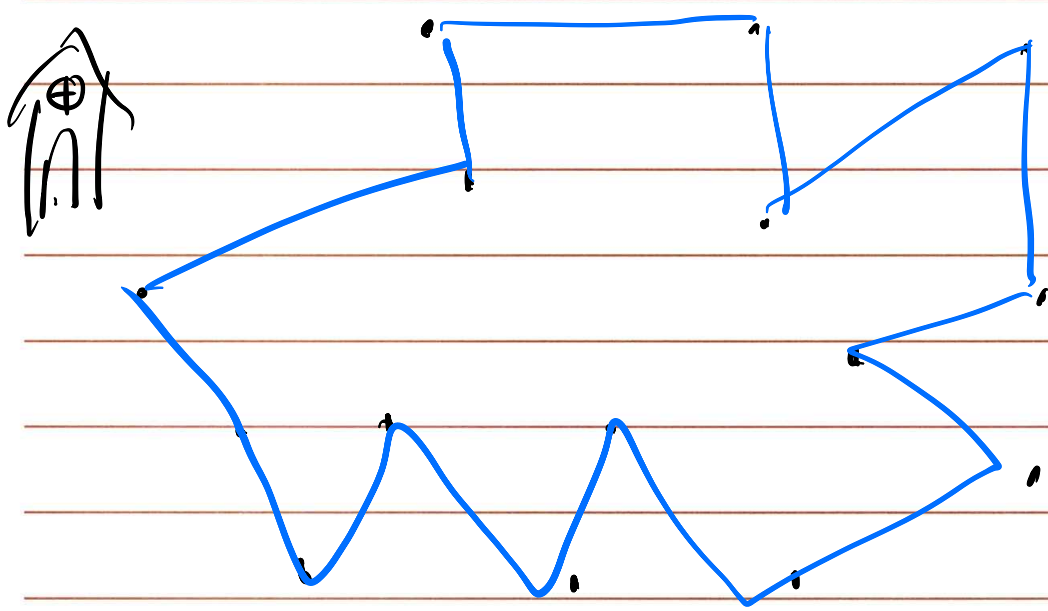
Traveling Salesman Problem (TSP)
&
Hamiltonian Cycle



Problem Statement

Given the set of distances, order $\underline{n}$ cities in a tour $V_{i_1}, V_{i_2}, \ldots V_{i_n}$ with $i_1 = 1$, so it minimizes

$$\sum d(V_{i_j}, V_{i_{j+1}}) + d(V_{i_n}, V_{i_1})$$

Decision version of TSP:

Given a set of distances on $\underline{n}$ cities and a bound $\underline{D}$, is there a tour of length/cost at most D?

Def. A cycle $\underline{C}$ in G is a Hamiltonian Cycle, if it visits each vertex exactly once.

Problem Statement:

Given an undirected graph G, is there a Hamiltonian cycle in G?

Show that the Hamiltonian Cycle Problem is NP-complete

1- Show HC is in NP

a. Certificate: ordered list of the vertices on the HC.

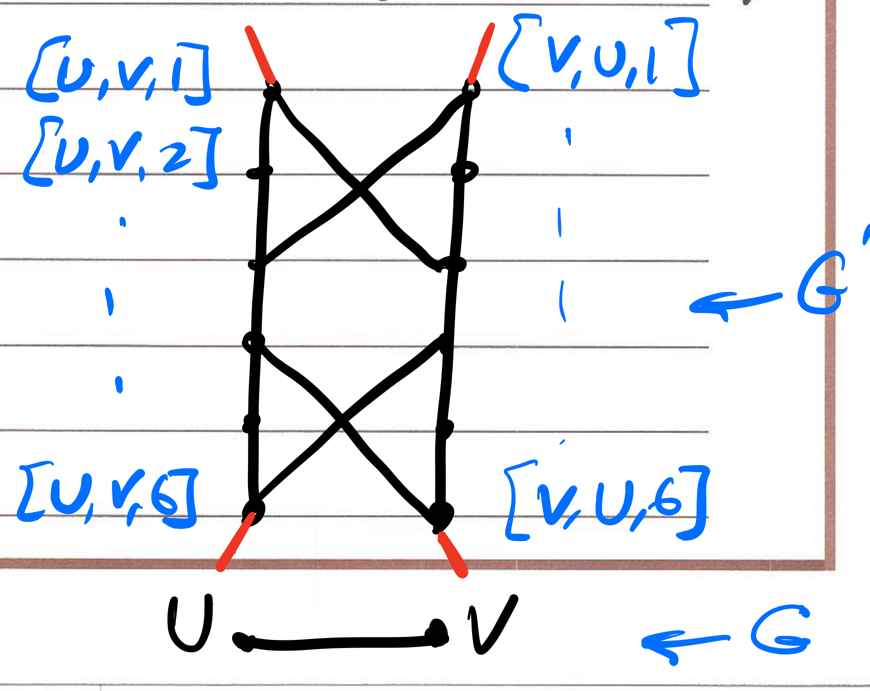
a. Certifier:
- Check all vertices are covered
- No duplicate nodes
- Check that all adjacent nodes in the list are connected by an edge.
- Check that last & first nodes are connected by an edge.

