FEM PROJECT – GROUP 3

Analysis of micropolar elastic beams

ME17BTECH11016	***CHANDRASHEKHAR***
ME19MTECH11025	***THANI ASWANTH***
ME19MTECH11030	***LOKESWAR***
ME17BTECH11035	***PANDHARE SHRIKANT***

Introduction

In this project, we have analysed the deformation of elastic beams using theory of linear micropolar elasticity based on micropolar continuum mechanics. A simple approximation is done by using power series expansion for the axial displacement and micro-rotation fields. The governing equations are derived by integrating the momentum and moment of momentum equations in the micropolar continuum theory. We can improve the micropolar beam theory by choosing more terms in the power series expansions of displacements and micro-rotation fields.

After some simplifications, this theory can be reduced to the well-known Timoshenko and Euler–Bernoulli beam theories. Also, we have calculated the deformation of a cantilever beam with transverse concentrated tip loading. The pattern of deflection of the beam is similar to the classical beam theories. The nature of flexural and longitudinal waves in the infinite length micropolar beam has been studied. This theory predicts the existence of micro-rotational waves which are not present in any of the known beam theories based on the classical continuum mechanics.

Theory of linear micropolar elasticity

Let 'u' denote the displacement vector of a macro-element in the continuum. Angles of rotation of the associated micro-structure constitute a rotation vector 'ϕ'. The components of infinitesimal micropolar strain tensor 'e' and infinitesimal wryness tensor 'κ' can be written as,

$$e_{ij} = u_{i,j} + \varepsilon_{ijk}\phi_k, \quad \kappa_{ij} = \phi_{j,i} \qquad \text{where i, j, k = x, y, z} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$\varepsilon_{ijk}$is the alternating tensor and it is given as $\varepsilon_{ijk} = \begin{cases} 1 \quad if \quad ijk = \{xyz, yzx, zxy\} \\ 0 \quad if \quad anyindexrepeats \\ -1 \quad if \quad ijk = \{xzy, zyx, yxz\} \end{cases}$

In the micropolar theory, balance of linear momentum and angular momentum are in the following form,

$$\sigma_{ji,j} + \rho(f_i - \ddot{u}_i) = 0 \cdots\cdots\cdots\cdots\cdots (2)_1, \quad m_{ji,j} + \varepsilon_{imn}\sigma_{mn} + \rho(l_i - J\ddot{\phi}_i) = 0 \cdots \cdots\cdots\cdots (2)_2$$

where, 'ρ' is the current density,

'f_i' is the body force density,

$\ddot{u}_i = \frac{\partial^2 u_i(x,t)}{\partial t^2}$ is the acceleration component in the i-direction,

'J' is the micro-inertia,

'l_i' is the body couple density and

$\ddot{\phi}_i = \frac{\partial^2 \phi_i(x,t)}{\partial t^2}$ is the micro-rotation acceleration component in the i-direction.

In the linear theory of micropolar elasticity, the constitutive equations are,

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + (\bar{\mu} + \eta)e_{ij} + \bar{\mu}e_{ji} \cdots\cdots\cdots\cdots (3)_1$$
$$m_{ij} = \alpha\kappa_{kk}\delta_{ij} + \beta\kappa_{ij} + \gamma\kappa_{ji} \cdots\cdots\cdots\cdots\cdots\cdots (3)_2$$

where, $\lambda = \frac{E\nu}{[(1-2\nu)(1+\nu)]}$, $\bar{\mu} = G - \frac{\eta}{2}$

$\lambda, \bar{\mu}, \eta, \alpha, \beta$ and γ are the elastic constants. E, ν, and G are respectively the Young modulus, Poisson ratio and Shear modulus.

Simplified equations of motion

Assumptions:-

1. The beam height 2h and its width b(x,z) are small as compared to the length L
2. The elastic constants are functions of x-coordinate
3. Distributions of the body force, body couple, surface traction, surface couple, strain, wryness, stress and couple stress components are independent of y-coordinate
4. The stress and displacement fields do not vary severely across the height
5. The loading is so that no torsion occurs in the beam
6. The non-zero components of stress, couple stress, body force and body couples are σ_{xx}, σ_{xz}, σ_{zx}, m_{xy}, m_{zy}, f_x, f_z and l_y

Under the assumptions mentioned above and setting i=x in equation $(2)_1$ we get,

$$\sigma_{jx,j} + \rho(f_x - \ddot{u}_x) = 0$$
$$\sigma_{xx,x} + \sigma_{yx,y} + \sigma_{zx,z} + \rho(f_x - \ddot{u}_x) = 0$$
$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z} + \rho(f_x - \ddot{u}_x) = 0 \qquad \left(but\ \sigma_{yx} = 0\ from\ assumptions\right)$$
$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{zx}}{\partial z} + \rho(f_x - \ddot{u}_x) = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)_1$$

