Supplementary material A

TABLE A.1: PARAMETERS OF THE COMPARED ALGORITHMS USED IN THE EXPERIMENTAL STUDIES.

Method	Parameter	Value
All the algorithms (if applicable)	Population size (N): 2-objective UF1-UF7 3-objective UF8-UF10	 100 150
	Number of function evaluations	300,000
	Number of decision variables(n)	30
	Distribution index for SBX	15
	Distribution index for polyn. mutation	15
	Polynomial mutation prob.(p_m)	1/n
	Scaling factor of DE(F)	0.5
	Crossover rate of DE (CR)	1
MOEA/DVA	Number of interaction analysis(NIA)	6
	Number of control property analysis(NCA)	20
MOEA/D	Size of neighbor subproblems (T)	0.1N
	Max. num. of solutions replaced(n_r)	0.01N
	Parent solution selection prob.(δ)	0.9
NSGA-III	Number of reference points	N
SMS-EMOA	The reference point	Maximal values of non-dominated solutions + 1
RMMEDA	Number of LPCA clusters(K)	5
	Extension rate	0.25
MrBOA	Selection portion of truncation selection	0.5
	Number of allowable parents	5
	Number of mixture components	3

	Threshold of leader algorithm	0.3
MTS	Size of population(N)	40
	Number of local search test	5
	Number of local search	45
	Number of solutions to do local search	5
DMOEADD	Number of multi-parent crossover	5
	Number of the new individuals by crossover	10
	Temperature(T)	10000
	Number of sub-domains	2
MOEA/D-M2M	Number of subpopulations(K): 2-objective UF1-UF7 3-objective UF8-UF10	 10 12
	Number of solutions in each subpopulation(S): 2-objective UF1-UF7 3-objective UF8-UF10	 10 12

Appendix A: PROOF THAT INDIVIDUAL FUNCTION IN MOST OF CONTINUOUS ZDT AND DTLZ PROBLEMS ARE SEPARABLE

1) Due to $f_1(\mathbf{x}) = x_1$ for ZDT1, it is obvious that $f_1(\mathbf{x})$ of ZDT1 is a separable function. For $f_2(\mathbf{x})$ of ZDT1, $\frac{\partial f_2(\mathbf{x})}{\partial x_i} = \left[1-\sqrt{x_1/g(\mathbf{x})}\right]\times\frac{9}{n-1}+\frac{1}{2\sqrt{x_1/g(\mathbf{x})}}\frac{x_1}{g(\mathbf{x})}\frac{9}{n-1} = \left[1-\frac{\sqrt{x_1/g(\mathbf{x})}}{2}\right]\times\frac{9}{n-1} > 0, i = 2,\ldots,n$ due to $x_1 \in (0,1], g(\mathbf{x}) \geq 1$. Therefore, the decision variable $x_i, i = 2,\ldots,n$ can be optimized independently of any other variables for $f_2(\mathbf{x})$ of ZDT1. For $x_1 \in (0,1]$, we have $\frac{\partial f_2(\mathbf{x})}{\partial x_1} = -\frac{1}{2\sqrt{x_1/g(\mathbf{x})}} < 0$ due to $g(\mathbf{x}) \geq 1$. Therefore, the decision variable x_1 can also

be optimized independently of any other variables for $f_2(\mathbf{x})$ of ZDT1. So, $f_2(\mathbf{x})$ of ZDT1 is a separable function.

2) Taking $f_1(\mathbf{x})$ of DTLZ1 problem for example, $\frac{\partial f_1(\mathbf{x})}{\partial x_i} = 0.5x_1 \ldots x_{i-1}x_{i+1} \ldots x_{m-1}(1+g(\mathbf{x}_m)) \geq 0, i = 1,\ldots,m-1$ due to $g(\mathbf{x}_m) \geq 0, x_i \geq 0$. Therefore, the decision variable $x_i, i = 1,\ldots,m-1$ can be optimized independently of any other variables for $f_1(\mathbf{x})$ of DTLZ1 problem. Independently of any other variables, $f_1(\mathbf{x}) = 0.5x_1 \ldots x_{m-1}(1+g(\mathbf{x}_m))$ reach the minimal value when $x_i = 0.5, i = m,\ldots,n$ due to $g(\mathbf{x}_m) \geq 0, x_i \geq 0$. That is to say,

$$\arg\min_{x_1,\ldots,x_n} f_1(x_1,\ldots,x_n) = \left(\arg\min_{x_1} f_1(x_1,\ldots),\ldots,\arg\min_{x_n} f_1(\ldots,x_n)\right) = \left(\underbrace{0,\ldots0}_{m-1},\underbrace{0.5,\ldots,0.5}_{n-m+1}\right) .$$

Therefore, $f_1(\mathbf{x})$ of DTLZ1 is a separable function. Similarly, $f_2(\mathbf{x}),\ldots,f_m(\mathbf{x})$ of DTLZ1 can also be proven to be separable functions.

3) Similarly to the proof of ZDT1, individual functions $f_1(\mathbf{x}), f_2(\mathbf{x})$ of ZDT2 and ZDT6 problems can also be proved to be separable functions. Similarly to the proof of DTLZ1, individual functions $f_1(\mathbf{x}), f_2(\mathbf{x})$ of ZDT3 and ZDT4 problems can also be proven to be separable functions.

Furthermore, similarly to the proof of DTLZ1, individual functions $f_1(\mathbf{x}),\ldots,f_m(\mathbf{x})$ of DTLZ2-DTLZ4 and DTLZ7 problems can also be proven to be separable functions.

APPENDIX B: PROOF OF LEMMA, COROLLARY AND THEOREMS ON S ECTION III-E

1) Proof of Lemma 3.3:

Proof: For arbitrary $x_1, x_2, \ldots, x_n$, there exists $x_1^*(x_2, x_3, \ldots, x_n)$ such that $f_i(x_1, x_2, x_3, \ldots, x_n) \geq f_i(x_1^*(x_2, x_3, \ldots, x_n), x_2, x_3, \ldots, x_n), i = 1,\ldots,m$ due to x_1 being a distance variable. $x_1^*(x_2, x_3, \ldots, x_n)$ represents its value to be dependent on the value of $(x_2, x_3, \ldots, x_n)$. According to the second condition that distance variables are dependent each other for each objective function $f_i(\mathbf{x}), i = 1,\ldots,m, \mathbf{x} \in \Omega$, $x_1^*(x_2, x_3, \ldots, x_n) = x_1^*$ is independent on the value of the other variables. Therefore, we have $f_i(x_1, x_2, x_3, ..., x_n) \geq f_i(x_1^*, x_2, x_3, ..., x_n), i = 1, ..., m$.

The same theory proves there exist $x_2^*, ..., x_n^*$ such that for arbitrary $x_1, x_2, ..., x_n$ and $i = 1, ..., m$, $f_i(x_1, x_2, x_3, ..., x_n) \geq f_i(x_1^*, x_2, x_3, ..., x_n) \geq f_i(x_1^*, x_2^*, x_3, ..., x_n) \geq \cdots \geq f_i(x_1^*, x_2^*, x_3^*, ..., x_n^*)$.

Therefore, this MOP only has one Pareto-optimal solution $\mathbf{F}(x_1^*, x_2^*, x_3^*, ..., x_n^*)$ in the objective space and the decision variables of this MOP can be optimized one by one.

2) Proof of Theorem 3.4:

Proof: As shown in Fig. 10, the values of diverse variables (position variables and mixed variables) of sub-MOP are constant. Therefore, each sub-MOP is a MOP with distance variables only. According to the condition, distance variables are independent of one another for each objective function $f_i(\mathbf{x}), i = 1, ..., m$. According to the Lemma 3.3, the distance variables of each sub-MOP can be optimized one by one and each sub-MOP of this MOP only has one Pareto-optimal solution in the objective space.

3) Proof of Theorem 3.5:

Proof: For arbitrary $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, ..., \mathbf{x}_c$, there exists $\bar{\mathbf{x}}_i, i = 1, ..., c$ such that $\mathbf{F}(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, ..., \mathbf{x}_c) \geq \mathbf{F}(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \bar{\mathbf{x}}_i, \mathbf{x}_{i+1}, ..., \mathbf{x}_c)$. Therefore, we have

$\mathbf{F}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_c) \geq \mathbf{F}(\bar{\mathbf{x}}_1, \mathbf{x}_2, ..., \mathbf{x}_c) \geq \mathbf{F}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, ..., \mathbf{x}_c) \geq \cdots \geq \mathbf{F}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, ..., \bar{\mathbf{x}}_c)$. Therefore, this MOP/sub-MOP only has one Pareto-optimal solution in the objective space and all subcomponents of distance variables of MOP/sub-MOP can be optimized one by one.

4) Proof of Theorem 3.6:

Proof: According to the suppose, each subcomponent $\mathbf{x}_i, i = 1, ..., c$ has only one optimal solution on the objective space. Therefore, for arbitrary $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_c$, there exists $\bar{\mathbf{x}}_i, 2 = 1, ..., c$ such that $\mathbf{F}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 ..., \mathbf{x}_c) \geq \mathbf{F}(\mathbf{x}_1, \bar{\mathbf{x}}_2, \mathbf{x}_3, ..., \mathbf{x}_c) \geq \mathbf{F}(\mathbf{x}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, ..., \mathbf{x}_c) \geq \cdots \geq \mathbf{F}(\mathbf{x}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, ..., \bar{\mathbf{x}}_c)$. So, if we fix the value of diverse variables $\mathbf{x}_1$, the dimension of PF of each sub-MOP is 0. Therefore, the PF of the original MOP is highly dependent on the diverse variables. If the number of diverse variables $\mathbf{x}_1$ is equal or lesser than the dimension of PF, then all of diverse variables including mixed variables have an important effect on PF/PS.