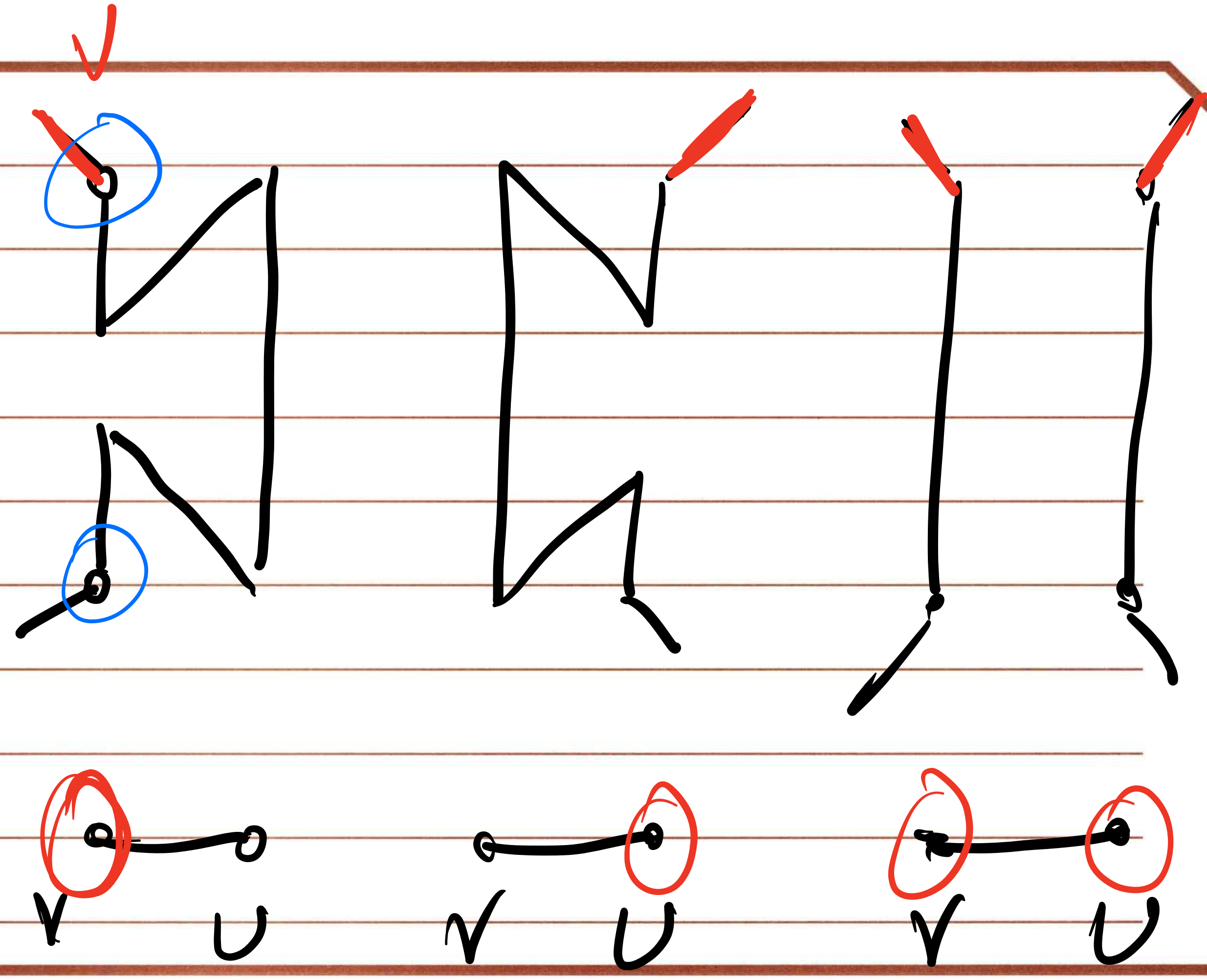
2- Choose Vertex Cover

Plan: Given an undirected graph $G = (V, E)$ and an integer $\underline{k}$, we construct $G' = (V', E')$ that has a Hamiltonian Cycle iff G has a vertex cover of size at most $\underline{k}$.

Construction of G'

For each edge (V, U) in G, G' will have one gadget W_{VU} with following node labeling:



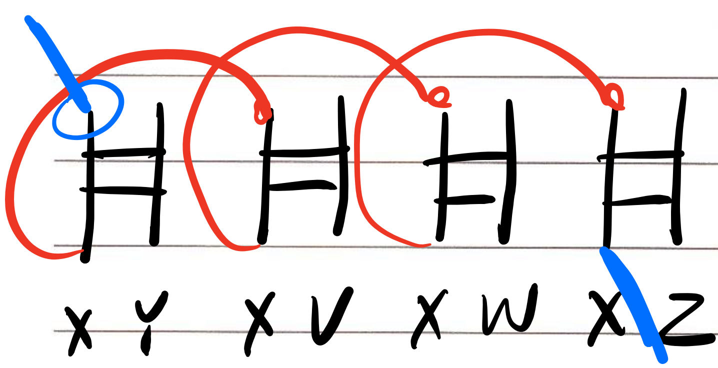


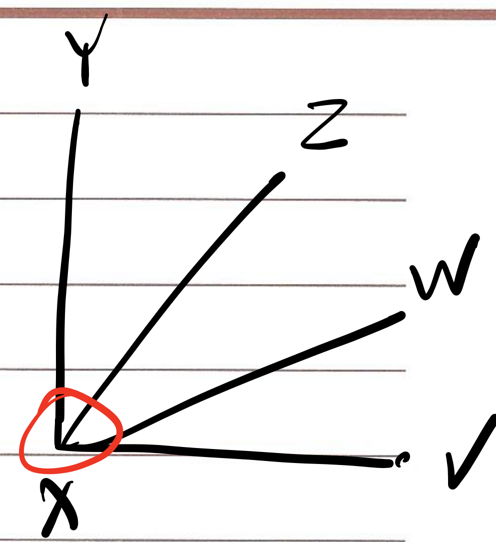
Other vertices in G'

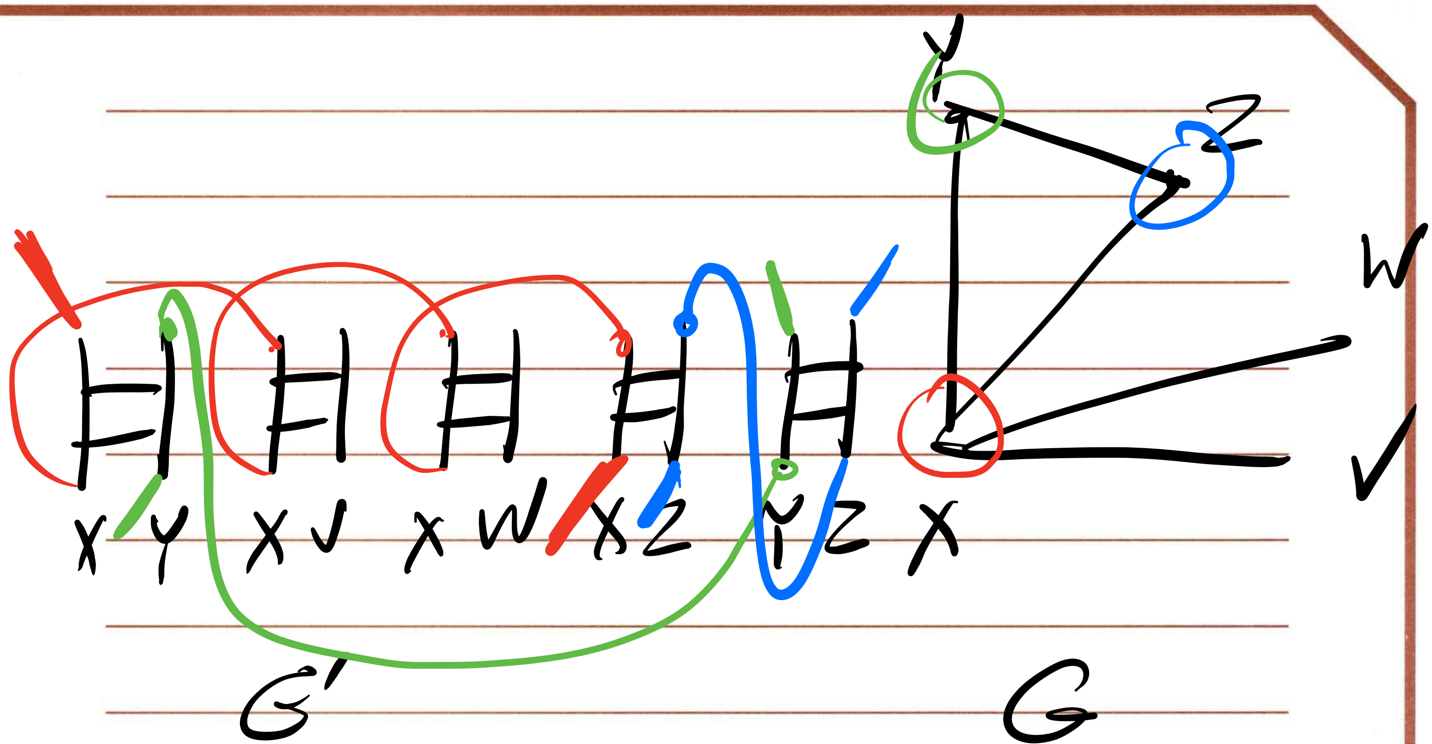
- Selector vertices: There are k selector vertices in G', $s_1, \ldots, s_k$

