AMS 301.2 (Spring, 2016) Estie Arkin

Exam 2 – Solutions

Mean 75.667, median 77, top quartile 87, bottom quartile 67, high 98, low 27.

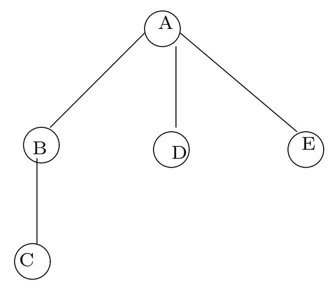
1. (5 points) Let T be a 10-ary tree with 2001 nodes in total. How many leaves does T have? (A correct guess with no work shown will receive very little partial credit.)

$n = mi + 1$, $n = 2001$, $m = 10$, so $i = 200$, $l = 2001 - 200 = 1801$

2. (13 points) (b). In what order were the edges insterted into the MST using Kruskal's algorithm?

(A,B) (C,D) (B,C) (A,E)

(c). Ignoring the costs on the edges, show the BFS tree of the graph starting at node A.



3. (12 points, 3 points for each part) Answer TRUE or FALSE: We are given a connected graph G with costs on edges. Assume all costs are positive integers and that there are no ties. A, B are nodes in the graph.

False If an edge e is part of a BFS Tree rooted at node A, then it must also be part of a BFS Tree rooted at B.

True A BFS on G rooted at node A has the same number of edges as a DFS on G rooted at node A.

False A quick TSP tour found (for the same graph) by two different students must have the same cost.

True A MST found (for the same graph) by two different students must have the same cost.

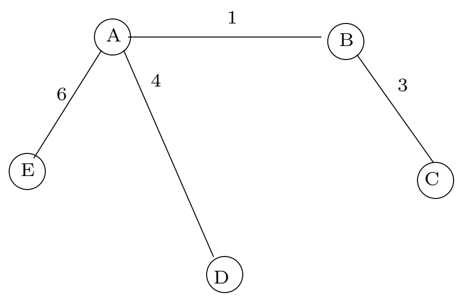
4. (12 points) Which of the following claims are true and which are false? Justify your answer by giving a short proof or a counterexample:

(a). If all arcs in a network have different costs, the network has a unique shortest path tree. False. Consider nodes A,B,C, cost of edge (A,B) is 3, edge (A,C) is 1 and (B,C) is 2. There are 2 shortest paths from A to B both of length 3.

(b). In a directed network with positive arc lengths, if we eliminate the direction on every arc (i.e., make it undirected) the shortest path distances will not change. False. Consider graph with nodes A,B,C, arcs (B,A), (A,C) and (C,B) have cost 1. In the directed graph, shortest distance from A to B is $1 + 1 = 2$ if we undircted, it becomes 1.

(c). In a shortest path problem, if each arcs length is increased by k units, the shortest path distances increase by a multiple of k. False. Undirected graph on nodes A,B,C. Edges (A,B) and (B,C) have cost 1 and edge (A,C) has cost 3. Length of shortest A to C path is 2. If each edge cost is increased by $k = 10$ the shortest path length becomes 13. But $13 - 2$ is not a multiple of 10.

5. (10 points) (a). Highlight the edges of the shortest path tree rooted at node A for the graph below.



(b). In what order do the nodes become permanent?

A,B,C,D,E or A,B,D,C,E.

6. (14 points) Consider a decimal sequence of length 8 (decimal meaning digits 0,1,2,...,9 may appear). Each of the following parts is independent of the others.

(a). How many such sequences start and end with at least two 3's? 10^4

(b). How many such sequences have exactly 2 different digits appearing (e.g. 05550005)? Choose the 2 digits, then make a sequence, subtracting 2 for a sequence with only 1 of the digits. $\binom{10}{2}(2^8-2)$

(c). How many such sequences have the digit 7 appearing at most 3 times? Cases 7 appears 0,1,2,3 times: $9^8 + \binom{8}{1}9^7 + \binom{8}{2}9^6 + \binom{8}{3}9^5$

7. (20 points) 40 identical balls are to be placed in 4 distinct boxes.

(a). How many different ways can the balls be placed?

$$\binom{40+4-1}{40} = \binom{43}{40}$$

(b). How many different ways can the balls be placed if every box gets at least 3 balls?

$$\binom{40-3\cdot 4+4-1}{40-3\cdot 4} = \binom{31}{28}$$

(c). How many different ways can the balls be placed if one of the boxes gets at most 2 balls, and the other boxes get at least 2 balls each?

$$4(\binom{40-3\cdot 2+3-1}{40-3\cdot 2} + \binom{39-3\cdot 2+3-1}{39-3\cdot 2} + \binom{38-3\cdot 2+3-1}{38-3\cdot 2}) = 4(\binom{36}{34} + \binom{35}{33} + \binom{34}{32})$$

(d). How many different ways can the balls be placed if each boxes get at least 2 balls each, but no box gets 18 or more balls?

$$\binom{40-4\cdot 2+4-1}{32} - 4\left(\binom{40-18-3\cdot 2+4-1}{16} - 3\right) - \binom{4}{} = \binom{35}{32} - 4\left(\binom{19}{16} - 3\right) - \binom{4}{2}$$

All outcomes (with 2 balls in each) minus those in which 1 box gets 18 or more, minus the outcome when exactly 2 boxes get 18.

8. (15 points). A deal of cards in bridge gives each of the 4 players (called North, South, East and West) 13 cards (from a standard deck of 52 cards).

(a). How many ways are there to deal the cards?

$$P(52; 13, 13, 13, 13) = \frac{52!}{(13!)^4}$$

(b). What is the probability that West has 4 spades, 2 hearts,4 diamonds and 3 clubs?

$$\binom{13}{4}^2 \binom{13}{2} \binom{13}{3} / \binom{52}{13}$$

(c). What is the probability of one of the players having all 4 aces?

$$4 \cdot \binom{48}{9} / \binom{52}{13}$$