5) Proof of Theorem 3.7:

Proof: According to the suppose, distance variables are dependent one another for each objective function. Therefore, if we fix the values of diverse variables, the dimension of PF of each sub-MOP is 0 based on the Theorem 3.4. Therefore, the PF of the original MOP is highly dependent on the diverse variables. If the number of diverse variables $\mathbf{x}_1$ is equal or lesser than the dimension of PF, then all of diverse variables including mixed variables have an important effect on PF/PS.

Supplementary material B

The function evaluations spending by MOEA/DVA are used on three parts: analyses of decision variables, subcomponent optimization and uniformity optimization. Someone may be interesting to see the distribution of FEs in each part of the proposed algorithm. In order to clearly see the computation cost caused by the proposed mechanisms, Table A.1 shows the function evaluations used by each parts and Fig. B.1 plots the rate of function evaluations spending by MOEA/DVA on the three parts. Fig. B.2 shows the run-time convergence analysis which is performed to compare with NSGA-III, SMS-EMOA and MOEA/D.

Firstly, we want to analyze the function evaluations spending by MOEA/DVA on analyses of decision variable. MOEA/DVA takes n*NCA+1.5*NIC*n*(n-1) function evaluation to perform the analyses of decision variables, where NCA=40 and NIC=6. Therefore, performing analyses of decision variables needs 30*40+1.5*6*30*29=9030 function evaluations on UF1-UF10 problems with 30 variables and 24*40+1.5*6*24*23=5928 function evaluations on WFG1-WFG9 problems with 24 variables. These data are shown in second column of Table B.1. From Fig. B.1, we can see that the rate of function evaluations spent by MOEA/DVA on analyses of decision variables is low (about 3% for UF problems and 6% for WFG problems).

In MOEA/DVA, subcomponent optimization is expected to speed up the convergence of algorithm. The third column of Table B.1 shows the average and standard deviation of function evaluations spent on subcomponent optimization. As shown in Fig. B.2, compared with NSGA-III, SMS-EMOA and MOEA/D, subcomponent optimization in MOEA/DVA can accelerate the convergence of the algorithm on most of UF problems. Compared with NSGA-III, SMS-EMOA and MOEA/D, MOEA/DVA needs additional function evaluations to analyze the decision variables. Some may wonder why MOEA/DVA provides better results with the same number of total function evaluations on most of UF problems. The reason is UF problems meeting the following condition: All the distance variables are independent of one another for each objective function as shown in Figs. 5 and 6. According to Theorem 3.4, all the distance variables of each sub-MOP can be independently optimized one by one. Roughly speaking, MOEA/DVA divides the search in *n*-dimensional decision space into two parts: 1) N*(n-m+1) one-dimensional search. There are N sub-MOPs/individuals needing to be optimized and each sub-MOP/individual has (n-m+1) subcomponents $\{x_m\},\{x_{m+1}\},...,\{x_n\}$ requiring to be optimized. 2) (m-1)-dimensional uniformity optimization for diverse variables $x_1,...,x_{m-1}$. On the contrary, NSGA-III, SMS-EMOA and MOEA/D need to search in *n*-dimensional space. N is the number of sub-MOPs or size of population, n is the number of decision variables and m is the number of objective functions. Why MOEA/DVA can reduce the dimensions of the search space? The reason is MOEA/DVA learning the variable linkages and dividing the distance variables with high dimension into several independent low-dimensional subcomponents to optimize.

MOEA/DVA does not work well on WFG problems. The reason may come from the

following aspects:
1) Most of WFG problems are not difficult MOPs. NSGA-III, SMS-EMOA and MOEA/D can converge rapidly at 20000~40000 function evaluations. Therefore, there is no enough space to show the effectiveness of subcomponent optimization of MOEA/DVA.
2) The mapping of MOP from PS to PF is highly biased and MOEA/DVA does not have enough function evaluations to perform uniformity optimization as shown Table B.1 and figure B.1. The example is WFG1 problem.

If there has rest function evaluations after subcomponent optimization, uniformity optimization will be performing.

Table B.1: The function evaluations used by three parts: analyses of decision variables, subcomponent optimization and uniformity optimization

	Analyses of decision variable	Subcomponent optimization	Uniformity optimization
UF1(30)	9.030e+3(0)	8.104e+4(1.223e+4)	2.099e+5(1.223e+4)
UF2(30)	9.030e+3(0)	9.777e+4(2.121e+4)	1.933e+5(2.121e+4)
UF3(30)	9.030e+3(0)	1.287e+5(2.263e+4)	1.623e+5(2.263e+4)
UF4(30)	9.030e+3(0)	9.337e+4(1.421e+4)	1.976e+5(1.421e+4)
UF5(30)	9.030e+3(0)	2.228e+5(3.014e+4)	6.820e+4(3.014e+4)
UF6(30)	9.030e+3(0)	1.925e+5(2.044e+4)	9.850e+4(2.044e+4)
UF7(30)	9.030e+3(0)	8.529e+4(1.442e+4)	2.057e+5(1.442e+4)
UF8(30)	9.030e+3(0)	2.886e+5(9.290e+3)	2.300e+3(9.290e+3)
UF9(30)	9.030e+3(0)	2.901e+5(3.270e+3)	9.000e+2(3.270e+3)
UF10(30)	9.030e+3(0)	2.910e+5(0)	0(0)
WFG1(3,24)	5.928e+3(0)	9.407e+4(0)	0 (0)
WFG2(3,24)	5.928e+3(0)	6.250e+4(1.202e+4)	3.157e+4(1.202e+4)
WFG3(3,24)	5.928e+3(0)	7.773e+4(1.319e+4)	1.634e+4(1.319e+4)
WFG4(3,24)	5.928e+3(0)	4.720e+4(7.435e+3)	4.687e+4(7.435e+3)
WFG5(3,24)	5.928e+3(0)	5.239e+4(3.247e+3)	4.169e+4(3.247e+3)
WFG6(3,24)	5.928e+3(0)	1.440e+4 (2.491e+3)	7.967e+4 (2.491e+3)
WFG7(3,24)	5.928e+3(0)	5.939e+4(2.511e+3)	3.468e+4(2.511e+3)
WFG8(3,24)	5.928e+3(0)	9.407e+4(0)	0 (0)
WFG9(3,24)	5.928e+3(0)	5.661e+3(3.685e+3)	8.741e+4(3.685e+3)

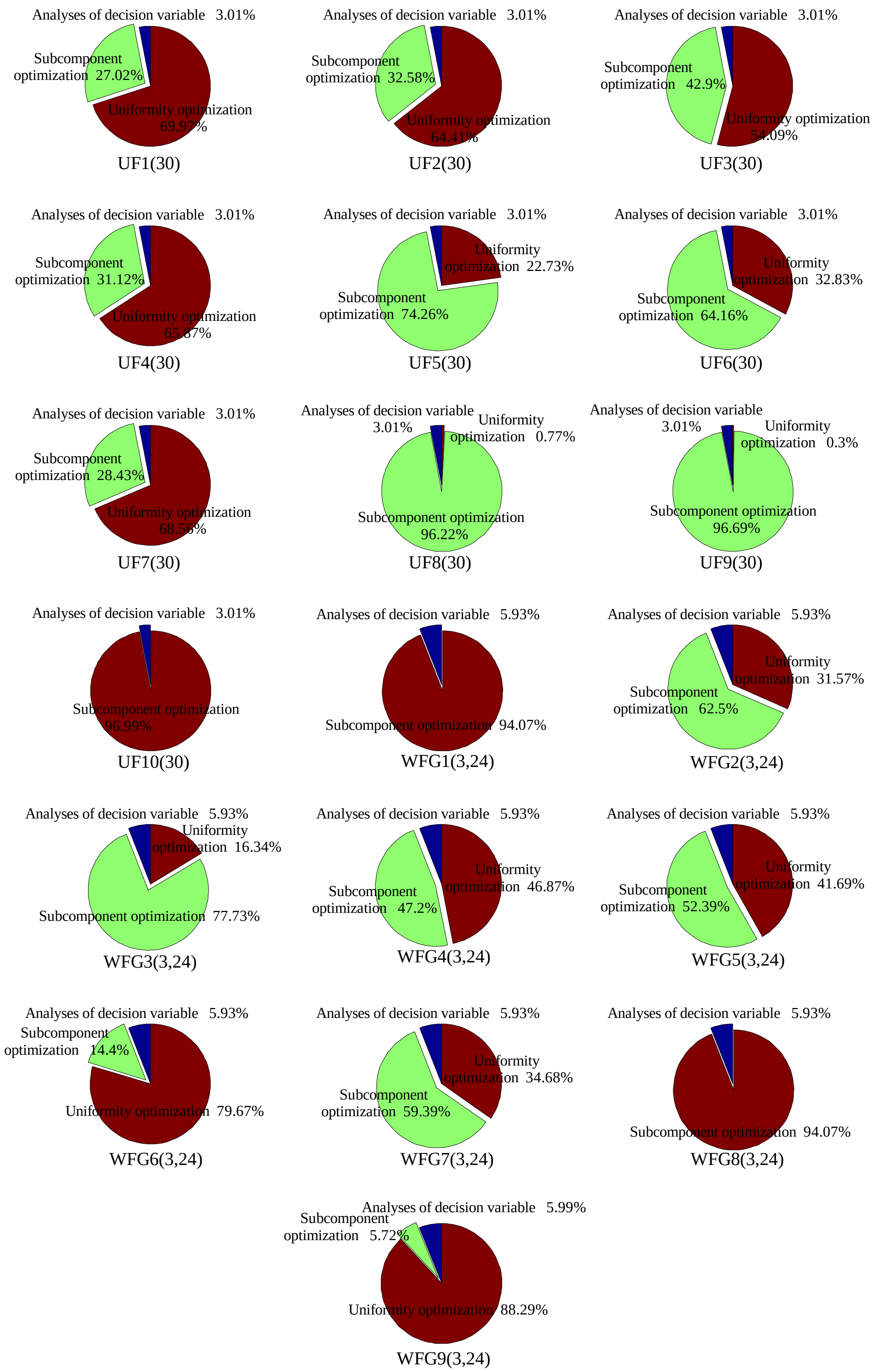


Figure B.1: Plot the pie chart of the function evaluations spending by MOEA/DVA on three parts: analyses of decision variables, subcomponent optimization and uniformity optimization.

