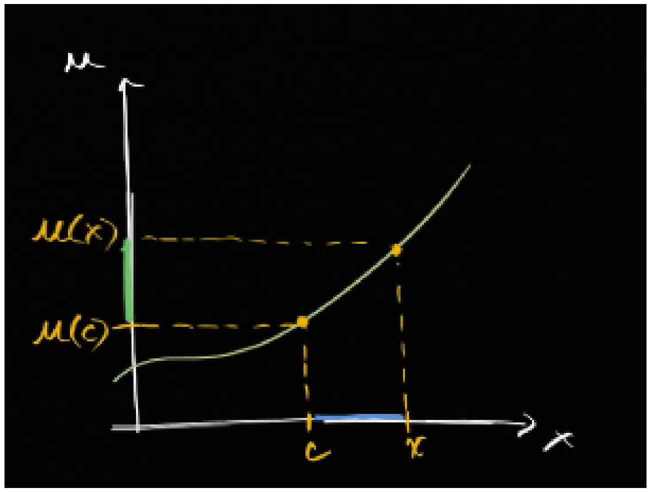

Figure 2: if function u is continuous at x, then $\Delta u \to 0$ as $\Delta x \to 0$

11.4 Implicit differentiation

In implicit differentiation, we differentiate each side of an equation with two variables, by treating one of the variables as function of the other.this calls for using chain rule.
example differentiating $x^2 + y^2 = 1$:
we treat y as an implicit function of x

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

11.5 Differentiating inverse functions

let f (x) a defined function let g (x) be $g(x) = f^{-1}(x)$ then $g(f(x)) = x$ and

$$\frac{d}{dx}[g(f(x))] = \frac{d}{dx}[x]$$
$$g'(f(x)) \cdot f'(x) = 1$$
$$f'(x) = \frac{1}{g'(f(x))}$$

11.6 Differentiating inverse trigonometric functions

let

$$\sin^2 y + \cos^2 y = 1$$

Inverse sine:
let $x = \sin y$

$$y = \sin^{-1} x$$
$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$
$$\cos y \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$
$$= \frac{1}{\sqrt{1 - (\sin y)^2}}$$
$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1 - x^2}}$$

Inverse cosine:
let $x = \cos y$

$$y = \cos^{-1} x$$
$$1 = (-\sin y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - P(\cos y)^2}} = -\frac{1}{\sqrt{1 - x^2}}$$
$$\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1 - x^2}}$$

Inverse tangent:
let $\frac{d}{dx}[\tan x] = \sec^2 x = \frac{1}{\cos^2 x}$

$$y = \tan^{-1} x$$

$$\frac{d}{dx}[\tan y] = \frac{d}{dx}$$

$$\frac{1}{\cos^2 y} \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \cos^2 y = \frac{\cos^2 y}{\cos^2 y + \sin^2 y} \cdot \frac{\frac{1}{\cos^2 y}}{\frac{1}{\cos^2 y}} = \frac{1}{1 + \frac{\sin y}{\cos y}^2} = \frac{1}{1 + \tan y^2}$$

$$= \frac{1}{1 + x^2}$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$$

11.7 Calculating higher-order derivatives

the second derivative of a function is the derivative of the function's derivative
let $f(x) = x^3 + 2x^2$.its first derivative is $f'(x) = 3x + 4x$, the second derivative of $f(x)$ would be the differentiation of $f'(x)$ which is:

$$f''(x) = 6x + 4$$

Notation for second derivatives:
leibniz's notation for second derivative is $\frac{d^2y}{dx^2}$
ex: leibniz notation of $x^3 + 2x^2$ is $\frac{d^2}{dx^2}(x^3 + 2x^2)$

12 Approximating values of a function using local linearity and linearization

12.1 Local linearity

let $f(x)$ be a defined function.
let (a, b) a defined point on the graph of the function $f(x)$.
the approximation of the point x_0.is given as

$$f(x_0) = L(x_0) = f(a) + f'(a)(x_0 - a)$$

12.2 local linearity and differentiability

if $f(x)$ is differentiable at x_0, then $f(x)$ is locally linear at that point.

