1 Defining average and instantaneous rates of change at a point

1.1 Derivative as a concept:

Derivative is defined as the instantaneous rate of change at a point, or the slope of a tangent line at that point.
which is defined as the $\lim_{x\to 0} \frac{dy}{dx} = f'(x)$.

1.2 Secant lines and average rate of change:

the average rate of change between two points in an interval $[a, b]$ of a curve is defined as the slope of the secant line that connects these points.

example:

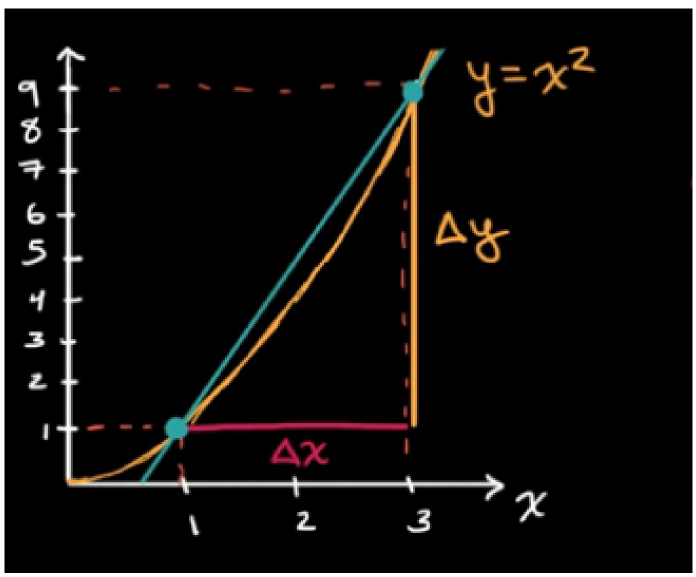

Figure 1: secant line

2 Defining the derivative of a function and using derivative notation

2.1 Formal definition of the derivative as a limit:

h represent some distance more than x. the derivative of a point x_0 formally defined as

$$f'(x_0) = lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

2.2 Alternate form of the derivative:

the derivative of a point x_0 in alternate form is written as

$$f'(x_0) = lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

3 Connecting differentiability and continuity

3.1 Differentiability and continuity

- if f(x) is differentiable at $x = c$, then f(x) is continuous at $x = c$
- if f(x) is not continuous at $x = c$, then f(x) is not differentiable at $x = c$
- if f(x) is not differentiable at a point, then f(x) may or may not be continuous at that point.

4 Power rule

let the function $f(x) = x^n, n \neq 0$, the derivative of $f(x)$ is given according to the power rule as

$$f'(x) = nx^{n-1}$$

5 Derivative rules: constant, sum, difference and constant multiple

5.1 Basic rules

- let A be a constant then $\frac{d}{dx}[A] = 0$.

- let $f(x)$ be a defined function then:

$$\frac{d}{dx}[Af(x)] = A\frac{d}{dx}[f(x)] = Af'(x)$$

- let g(x) be a defined function then:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = f'(x) + g'(x)$$

6 Derivatives of $cos(x)$, $sin(x)$, e^x and $\ln(x)$

6.1 Derivatives of $sin(x)$ and $cos(x)$

- $\frac{d}{dx}[sin(x)] = cos(x)$
- $\frac{d}{dx}[cos(x)] = -sin(x)$

6.2 Derivative of e^x

$\frac{d}{dx}[e^x] = e^x$

6.3 Derivative of $\ln(x)$

$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

7 The product rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

8 The Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$$

9 Derivatives of $tan(x)$ and $cot(x)$

tan(x):

$$\frac{d}{dx}[tan(x)] = \frac{d}{dx}\left[\frac{sin(x)}{cos(x)}\right] = \frac{cos(x) \cdot cos(x) + sin(x) \cdot sin(x)}{cos(x)^2} = \frac{1}{cos(x)^2} = sec(x)^2$$

cot(x):

$$\frac{d}{dx}[cot(x)] = \frac{d}{dx}\left[\frac{cos(x)}{sin(x)}\right] = \frac{-sin(x) \cdot sin(x) - cos(x) \cdot cos(x)}{sin(x)^2} = -\frac{1}{sin(x)^2} = -csc(x)^2$$