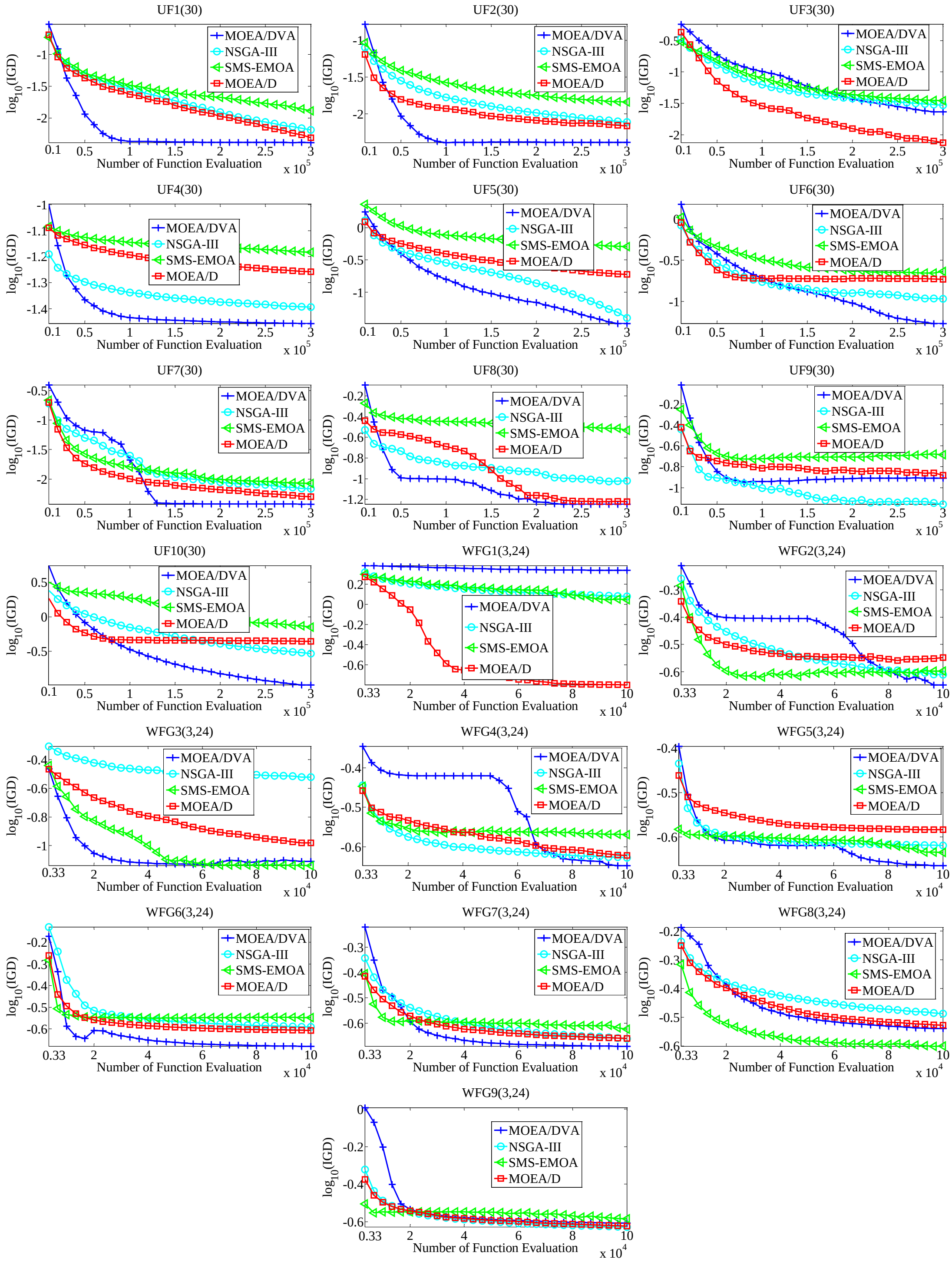


Figure B.2: Plot the evolution process of the mean of IGD-metric values by MOEA/DVA, NSGA-III, SMS-EMOA and MOEA/D on UF1-UF10 and WFG1-WFG9 problems.

Supplementary material C: Statically valid experiments on UF1-UF7 with a number of variables that range from 10 to 1000

Recently, researchers (Durillo et al., 2010) suggested that performing a fixed number of function evaluations may not offer information about the effort needed by an algorithm to obtain satisfactory solutions. In this part, the effect of parameter scalability by MOEA/DVA, NSGA-III, SMS-EMOA and MOEA/D is studied. The parameter setting for the compared algorithms is introducing in Section IV.

UF1-UF7 problems are used as test MOPs since they have different properties: convex, non-convex, disconnected, multi-frontal. The behaviors of four compared algorithms on UF1-UF7 problems with 10, 30, 100, 300 and1000 variables are investigated. By this way, we can study which algorithm behaves more efficiently when solving problems with an increasing number of variables.

The compared algorithms stop either when they obtain a set of solutions with a value of IGD metric less than or equal to 0.04 to the true PF or when they have used 10,000,000 function evaluations. 40 independent runs have been performed for each compared algorithm and each MOP. According to Wilcoxon's rank sum test at 0.05 significance level, "+" means that the metric values of the best algorithm are importantly better than the compared algorithm.

Tables C.1–C.7 respectively show the number of function evaluations required by the compared algorithms on UF1-UF7. If all the 40 independent runs have been successful, the mean and standard deviation of function evaluations are shown. This also means a hit rate of 100%. When an algorithm can not obtain good solutions within 10,000,000 function evaluations for all the 40 independent runs, its cell in the tables is marked as "−". For these cases, the hit rate is less than 100%. Moreover, we also present the hit rate in the same cell. Therefore, "−/25%" means an algorithm has 10 of 40 independent runs obtaining good solutions, whose IGD value less than 0.04, within 10000000 function evaluations. According to Wilcoxon's rank sum test at 0.05 significance level, "+" in the last row of table is that the number of function evaluations of the best algorithm are importantly better than other compared algorithms.

To have a clearer result, figures summarizing the values are given by using a logarithmic scale. Figure C.1 gives the number of average function evaluations required to get a satisfactory set of solutions (IGD<=0.04) on different number of decision variables.

(1) For UF1 and UF2 problems, Tables C.1 and C.2 show that all the compared algorithms have success in the cases with 10, 30, 100 and 300 decision variables. However, only MOEA/DVA requires the number of function evaluations lower than 10,000,000 for 1000 variables on UF1 and UF2 problems. Let us examine the speed, i.e., the number of function evaluations required by the compared algorithms to approximate the Pareto front based on our success condition. Figure C.1 plots the results using a logarithmic scale. We can clearly see from Fig. C.1 and Tables C.1-C.2 that the fastest algorithm is MOEA/DVA.

(2) For UF3 problem, Table C.3 presents that all the compared algorithms have a 100% success rate in all the problems. For the convergence speed of algorithm, we can see from Fig. C.1 and Table C.3 that MOEA/D is the fastest algorithm from 10 to 100 variables, while SMS-EMOA scales better from 300 to 1000 variables.

(3) For UF4 problem, Table C.4 shows that all the compared algorithms have success in the case with

10 decision variables. However, only MOEA/DVA needs the number of function evaluations lower than 10,000,000 for 30, 100, 300 and 1000 variables on UF3 problem. For the convergence speed of algorithm, we can see from Fig. C.1 that MOEA/DVA is the fastest algorithm.

(4) For UF5 problem, there is no algorithm having 100% hit rate to solve the problem successfully with 1000 variables within 10,000,000 function evaluations. Only MOEA/DVA requires the number of function evaluations lower than 10,000,000 for 100 and 300 variables on UF5 problem. SMS-EMOA cannot solve any of instances within maximal function evaluations. MOEA/D and NSGA-III can solve UF5 problem with 10 and 30 variables. From the view of speed, Fig C.1 shows the fastest algorithm is MOEA/DVA.

(5) There is no algorithm having 100% hit rate on all the instances of UF6 problem. NSGA-III and SMS-EMOA do not achieve a 100% of success on any instance of UF6 problem. MOEA/D only have 100% hit rate on UF6 problem with 10 variables. MOEA/DVA have 100% hit rate on UF6 problem with 30, 100, 300 and 1000 variables. For the convergence speed of algorithm, we can see from Fig. C.1 and Table C.6 that MOEA/D is the fastest algorithm on 10 variables, while SMS-EMOA is the fastest algorithm from 30 to 1000 variables. Some may wonder why MOEA/DVA can achieve 100% of success on UF6 with many variables but not get 100% of success on UF6 with 10 variables? The function evaluations spending by MOEA/DVA are used on three parts: analyses of decision variables, subcomponent optimization and uniformity optimization. Table C.8 shows the distribution of function evaluations in each part of MOEA/DVA. MOEA/DVA takes n*NCA+1.5*NIC*n*(n-1) function evaluation to perform the analyses of decision variables. Taking UF6 problem with 10 variables for example, MOEA/DVA spends 10*40+1.5*6*10*9=1210 function evaluations to analyze the decision variables. As shown in Table C.8, MOEA/DVA just uses 22,380 function evaluations on average to do subcomponent optimization on UF6 problems with 10 variables and uses the rest function evaluations for do uniformity optimization by MOEA/D. However, UF6 is difficult problem and 22,380 function evaluations are impractical to make MOEA/DVA convergent to PF. Moreover, performing uniformity optimization by MOEA/D fail to make the algorithm convergent to PF as shown Table C.6.

