1 Defining average and instantaneous rates of change at a point

1.1 Derivative as a concept:

Derivative is defined as the instantaneous rate of change at a point, or the slope of a tangent line at that point.
which is defined as the $\lim_{x\to 0} \frac{dy}{dx} = f'(x)$.

1.2 Secant lines and average rate of change:

the average rate of change between two points in an interval $[a, b]$ of a curve is defined as the slope of the secant line that connects these points.

example:

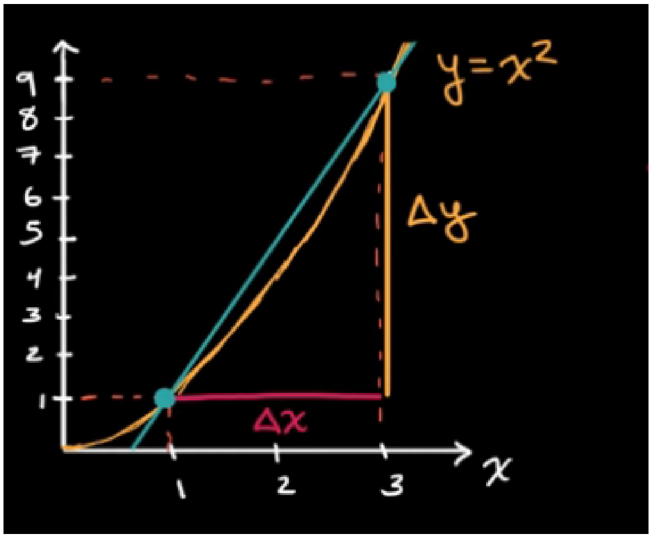

Figure 1: secant line

2 Defining the derivative of a function and using derivative notation

2.1 Formal definition of the derivative as a limit:

h represent some distance more than x. the derivative of a point x_0 formally defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

2.2 Alternate form of the derivative:

the derivative of a point x_0 in alternate form is written as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

3 Connecting differentiability and continuity

3.1 Differentiability and continuity

- if f (x) is differentiable at $x = c$, then f (x) is continuous at $x = c$
- if f (x) is not continuous at $x = c$, then f (x) is not differentiable at $x = c$
- if f (x) is not differentiable at a point, then f (x) may or may not be continuous at that point.

4 Power rule

let the function $f(x) = x^n, n \neq 0$, the derivative of $f(x)$ is given according to the power rule as

$$f'(x) = nx^{n-1}$$

5 Derivative rules: constant, sum, difference and constant multiple

5.1 Basic rules

- let A be a constant then $\frac{d}{dx}[A] = 0$.
- let $f(x)$ be a defined function then:

$$\frac{d}{dx}[Af(x)] = A\frac{d}{dx}[f(x)] = Af'(x)$$

- let g (x) be a defined function then:

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = f'(x) + g'(x)$$

6 Derivatives of $\cos(x)$, $\sin(x)$, e^x and $\ln(x)$

6.1 Derivatives of $\sin(x)$ and $\cos(x)$

- $\frac{d}{dx}[\sin(x)] = \cos(x)$
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$

6.2 Derivative of e^x

$\frac{d}{dx}[e^x] = e^x$

6.3 Derivative of $\ln(x)$

$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

7 The product rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

8 The Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{g(x)^2}$$

9 Derivatives of $\tan(x)$ and $\cot(x)$

tan (x):

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cdot\cos(x) + \sin(x)\cdot\sin(x)}{\cos x^2} = \frac{1}{\cos x^2} = \sec x^2$$

cot (x):

$$\frac{d}{dx}[\cot(x)] = \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] = \frac{-\sin(x)\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\sin x^2} = -\frac{1}{\sin x^2} = -\csc x^2$$

10 Chain rule

$$\frac{d}{dx}[f\Big(g(x)\Big)] = f'\Big(g(x)\Big) \cdot g'(x)$$

11 The chain rule: further practice

11.1 Derivative of a^x(for any positive base a)

let $a = e^{\ln a}$ then:

$$\frac{d}{dx}[a^x] = \frac{d}{dx}\left[\left(e^{\ln a}\right)^x\right] = e^{(\ln a)\cdot x} \cdot \ln a = (\ln a) \cdot a^x$$

11.2 Derivative of $\log_a x$ (for any positive base $a \neq 1$)

let $\frac{d}{dx}[\ln x] = \frac{1}{x}$ and $\log_a b = \frac{\log_c b}{\log_c a}$ then:

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a} \cdot \ln x\right] = \frac{1}{\ln a} \cdot \frac{d}{dx}[\ln x] = \frac{1}{(\ln a)x}$$

11.3 Proving the chain rule

- $u(x)$ continuous at $x = c$ implies that $\Delta u \to 0$ as $\Delta x \to 0$
- $u(x)$ is continuous $\iff$ $\lim_{x\to c} u(x) = u(c) \equiv \lim_{x\to c}(u(x) - u(c)) = 0$

we have:

- $\Delta u = u(x) - u(c)$
- $\Delta x = x - c$
- $\lim_{\Delta x\to 0} \Delta u = 0$

chain rule prove:
assume y, u differentiable at x.

$$\frac{d}{dx}[y(u(x))] = \frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{dy}{dx}$$

$$\frac{dx}{dy} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \left(\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(\lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x}\right)$$

$$= \left(\lim_{\Delta u\to 0} \frac{\Delta y}{\Delta u}\right) \cdot \frac{du}{dx} = \frac{dy}{dy} \cdot \frac{du}{dx}$$

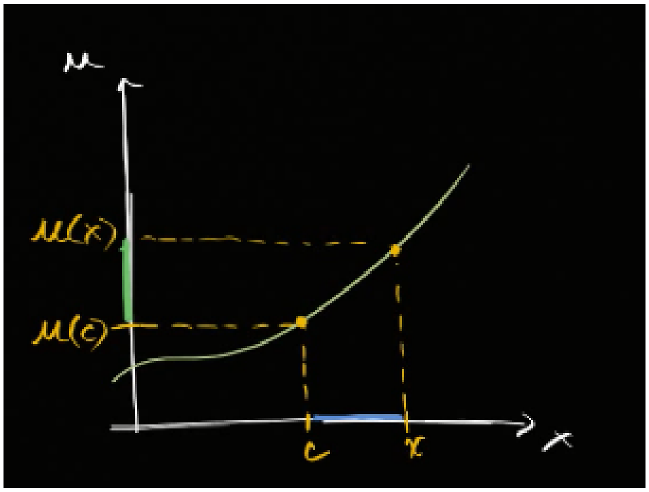

Figure 2: if function u is continuous at x, then $\Delta u \to 0$ as $\Delta x \to 0$