Other edges in G'

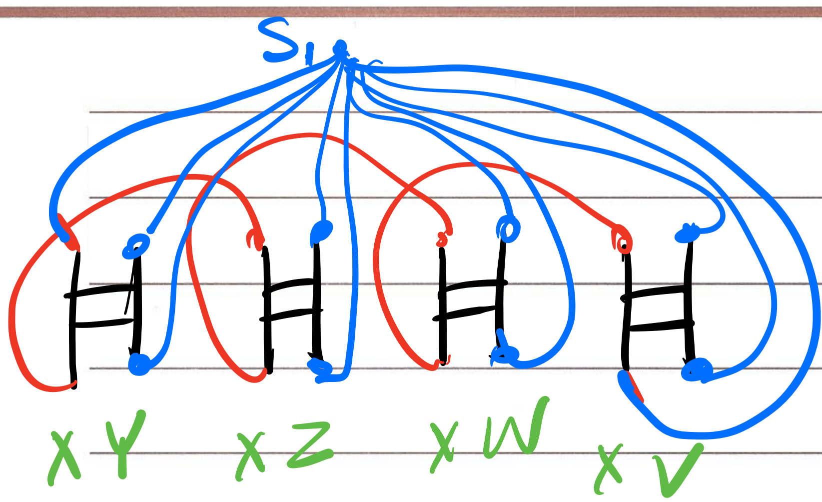
1. For each vertex $v \in V$ we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on v in G.



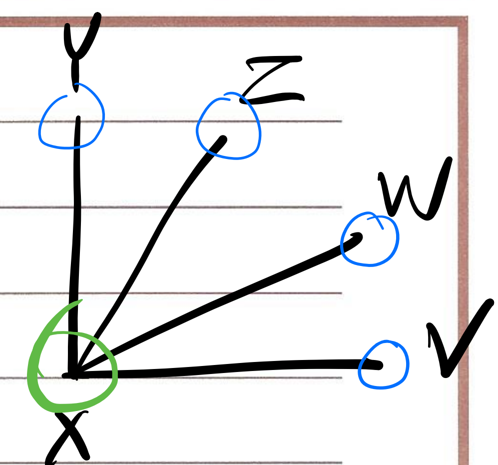



Y
Z
W
V
X Y X V X W X Z Y Z X
G G

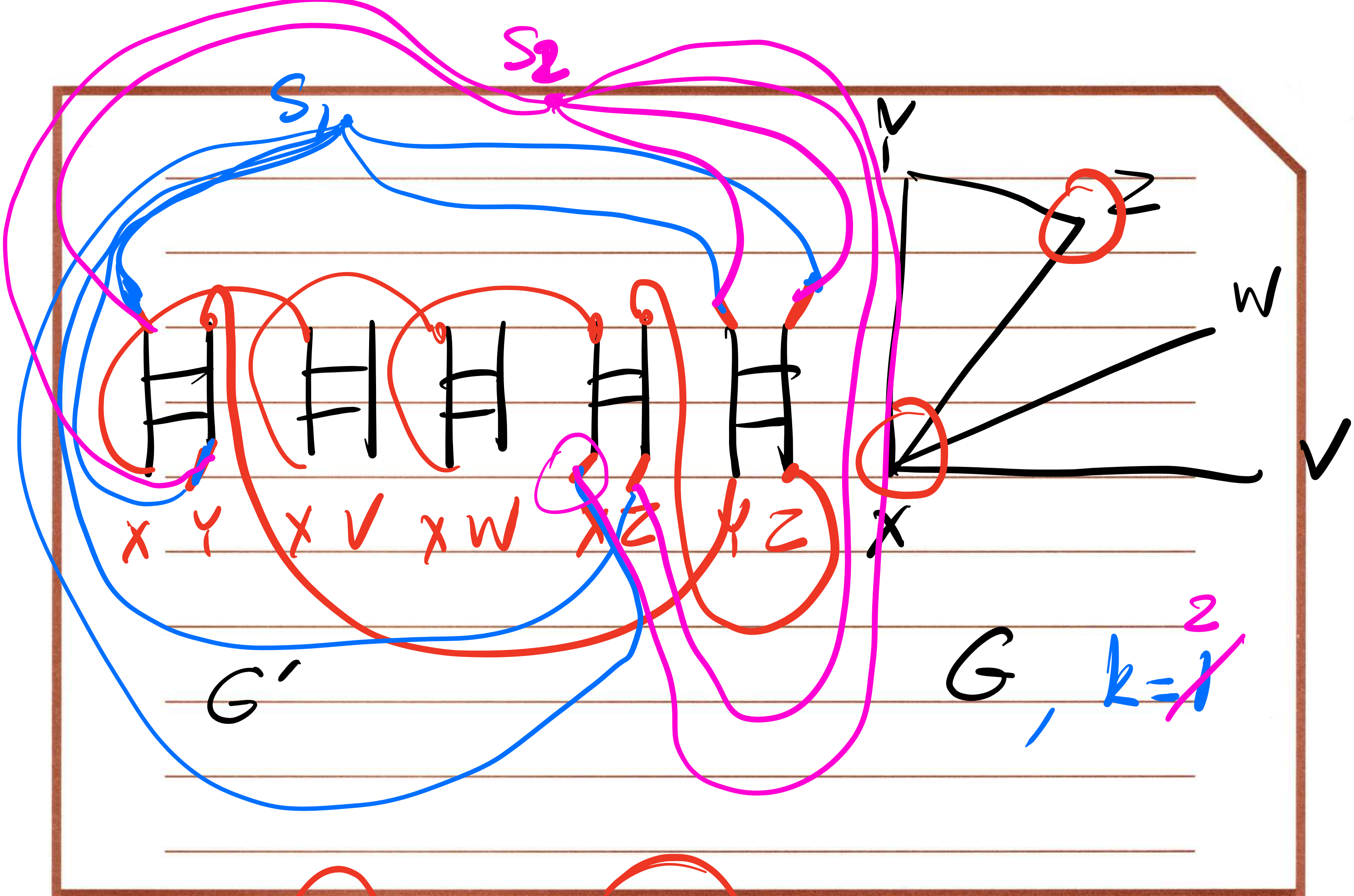
2- Final set of edges in G' join the first vertex $[X, Y, 1]$ and last vertex $[X, Y_{(deg(x))}, 6]$ of each of these paths to each of the selector vertices.