(6) The hit rate indicator presents that SMS-EMOA and MOEA/D can solve different instances of UF7 problem, while the hit rates of MOEA/DVA and NSGA-III on UF7 problem with 1000 variables are less than 100%. For the convergence speed of algorithm, Fig. C.1 shows that NSGA-III is the fastest algorithm on UF7 problem with 10 variables, while SMS-EMOA scales better from 30 to 1000 variables.

TABLE C.1: FUNCTION EVALUATIONS ON UF1

UF1	10 variables	30 variables	100 variables	300 variables	1000 variables
MOEA/DVA	**9.38e+3**(5.72e+2)	**3.66e+4**(1.25e+3)	**1.49e+5**(2.22e+3)	**8.15e+5**(3.10e+4)	**9.01e+6**(8.54e+4)
NSGA-III	1.37e+4(6.68e+3)	8.52e+4(2.94e+4)	8.60e+5(2.42e+5)	6.83e+6(1.28e+6)	−/0%
SMS-EMOA	1.33e+4(4.81e+3)	9.77e+4(2.31e+4)	7.23e+5(1.29e+5)	4.24e+6(1.00e+5)	−/0%
MOEA/D	2.15e+4(4.12e+3)	8.97e+4(1.98e+4)	7.87e+5(1.57e+5)	4.72e+6(9.13+5)	−/0%
	+	+	+	+	+

TABLE C.2: FUNCTION EVALUATIONS ON UF2

UF2	10 variables	30 variables	100 variables	300 variables	1000 variables
MOEA/DVA	**7.83e+3**(6.97e+2)	**3.08e+4**(1.17e+3)	**1.26e+5**(1.72e+3)	**8.74e+5**(5.82e+4)	**8.97e+6**(7.63e+4)
NSGA-III	8.55e+3(1.28e+3)	3.85e+4(5.74e+3)	3.39e+5(3.30e+4)	2.91e+6(1.85e+5)	−/0%

SMS-EMOA	8.00e+3(1.33e+3)	5.28e+4(1.15e+4)	3.67e+5(5.10e+4)	3.76e+6(4.61e+5)	−/0%
MOEA/D	1.49e+4(1.65e+3)	5.80e+4(5.94e+3)	2.73e+5(2.40e+4)	1.62e+6(1.12e+5)	−/0%
	+	+	+	+	+

TABLE C.3: FUNCTION EVALUATIONS ON UF3

UF3	10 variables	30 variables	100 variables	300 variables	1000 variables
MOEA/DVA	1.50e+5(1.55e+4)	1.87e+5(4.08e+4)	2.71e+5(5.90e+3)	9.67e+5(1.71e+3)	9.12e+6(9.87e+4)
NSGA-III	2.02e+5(1.47e+5)	1.98e+5(1.24e+5)	2.90e+5(7.15e+4)	7.17e+5(1.49e+5)	2.93e+6(2.29e+5)
SMS-EMOA	2.37e+5(1.44e+5)	2.58e+5(2.25e+5)	2.66e+5(6.77e+4)	**3.82e+5**(3.41e+4)	**1.62e+6**(8.20e+4)
MOEA/D	**1.35e+5**(9.10e+4)	**1.74e+5**(6.77e+4)	**2.43e+5**(3.34e+4)	5.56e+5(4.41e+4)	2.04e+6(1.10e+6)
	+	+	+	+	+

TABLE C.4: FUNCTION EVALUATIONS ON UF4

UF4	10 variables	30 variables	100 variables	300 variables	1000 variables
MOEA/DVA	**2.71e+4**(4.75e+3)	**1.03e+5**(1.76e+4)	**4.56e+5**(5.40e+4)	**1.64e+6**(5.24e+4)	**9.20e+6**(9.92e+4)
NSGA-III	4.10e+5(2.09e+5)	−/20%	−/0%	−/0%	−/0%
SMS-EMOA	4.97e+5(7.25e+5)	−/17.5%	−/0%	−/0%	−/0%
MOEA/D	1.59e+5(1.28e+5)	−/25%	−/0%	−/0%	−/0%
	+	+	+	+	+

TABLE C.5: FUNCTION EVALUATIONS ON UF5

UF5	10 variables	30 variables	100 variables	300 variables	1000 variables
MOEA/DVA	**6.57e+4**(6.19e+3)	**2.64e+5**(2.53e+4)	**1.11e+6**(9.12e+4)	**4.43e+6**(6.31e+5)	−/0%
NSGA-III	5.46e+5(2.20e+5)	2.36e+6(1.31e+6)	−/0%	−/0%	−/0%
SMS-EMOA	−/0%	−/0%	−/0%	−/0%	−/0%
MOEA/D	4.66e+5(2.56e+5)	2.85e+6(2.90e+6)	−/0%	−/0%	−/0%
	+	+	+	+	

TABLE C.6: FUNCTION EVALUATIONS FOR UF6

UF6	10 variables	30 variables	100 variables	300 variables	1000 variables
MOEA/DVA	−/90%	**7.37e+5**(1.74e+5)	**8.23e+5**(6.26e+5)	**1.44e+6**(4.17e+4)	**9.28e+6**(6.53e+4)
NSGA-III	−/70%	−/60%	−/0%	−/0%	−/0%
SMS-EMOA	−/50%	−/40%	−/0%	−/0%	−/0%
MOEA/D	**4.72e+6**(1.81e+6)	−/55%	−/0%	−/0%	−/0%
		+	+	+	+

TABLE C.7: FUNCTION EVALUATIONS FOR UF7

UF7	10 variables	30 variables	100 variables	300 variables	1000 variables
MOEA/DVA	2.55e+4(3.03e+3)	9.35e+4(1.44e+4)	4.04e+5(5.36e+4)	1.78e+6(1.20e+5)	−/0%
NSGA-III	**5.72e+3**(1.26e+3)	3.66e+4(5.14e+4)	2.32e+5(1.59e+5)	1.07e+6(2.38e+5)	−/80%
SMS-EMOA	6.27e+3(1.03e+3)	**3.27e+4**(4.36e+3)	**2.03e+5**(4.14e+4)	**8.66e+5**(1.62e+5)	**4.99e+6**(7.05e+5)
MOEA/D	1.32e+4(2.03e+3)	4.66e+4(6.16e+3)	2.53e+5(4.10e+4)	1.16e+6(1.25e+5)	6.94e+6(5.76e+5)
	+	+	+	+	+

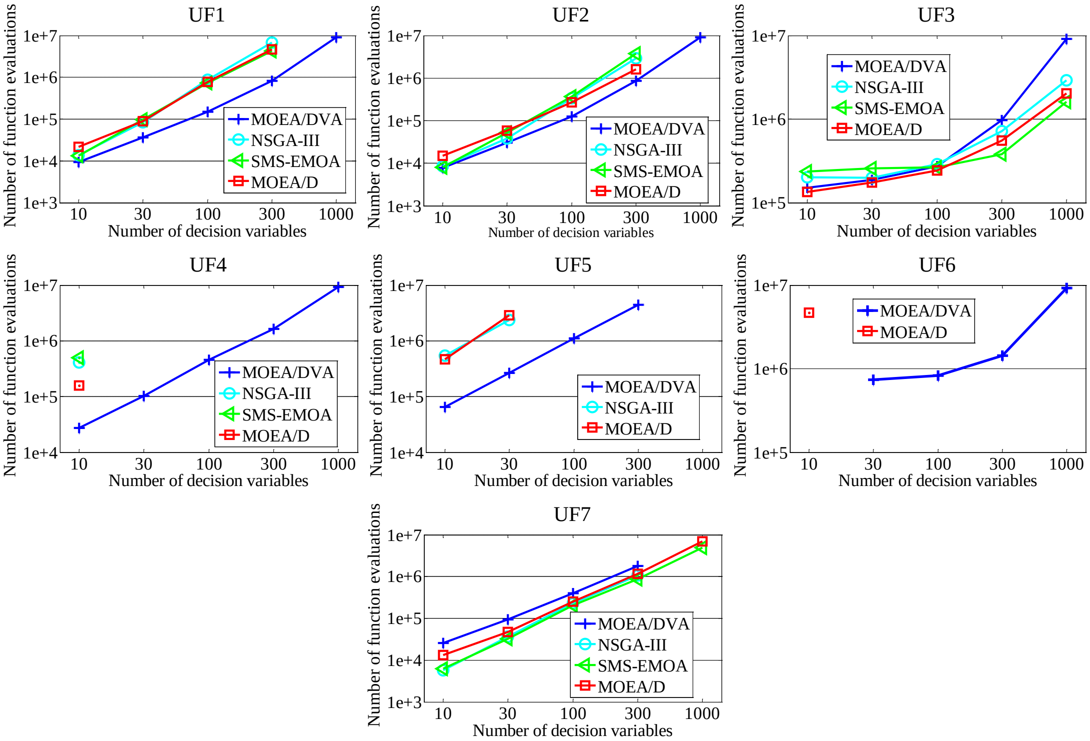


Figure C.1: Plot the number of function evaluations when solving UF1-UF7 problems.