10 Chain rule

$$\frac{d}{dx}[f\Big(g(x)\Big)] = f'\Big(g(x)\Big) \cdot g'(x)$$

11 The chain rule: further practice

11.1 Derivative of a^x(for any positive base a)

let $a = e^{\ln a}$ then:

$$\frac{d}{dx}[a^x] = \frac{d}{dx}\Big[\Big(e^{\ln a}\Big)^x\Big] = e^{(\ln a)\cdot x} \cdot \ln a = (\ln a) \cdot a^x$$

11.2 Derivative of $\log_a x$ (for any positive base $a \neq 1$)

let $\frac{d}{dx}[\ln x] = \frac{1}{x}$ and $\log_a b = \frac{\log_c b}{\log_c a}$ then:

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\Big[\frac{1}{\ln a} \cdot \ln x\Big] = \frac{1}{\ln a} \cdot \frac{d}{dx}[\ln x] = \frac{1}{(\ln a)x}$$

11.3 Proving the chain rule

- $u(x)$ continuous at $x = c$ implies that $\Delta u \to 0$ as $\Delta x \to 0$
- $u(x)$ is continuous $\iff lim_{x\to c}u(x) = u(c) \equiv lim_{x\to c}(u(x) - u(c)) = 0$

we have:

- $\Delta u = u(x) - u(c)$

- $\Delta x = x - c$
- $lim_{\Delta x \to 0} \Delta u = 0$

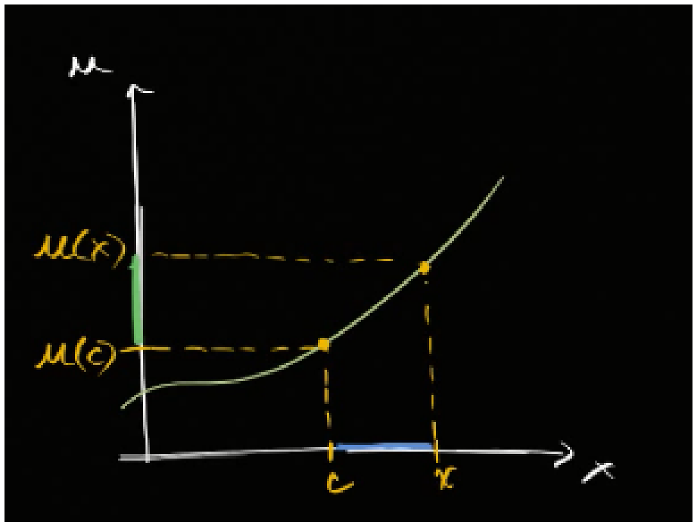

Figure 2: if function u is continuous at x, then $\Delta u \to 0$ as $\Delta x \to 0$

chain rule prove:
assume y, u differentiable at x.

$$\frac{d}{dx}[y(u(x))] = \frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{dy}{dx}$$

$$\frac{dx}{dy} = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \left(lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

$$= \left(lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right) \cdot \frac{du}{dx} = \frac{dy}{dy} \cdot \frac{du}{dx}$$

11.4 Implicit differentiation

In implicit differentiation, we differentiate each side of an equation with two variables, by treating one of the variables as function of the other. this calls for using chain rule.
example differentiating $x^2 + y^2 = 1$:
we treat y as an implicit function of x

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$
$$2x + 2y \cdot \frac{dy}{dx} = 0$$
$$2y \cdot \frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

11.5 Differentiating inverse functions

let f(x) a defined function let g(x) be $g(x) = f^{-1}(x)$ then $g(f(x)) = x$ and

$$\frac{d}{dx}[g(f(x))] = \frac{d}{dx}[x]$$
$$g'(f(x)) \cdot f'(x) = 1$$
$$f'(x) = \frac{1}{g'(f(x))}$$