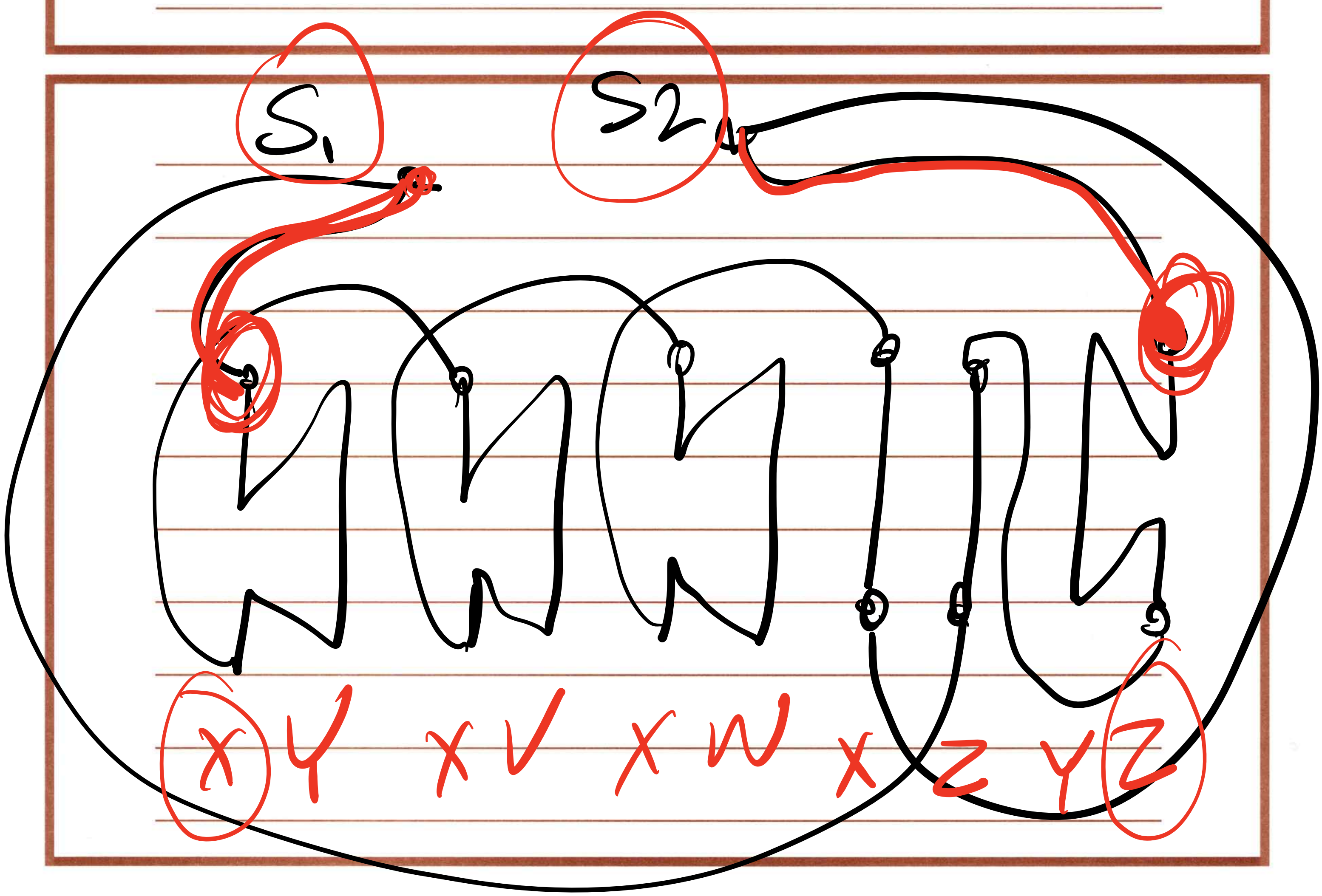


G'



G, k=1

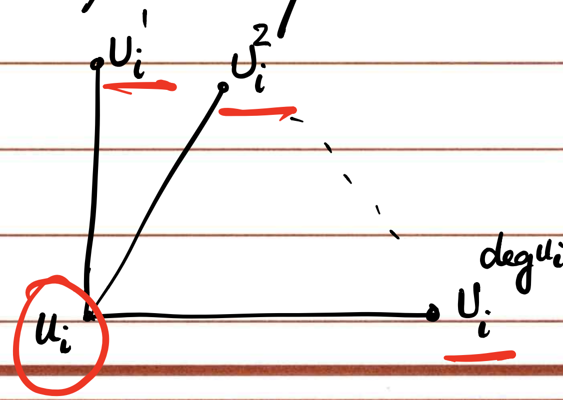

S1
S2
X Y X V X W X Z Y Z
Y
Z
W
V
X
G'
G
k = ~~1~~ 2


S1
S2
X Y X V X W X Z Y Z

Proof: A) Suppose that $G = (V, E)$ has a vertex cover of size $\underline{k}$. Let the vertex cover set be

$$S = \{u_1, u_2, \ldots u_k\}$$

We will identify neighbors of u_i as shown here:



Form a Ham. Cycle in G' by following the nodes in G in this order:

start at s_1 and go to

$[u_1, u_1^1, 1]$ - - - - $[u_1, u_1^1, 6]$

$[u_1, u_1^2, 1]$ - - - - $[u_1, u_1^2, 6]$

$\vdots$

$[u_1, u_1^{\deg u_1}, 1]$ - - - - $[u_1, u_1^{\deg u_1}, 6]$

Path corresponding to edges that are incident on U_2

Then go to S_2 and follow the nodes

$[U_2, U_2^1, 1]$ - - - - - $[U_2, U_2^1, 6]$

$[U_2, U_2^2, 1]$ - - - - $[U_2, U_2^2, 6]$

$\vdots$

$[U_2, U_2^{\deg U_2}, 1]$ - - . $[U_2, U_2^{\deg U_2}, 6]$

Then go to S_3 - - - .