13 Using L'Hôpital's rule for finding limits of indeterminate forms

13.1 L'Hôpital's rule introduction

- case 1:
  $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$
- case 2:
  $\lim_{x \to c} f(x) = \pm\infty$ and $\lim_{x \to c} g(x) = \pm\infty$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$

13.2 Proof of special case of l'Hôpital's rule

if $f(a) = 0$, $g(a) = 0$ and $f'(a)$, $g'(a)$ exist. then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

proof:

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

14 Using the mean value theorem

14.1 mean value theorem

for a function f that's differentiable over an open interval from (a, b), and continuous over the closed interval $[a, b]$ that there exists a number c on that interval such that $f'(c)$ is equal to the function's average rate of change over the interval

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

graphically the tangent line at c is parallel to the secant line going through a and b.

15 Extreme value theorem, global vs local extrema, and critical points

15.1 Extreme value theorem

f is continuous function over $[a, b]$ then
$\exists$ an absolute maximum and and absolute minimum over that interval.

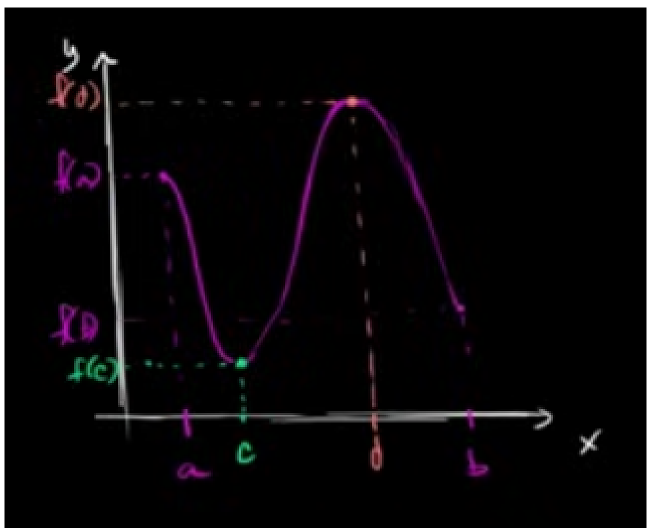

Figure 3: secant line

$$\exists\ c, d \in [a, b] : f(c) \leq f(x) \leq f(d)\ \forall\ x \in [a, b]$$

15.2 Critical points introduction

let x_0, x_1, x_2 be non endpoints maximum or minimum points of $f(x)$, then the derivative of $f'(x_0)$, $f'(x_1)$, $f'(x_2)$ is either going to be 0 or undefined.
A critical point is a point where f' is equal to 0 or undefined.
A critical point is not necessarily an extreme point, the reverse is true.

16 Determining intervals on which a function is decreasing or increasing

16.1 Increasing and decreasing intervals

The intervals where a function $f(x)$ is increasing (or decreasing) correspond to the intervals where its derivative is positive (or negative) $f'(x) < 0$ or $f'(x) > 0$.
the derivative of function changes sign at each critical point.

17 Using the first derivative test to find relative (local) extrema

17.1 first derivative test

If a is min/max value of $f(x)$ at $x = a$ then a is a critical point.
If a is critical point and in the domain of definition of f.then a is a maximum point of $f(x)$, if $f'(x)$ switches sign from positive to negative as $f'(x)$ cross $x = a$.
If a is a critical point and in the domain of definition of f.then a is a minimum point of $f(x)$, if $f'(x)$ switches sign from negative to positive as $f'(x)$ cross $x = a$.