11.6 Differentiating inverse trigonometric functions

let

$$\sin^2 y + \cos^2 y = 1$$

Inverse sine:
let $x = \sin y$

$$y = \sin^{-1} x$$
$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$
$$\cos y \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$
$$= \frac{1}{\sqrt{1 - (\sin y)^2}}$$

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$

Inverse cosine:
let $x = \cos y$

$$y = \cos^{-1} x$$

$$1 = (-\sin y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-(\cos y)^2}} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$$

Inverse tangent:
let $\frac{d}{dx}[\tan x] = \sec^2 x = \frac{1}{\cos^2 x}$

$$y = \tan^{-1} x$$

$$\frac{d}{dx}[\tan y] = \frac{d}{dx}$$

$$\frac{1}{\cos^2 y} \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \cos^2 y = \frac{\cos^2 y}{\cos^2 y + \sin^2 y} \cdot \frac{\frac{1}{\cos^2 y}}{\frac{1}{\cos^2 y}} = \frac{1}{1+(\frac{\sin y}{\cos y})^2} = \frac{1}{1+(\tan y)^2}$$

$$= \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$

11.7 Calculating higher-order derivatives

the second derivative of a function is the derivative of the function's derivative
let $f(x) = x^3+2x^2$. its first derivative is $f'(x) = 3x+4x$, the second derivative of $f(x)$ would be the differentiation of $f'(x)$ which is:

$$f''(x) = 6x + 4$$

Notation for second derivatives:
leibniz's notation for second derivative is $\frac{d^2y}{dx^2}$
ex: leibniz notation of $x^3 + 2x^2$ is $\frac{d^2}{dx^2}(x^3 + 2x^2)$

12 Approximating values of a function using local linearity and linearization

12.1 Local linearity

let $f(x)$ be a defined function.
let (a, b) a defined point on the graph of the function $f(x)$.
the approximation of the point x_0. is given as

$$f(x_0) = L(x_0) = f(a) + f'(a)(x_0 - a)$$

12.2 local linearity and differentiability

if $f(x)$ is differentiable at x_0, then $f(x)$ is locally linear at that point.

13 Using L'Hôpital's rule for finding limits of indeterminate forms

13.1 L'Hôpital's rule introduction

- case 1:
  $lim_{x\to c} f(x) = 0$ and $lim_{x\to c} g(x) = 0$ and $lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ then $lim_{x\to c} \frac{f(x)}{g(x)} = L$
- case 2:
  $lim_{x\to c} f(x) = \pm\infty$ and $lim_{x\to c} g(x) = \pm\infty$ and $lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ then $lim_{x\to c} \frac{f(x)}{g(x)} = L$

13.2 Proof of special case of l'Hôpital's rule

if $f(a) = 0$, $g(a) = 0$ and $f'(a)$, $g'(a)$ exist. then

$$lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

proof:

$$\frac{f'(a)}{g'(a)} = \frac{lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = lim_{x\to a} \frac{f(x)}{g(x)}$$

14 Using the mean value theorem

14.1 mean value theorem

for a function f that's differentiable over an open interval from (a, b), and continuous over the closed interval $[a, b]$ that there exists a number c on that interval such that $f'(c)$ is equal to the function's average rate of change over the interval

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

graphically the tangent line at c is parallel to the secant line going through a and b.

15 Extreme value theorem, global vs local extrema, and critical points

15.1 Extreme value theorem

f is continuous function over $[a, b]$ then
$\exists$ an absolute maximum and and absolute minimum over that interval.