$\vdots$

$\vdots$

$[U_k, U_k^1, 1]$ - - - $[U_k, U_k^1, 6]$

$[U_k, U_k^2, 1]$ - - - $[U_k, U_k^2, 6]$

$\vdots$

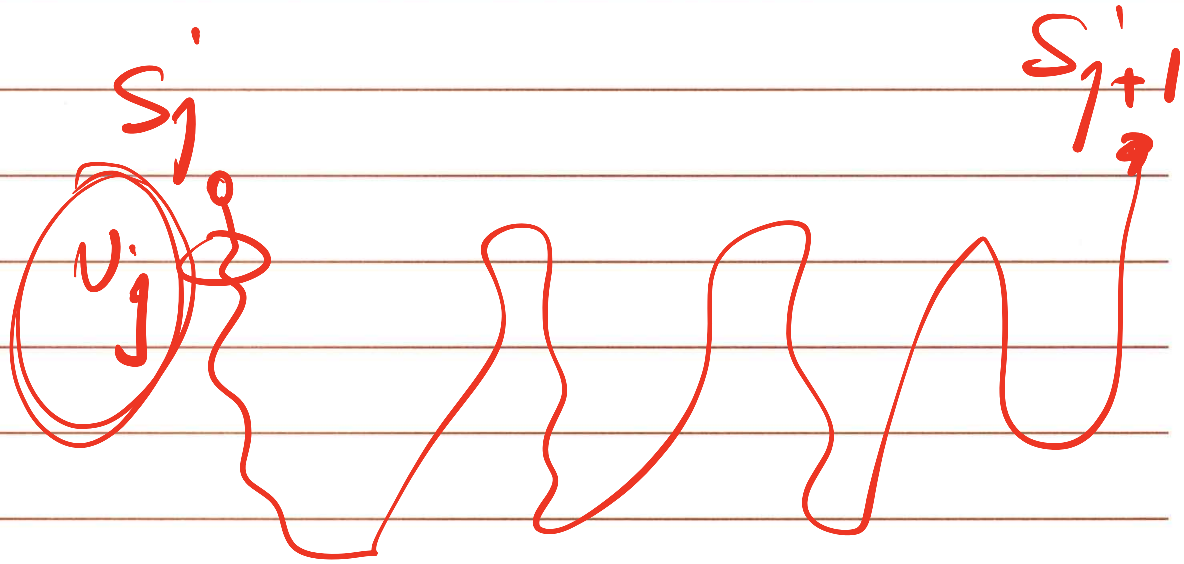
$[U_k, U_k^{\deg U_k}, 1]$ - - - $[U_k, U_k^{\deg U_k}, 6]$

Then return back to S_1.

B) Suppose G' has a Hamiltonian cycle C, then the set

$$S = \{ u_j \in V : (s_j, [u_j, u_j', 1]) \in C \text{ for some } 1 \leq j \leq k \}$$

will be a vertex cover set in G.



We Prove that TSP is NP-Complete

1- Show that TSP $\in$ NP

Certificate: Tour of cost $\leq C$

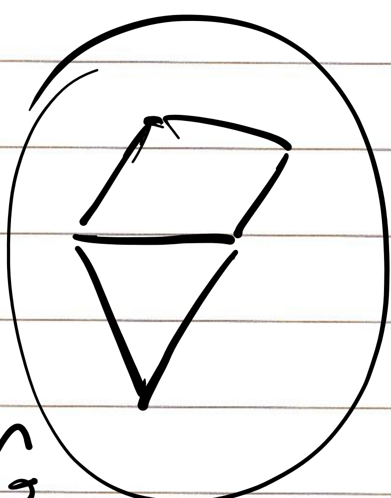
Certifier: Everything we did for HC certifier +

check total cost of the tour $\leq C$

2- Choose an NP-Complete problem:

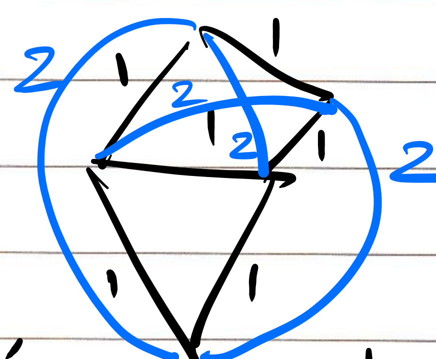
Hamiltonian Cycle.

3- Prove that Ham. Cycle $\leq_p$ TSP



G

no. of nodes in G



G'

Is there a tour of cost $\leq n$ in G'?

Proof:

a) If we have a HC in G

$\Rightarrow$ I can create a tour of Cost n in G'

b) If we have a tour of Cost $\underline{n}$ in G'