11.4 Implicit differentiation

In implicit differentiation, we differentiate each side of an equation with two variables, by treating one of the variables as function of the other.this calls for using chain rule.
example differentiating $x^2 + y^2 = 1$:
we treat y as an implicit function of x

$$x^2 + y^2 = 1$$
$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$
$$2x + 2y \cdot \frac{dy}{dx} = 0$$
$$2y \cdot \frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

11.5 Differentiating inverse functions

let f (x) a defined function let g (x) be $g(x) = f^{-1}(x)$ then $g(f(x)) = x$ and

$$\frac{d}{dx}[g(f(x))] = \frac{d}{dx}[x]$$

$$g'(f(x)) \cdot f'(x) = 1$$

$$f'(x) = \frac{1}{g'(f(x))}$$

11.6 Differentiating inverse trigonometric functions

let

$$\sin^2 y + \cos^2 y = 1$$

Inverse sine:
let $x = \sin y$

$$y = \sin^{-1} x$$

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1 - (\sin y)^2}}$$

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1 - x^2}}$$

Inverse cosine:
let $x = \cos y$

$$y = \cos^{-1} x$$

$$1 = (-\sin y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - P(\cos y)^2}} = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$$

Inverse tangent:
let $\frac{d}{dx}[\tan x] = \sec^2 x = \frac{1}{\cos^2 x}$

$$y = \tan^{-1} x$$
$$\frac{d}{dx}[\tan y] = \frac{d}{dx}$$
$$\frac{1}{\cos^2 y} \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \cos^2 y = \frac{\cos^2 y}{\cos^2 y + \sin^2 y} \cdot \frac{\frac{1}{\cos^2 y}}{\frac{1}{\cos^2 y}} = \frac{1}{1 + \frac{\sin y}{\cos y}^2} = \frac{1}{1 + \tan y^2}$$
$$= \frac{1}{1+x^2}$$
$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$

11.7 Calculating higher-order derivatives

the second derivative of a function is the derivative of the function's derivative
let $f(x) = x^3 + 2x^2$.its first derivative is $f'(x) = 3x + 4x$, the second derivative of $f(x)$ would be the differentiation of $f'(x)$ which is:

$$f''(x) = 6x + 4$$

Notation for second derivatives:
leibniz's notation for second derivative is $\frac{d^2y}{dx^2}$
ex: leibniz notation of $x^3 + 2x^2$ is $\frac{d^2}{dx^2}(x^3 + 2x^2)$

12 Approximating values of a function using local linearity and linearization

12.1 Local linearity

let $f(x)$ be a defined function.
let (a, b) a defined point on the graph of the function $f(x)$.
the approximation of the point x_0.is given as

$$f(x_0) = L(x_0) = f(a) + f'(a)(x_0 - a)$$

12.2 local linearity and differentiability

if $f(x)$ is differentiable at x_0, then $f(x)$ is locally linear at that point.

13 Using L'Hôpital's rule for finding limits of indeterminate forms

13.1 L'Hôpital's rule introduction

- case 1:
  $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$
- case 2:
  $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$

13.2 Proof of special case of l'Hôpital's rule

if $f(a) = 0$, $g(a) = 0$ and $f'(a)$, $g'(a)$ exist. then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

proof:

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

14 Using the mean value theorem

14.1 mean value theorem

for a function f that's differentiable over an open interval from (a, b), and continuous over the closed interval $[a, b]$ that there exists a number c on that interval such that $f'(c)$ is equal to the function's average rate of change over the interval

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

graphically the tangent line at c is parallel to the secant line going through a and b.

15 Extreme value theorem, global vs local extrema, and critical points

15.1 Extreme value theorem

f is continuous function over $[a, b]$ then
$\exists$ an absolute maximum and and absolute minimum over that interval.

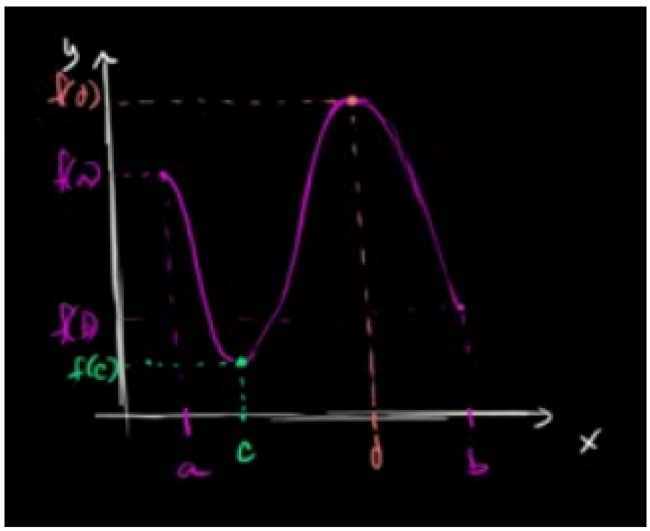

Figure 3: secant line

$$\exists\ c, d \in [a, b] : f(c) \leq f(x) \leq f(d)\ \forall\ x \in [a, b]$$

15.2 Critical points introduction

let x_0, x_1, x_2 be non endpoints maximum or minimum points of $f(x)$, then the derivative of $f'(x_0)$, $f'(x_1)$, $f'(x_2)$ is either going to be 0 or undefined.
A critical point is a point where f' is equal to 0 or undefined.
A critical point is not necessarily an extreme point, the reverse is true.

16 Determining intervals on which a function is decreasing or increasing

16.1 Increasing and decreasing intervals

The intervals where a function $f(x)$ is increasing (or decreasing) correspond to the intervals where its derivative is positive (or negative) $f'(x) < 0$ or $f'(x) > 0$.
the derivative of function changes sign at each critical point.

17 Using the first derivative test to find relative (local) extrema

17.1 first derivative test

If a is min/max value of $f(x)$ at $x = a$ then a is a critical point.
If a is critical point and in the domain of definition of f.then a is a maximum point of $f(x)$, if $f'(x)$ switches sign from positive to negative as $f'(x)$ cross $x = a$.
If a is a critical point and in the domain of definition of f.then a is a minimum point of $f(x)$, if $f'(x)$ switches sign from negative to positive as $f'(x)$ cross $x = a$.