TABLE C.8: FUNCTION EVALUATIONS FOR UF6

Function evaluations	Analyses of decision variable [n*NCA+1.5*NIC*n*(n-1)]	Subcomponent optimization	Uniformity optimization
UF6(10)	1210	2.238e+4(7.935e+3)	Using other FEs
UF6(30)	9030	1.921e+5(2.280e+4)	Using other FEs
UF6(100)	93100	6.010e+5(1.007e+5)	Using other FEs
UF6(300)	819300	2.154e+6(4.442e+5)	Using other FEs

References:

[1] Durillo, J. J., Nebro, A. J., Coello, C. A. C., García-Nieto, J., Luna, F., & Alba, E. (2010). A Study of Multiobjective Metaheuristics When Solving Parameter Scalable Problems. IEEE Transactions on Evolutionary Computation, 14(4), 618–635.

Supplementary material D

Section IV-D: MOEA/DVA VS. Other MOEAs Based on Linkage Learning

In order to study the effectiveness of the linkage learning method used in this paper, MOEA/DVA is compared with MTS, MrBOA, MIDEA and RMMEDA. In this part of the experiments, UF1-UF10 problems with30 variables are used as the test problems. The maximumnumber of function evaluations is set as 300,000 forUF1-UF10 problems.

Table D.1 shows the average and standard deviation of I_{ε^+} -metric and IGD-metric values of the obtained solutions by each algorithm. Table VIII provides the statistics that summarize these performance comparisons between the proposed MOEA/DVA and the other compared algorithms. Fig. D.1 illustrates the objective space, the distribution of solution set with the median IGD values found by MOEA/DVA, MTS, MrBOA, MIDEA and RMMEDA.

From Table D.1, in terms of average I_{ε^+} -metric and IGD-metric, we can see that the proposed MOEA/DVA is the best on UF1-UF3, UF6-UF10 problems, while MTS performs best on the two-objective UF4-UF5 problems. As shown in Table VIII, our proposed algorithm significantly outperforms MTS in 8 out of 10 comparisons based on I_{ε^+} and IGD metrics. According to Wilcoxon's rank sum test at 0.05 level, MOEA/DVA significantly outperforms MrBOA, MIDEA and RMMEDA respectively in 10 out of 10 comparisons based on I_{ε^+} and IGD metrics. As shown in Tables C.1, the performance of RMMEDA is not good on UF problems. The reason is that RMMEDA uses linear distribution model (Local PCA) to grasp the linkages among decision variables. Unfortunately, the variable linkages in UF problems are nonlinear.

Table IX gives the mean CPU times (in seconds) spent by MOEA/DVA, MTS, RMMEDA and MrBOA. Table X gives the comparison of CPU times between MOEA/DVA and MTS, RMMEDA and MrBOA. Roughly speaking, the CPU times spent by the four compared algorithms has the relationship: MOEA/DVA $\approx$ MTS<MIDEA<MrBOA< RMMEDA.

As shown in Fig. D.1, for UF1-UF3 and UF6-UF10 problems, MOEA/DVA obtains better diversity and approximation to the PF than MTS, MrBOA and RMMEDA. The key to the success of MOEA/DVA is in the used linkage learning/subcomponent optimization. Based on the learned linkages, MOEA/DVA divides distance variables into a set of low-dimensional components. Each sub-MOP independently optimizes subcomponents one by one. Therefore, MOEA/DVA is expected to have a quicker convergence speed. Additionally, the obtained solutions by MTS have better diversity and approximation quality than those found by MOEA/DVA, MrBOA, MIDEA and RMMEDA on complicated UF4-UF5 problems. Table D.2 gives the differences between MOEA/DVA and MTS. RMMEDA does not work well on UF problems because the linear distribution model used by RMMEDA is not suitable for nonlinear variable linkages in UF problems.

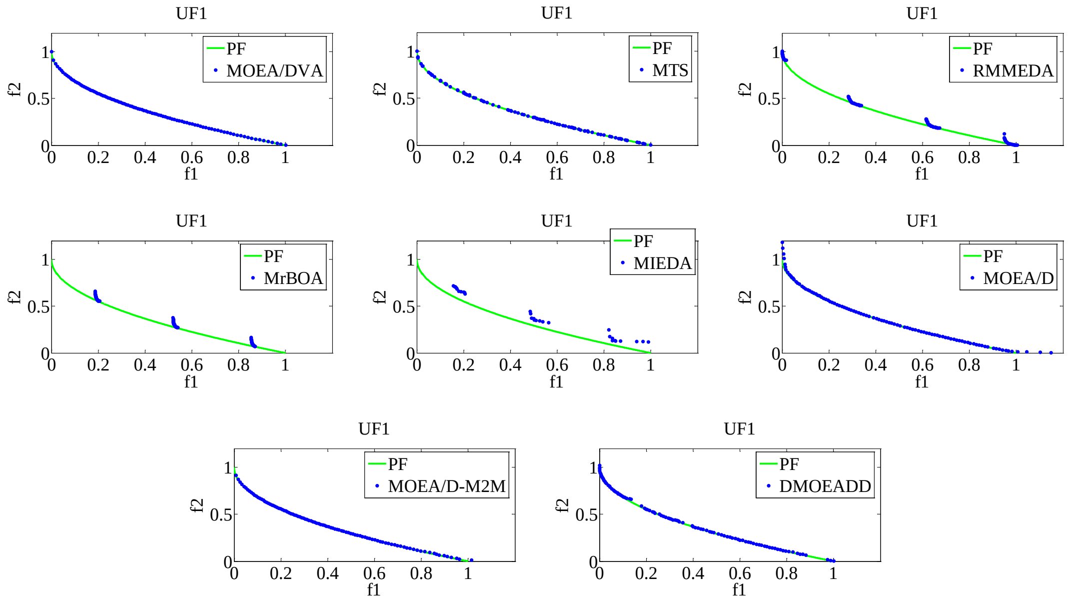
TABLE D.1 MEAN AND STANDARD DEVIATION OF I_{ε^+} AND IGD METRIC METRIC VALUES FOUND BY THE COMPARED ALGORITHMS ON UF1-UF10 PROBLEMS. THE VALUE WITHIN PARENTHESES IS THE DEVIATION OF METRIC.