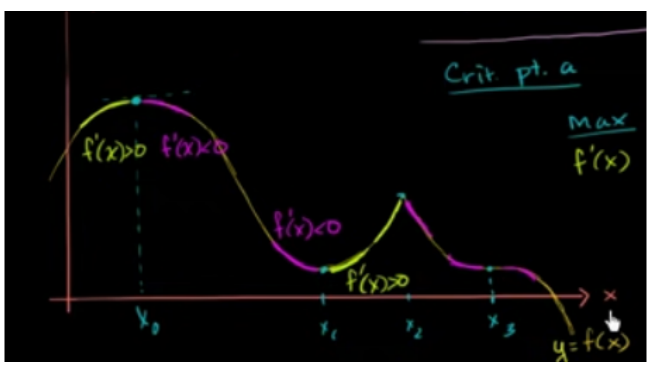

Figure 4: secant line

18 Using the candidates test to find absolute (global) extrema

18.1 Absolute minima and maxima

let $f(x)$ be defined function over the interval $x \in [a, b]$.
let x_0, x_1 x_2 be critical points and maximum points of $f(x)$, where $f'(x_0) = 0 \mid undefined$, $f'(x_1) = 0 \mid undefined$, $f'(x_2) = 0 \mid undefined$.
x_1 is an absolute maximum of $f(x)$ if and only if $f(x_1) > f(x_0)$ *and* $f(x_1) > f(x_2)$ *and* $f(x_1) > f(a)$ *and* $f(x_1) > f(b)$.
let x_0, x_1 x_2 be critical points and minimum points of $f(x)$, where $f'(x_0) = 0|undefined$, $f'(x_1) = 0|undefined$, $f'(x_2) = 0|undefined$.
x_1 is an absolute minimum of $f(x)$ if and only if $f(x_1) < f(x_0)$ *and* $f(x_1) < f(x_2)$ *and* $f(x_1) < f(a)$ *and* $f(x_1) < f(b)$.

19 Determining concavity of intervals and finding points of inflection

19.1 Concavity introduction

let $f(x)$ be continuous and defined over the interval $[a, b]$.
$f(x)$ is concave downards on a sub interval of $[a, b]$ and has maximum point at x_0 where $f'(x_0) = 0$ iff:

- slope is decreasing: $f'(x)$ is decreasing
- the second derivative is negative: $f''(x) < 0$

$f(x)$ is concave upwards on a sub interval of $[a, b]$ and has minimum-point at x_0 where $f'(x_0) = 0$ iff:

- slope is increasing: $f'(x)$ is increasing
- the second derivative is positive: $f''(x) > 0$

19.2 Inflectoin point introduction

Inflection points is a point where the second derivative switches signs
x_1 is an inflection point iff:

- for $x < x_1$: $f''(x) < 0$ and for $x > x_1$: $f''(x) > 0$
- or for $x < x_1$: $f''(x) > 0$ and for $x > x_1$: $f''(x) < 0$

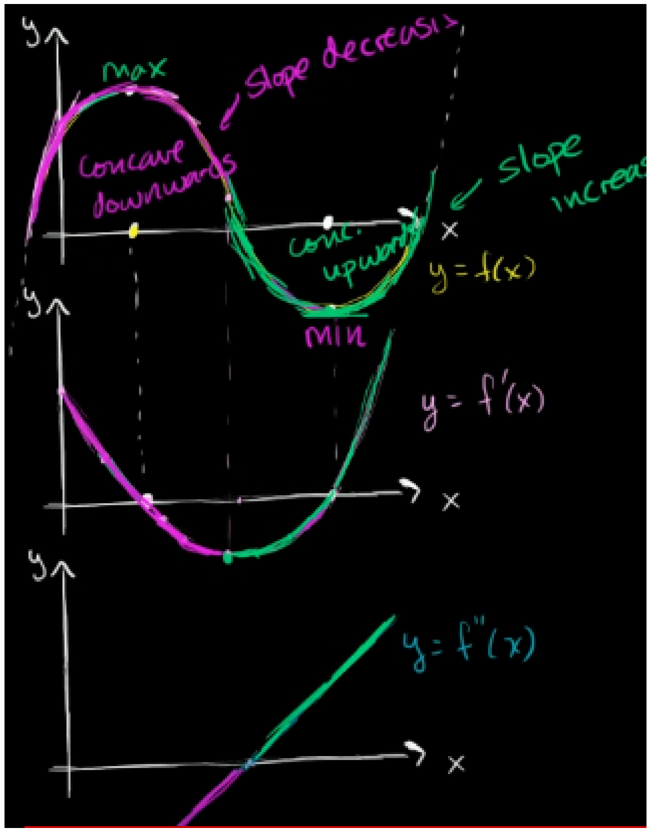

Figure 5: secant line

20 Second derivative test

let $f'(c) = 0$, f' exists in neighborhoud around $x = c$
let $f''(c)$ exists then