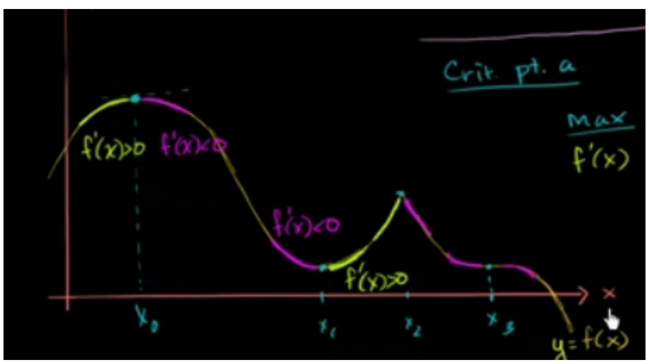

Figure 4: secant line

18 Using the candidates test to find absolute (global) extrema

18.1 Absolute minima and maxima

let $f(x)$ be defined function over the interval $x \in [a, b]$.
let x_0, x_1 x_2 be critical points and maximum points of $f(x)$, where $f'(x_0) = 0 \mid undefined$, $f'(x_1) = 0 \mid undefined$, $f'(x_2) = 0 \mid undefined$.
x_1 is an absolute maximum of $f(x)$ if and only if $f(x_1) > f(x_0)$ *and* $f(x_1) > f(x_2)$ *and* $f(x_1) > f(a)$ *and* $f(x_1) > f(b)$.
let x_0, x_1 x_2 be critical points and minimum points of $f(x)$, where $f'(x_0) = 0|undefined$, $f'(x_1) = 0|undefined$, $f'(x_2) = 0|undefined$.
x_1 is an absolute minimum of $f(x)$ if and only if $f(x_1) < f(x_0)$ *and* $f(x_1) < f(x_2)$ *and* $f(x_1) < f(a)$ *and* $f(x_1) < f(b)$.

19 Determining concavity of intervals and finding points of inflection

19.1 Concavity introduction

let $f(x)$ be continuous and defined over the interval $[a, b]$.
$f(x)$ is concave downards on a sub interval of $[a, b]$ and has maximum point at x_0 where $f'(x_0) = 0$ iff:

- slope is decreasing: $f'(x)$ is decreasing
- the second derivative is negative: $f''(x) < 0$

$f(x)$ is concave upwards on a sub interval of $[a, b]$ and has minimum-point at x_0 where $f'(x_0) = 0$ iff:

- slope is increasing: $f'(x)$ is increasing
- the second derivative is positive: $f''(x) > 0$

19.2 Inflectoin point introduction

Inflection points is a point where the second derivative switches signs
x_1 is an inflection point iff:

- for $x < x_1$: $f''(x) < 0$ and for $x > x_1$: $f''(x) > 0$
- or for $x < x_1$: $f''(x) > 0$ and for $x > x_1$: $f''(x) < 0$

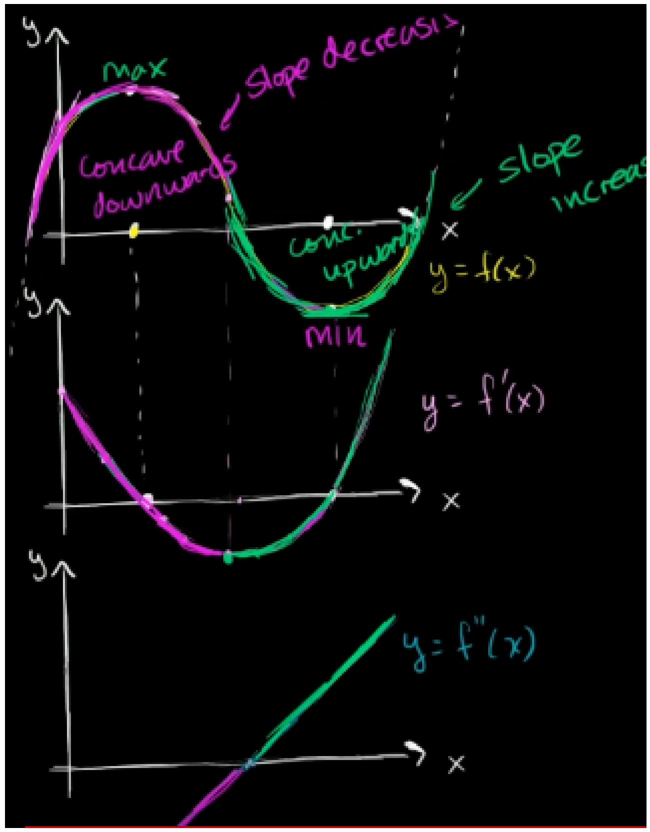

Figure 5: secant line

20 Second derivative test

let $f'(c) = 0$, f' exists in neighborhoud around $x = c$
let $f''(c)$ exists then

- if $f''(c) < 0$ then f has relative maximum at $x = c$
- if $f''(c) = 0$ then $x = c$ is inconclusive
- if $f''(c) > 0$ then f has relative minimum at $x = c$

21 Exploring accumulations of change

21.1 Intro to integral calculus

let $f(x)$ be a defined function.
let δx_i be an infinitesimally small distance of the interval $[a, b]$.
the area under the curve of $f(x)$ between the interval $[a, b]$ is represented as:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \delta x_i = \int_a^b f(x) dx$$

$\int_a^b$ represents the integral of the function $f(x)$

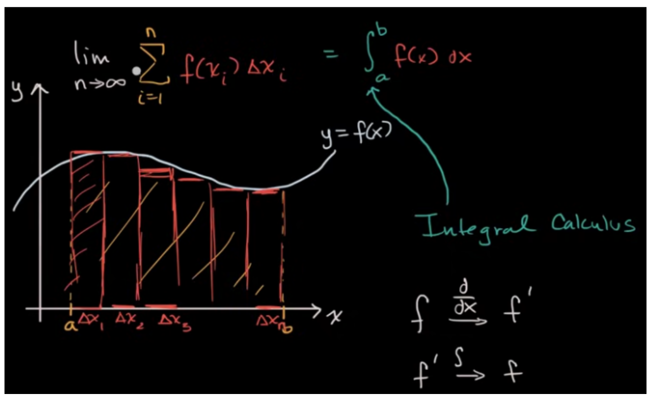

Figure 6: integrals

21.2 Definite integrals intro

The area under the curve between the interval $[a, b]$ is denoted as:

$$\int_a^b f(x) dx$$

which is the definite integral of $f(x)$ between two bounds.

