5MC10: Combinatorial Algorithms
NP-complete problems

AmirHossein Ghamarian

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1. Show that **VERTEX-COVER** remains NP-Complete even if all vertices in the input graph are restricted to have even degrees. Hint: The sum of the degrees of the vertices in an undirected graph is always an even number.

   hint: you can use the vertex cover problem for the reduction.
   VERTEX COVER
   INSTANCE: A graph $G = (V, E)$ and a positive integer $K \leq |V|$.
   QUESTION: Is there a *vertex cover* of size K or less for G, that is, a subset $V' \subseteq V$ such that $|V'| \leq K$ and, for each edge $\{u, v\} \in E$, at least one of u and v belongs to V.

   Solution: The reduction is from the vertex cover problem. Let $< G, K >$ be an input instance of VERTEX-COVER, where $G = (V, E)$. Because each edge in E contributes a count of 1 to the degree of each of the vertices with which it is incident, the sum of the degrees of the vertices is exactly $2.|E|$, an even number. Hence, there is an even number of vertices in G that have odd degrees.

   Let U be a the subset of vertices with odd degrees in G; $|U|$ is even. Construct a new instance $< G', k + 2 >$ of VERTEX-COVER, where $G' = (V', E')$ with $V' = V \cup \{x, y, z\}$ (three new vertices), and

   $$E' = E \cup \{\{x, y\}, \{x, z\}, \{y, z\}\} \cup \{\{x, v\} | v \in U\}.$$

   In words, we make a triangle with the three new vertices, and then connect one of them (say x) to all the vertices in U. The degree of every vertex in V' is even. Because a vertex cover for a triangle is of (minimum) size 2, it is clear that G' has a vertex cover of size $k + 2$ iff G has a vertex cover of size k.

2. **TRAVELING SALESMAN PROBLEM**
   INSTANCE: A finite set $C = \{c_1, c_2, \ldots, c_m\}$ of "cities," a "distance" $d(c_i, c_j) \in \mathbb{N}$ for each pair of cities $c_i, c_j \in C$, and a bound $B \in Z^+$ (where $\mathbb{N}$ denotes the positive integers).
   QUESTION: Is there a "tour" of all the cities in C having total length no more than B, that is, an ordering $< c_{\pi(1)}, c_{\pi(2)}, \ldots c_{\pi(m)} >$ of C such that

   $$\left(\sum_{i=1}^{m-1} d(c_{\pi(i)}, c_{\pi(i+1)}) \right) + d(c_{\pi(m)}, c_{\pi(1)}) \leq B?$$

hint: you can use the hamiltonian circuit problem for the reduction.
HAMILTONIAN CIRCUIT
INSTANCE: A graph $G = (V, E)$.
QUESTION: Does G contain a HAMILTONIAN CIRCUIT, that is, an ordering $< v_1, v_2, \ldots, v_n >$ of vertices of G, where $n = |V|$, such that $\{v_n, v_1\} \in E$ and $\{v_i, v_{i+1}\} \in E$ for all i, $1 \leq i \leq n$? edge E?

Solution: The proof that Traveling Salesman Problem (TSP) belongs to NP is trivial. We use HAMILTONIAN CIRCUIT (HC) problem, for the reduction. Let $G = (V, E)$ be an input instance of HC problem. Construct a new complete graph $G' = (V, E')$ as a new instance of TSP. We also assign weight 1 to every edge $e \in E$ and 2 to every edge $e \notin E$. We also set B to $|V|$. It is clear that G has a HC if and only if $< G', |V| >$ has a TSP with size $|V|$.

3. **COMMUNICATION SCHEDULING PROBLEM**
INSTANCE: A finite set T of tuples $t_i = (s_i, e_i, d_i)$ where $s_i, e_i \in \mathbb{N}$ are the earliest start time and the latest end time of transactions and $d_i \in \mathbb{R}$ represents the amount of data that is to be transferred. There is also a bound $B \in \mathbb{R}$ which represents the maximum amount of data which can be transferred at each time instance.
QUESTION: Is there a tuple $(\alpha_i, \beta_i, \gamma_i)$ for each t_i such that:

- $\alpha_i \geq s_i$,
- $\beta_i \leq e_i$,
- $\sum_{\{i|\alpha_i \leq t \leq \beta_i\}} \gamma_i \leq B$, for any given time t,
- $(\beta_i - \alpha_i) \cdot \gamma_i \geq d_i$.

hint: you can use the partition problem for the reduction:
PARTITION
INSTANCE: A finite set A and a "size" $s(a) \in \mathbb{N}$ for each $a \in A$.
QUESTION: Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} S(a) = \sum_{a \in A - A'} s(a)?$$

Solution: Checking that COMMUNICATION SCHEDULING PROBLEM (CSP) belongs to NP is trivial. Now we need to prove that CSP is NP-complete by using a reduction from the PARTITION problem. Suppose A is an input instance of the PARTITION problem. Let $S = \sum_{a \in A} s(a)$. Now we construct a new instance of the CSP problem by creating $t_a = (1, S+1, a)$ for every $a \in A$. We also add an extra transaction $t = (\lfloor S/2 \rfloor, \lfloor S/2 \rfloor + 1, 1)$. The bandwith bound is also set to $B = 1$. It is clear that CSP has an schedule if and only if there is a partition for set A. This is because of the extra transaction which is forced to placed in the middle of the bandwidth.