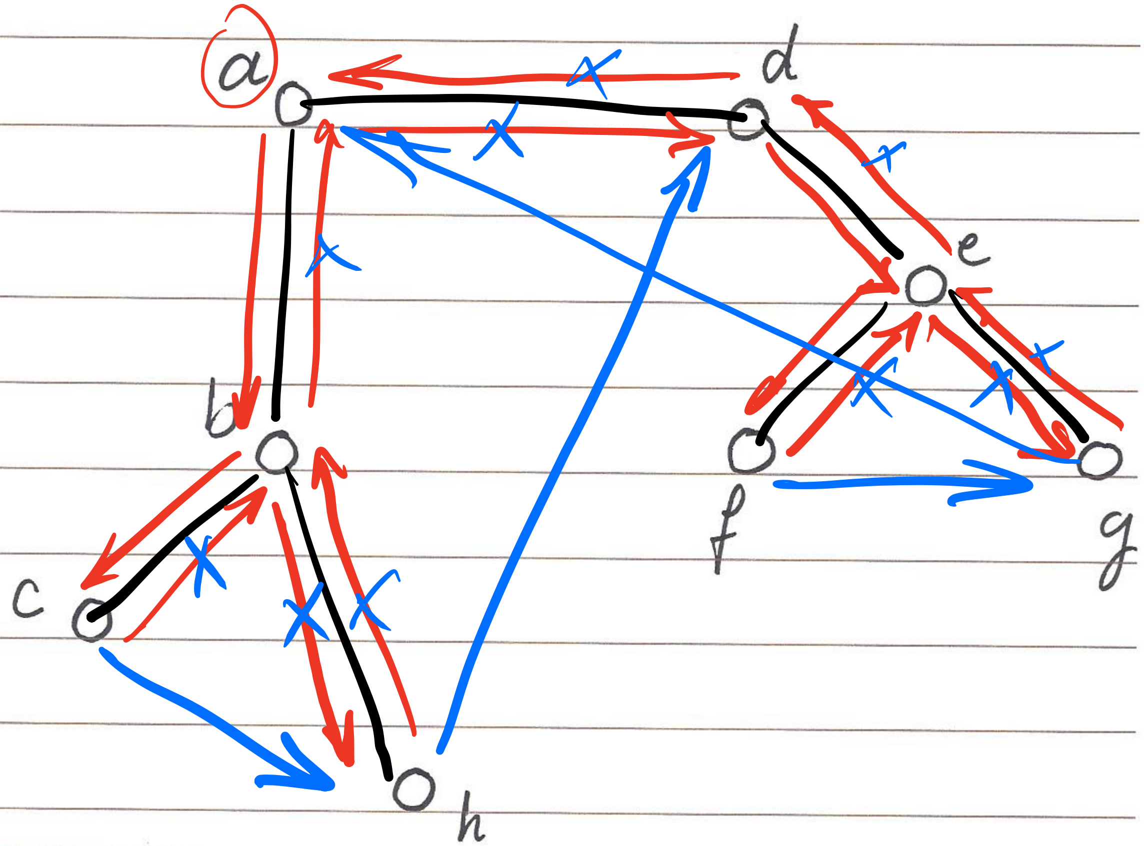
$\Rightarrow$

Our Collection of NP-complete problems:

3SAT, Ind. Set, Vertex Cover, Set Cover, Ham. Cycle, TSP, 0-1 Knapsack, Subset Sum, Graph 3-Coloring

Traveling Salesman Problem

(w/ Triangle Inequalities)



Cost of initial tour = 2 * Cost of MST

(Cost of MST ≤ Cost of opt tour

Cost of our approx. sol. ≤ 2 * Cost of MST

Cost of our sol. ≤ 2 * Cost of opt. sol.

This is a 2-approximation alg.

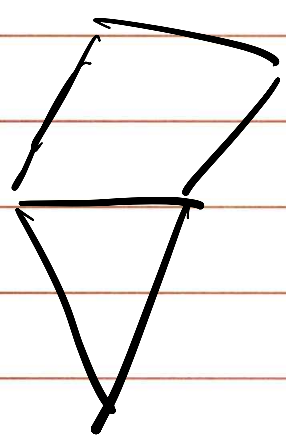
General TSP

Theorem: if $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP

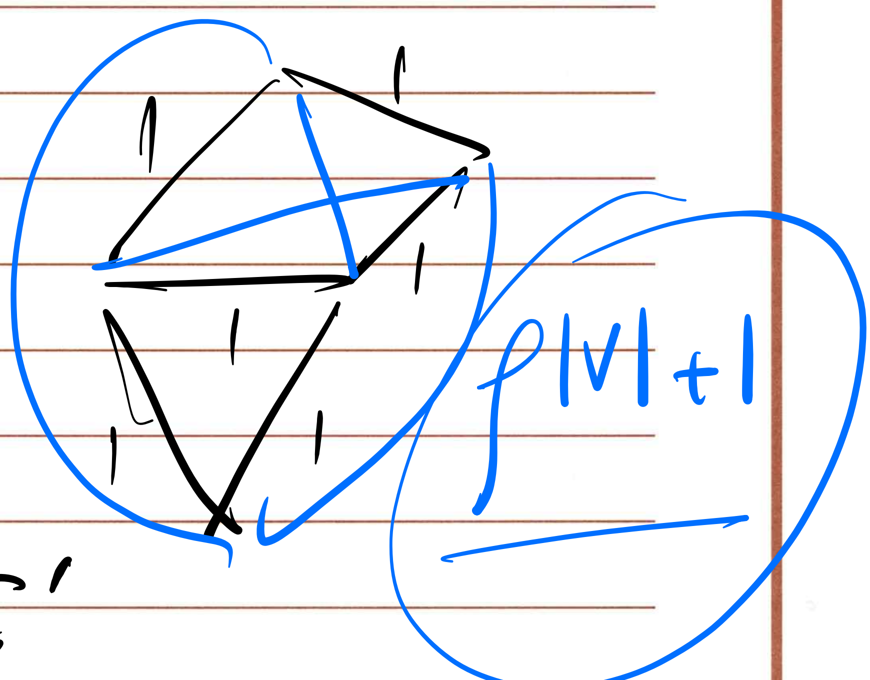
Plan: We will assume that such an approximation algorithm exists. We will then use it to solve the HC problem.

Given an instance of the HC problem on graph G, we will construct G' as follows:

- G' has the same set nodes as in G
- G' is a fully connected graph.
- Edges in G' that are also in G have a cost of 1.
- Other edges in G' have a cost of $\rho|V|+1$



G



G'

a) If there is a HC in G

$\Rightarrow$ cost of opt. tour in $G' = |V|$

approx. alg. should find a tour

of cost $\leq \rho |V|$

b) if there is a tour of cost $\leq \rho |V|$

in G'

$\Longrightarrow$ G has a HC!

Discussion 11

1. In the *Min-Cost Fast Path* problem, we are given a directed graph $G=(V,E)$ along with positive integer times t_e and positive costs c_e on each edge. The goal is to determine if there is a path P from s to t such that the total time on the path is at most T and the total cost is at most C (both T and C are parameters to the problem). Prove that this problem is **NP**-complete.

2. We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

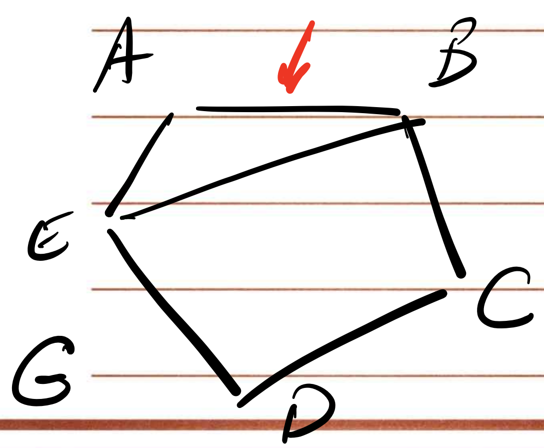
3. Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.
Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.