Under the assumptions mentioned above and setting i=z in equation $(2)_1$ we get,

$$\sigma_{jz,j} + \rho(f_z - \ddot{u}_z) = 0$$
$$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + \rho(f_z - \ddot{u}_z) = 0$$
$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + \rho(f_z - \ddot{u}_z) = 0 \qquad \left(but\ \sigma_{yz} = 0\ from\ assumptions\right)$$
$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{zz}}{\partial z} + \rho(f_z - \ddot{u}_z) = 0 \cdots\cdots\cdots\cdots (4)_2$$

Under the assumptions mentioned above and setting i=y in equation $(2)_2$ we get,

$$m_{jy,j} + \varepsilon_{ymn}\sigma_{mn} + \rho(l_y - J\ddot{\phi}_y) = 0$$
$$m_{xy,x} + m_{yy,y} + m_{zy,z} + \sum_{m=x}^{z}\sum_{n=x}^{z}\varepsilon_{ymn}\sigma_{mn} + \rho(l_y - J\ddot{\phi}_y) = 0$$
$$\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{zy}}{\partial z} + \sigma_{zx} - \sigma_{xz} + \rho(l_y - J\ddot{\phi}_y) = 0 \qquad \left(but\ m_{yy} = 0\ from\ assumptions\right)$$
$$\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{zy}}{\partial z} + \sigma_{zx} - \sigma_{xz} + \rho(l_y - J\ddot{\phi}_y) = 0 \cdots\cdots\cdots\cdots (5)$$

In equations $(4)_1$, $(4)_2$ and (5), u_x, u_z are the displacement components along x-axis and z-axis respectively and ϕ_y is the micro-rotation about the y-axis.

The upper and lower faces of the beam are subjected to traction vectors $t_{top}^{(k)}$, $t_{bottom}^{(-k)}$, and to couple vectors $m_{top}^{(k)}$, $m_{bottom}^{(-k)}$ respectively. The resultant traction, $\tau$and resultant couple vector,μ are:

$$\tau = t_{top}^{(k)} - t_{bottom}^{(-k)} = [\tau_{top} - \tau_{bottom}]\overset{\Lambda}{i} + [p_{top} - p_{bottom}]\overset{\Lambda}{k}$$
$$= \tau\overset{\Lambda}{i} + p\overset{\Lambda}{k} \cdots\cdots\cdots (6)$$
$$\mu = m_{top}^{(k)} - m_{bottom}^{(-k)} = [m_{top} - m_{bottom}]\overset{\Lambda}{j} = m\overset{\Lambda}{j} \cdots\cdots\cdots\cdots (7)$$

Approximate displacement and micro-rotation fields

We assume power series expansion for the displacement component ‘u_x’ and the micro-rotation field ‘ϕ_y’ in the following form,

$$u_x(x,z,t) \approx \sum_{i=0}^{\infty} z^i \psi_{(i)}(x,t), \qquad \phi_y(x,z,t) \approx \sum_{i=0}^{\infty} z^i \phi_{(i)}(x,t) \cdots\cdots\cdots\cdots (8)$$

Similar to the classical theory, we do not consider any power series expansion for the displacement component ’ and it is given in the following simple approximation as,

$$u_z(x,z,t) \approx w(x,t) \cdots\cdots\cdots\cdots (9)$$

In general, one may construct a (r,s) approximate theory by considering the following approximations for u_x and ϕ_y

$$u_x(x,z,t) \approx \sum_{i=0}^{r\geq 0} z^i \psi_{(i)}(x,t), \qquad \phi_y(x,z,t) \approx \sum_{i=0}^{s\geq 0} z^i \phi_{(i)}(x,t) \cdots\cdots\cdots\cdots (10)$$

Setting (i=j=k) in equation $(3)_1$, we get,

$$\sigma_{kk} = \lambda e_{kk}\delta_{kk} + (\bar{\mu}+\eta)e_{kk} + \bar{\mu}e_{kk}$$
$$\sigma_{kk} = (\lambda\delta_{kk} + 2\bar{\mu} + \eta)e_{kk}$$

We know that $\delta_{kk} = \delta_{xx} + \delta_{yy} + \delta_{zz} = 3$, then the above equation will be,

$$e_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\bar{\mu} + \eta} \cdots\cdots\cdots\cdots (11)$$