21.3 Exploring accumulations of change

The definite integral can be used to express information about accumulation and net change in applied contexts. the definite integral always gives us the net change in a quantity, not the actual value of

that quantity
In differential calculus, the derivative f' of a function f gives the instantaneous rate of change of f for a given input.
for any rate function f, its antiderivative F gives the accumulated value of the quantity whose rate is described by f.

	Quantity	Rate
Differential calculus	$f(x)$	$f'(x)$
Integral calculus	$F(x) = \int_a^x f(t)dt$	$f(x)$

22 Approximating areas with Riemann sums

22.1 RRiemann approximation introduction

A Riemann sum is an approximation of the area under a curve by dividing it into multiple simple shapes (like rectangles or trapezoids).

In a left Riemann sum, we approximate the area using rectangles (usually of equal width), where the height of each rectange is equal to the value of function at the left endpoint of its base, this type of approximation is considered underestimate.

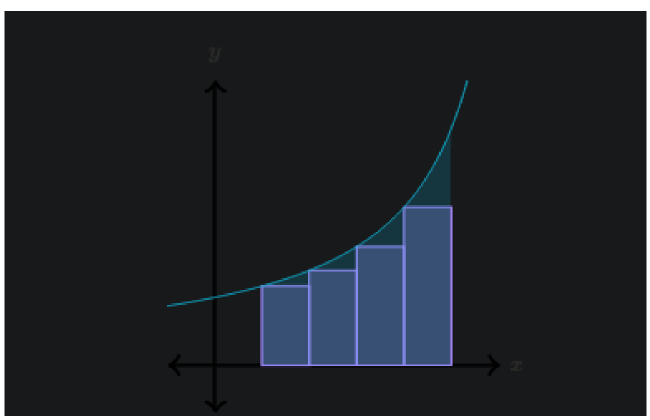

Figure 7: integrals

In a right Riemann sum, the height of each rectangle is equal to the value of the function at the right endpoints of its base.this type of approximation is considered overestimate

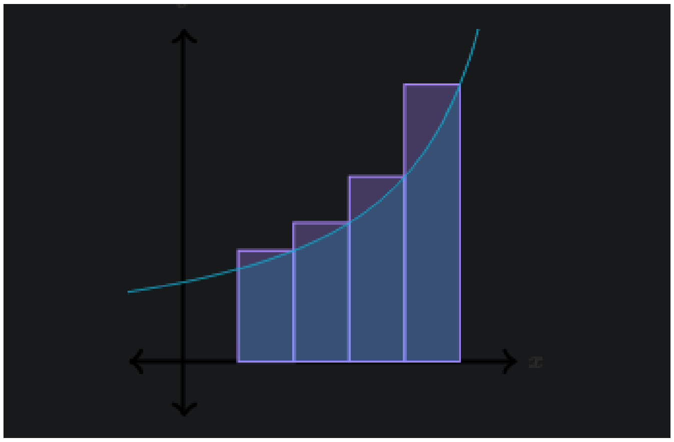

Figure 8: integrals

In a midpoint Riemann sum, the height of each rectangle is equal to the value of the function at the midpoint of its base.

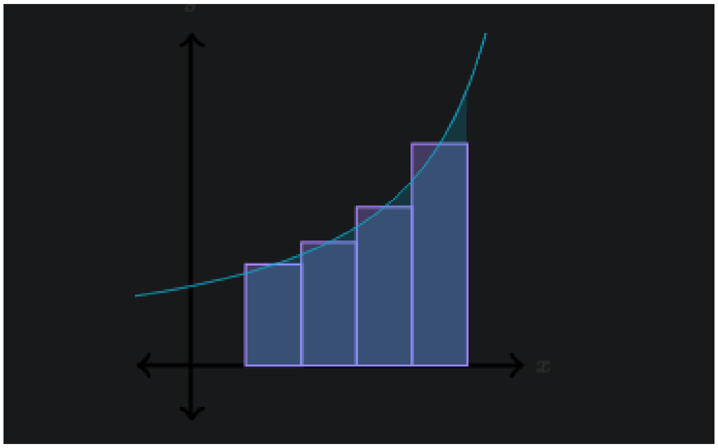

Figure 9: integrals

We can also use trapezoids to approximate the rea (this is called trapezoidal rule). In this case, each trapezoid touches the curve at both of its top vertices.

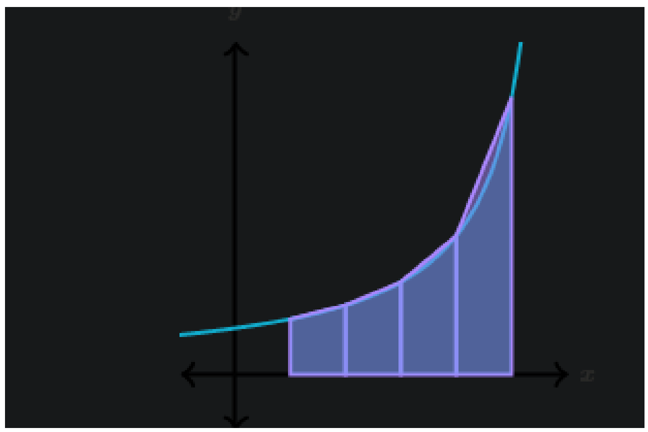

Figure 10: integrals

23 Riemann sums, summation notation and definite integral notation

23.1 Riemann sums in summation notation

Imagine we want to approximate the area under the graph of f over the interval $[a, b]$ with n equal subdivisions.
Define δx: let δx denote the width of each rectangle, then $\delta x = \frac{b-a}{n}$
Define x_i: Let x_i denote the right endpoint of each rectangle, then $x_i = a + \delta x \cdot i$.
Define the area of i^{th}: The height of each rectangle is then $f(x_i)$, and are of each rectangle is $\delta x \cdot f(x_i)$.
sum the rectangles: Now we use summation notation to add all the areas. The values we use for i are different for left and right Riemann sums:

- When we are writing a right Riemann sum, we will take values of i from 1 to n

- However, when we are writing a left Riemann sum, we will take values of i from 0 to $n-1$(this will give us the value of f at the left endpoint of each rectangle).