- if $f''(c) < 0$ then f has relative maximum at $x = c$
- if $f''(c) = 0$ then $x = c$ is inconclusive
- if $f''(c) > 0$ then f has relative minimum at $x = c$

21 Exploring accumulations of change

21.1 Intro to integral calculus

let $f(x)$ be a defined function.
let δx_i be an infinitesimally small distance of the interval $[a, b]$.
the area under the curve of $f(x)$ between the interval $[a, b]$ is represented as:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\delta x_i = \int_a^b f(x)dx$$

$\int_a^b$ represents the integral of the function $f(x)$

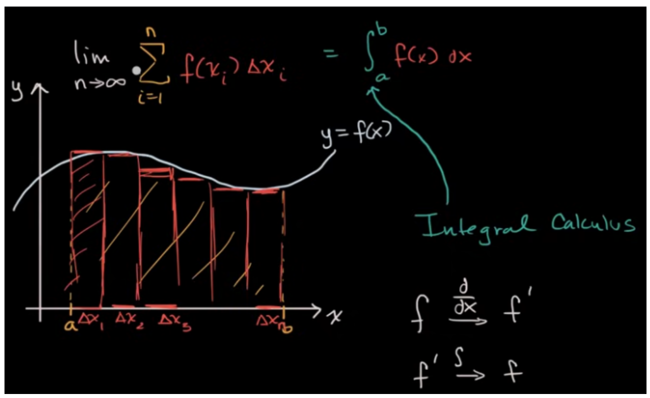

Figure 6: integrals

21.2 Definite integrals intro

The area under the curve between the interval $[a, b]$ is denoted as:

$$\int_a^b f(x)dx$$

which is the definite integral of $f(x)$ between two bounds.

21.3 Exploring accumulations of change

The definite integral can be used to express information about accumulation and net change in applied contexts. the definite integral always gives us the net change in a quantity, not the actual value of that quantity

In differential calculus, the derivative f' of a function f gives the instantaneous rate of change of f for a given input.
for any rate function f, its antiderivative F gives the accumulated value of the quantity whose rate is described by f.

	Quantity	Rate
Differential calculus	$f(x)$	$f'(x)$
Integral calculus	$F(x) = \int_a^x f(t)dt$	$f(x)$

22 Approximating areas with Riemann sums

22.1 RRiemann approximation introduction

A Riemann sum is an approximation of the area under a curve by dividing it into multiple simple shapes (like rectangles or trapezoids).

In a left Riemann sum, we approximate the area using rectangles (usually of equal width), where the height of each rectange is equal to the value of function at the left endpoint of its base, this type of approximation is considered underestimate.

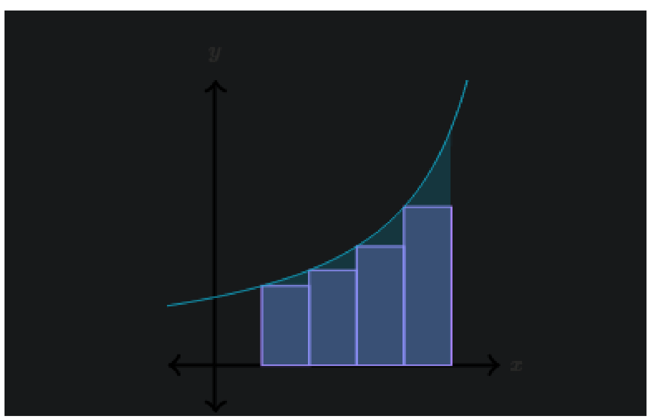

Figure 7: integrals

In a right Riemann sum, the height of each rectangle is equal to the value of the function at the right endpoints of its base.this type of approximation is considered overestimate