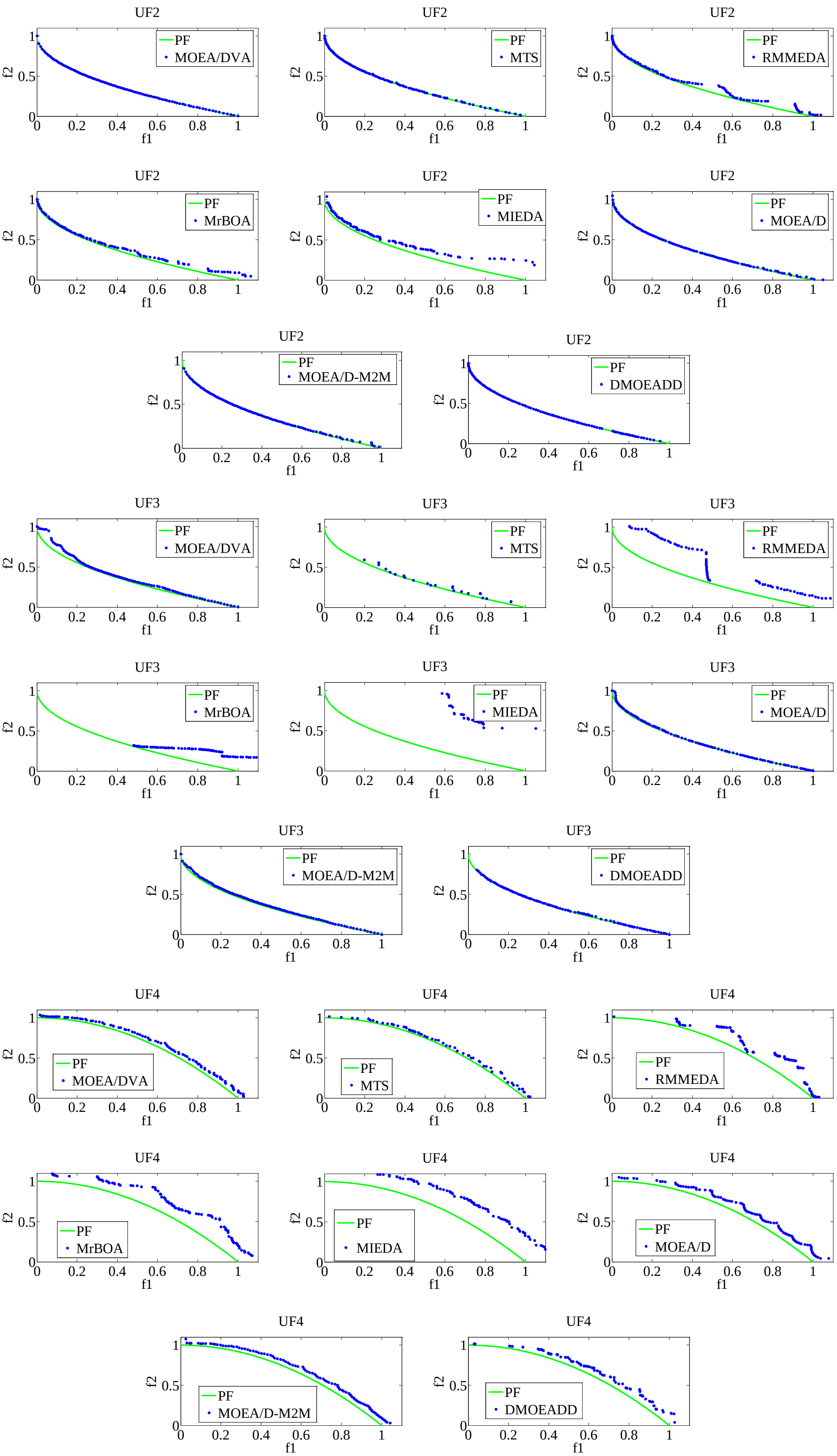
	MOEA/DVA		MTS		RMMEDA		MrBOA		MIEDA	
	IGD	I_{ε^+}	IGD	I_{ε^+}	IGD	I_{ε^+}	IGD	I_{ε^+}	IGD	I_{ε^+}
UF1	**4.1350e-3** (9.9053e-5)	**9.2148e-3** (9.6139e-4)	6.4632e-3+ (3.8197e-4)	2.0264e-2+ (4.4763e-3)	7.5951e-2+ (1.2036e-3)	1.9885e-1+ (5.6485e-4)	9.4294e-2+ (1.2867e-2)	1.9767e-1+ (3.4624e-2)	1.0771e-1 (1.1058e-2)	2.1996e-1 (3.0779e-2)
UF2	**4.1065e-3** (4.9092e-5)	**9.0603e-3**(5.7421e-4)	6.4310e-3+ (7.3767e-4)	2.2863e-2+ (5.7479e-3)	3.3254e-2+ (1.4560e-3)	9.5070e-2+ (5.4562e-3)	3.7779e-2+ (2.2820e-2)	9.6175e-2+ (5.3479e-2)	8.4453e-2 (2.7805e-3)	1.8737e-1 (1.4809e-2)
UF3	**2.2714e-2** (7.2599e-3)	**5.6724e-2** (1.3794e-2)	5.6533e-2+ (9.9795e-3)	1.8107e-1+ (4.2779e-2)	1.2031e-1+ (4.8243e-2)	2.0973e-1+ (6.9397e-2)	3.0213e-1+ (3.4797e-2)	4.9407e-1+ (1.8198e-2)	4.5684e-1 (1.1836e-2)	5.8660e-1 (1.9861e-2)
UF4	3.5067e-2+ (1.0070e-3)	4.3763e-2+ (2.8428e-3)	**2.3650e-2** (9.3916e-4)	**4.1821e-2** (6.1853e-3)	8.0043e-2+ (4.6874e-3)	1.1906e-1+ (3.4299e-3)	1.0324e-1+ (9.4701e-3)	1.2701e-1+ (6.5195e-3)	1.5161e-1 (1.6171e-3)	1.3388e-1 (1.9164e-3)
UF5	3.2592e-2+ (4.6786e-3)	6.9379e-2+ (1.5777e-2)	**1.4698e-2** (3.2434e-3)	**4.5407e-2** (1.3503e-2)	4.8447e-1+ (1.1350e-1)	6.3276e-1+ (1.3497e-1)	2.2734e-1+ (7.1143e-2)	3.9508e-1+ (1.3465e-1)	1.2134 (1.0324e-1)	1.1865 (4.1494e-2)
UF6	**5.6134e-2** (1.3729e-2)	**9.3159e-2** (1.8142e-2)	6.4838e-2+ (1.6586e-2)	9.4260e-2+ (3.5883e-2)	2.6703e-1+ (3.7181e-2)	3.7696e-1+ (4.9804e-2)	3.8840e-1+ (1.0310e-1)	5.7866e-1+ (1.4603e-1)	6.3320e-1 (5.2263e-2)	7.3986e-1 (1.0555e-1)
UF7	**3.7667e-3** (4.6437e-5)	**8.8102e-3** (1.4653e-3)	4.0545e-2+ (1.3300e-2)	1.5228e-1+ (4.7261e-2)	4.6426e-2+ (2.3175e-3)	1.8730e-1+ (1.5355e-3)	2.7530e-1+ (8.7059e-2)	5.9676e-1+ (1.5691e-1)	2.4237e-1 (1.0747e-1)	5.3685e-1 (2.2181e-1)
UF8	**5.7788e-2** (1.1960e-2)	**1.4471e-1** (3.9164e-2)	1.3501e-1+ (2.2413e-2)	4.6685e-1+ (1.0837e-1)	1.7546e-1+ (5.3099e-3)	5.3492e-1+ (2.1404e-3)	3.8102e-1+ (1.3964e-1)	7.8402e-1+ (1.0659e-1)	3.4386e-1 (1.5222e-2)	7.9257e-1 (1.9024e-2)
UF9	**1.2333e-1** (1.6254e-1)	**1.8964e-1** (2.0914e-1)	1.7404e-1+ (4.3008e-2)	4.1020e-1+ (7.7031e-2)	1.6712e-1+ (2.0321e-2)	3.8596e-1+ (1.6017e-2)	6.0469e-1+ (1.1503e-1)	7.6973e-1+ (1.2408e-1)	5.7649e-1 (3.3360e-2)	7.6305e-1 (2.5516e-2)
UF10	**1.0352e-1** (3.3009e-3)	**2.0215e-1** (2.5121e-2)	1.8802e-1+ (2.9794e-2)	5.4828e-1+ (9.4441e-2)	3.3216e-1+ (1.3907e-2)	7.6843e-1+ (3.5360e-2)	5.8698e-1+ (1.8431e-1)	9.8460e-1+ (1.0084e-1)	1.5946 (1.6337e-1)	1.8735 (9.4517e-2)

Finally, we will explain why MOEA/DVA learns the variable linkages in advance but the uniformity of its obtained solutions may not be very good on some MOPs. The possible reasons come from two aspects. One is to fix the values of diverse variables of the population in the early stages of evolution. The other is to assign the same computing resource for each sub-MOP and subcomponent respectively in the stage of subcomponent optimization. For example, MOEA/DVA decomposes two-objective UF1-UF7 problems with 30 variables into 100 individuals/sub-MOPs. Each individual/sub-MOP has 29 variable subcomponents {x2},...,{x30} needing to be optimized and the maximum number of function evaluations is set as 300,000. Therefore, roughly speaking, each individual/sub-MOP has spent 300000/(100 *29) $\approx$ 100 function evaluations optimizing each distance variable $x_i, i = 1, 2, \ldots, 30$.

TABLED.2: DIFFERENCES BETWEEN MOEA/DVA AND MTS.

	MOEA/DVA	MTS	MOEA/DVA VS. MTS
Variable linkage analysis	Based on good definition (Definition 2)	Suppose no variable linkage or average 1/4 variables to be interacted for all the MOPs	Better
Control analysis of variable	Has	None	Better
Number of operators	Only one operator on the different stage of evolution	Three local search operators with operation adaptation	Worse
Computing resources assignment	Treat all the individuals/sub-MOPs equally	Select 5 individuals with highest fitness improvements in recent preference	Worse
Diversity maintaining of evoltuionary population	On the early evolution, keep the diversity of diverse variable of population. On the late evolution, introduce uniformity optimization	None	better
External population	None	Has	Worse

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PF
MOEA/DVA
MTS
RMMEDA
MrBOA
MIEDA
MOEA/D
MOEA/D-M2M
DMOEADD
UF3
PF
MOEA/DVA
MTS
RMMEDA
MrBOA
MIEDA
MOEA/D
MOEA/D-M2M
DMOEADD
UF4
PF
MOEA/DVA
MTS
RMMEDA
MrBOA
MIEDA
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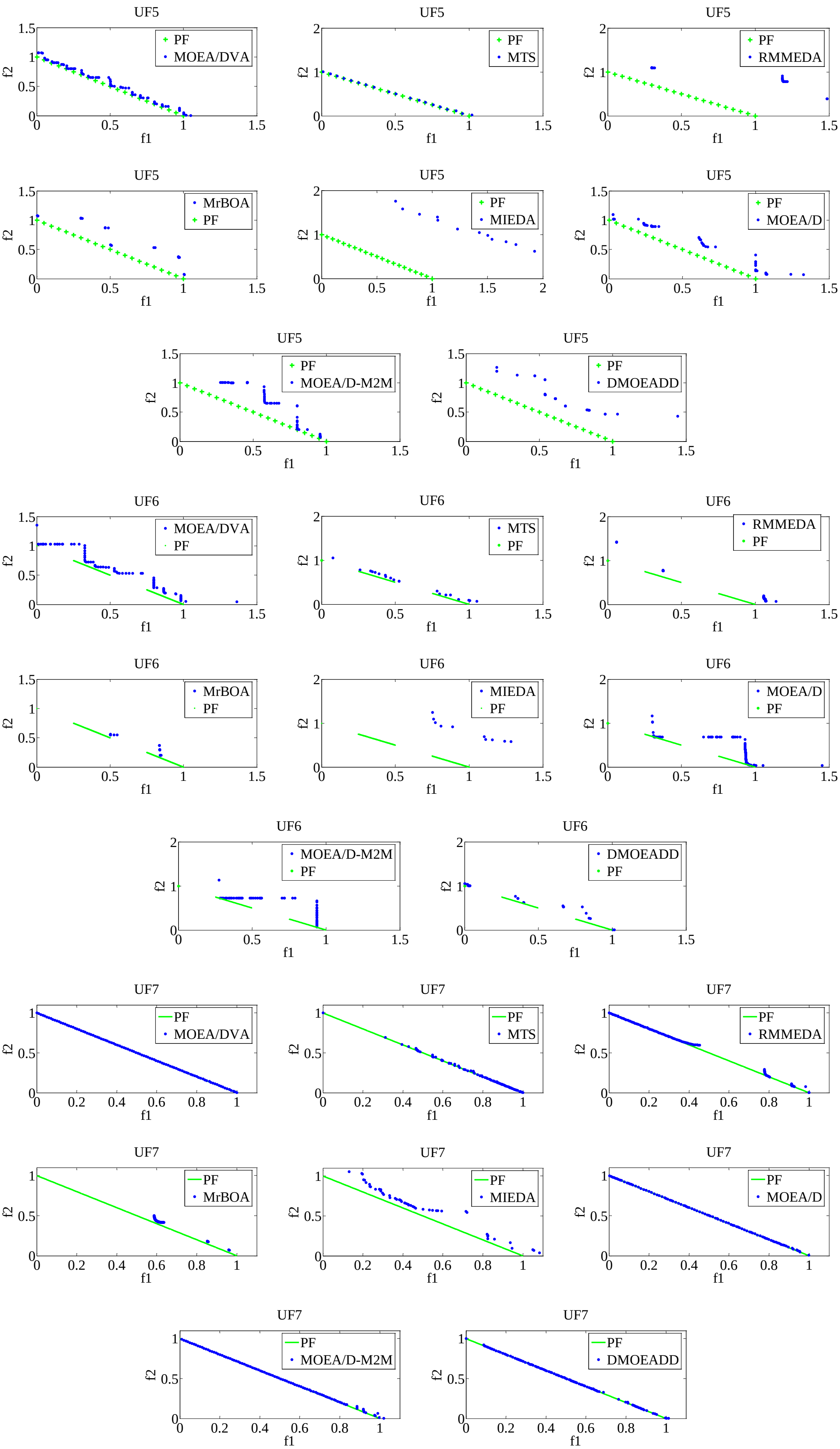
DMOEADD
f1
f2