Left Riemann sum	Right Riemann sum
$\sum_{i=0}^{n-1} \delta x \cdot f(x_i)$	$\sum_{i=1}^{n} \delta x \cdot f(x_i)$

23.2 Definite integral as the limit of a riemann sum

The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is the limit of a Riemann sum as the number of subdivisions approches infinity.that is,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \delta x \cdot f(x_i)$$

where $\delta x = \frac{b-a}{n}$ and $x_i = a + \delta x \cdot i$

24 The fundamental theorem of calculus and accumulation functions

24.1 the fundamental theorem of calculus

let f be continuous function over the interval $[a, b]$.
let $F(x) = \int_a^x f(t)dt$, where x is in $[a, b]$.
the fundamental theorem of calculus states that:

$$\frac{dF}{dx} = \frac{d}{dx}[\int_a^x f(t)dt] = f(x)$$

- Every continuous function f has an antiderivative $F(x)$.
- the FTOC connects integration and differentiation
- $F(x)$ is an antiderivative of f.

25 Applying properties of definite integrals

25.1 Negative definite integrals

let f be continuous defined function.
let $f([a, b]) < 0$. then

$$\int_a^b f(x)dx < 0$$

25.2 Definite integrals properties

Sum/Difference:

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Constant multiple:

$$\int_a^b k \cdot f(x)dx = k \int_a^b f(x)dx$$

Reverse interval:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Zero-length interval:

$$\int_a^a f(x)dx = 0$$

Adding intervals:

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

26 The fundamental theorem of calculus and definite integrals II

26.1 the fundamental theorem of calculus II

let $F(x) = \int_c^x f(t)dt$ and $F'(x) = f(x)$. then

$$F(b) - F(a) = \int_c^b f(t)dt - \int_c^a f(t)dt = \int_a^b f(t)dt$$

OR

$$\int_a^b f(t)dt = F(b) - F(a)$$

26.2 Antiderivative and indefinite integrals

The antiderivative of $2x$ is $\int 2xdx = x^2 + c$.
ther term $\int 2xdx$ is called an indefinite integral.

26.3 Proof of the fundamental theorem of calculus

let $F(x) = \int_a^x f(t)dt$ where $a \leq x \leq b$.

$$F'(x) = \lim_{\delta x \to 0} \frac{F(x+\delta x) - F(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\int_a^{x+\delta x} f(t)dt - \int_a^x f(t)dt}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \int_x^{x+\delta x} f(t)dt$$

according to the mean value theorem of definite integral, there exists a c (where $x \leq c \leq x + \delta x$) such that:

$$f(c)\delta x = \int_x^{x+\delta x} f(t)dt$$

$$f(c) = \frac{1}{\delta x} \int_x^{x+\delta x} f(t)dt$$

as consequence and according to the squeeze theorem:
$c \to x$ as $\delta x \to 0$.
$f(c) \to f(x)$ as $\delta x \to 0$

$$x \leq c(\delta x) \leq x + \delta x$$

$\lim_{\delta x \to 0} x = x$, $\lim_{\delta x \to 0} c(\delta x) = x$, $\lim_{\delta x \to 0} x + \delta x = x$

27 Finding antiderivatives and indefinite integrals: basic rules and notation: reverse power rule

27.1 Reverse power rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

where $n \neq -1$ and c is some constant.

27.2 Indefinite integrals: sum and multiples

- $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
- $\int cf(x)dx = c \int f(x)dx$.

28 finding antiderivative and indefinite integrals: basic rules and notation: common indefinite integrals

28.1 Polynomials

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

28.2 Radicals

$$\int \sqrt[m]{x^n} dx = \int x^{\frac{n}{m}} dx$$

$$= \frac{x^{\frac{n}{m}+1}}{\frac{n}{m}+1} + c$$

28.3 Trigonometric functions

$$\int \sin(x)dx = -\cos(x) + c$$

$$\int \cos(x)dx = \sin(x) + c$$

$$\int \sec^2(x)dx = \tan(x) + c$$

$$\int \csc^2(x)dx = -\cot(x) + c$$

$$\int \sec(x)\tan(x)dx = \sec(x) + c$$

$$\int \csc(x)\cot(x)dx = -\csc(x) + c$$

28.4 Exponential functions

$$\int e^x dx = e^x + c$$
$$\int a^x dx = \frac{a^x}{\ln(a)} + c$$

28.5 Integrals that are lograithmic functions

$$\int \frac{1}{x} dx = \ln|x| + c$$

28.6 Integrals that are inverse trigonometric functions

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin(\frac{x}{a}) + c$$
$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan(\frac{x}{a}) + c$$

29 Integrating using substitution

29.1 u-substitution

u-substitution is about reversing the chain rule:

- according to the chain rule, the derivative of $w(u(x))$ is $w'(u(x)) \cdot u'(x)$.
- In u-substitution, we take an expression of the form $w'(u(x)) \cdot u'(x)$ and find its antiderivative $w(u(x))$

u-substitution helps us take an expression and simplify it b making the "inner" function the variable. example:

$$\int 2x \cos(x^2) dx$$

where $u(x) = x^2$ and $w(x) = \cos(x)$ sometimes we need to multiply/divide the integral by a constant, to get the derivative of $u(x)$. ex:

$$\int \sin(3x + 5) dx = \frac{1}{3} \int \sin(3x + 5) 3dx$$

29.2 u-substitution with definite integrals

when performing u-substitution for definite integral we need to account for the limits of integration, ex:
let $\int_1^2 2x(x^2+1)^3 dx$.
$2x$ is the derivative of x^2+1, so $u = x^2+1$ and $du = 2xdx$.

$$\int_1^2 2x(x^2+1)^3 dx = \int_1^2 u^3 du$$

since the integration limit are fitted for x when need to fit it for u.
since $u = x^2+1$ the new bounds will be:

- Lower bound: $(1)^2 + 1 = 2$
- Upper bound: $(2)^2 + 1 = 5$

$$\int_1^2 2x(x^2+1)^3 dx = \int_2^5 u^3 du$$

alternative way: keep the limits of integration, but substitute back to x before calculating defninite integral, ex:

$$\int_1^2 2x(x^2+1)^3 dx$$

$$= \int_{x=2}^{x=1} (u)^3 du$$

$$= [\frac{u^4}{4}]_{x=1}^{x=2}$$

$$= [\frac{(x^2+1)}{4}]_{x=1}^{x=2}$$

$$= \frac{[(2^2+1)]^4}{4} - \frac{[(1^2+1)]^4}{4}$$

$$= 152.25$$

30 Differential Equations

30.1 Introduction

A differential equation is an equation that elates one or more unknown functions and their derivatives.

$$y'' + 2y' = 3y$$

$$f''(x) + 2f'(x) = 3f(x)$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3y$$

The solution to a differential equation is a function or a class of functions,

31 Slope fields

31.1 Introudction

A slope field is a graphical representation of the solutions to a first-order differential equation.