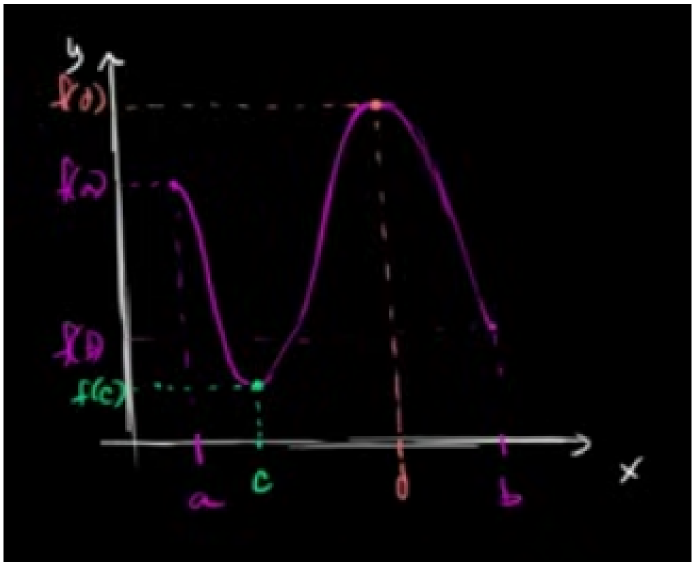

Figure 3: secant line

$$\exists\ c, d \in [a, b] : f(c) \leq f(x) \leq f(d)\ \forall\ x \in [a, b]$$

15.2 Critical points introduction

let x_0, x_1, x_2 be non endpoints maximum or minimum points of $f(x)$, then the derivative of $f'(x_0)$, $f'(x_1)$, $f'(x_2)$ is either going to be 0 or undefined.
A critical point is a point where f' is equal to 0 or undefined.
A critical point is not necessarily an extreme point, the reverse is true.

16 Determining intervals on which a function is decreasing or increasing

16.1 Increasing and decreasing intervals

The intervals where a function $f(x)$ is increasing (or decreasing) correspond to the intervals where its derivative is positive (or negative) $f'(x) <)$ or $f'(x) > 0)$.
the derivative of function changes sign at each critical point.

17 Using the first derivative test to find relative (local) extrema

17.1 first derivative test

If a is min/max value of $f(x)$ at $x = a$ then a is a critical point.
If a is critical point and in the domain of definition of f. then a is a maximum point of $f(x)$, if $f'(x)$ switches sign from positive to negative as $f'(x)$ cross $x = a$.
If a is a critical point and in the domain of definition of f. then a is a minimum point of $f(x)$, if $f'(x)$ switches sign from negative to positive as $f'(x)$ cross $x = a$.

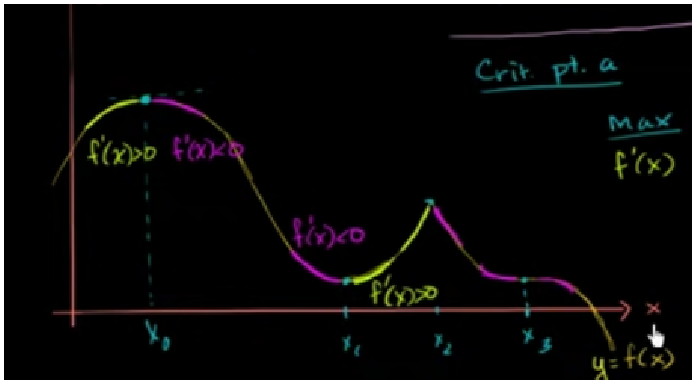

Figure 4: secant line

18 Using the candidates test to find absolute (global) extrema

18.1 Absolute minima and maxima

let $f(x)$ be defined function over the interval $x \in [a, b]$.
let x_0, x_1 x_2 be critical points and maximum points of $f(x)$, where $f'(x_0) = 0 \mid undefined$, $f'(x_1) = 0 \mid undefined$, $f'(x_2) = 0 \mid undefined$.
x_1 is an absolute maximum of $f(x)$ if and only if $f(x_1) > f(x_0)$ *and* $f(x_1) > f(x_2)$ *and* $f(x_1) > f(a)$ *and* $f(x_1) > f(b)$.
let x_0, x_1 x_2 be critical points and minimum points of $f(x)$, where $f'(x_0) = 0|undefined$, $f'(x_1) = 0|undefined$, $f'(x_2) = 0|undefined$.
x_1 is an absolute minimum of $f(x)$ if and only if $f(x_1) < f(x_0)$ *and* $f(x_1) < f(x_2)$ *and* $f(x_1) < f(a)$ *and* $f(x_1) < f(b)$.