UF5
PF
MOEA/DVA
MTS
RMMEDA
MrBOA
MIEDA
MOEA/D
MOEA/D-M2M
DMOEADD
UF6
UF7
f1
f2

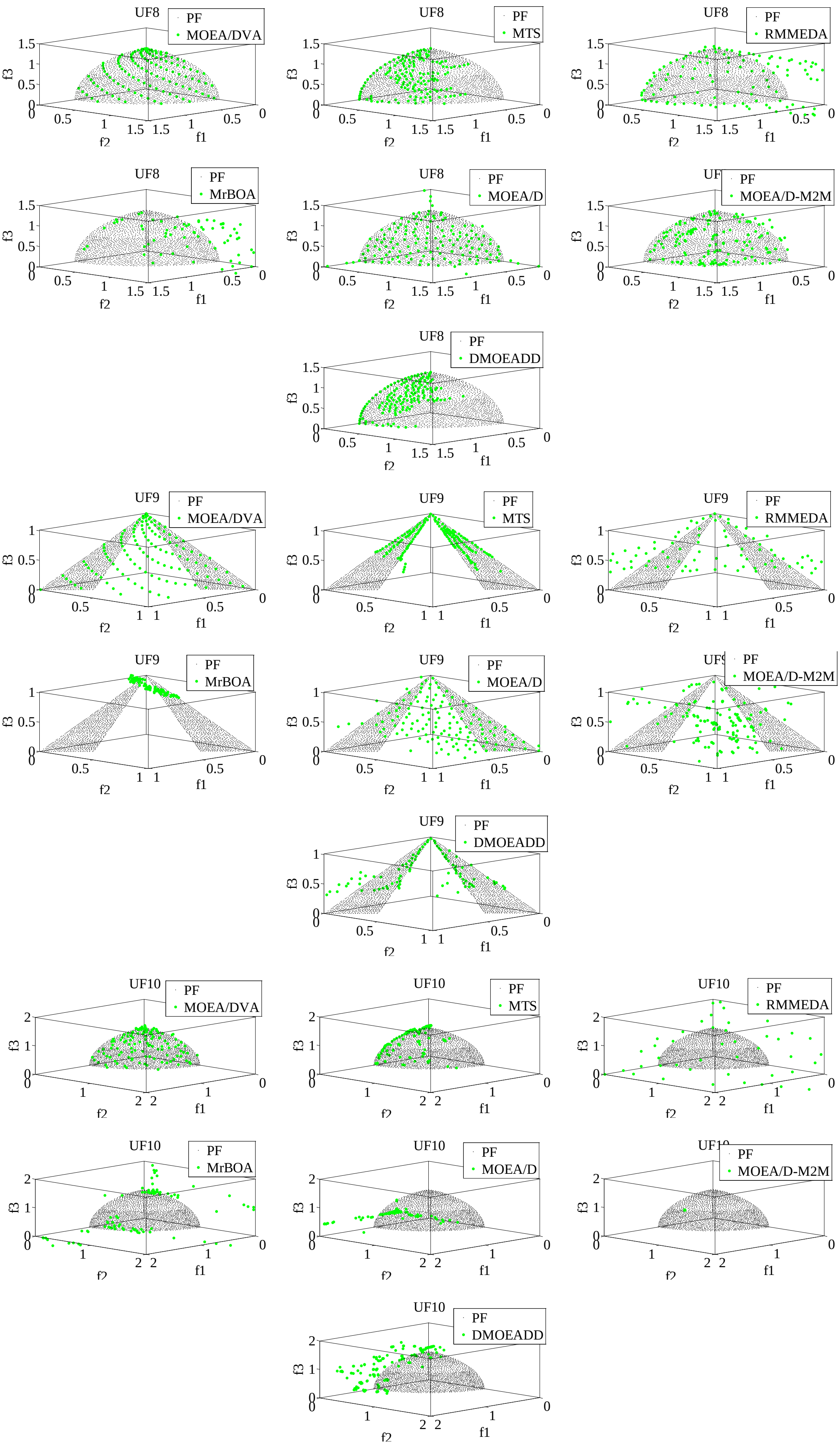

UF8
PF
MOEA/DVA
MTS
RMMEDA
MrBOA
MOEA/D
MOEA/D-M2M
DMOEADD
UF9
UF10
f1
f2
f3

Fig. D.1: The solution set with the median IGD values found by the compared algorithms.

Section IV-E: MOEA/DVA VS. Other MOEAs Based on Decomposition

To observe the effectiveness of the proposed MOP decomposition based on diverse variables, MOEA/DVA is compared with MOEA/D, MOEA/D-M2M and DMOEADD. UF1-UF10 problems with 30 variables and 300,000 function evaluations are used as the test problems.

Table D.3 shows the average and standard deviation of I_{ε^+}-metric and IGD-metric values of the obtained solutions by each algorithm on UF1-UF10 problems. Based on Wilcoxon rank sum test at 0.05 level, Table VIII provides the statistical results on the performance comparisons between MOEA/DVA and the other compared algorithms. As shown in Figs. 16 and 17, they plot the solution set with the median IGD values obtained by MOEA/DVA, MOEA/D, MOEA/D-M2M and DMOEADD on UF problems.

From Table D.3, in terms of average I_{ε^+}-metric and IGD-metric, we can see that MOEA/DVA is better than MOEA/D, MOEA/D-M2M and DMOEADD on UF1-UF2 and UF4-UF10 problems, while MOEA/D performs best on UF3 problem. As shown in Table VIII, our proposed algorithm significantly outperforms MOEA/D in 9 out of 10 comparisons based on I_{ε^+} and IGD metrics. According to Wilcoxon's rank sum test at 0.05 significance level, MOEA/DVA significantly outperforms MOEA/D-M2M in 9 out of 10 comparisons and performs better than DMOEADD in 10 out of 10 comparisons.

TABLE D.3 MEAN AND STANDARD DEVIATION OF I_{ε^+} METRIC AND IGD METRIC VALUES OBTAINED BY THE COMPARED ALGORITHMS ON UF1-UF10 PROBLEMS. THE VALUE WITHIN PARENTHESES IS THE DEVIATION OF METRIC.

	MOEA/DVA		MOEA/D		MOEA/D-M2M		DMOEADD	
	IGD	I_{ε^+}	IGD	I_{ε^+}	IGD	I_{ε^+}	IGD	I_{ε^+}
UF1	**4.1350e-3** (9.9053e-5)	**9.2148e-3**(9.6139e-4)	5.0808e-3+ (1.0462e-1)	1.5021e-2+ (1.0057e-2)	5.2429e-3+ (5.9142e-4)	1.8962e-2+ (7.5582e-3)	8.3725e-3+ (2.1917e-3)	3.5797e-2+ (1.3387e-2)
UF2	**4.1065e-3**(4.9092e-5)	**9.0603e-3**(5.7421e-4)	6.1240e-3+ (5.1602e-4)	2.5338e-2+ (6.1677e-3)	7.0055e-3+ (8.5850e-4)	2.8356e-2+ (6.3818e-3)	5.9688e-3+ (1.1542e-1)	2.7743e-2+ (9.2655e-3)
UF3	2.2714e-2+ (7.2599e-3)	5.6724e-2+(1.3794e-2)	**7.4994e-3** (3.6372e-3)	**2.1551e-2** (1.1455e-2)	1.5595e-2+ (1.4647e-2)	4.2396e-2+ (5.3925e-2)	3.2785e-2+ (5.0361e-3)	5.7202e-2+ (1.7281e-2)
UF4	**3.5067e-2** (1.0070e-3)	**4.3763e-2**(2.8428e-3)	5.5163e-2+ (3.4085e-3)	6.8570e-2+ (7.1297e-3)	4.2163e-2+ (7.4026e-4)	4.8188e-2+ (2.3564e-3)	4.9357e-2+ (2.0535e-3)	6.0771e-2+ (3.1612e-3)
UF5	**3.2592e-2**(4.6786e-3)	**6.9379e-2**(1.5777e-2	1.8845e-1+ (7.9539e-2)	3.1415e-1+ (1.3013e-1)	1.6226e-1+ (2.2077e-2)	2.8783e-1+ (5.0966e-2)	2.8821e-1+ (3.7242e-2)	3.9314e-1+ (5.9637e-2)
UF6	**5.6134e-2** (1.3729e-2)	**9.3159e-2)**(1.8142e-2)	1.8611e-1+ (1.7664e-1)	3.8414e-1+ (2.4237e-1)	1.6680e-1+ (7.2720e-2)	2.9963e-1+ (9.4474e-2)	7.1190e-2+ (4.1361e-2)	1.9101e-1+ (7.9887e-2)
UF7	**3.7667e-3**(4.6437e-5)	**8.8102e-3**(1.4653e-3)	5.0099e-3+ (3.3674e-4)	2.2023e-2+ (6.7085e-3)	5.9443e-3+ (6.7321e-4)	2.8880e-2+ (4.9815e-3)	9.5198e-3+ (2.3555e-3)	5.2256e-2+ (1.5997e-2)