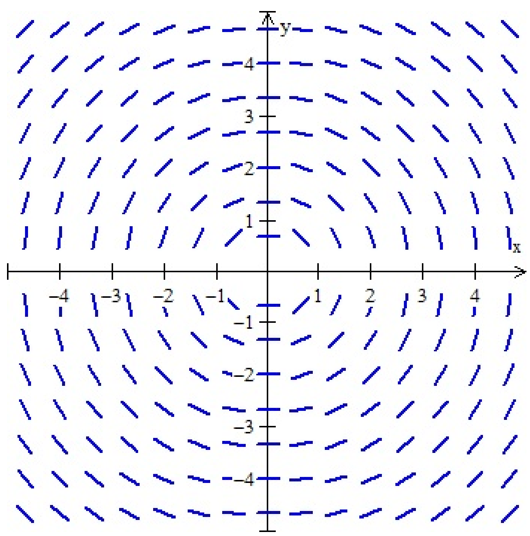

Figure 11: slope field

32 Separable differential equation

32.1 Intro

Separable differential equation: are equations that can be solved using separation of varibales.

To solve a differential equation using separatoin of varibales.we must be able to bring it to the form $f(y)dy = g(x)dx.f(y)$ is an expressoin that doesn't conatain y.and $g(x)$ is an expression that doesn't contain y

33 Finding the average value of a function on an interval

33.1 Mean value theorem for integrals

the average value of a function f over the interval $[a, b]$ is given as:

$$\frac{\int_a^b f(x)dx}{b-a}$$

34 Connecting position, velocity and acceleration functions using integrals

34.1 motion problems

- Displacement: 'what is the particle's displacement between... and... ' or 'What is the change in the particle's position between... and... '

$$\int_a^b v(t)dt$$

- Total distance: 'what is the total distance the particle has traveled between... and... '

$$\int_a^b |v(t)|dt$$

- Actual position: 'What is the particle's position at... '

$$C + \int_a^b v(t)dt$$

35 Washer method

Washer method is a technique in calculus used for finding the volume of a solid or revolution when the solid has a hole in the middle, it's an extension of the disk method and it's used when you're rotating a region aroun an axis and that region is bounded by two functions: an outer radius and an inner radius.

35.1 Formula round the x-axis

if rotating around the x-axis,and the region is between two curves $y = R(x)$ (outer) and $y = r(x)$ (inner) from $x = a$ to $x = b$ then the volume is:

$$V = \pi \int_a^b [R(x)^2 - r(x)^2] dx$$

- $R(x)$: outer radius (from axis to outer curve)
- $r(x)$: inner raius (from axis to inner curve)

35.2 Formula about the y-axis (solve for x)

$$V = \pi \int_a^b [R(y)^2 - r(y)^2]$$

35.3 When to use washer method

- when rotating around a horizontal or vertical axis
- when region lies between two curves
- solid has a hole in the middle

36 Disk Method

the Disk method is a technique used to find thhe volume of a solid of revolution when a region is rotated around an axis

36.1 Formula (rotating around x-axis)

$$V = \pi \int_a^b [R(x)^2] dx$$

where:

- $R(x)$ is the radius of the disk (distance from the axis to the curve)

- a to b are the limits of integration

36.2 When to use disk method

- the region is being rotated around the x-axis or the y-axis
- the solid has no hole in the middle
- the outer edge is defined by a single function

37 Shell method

shell method is a technique used to find the volume of a solid of revolution

37.1 shell method formula (about the y-axis)

if a region between $x = a$ and $x = b$ is rotated about the y-axis the volume is:

$$V = 2\pi \int_a^b (radius \cdot height)dx$$

where:

- $radius = distance from the axis of rotation \to x$
- $height = value of the function \to usually f(x)$

shell method around x-axis:

$$V = 2\pi \int_c^d (radius \cdot height)dy$$

37.2 when to use shell Method

- When it's easier to integrate with respect to x, but the axis of rotation is vertical (like the y-axis).
- When using the disk or washer method would require splitting the region into multiple parts or solving for inverse functions.

38 Integration by parts

Integration by parts is used to find the integral of products:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

$$\int udv = uv - \int vdu$$

this method is considred as the reverse power product rule.

38.1 Using Liate rule to pick u

Priority	**Type**	**Example**
L	Logarithmic	$\ln x$, $\log_a x$
I	Inverse Trig	$\tan^{-1} x$, $\sin^{-1} x$
A	Algebraic	x, x^2, x^3, etc.
T	Trigonometric	$\sin x$, $\cos x$, $\tan x$
E	Exponential	e^x, 2^x, a^x

Table 1: LIATE rule for choosing u in integration by parts

39 Improper integrals

Improper integrals are definite integrals that cover an unbounded area.
Two type of imporper integrals:

- an integral where one of the endpoints is extended to infinity ex:

  $$\int_1^\infty \frac{1}{x^2}dx$$

  which is viewed as:

  $$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx$$

- an integral whose endpoints are finite, but the integrated function is unbounded at one (or two) of the endpoints ex:

  $$\int_0^1 \frac{1}{\sqrt{x}}dx$$

it can be viewed as the limit:

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}dx}$$

When the limit exists we say the integral is **converget**, and when it doesn't we say it's **divergent**.

40 Approximating solution using Euler's method differential equation

let $\frac{dy}{dx} = x + y$ and $f(1) = 2$.
$f(3) \approx ?$
using Euler's method we chose step size of $\Delta x = 1$.
we let $\frac{dy}{dx} = F(x, y) = x + y$ to calculate Δy so that

$$\Delta y_{n+1} \approx y_n + \Delta y_n$$

Then use Euler's method to determine the next y-value.

$$y_{n+1} = y_n + \Delta y_n$$

We'll need to repeat the process-iterate-two times to find the desired value.
We are given $(x_0, y_0) = (1, 2)$, the starting point. Let $n = 1$.

$$x_1 = x_0 + \Delta x \to x_1 = 1 + 1 = 2$$

$$y_1 = y_0 + \Delta y_0 \to y_1 = 2 + 3 \cdot 1 = 5$$

Thus $(x_1, y_1) = (2, 5)$. Now let $n = 2$.

$$x_2 = x_1 + \Delta x \to x_2 = 2 + 1 = 3$$

$$y_2 = y_1 + \Delta y_1 \to y_2 = 5 + 7 \cdot 1 = 12$$

So $f(3) \approx 12$.