Interchanging ‘i’ and ‘j’ in equation $(3)_1$, we get

$$\sigma_{ji} = \lambda e_{kk}\delta_{ji} + (\bar{\mu}+\eta)e_{ji} + \bar{\mu}e_{ij}$$

$$\sigma_{ji} = \lambda e_{kk}\delta_{ij} + (\bar{\mu}+\eta)e_{ji} + \bar{\mu}e_{ij} \qquad (\because \delta_{ji} = \delta_{ij})$$

Now multiplying equation $(3)_1$ with $(\bar{\mu}+\eta)$ and the above equation with $\bar{\mu}$ and then subtracting them we will obtain,

$$(\bar{\mu}+\eta)\sigma_{ij}-\bar{\mu}\sigma_{ji}=\lambda\eta e_{kk}\delta_{ij}+[(\bar{\mu}+\eta)^2-\bar{\mu}^2]e_{ij}$$

Substituting the value of e_{kk} obtained from equation (11) in the above equation and rearranging the terms we get,

$$\left[\eta^2+2\bar{\mu}\eta\right]e_{ij}=(\bar{\mu}+\eta)\sigma_{ij}-\bar{\mu}\sigma_{ji}-\lambda\eta\frac{\sigma_{kk}}{3\lambda+2\bar{\mu}+\eta}\ \delta_{ij}$$

$$e_{ij}=\frac{1}{\eta(\eta+2\bar{\mu})}\left[(\bar{\mu}+\eta)\sigma_{ij}-\bar{\mu}\sigma_{ji}-\frac{\lambda\eta}{3\lambda+2\bar{\mu}+\eta}\ \sigma_{kk}\delta_{ij}\right]\cdots\cdots\cdots\cdots(12)$$

where (i, j, k = x, y, z)

Setting i=x and j=x in equation (12) and using the assumption$\sigma_{yy}=\sigma_{zz}=0$, we get,

$$e_{xx}=\frac{1}{\eta(\eta+2\bar{\mu})}\left[(\bar{\mu}+\eta)\sigma_{xx}-\bar{\mu}\sigma_{xx}-\frac{\lambda\eta}{3\lambda+2\bar{\mu}+\eta}\sigma_{kk}\delta_{xx}\right]$$

We know $\sigma_{kk}=\sigma_{xx}+\sigma_{yy}+\sigma_{zz}=\sigma_{xx}+0+0=\sigma_{xx}$ and $\delta_{xx}=\delta_{yy}=\delta_{zz}=1$

$$e_{xx}=\frac{1}{\eta(\eta+2\bar{\mu})}\left[\eta\sigma_{xx}-\frac{\lambda\eta}{3\lambda+2\bar{\mu}+\eta}\sigma_{xx}\right]$$

$$e_{xx}=\frac{1}{(\eta+2\bar{\mu})}\left[1-\frac{\lambda}{3\lambda+2\bar{\mu}+\eta}\right]\sigma_{xx}$$

$$e_{xx}=\frac{2\lambda+2\bar{\mu}+\eta}{(\eta+2\bar{\mu})(3\lambda+2\bar{\mu}+\eta)}\sigma_{xx}\cdots\cdots\cdots\cdots(13)_1$$

Setting i=j=y in equation (12) and using the assumption$\sigma_{yy}=\sigma_{zz}=0$, we get

$$e_{yy}=\frac{1}{\eta(\eta+2\bar{\mu})}\left[(\bar{\mu}+\eta)\sigma_{yy}-\bar{\mu}\sigma_{yy}-\frac{\lambda\eta}{3\lambda+2\bar{\mu}+\eta}\sigma_{kk}\delta_{yy}\right]$$

$$e_{yy}=\frac{-\lambda}{(\eta+2\bar{\mu})(3\lambda+2\bar{\mu}+\eta)}\sigma_{xx}$$

Similarly setting i=j=z in equation (12) and using the assumption$\sigma_{yy} = \sigma_{zz} = 0$, we get

$$e_{zz} = \frac{-\lambda}{(\eta + 2\bar{\mu})(3\lambda + 2\bar{\mu} + \eta)} \sigma_{xx}$$

We have,

$$e_{yy} = e_{zz} = \frac{-\lambda}{(\eta + 2\bar{\mu})(3\lambda + 2\bar{\mu} + \eta)} \sigma_{xx} = -\nu e_{xx} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (13)_2$$

In an analogy with the classical beam theory, equation $(13)_1$ may be written as $e_{xx} = \frac{\sigma_{xx}}{\bar{E}}$

where, $\bar{E} = \frac{(\eta+2\bar{\mu})(3\lambda+2\bar{\mu}+\eta)}{2\lambda+2\bar{\mu}+\eta} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (14)$