19 Determining concavity of intervals and finding points of inflection

19.1 Concavity introduction

let $f(x)$ be continuous and defined over the interval $[a, b]$.
$f(x)$ is concave downards on a sub interval of $[a, b]$ and has maximum point at x_0 where $f'(x_0) = 0$ iff:

- slope is decreasing: $f'(x)$ is decreasing
- the second derivative is negative: $f''(x) < 0$

$f(x)$ is concave upwards on a sub interval of $[a, b]$ and has minimum-point at x_0 where $f'(x_0) = 0$ iff:

- slope is increasing: $f'(x)$ is increasing
- the second derivative is positive: $f''(x) > 0$

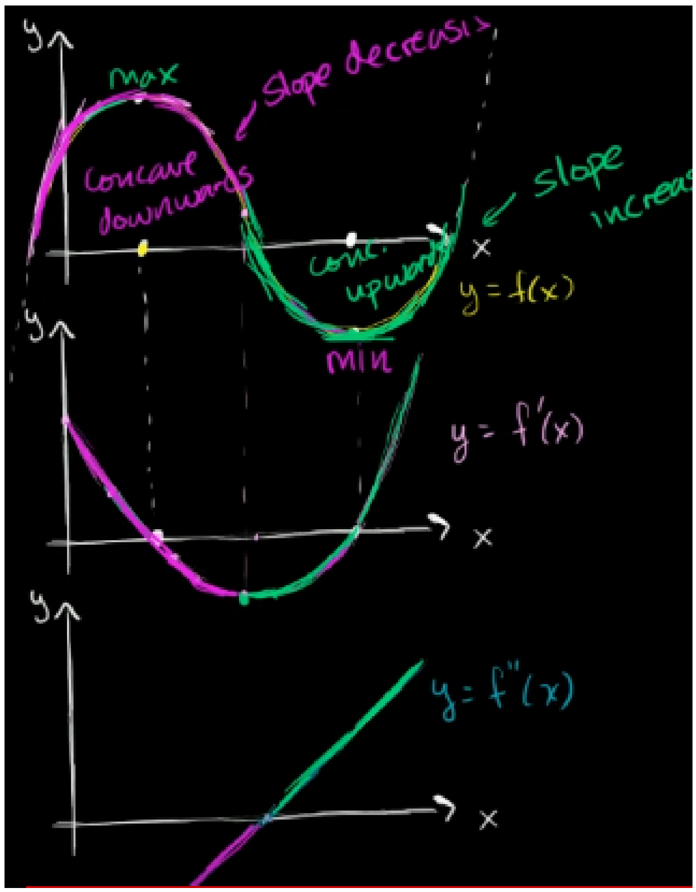

Figure 5: secant line

19.2 Inflectoin point introduction

Inflection points is a point where the second derivative switches signs x_1 is an inflection point iff:

- for $x < x_1$: $f''(x) < 0$ and for $x > x_1$: $f''(x) > 0$
- or for $x < x_1$: $f''(x) > 0$ and for $x > x_1$: $f''(x) < 0$

20 Second derivative test

let $f'(c) = 0$, f' exists in neighborhoud around $x = c$
let $f''(c)$ exists then

- if $f''(c) < 0$ then f has relative maximum at $x = c$
- if $f''(c) = 0$ then $x = c$ is inconclusive
- if $f''(c) > 0$ then f has relative minimum at $x = c$