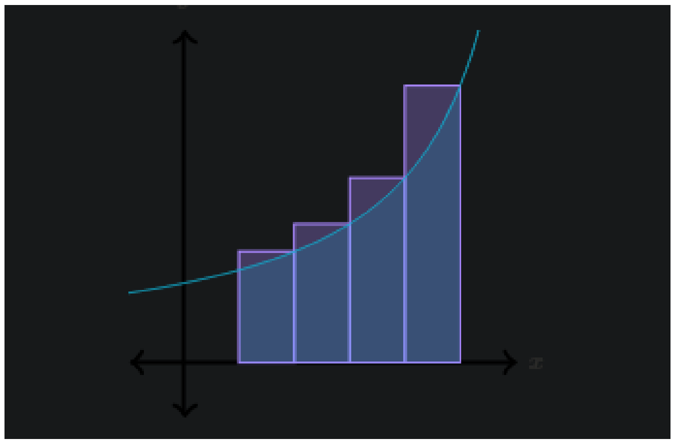

Figure 8: integrals

In a midpoint Riemann sum, the height of each rectangle is equal to the value of the function at the midpoint of its base.

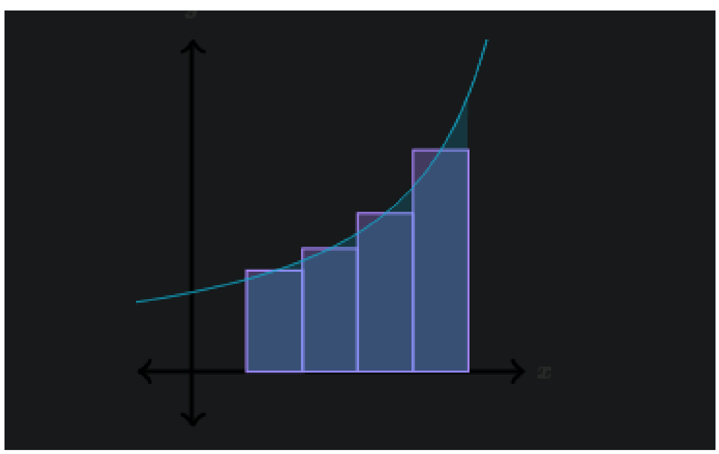

Figure 9: integrals

We can also use trapezoids to approximate the rea (this is called trapezoidal rule). In this case, each trapezoid touches the curve at both of its top vertices.

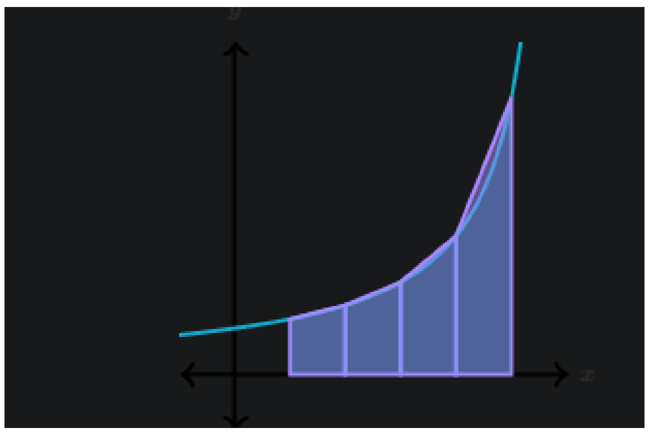

Figure 10: integrals

23 Riemann sums, summation notation and definite integral notation

23.1 Riemann sums in summation notation

Imagine we want to approximate the area under the graph of f over the interval $[a, b]$ with n equal subdivisions.
Define δx: let δx denote the width of each rectangle, then $\delta x = \frac{b-a}{n}$
Define x_i: Let x_i denote the right endpoint of each rectangle, then $x_i = a + \delta x \cdot i$.
Define the area of i^{th}: The height of each rectangle is then $f(x_i)$, and are of each rectangle is $\delta x \cdot f(x_i)$.
sum the rectangles: Now we use summation notation to add all the areas. The values we use for i are different for left and right Riemann sums:

- When we are writing a right Riemann sum, we will take values of i from 1 to n

- However, when we are writing a left Riemann sum, we will take values of i from 0 to $n-1$(this will give us the value of f at the left endpoint of each rectangle).

Left Riemann sum	Right Riemann sum
$\sum_{i=0}^{n-1} \delta x \cdot f(x_i)$	$\sum_{i=1}^{n} \delta x \cdot f(x_i)$