UF8	**5.7788e-2** (1.1960e-2)	**1.4471e-1** (3.9164e-2)	5.9993e-2 (1.2990e-2)	1.6221e-1 (3.8858e-2)	1.0474e-1+ (3.9437e-3)	2.8104e-1+ (1.1950e-1)	1.5142e-1+ (2.8463e-2)	4.9857e-1+ (6.2532e-2)
UF9	**1.2333e-1** (1.6254e-1)	**1.8964e-1** (2.0914e-1)	1.3161e-1+ (7.1251e-2)	3.0016e-1+ (1.7651e-1)	1.8360e-1+ (5.7874e-2)	3.6617e-1+ (1.0438e-1)	1.6640e-1+ (1.4986e-2)	3.8458e-1+ (2.5221e-3)
UF10	**1.0352e-1** (3.3009e-3)	**2.0215e-1**(2.5121e-2)	4.4116e-1+ (4.0000e-2)	8.1656e-1+ (7.0977e-2)	5.2980e-1+ (1.6245e-1)	7.9019e-1+ (8.7734e-2)	4.2270e-1+ (4.4254e-2)	8.0717e-1+ (3.3346e-2)

Table IX gives the average CPU time (in seconds) spent by MOEA/DVA, MOEA/D, MOEA/D-M2M and DMOEADD. Table X gives the comparison of CPU time between MOEA/DVA and MOEA/D, MOEA/D-M2M and DMOEADD. Roughly speaking, the CPU time spent by the four compared algorithms has the relationship: MOEA/DVA < MOEA/D < MOEA/D-M2M < DMOEADD.

From Fig. D.1, we can see that based on the uniformity and approximation quality of the obtained solutions, MOEA/DVA obtains better performance than the other compared algorithms on UF1-UF2 and UF4-UF10 problems, while MOEA/D has the best coverage on UF3 problem. From Fig. D.1, we can find that the solutions obtained by MOEA/DVA are not very uniform for UF8-UF10 problems. The reason is that MOEA/DVA fixes the values of the diverse variables (position variables and mixed variables) of the population in the early stages of evolution and subcomponent optimization always works (utility≥threshold) before the algorithm stops. Therefore, the distribution of the solutions found by MOEA/DVA is dependent on the mapping from PS of MOP to its PF. One may consider the uniformity of obtained solutions based on domain decomposition techniques, which will be a research issue for our further work.

Compared with MOEA/D-M2M and DMOEADD which are based on Pareto dominance and domain decomposition, MOEA/DVA has better convergence on UF1-UF2 and UF4-UF10 problems as shown in figure D.1. The reason is that MOEA/DVA uses subcomponent optimization to improve the convergence of the population. Moreover, the solutions obtained by MOEA/DVA have better uniformity than MOEA/D-M2M and DMOEADD on UF1-UF2 and UF4-UF10 problems. The reason is that MOEA/DVA optimizes the uniformity of the population in the late stages of evolution.

Compared with MOEA/D, MOEA/DVA has better convergence on UF problems except for UF3 problem as shown in figure D.1. The reason is that MOEA/DVA uses DVA to learn the variable linkages and each individual/sub-MOP independently optimizes low-dimensional subcomponents one by one. Conversely, the MOEA/D has better uniformity than MOEA/DVA on some problems. This is because MOEA/D uses uniformly fixed weight vectors to obtain evenly distributed Pareto-optimal solutions in the objective space. MOEA/DVA fixes the diverse variables of the population with uniformly distributed values in the stage of subcomponent optimization. Therefore, the distribution of the solutions found by MOEA/DVA is dependent on the mapping from PS of MOP to PF in the early stages of evolution.

Section IV-F. Sensitivity Analysis of Parameters

There are two control parameters in dividing of distance variables in MOEA/DVA:

1) NCA represents the number of sampled solutions to recognize the control property of

decision variables. NCA affects the precision of control analysis of decision variables.

2) NIA is the maximum number of tries required to judge the interaction between two variables. NIA determines the precision of learned variable linkages.

To study how MOEA/DVA is sensitive to the above two parameters, multiple data for each parameter have been tried. Three values of NCA are set as 20, 40 and 80. Three values of NIA are set as 6, 12 and 24. UF1, UF8, three-objective WFG1 and WFG2 with 30 variables are used as the test problems. 10,000 independent runs have been conducted for each configuration on each MOP. MOEA/DVA analyzes the decision variables accurately if the independent run meets the following three conditions simultaneously:

1) Diverse variables (position variables and mixed variables) are recognized as diverse variables.

2) Distance variables are judged as distance variables.

3) Based on learning variable linkages, the distance variables are divided accurately.

We first suppose all decision variables are independent of one another. If the sampling data meets the condition in Definition 2, we judge two variables to have interaction. Fig. D.2 plots the success rate of analyzing the decision variables. For UF1 and WFG1, the success rates of analyzing decision variables are close to 100% on the nine configurations. The reason is that UF1 and WFG1 problems have no variable linkage among distance variables. For UF8 problem, the success rates of analyzing decision variables are close to 87% for NCA=20, 90% for NCA=40 and 92% for NCA=80.

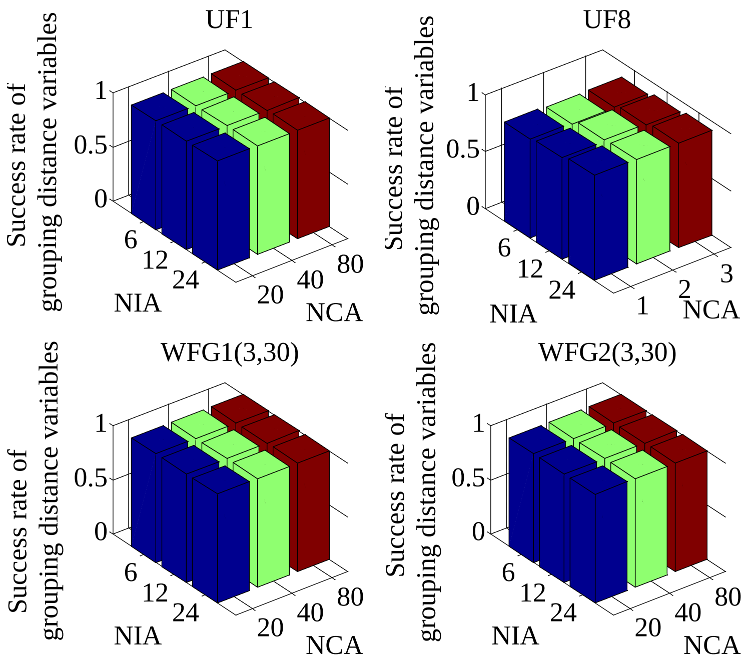


Figure D.2: Success rate of analyzing decision variables by MOEA/DVA with 9 different combinations of *NCA* and *NIA* on UF1, UF8, WFG1 and WFG2 problems.

The failure reason in analyzing decision variables for UF8 problem is that the mixed variable x2 is sometimes misidentified as distance variable. For WFG2 problem whose distance variables have the linkages, the success rates of analyzing decision variables are also close to 100% on the nine configurations. The reason is that the variable linkage is easy to learn on WFG2 problem.

Some may be interested in the case where the proposed algorithms 1, 2 and 3 may not work well. UF8 problem is taken for example. If algorithms 1, 2 and 3 can work well, the solutions obtained by MOEA/DVA are plotted in the Fig. D.3 (left). On the contrary, if the two following conditions are established:

(1) Mixed variable x2 is misidentified as distance variable.

(2) During the whole evolution, the values of the diverse variables are fixed.

The obtained solutions are shown in the Fig. D.3 (right). If there are remaining function evaluations after subcomponent optimization, uniformity optimization of MOEA/DVA is expected to deal with this case. As shown in Fig. D.3, we can see that the solutions plotted on the left are distributed on a 2-dimensional surface while the solutions plotted on the right are mainly distributed on a 1-dimensional curve.

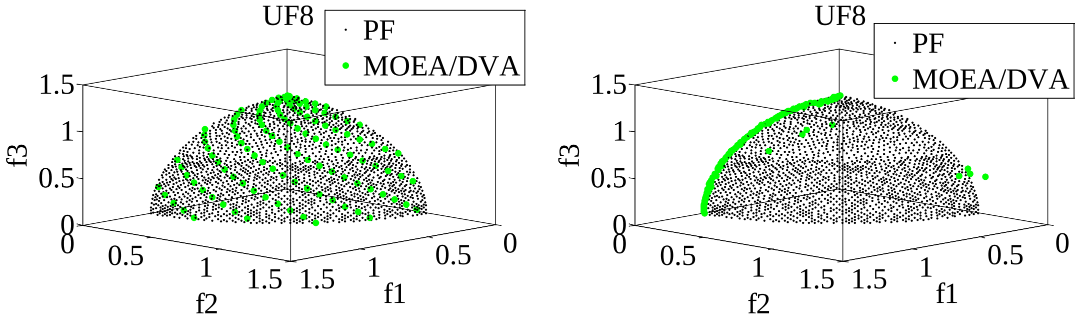


Figure D.3:Plot the solutions obtained by MOEA/DVA for two cases of UF8 problem.