41 Logistic models with differential equations

41.1 The logistic growth model

let p be the population. the rate of change of the population with respect to time is proportional to the population times $1 - \frac{p}{k}$ where

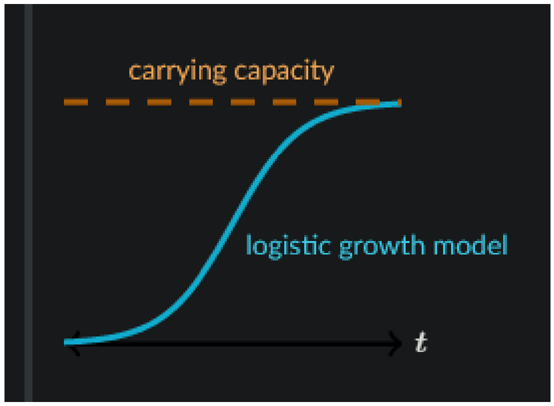

Figure 12: slope field

k is the upper limit that the population can get and $p = 0$ is the lowest limit the polpulatoin can get.
when the population at it's max limit k $\frac{dp}{dt} = 0$.
the logistic differential equation is given as:

$$\frac{dp}{dt} = rp(1 - \frac{p}{k})$$

Logistic growth mean initially growing almost exponentially, but the growth slows as the function approaches a horizontal asymptote.
In logistic growth, the carrying capacity is the upper limit the quantity grows towards $\lim_{t\to\infty} P(t)$.
for any logistic model P, the limit $\lim_{t\to\infty} \frac{dP}{dt} = 0$.
so at the carrying capacity $\frac{dP}{dt} = 0$.

42 Arc length

let ds be small change in the arc length and dx, dy small change in x, y with respect to ds. the arc length over the interval $[a, b]$ is given as:

$$\int ds = \int \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{dx^2(1 + (\frac{dy}{dx})^2)}$$

$$\boxed{= \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx}$$

43 Defining and differentiating parametric equations

A parametric equation expresses coordinates (like x and y) as functions of a third variable, called a parameter (often represented by t) this is a way to define a curve or surface by describing how each coordinates changes with respect to the parameter. ex:
line $x = at + b$, $y = ct + d$ where a, b,c, and d are constants

43.1 Parametric equation differentiation

to defferentiate a function that is defined parametrically by the equation $x = u(t)$ and $y = v(t)$. (where u and v are any function of t) we use the following rule:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{u'(t)}{v'(t)}$$

44 Second derivatives of parametric equations

the second derivative is found with the following rule.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{(\frac{dx}{dt})} = \frac{\frac{d}{dt}(\frac{v'(t)}{u'(t)})}{u'(t)}$$

45 Parametric curve arc length

Arc length of a parametric curve over the interval $[a, b]$:

$$\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = \int_a^b \sqrt{u'(t)^2 + v'(t)^2} dt$$

46 Defining and differentiating vector-valued functions

A vector-valued function is a function that takes one or more variables as input and ouputs a vector.
General form in 2d:

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

or

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

46.1 vector-valued functions differentiation

the derivative of vvf is the derivative of it's compnents.

$$\vec{r'}(t) = \langle x'(t), y'(t) \rangle$$

47 Solving motion prolbems using vvf

the vvf of velocity is given as

$$\vec{v}(t) = (\frac{dx}{dt}, \frac{dy}{dt})$$

the vvf of acceleratoin is given assume

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = (\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2})$$

the magnitude of displacement is given as

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

where

$$\Delta x = \int_a^b x(t)dt$$

$$\Delta y = \int_a^b y(t)dt$$

48 Defining polar coordinates and differentiating in polar form

A polar function is defined using polar coordinates instead of cartesian (rectangular) coordinates.
polar coordinates:

- r: the distance from the origin (called the pole)
- θ: the angel from the positive x-axis (called the polar axis) measured in radians or degrees.

we write (r, θ) instead of (x, y), polar function expresses r as a function of θ.

$$r = f(\theta)$$

to convert between cartesian and polar:

- From polar to cartesian: $x = r\cos\theta, y = r\sin\theta$
- from Cartesian to polar $r = \sqrt{x^2 + y^2}, \theta = \arctan(\frac{y}{x})$

polar function differentiation:

$$\frac{d}{d\theta}y(\theta) = r(\theta)\sin\theta$$

$$\frac{d}{d\theta}x(\theta) = r(\theta)\cos\theta$$

48.1 Formula for slope in polar coordinates

$$\frac{dy}{dx} = \frac{r'\sin(\theta) + r\cos(\theta)}{r'\cos(\theta) - r\sin(\theta)}$$

48.2 Polar area formula

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r(\theta)^2 d\theta$$

48.3 General rule for finding (α, β)

- A:find where the curve starts/ends.
  - solve $r(\theta) = 0$.
  - values of θ are where the curve crosses the pole (origin).
  - they often serve as the bounds for on loop.
- B:look at symmetry
  - many polar curves are symmetric.
  - symmetric about the x-axis $\rightarrow$ integrate from 0 to $\frac{\pi}{2}$ and multiply by 2.
  - symmetric about the y-axis $\rightarrow$ integrate from 0 to $\frac{\pi}{2}$ and multiply by 2.
- C:make sure $r \geq 0$
  - if $r(\theta)$ becomes negative, the curve is traced on the opposite side.
  - either adjust limits or take absolute value in the integral

49 Convergent and divergent sequences

A sequence a_n converges to L if for every small number $\beta > 0$, there exists N such that for all $n > N$, $|a_n - L| < \beta$.
Examples:

- $\frac{1}{n} \rightarrow 0$
- $\frac{2n}{n+1} \rightarrow 2$
- $\left(\frac{1}{n}\right)^2 \rightarrow 0$

A sequence is divergent if it does not converge (diverge) to a finite number.
A sequence diverges when it:

- Goes to infinity: $a_n = n \rightarrow \infty$
- Oscillates without approaching a single value: $a_n = (-1)^2$

49.1 Partial Sum

The n-th partial sum of a series is the sum of the first n terms:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

so

- $S_1 = a_1$
- $S_2 = a_1 + a_2$
- $S_3 = a_1 + a_2 + a_3$

the sequence of partial sum S_n tells us if the series converges or diverges:

- if $\lim_{n\to\infty} S_n = S$ (a finite number), then the series $\sum a_n$ converges to S.
- if the limit does not exist (goes to $\infty, -\infty$ or oscillates), the series diverges.