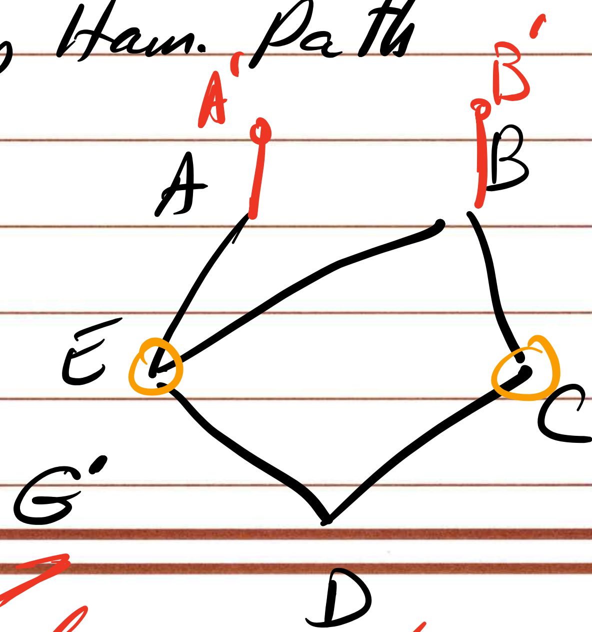
2. We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

1- Skip

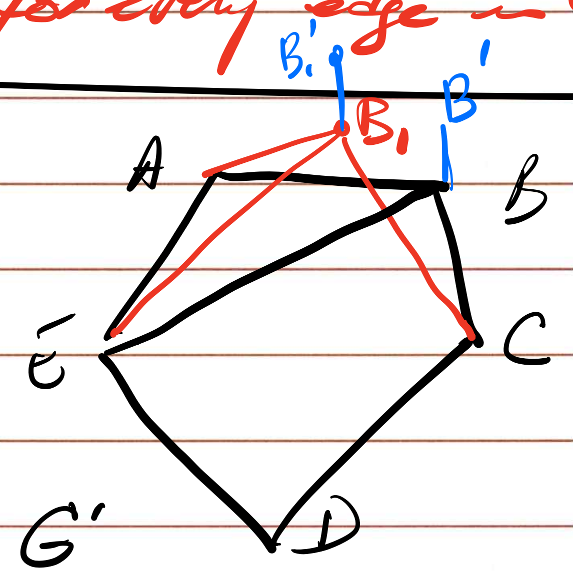
2- Choose Ham. Cycle.

3- Show HC $\le_p$ Ham. Path





repeat for every edge in G.



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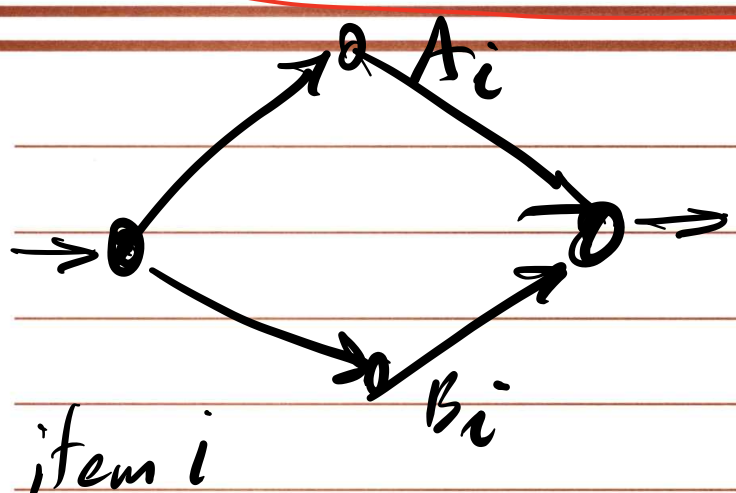
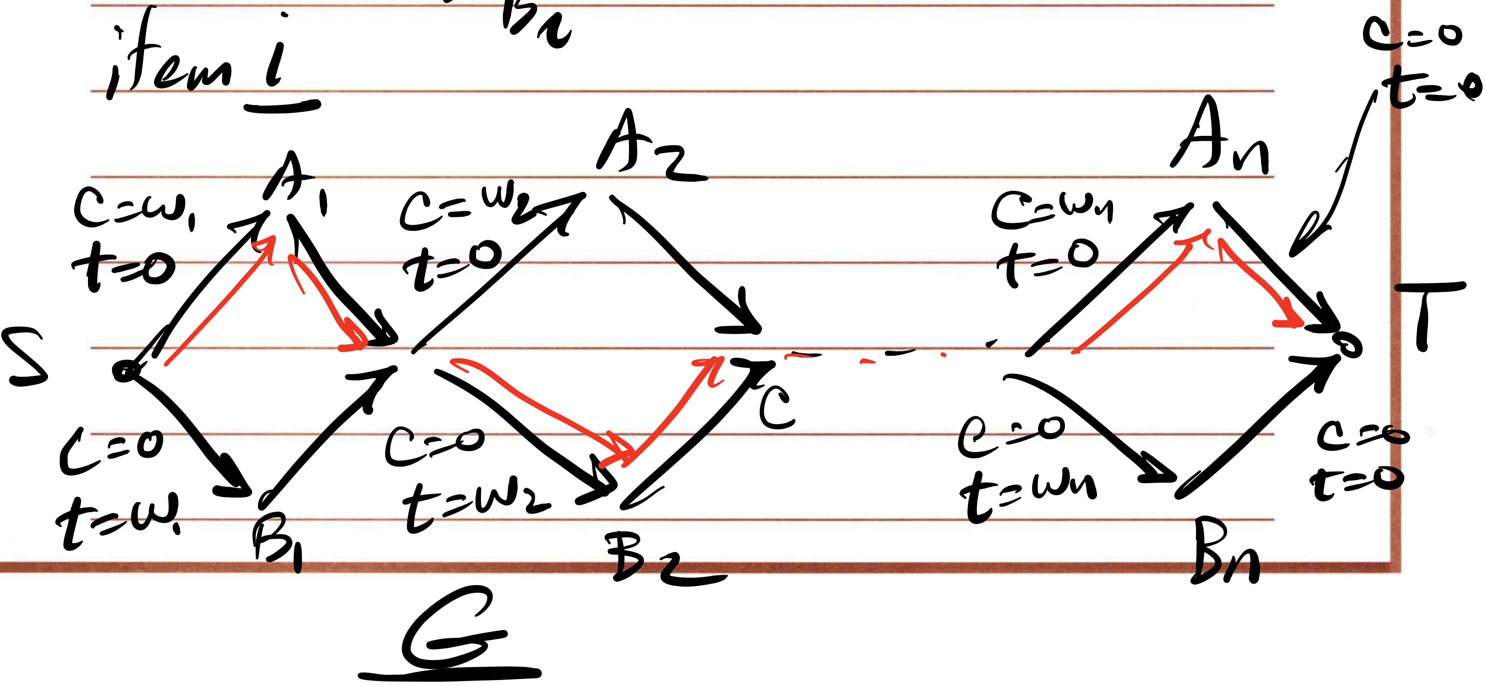
1- Skip
2- Choose Subset sum
3- Show Subset sum $\leq_p$ MCFP

Given a set of n items with weights w_i
is there a subset of them with
total weight $\leq W$ &
total " $\geq M$