49.2 Partial sum formula for common type of series

Arithmetic series (linear sequence)
if a_k is arithmetic:

$$a_k = a_1 + (k-1)d$$

the partial sum is:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

or equivalently:

$$S_n = \frac{n}{2}(a_1 + a_2)$$

Geometric Series
if a_k is geometric:

$$a_k = a_1 r^{k-1}$$

the partial sum is:

$$S_n = a_1 \frac{1-r^n}{1-r}, r \neq 1$$

50 Working with geometric series

let a_n be a gemoetric series, and let $\sum_{k=0}^{\infty} a_n(r^k) = \frac{a_1}{1-r}$.
a_n converges iff $|r| < 1$.

51 nth-term divergence test

if $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then the serie $\sum_{n=1}^{\infty} a_n$ diverges.

- nth test can only prove divergence, never convergence
- if $\lim_{n\to\infty} a_n = 0$, the series might converge or diverge

52 Integral test

let

$$\sum_{n=1}^{\infty} a_n$$

if $a_n = f(n)$ where $f(x)$ is:

- continuous
- positive
- decreasing

for all $x \geq N$ (some starting point),
then: $\sum_{n=1}^{\infty} a_n$ converges iff $\int_N^{\infty} f(x)dx$ converges.

53 Harmonic series and p-series

53.1 harmonic series

the harmonic series is the infinte series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

it's called harmonic because th terms are the reciprocals of the positive integers, which are related to the harmonic mean in mathematics.
Key properties:

- it is a p-series with p = 1
- the harmonic series diverges, even though $a_n \to 0$.

53.2 P-series

A p-series is an infinite series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where $p > 0$ is a real number.
Convergence Test: the convergence depends entirely on the value of p:

- if $p > 1$, the series converges.
- if $0 < p \leq 1$, the series diverges.

54 Comparison tests for convergence

let $\sum a_n$ and $\sum b_n$ with $a_n \geq 0$ and $b_n \geq 0$ for all n (nonnegative terms).
The Test
if $0 \leq a_n \leq b_n$ for all n (or for sufficiently large n)

- if $\sum b_n$ converges, then $\sum a_n$ also converges.
- if $\sum a_n$ diverges, then $\sum b_n$ also diverges.

54.1 Limit test comparison

let $\sum a_n$ and $\sum b_n$ where $a_n > 0$ and $b_n > 0$. let:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

- if $0 < L < \infty$, then both series either converge or diverge.
- if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges
- if $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

55 Alternating series test

An alternating series is one where the signs of the terms switch between positive and negative:

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n$$

55.1 the test

the series

$$\sum(-1)^{n}b_n$$

converges if BOTH of these condition hold:

- $b_{n+1} \leq b_n$ for all n (the terms are monotonically decreasing)
- $\lim_{n\to\infty} b_n = 0$

if both conditions are satisfied, the series converges.

56 Taylor and Maclaurin polynomials

56.1 Taylor Polynomial (centered at a)

let $f(x)$ be a function that is n-times differentiable at $x = a$. The Taylor polynomial of degree n centered at a is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdot + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or in summation form:

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

56.2 Maclaurin polynomial

the Maclaurin polynomial is just a special case of the Taylor polynomial with $a = 0$.

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdot + \frac{f^{(n)}(0)}{n!}x^n$$

Or equivalently:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

56.3 Key point

- Taylor polynomial approximates a function near a general point a.
- Maclaurin polynomial is just a Taylor polynomial centered at 0.

57 Largrange error bound

57.1 Taylor polynomial remainder

if f is $(n+1)$-times differentiable at a, the taylor polynomial of degree n around a is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdot + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Then,

$$f(x) = T_n(x) + R_n(x)$$

where $R_n(x)$ is the remainder term.
Forms of the Remainder

- Largrange's Form (most common in analysis)

  $$R_n(x) = \frac{f^{(n+1)}(\beta)}{(n+1)!}(x-a)^{(n+1)}$$

  for some β between a and x.
- Integral Form

  $$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt$$

57.2 Lagrange error bound

is thestandard practical way to bound Taylor remainder.
if f is $(n+1)$-tiems continously differentiable on an interval containing a and x, then the Taylor remainder after degree n satisfies the lagrange form

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where M is any constant with

$$M \geq \max_{t \subset I} |f^{(n+1)}(t)$$

58 Power series

A power series is an infinte series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

where

- c_n are coefficients
- a is the center of the series
- x is the variable

58.1 Convergence of a power series

for a given x, the series may converge or diverge.
There is always a radius of convergence $R \geq 0$ such that:

- The series converges for all $|x-a| < R$
- Diverges for all $|x-a| > R$.
- On the boundary $|x-a| = R$, each case must be checked separately.

58.2 Finding the radius of convergence

use the ration test or root test

$$R = \lim_{n\to\infty} |\frac{c_n}{c_n + 1}|$$

or

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|c_n|}$$

59 Finding Taylor or Maclaurin series for a function

59.1 Function as a geometric series

A geometric series representation of a function is when a function $f(x)$ can be written in the form

$$f(x) = \sum_{n=0}^{\infty} ar(x)^n$$

where:

- a is a constant
- $r(x)$ is an expression in x that plays the role of the ratio
- the series converges when $|r(x)| < x$

Most common Case
the basic geometric series identity is

$$\frac{1}{1 - r(x)} = \sum_{n=0}^{\infty} r(x)^n, |r(x)| < 1$$

this is the formal definition of expanding functions as geometric series

59.2 Power series and Maclaurin expansions of common functions

- Geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$$

- Exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, R = \infty$$

- Sine Function

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}, R = \infty$$

- Cosine Function

$$\cos(x) = \frac{n=0}{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$$

- Natural Logarithm

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, |x| < 1, x \neq -1$$

- Arctangent

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, |x| \leq 1$$

Note:
when finding power series of function we use algebraic manipulation to derive the series, when finding Maclaurin series of a function we use the forumla for Maclaurin series, common function have the equal power serie and maclaurin series expansions.