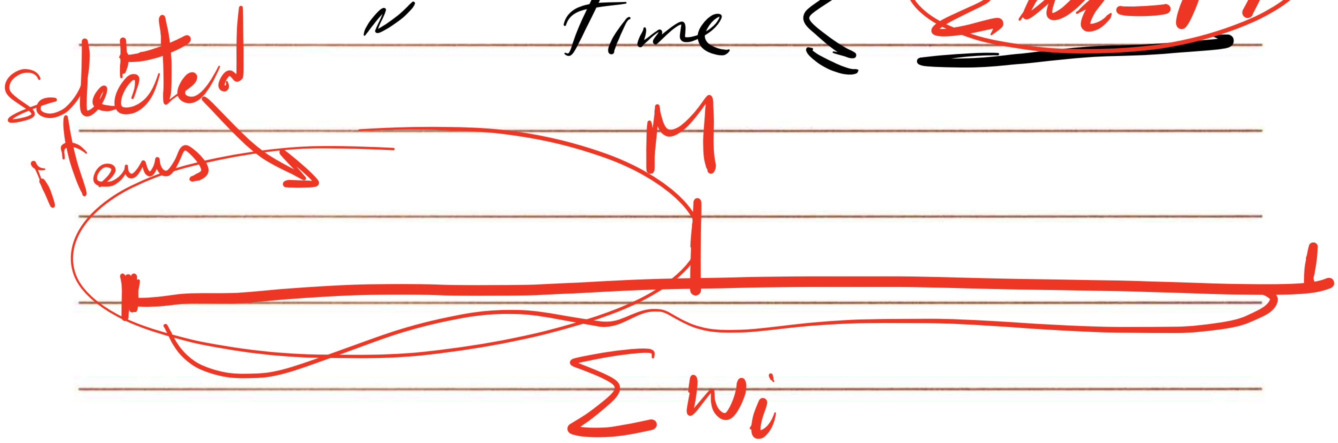




Is there a MCFP from s to t

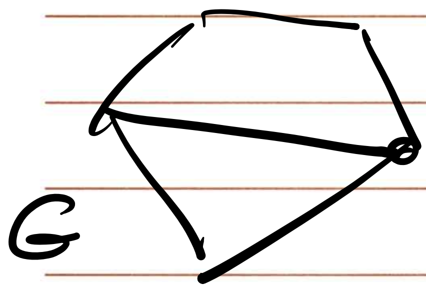
w/ total cost $\leq W$

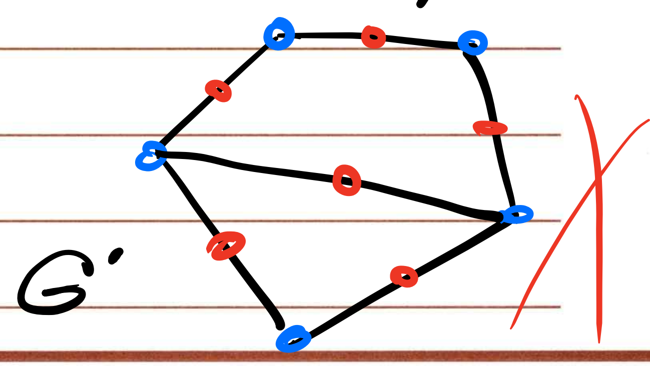
" time $\leq \sum w_i - M$

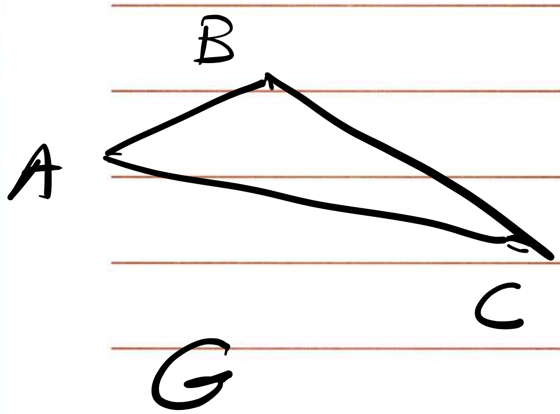


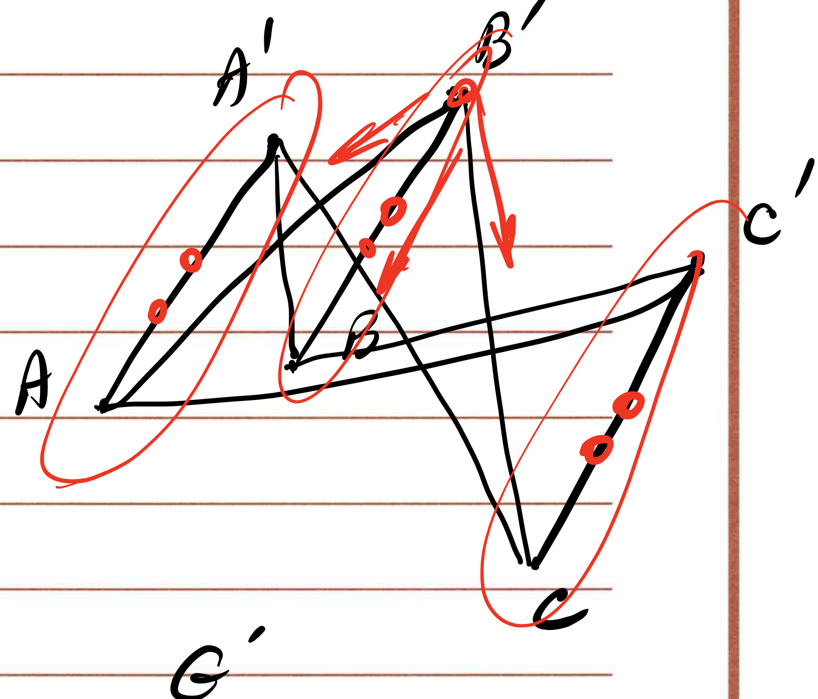
3. Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.
Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.

1- Skip
2- Choose HC
3- Show $HC \leq_p HC$ in a bipartite graph









ABCA

$AA'BB'CC'A$

$\underbrace{CC'}_{C}\underbrace{AA'}_{A}\underbrace{BB'}_{B}C$

C ABC

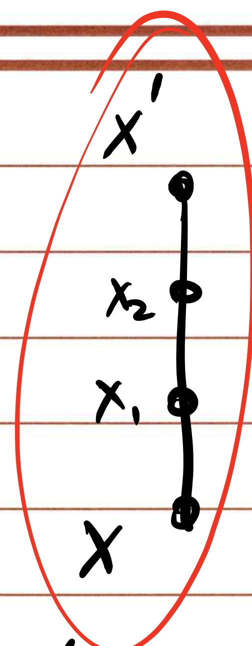
?

$AB'CA'BC'A$

$\underbrace{AA'}_{A}\underbrace{BB'}_{B}\underbrace{CC'}_{C}A$

B'

A node in $G(X)$ →



An edge in $G(XY)$ →