Setting i=j=x in equation (1) and using equation (10) we get,

$$e_{xx} = u_{x,x} + \varepsilon_{xxk}\phi_k$$
$$e_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=0}^{r} z^i \psi_{(i)}(x,t)\right)$$
$$e_{xx} = \frac{\partial u_x}{\partial x} = \sum_{i=0}^{r} z^i \frac{\partial \psi_{(i)}}{\partial x} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (15)_1$$

Setting i=z and j=x in equation (1) and using equations (9) and (10) we get,

$$e_{zx} = u_{z,x} + \varepsilon_{zxk}\phi_k$$
$$e_{zx} = u_{z,x} + \varepsilon_{zxx}\phi_x + \varepsilon_{zxy}\phi_y + \varepsilon_{zxz}\phi_z$$
$$e_{zx} = u_{z,x} + 0 + \phi_y + 0$$
$$e_{zx} = \frac{\partial u_z}{\partial x} + \phi_y = \frac{\partial w}{\partial x} + \sum_{i=0}^{s} z^i \phi_{(i)} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (15)_2$$

Setting i=x and j=z in equation (1) and using equation (10) we get,

$$e_{xz} = u_{x,z} + \varepsilon_{xzk}\phi_k$$
$$e_{xz} = u_{x,z} + \varepsilon_{xzx}\phi_x + \varepsilon_{xzy}\phi_y + \varepsilon_{xzz}\phi_z$$
$$e_{xz} = u_{x,z} + 0 - \phi_y + 0$$

$$e_{xz} = \frac{\partial u_x}{\partial z} - \phi_y = \frac{\partial}{\partial z}\left(\sum_{i=0}^{r} z^i \psi_{(i)}(x,t)\right) - \sum_{i=0}^{s} z^i \phi_{(i)}$$

$$e_{xz} = \frac{\partial u_x}{\partial z} - \phi_y = \sum_{i=0}^{r\geq 1} i z^{i-1} \psi_{(i)} - \sum_{i=0}^{s} z^i \phi_{(i)} \cdots\cdots\cdots\cdots (16)$$

We know $e_{xx} = \frac{\sigma_{xx}}{\bar{E}}$, therefore using equation $(15)_1$ we get

$$\sigma_{xx} = \bar{E} e_{xx}$$

$$\sigma_{xx} = \bar{E} \sum_{i=0}^{r} z^i \frac{\partial \psi_{(i)}}{\partial x} \cdots\cdots\cdots\cdots (17)_1$$

Setting i=z and j=x in equation $(3)_1$ and using equations $(15)_2$ and (16) we obtain,

$$\sigma_{zx} = \lambda e_{kk} \delta_{zx} + (\bar{\mu} + \eta) e_{zx} + \bar{\mu} e_{xz}$$

$$\sigma_{zx} = 0 + (\bar{\mu} + \eta) e_{zx} + \bar{\mu} e_{xz} \qquad (\because \delta_{zx} = 0)$$

$$\sigma_{zx} = (\bar{\mu} + \eta) e_{zx} + \bar{\mu} e_{xz}$$

$$\sigma_{zx} = (\bar{\mu} + \eta)\left[\frac{\partial w}{\partial x} + \sum_{i=0}^{s} z^i \phi_{(i)}\right] + \bar{\mu}\left[\sum_{i=0}^{r\geq 1} i z^{i-1} \psi_{(i)} - \sum_{i=0}^{s} z^i \phi_{(i)}\right]$$

$$\sigma_{zx} = (\bar{\mu} + \eta)\frac{\partial w}{\partial x} + \bar{\mu} \sum_{i=0}^{r\geq 1} i z^{i-1} \psi_{(i)} + \eta \sum_{i=0}^{s} z^i \phi_{(i)} \cdots\cdots\cdots\cdots (17)_2$$

Setting i=x and j=z in equation $(3)_1$ and using equations $(15)_2$ and (16) we obtain,

$$\sigma_{xz} = \lambda e_{kk} \delta_{xz} + (\bar{\mu} + \eta) e_{xz} + \bar{\mu} e_{zx}$$

$$\sigma_{xz} = 0 + (\bar{\mu} + \eta) e_{xz} + \bar{\mu} e_{zx} \qquad (\because \delta_{xz} = 0)$$

$$\sigma_{xz} = (\bar{\mu} + \eta) e_{xz} + \bar{\mu} e_{zx}$$

$$\sigma_{xz} = (\bar{\mu} + \eta)\left[\sum_{i=0}^{r\geq 1} i z^{i-1} \psi_{(i)} - \sum_{i=0}^{s} z^i \phi_{(i)}\right] + \bar{\mu}\left[\frac{\partial w}{\partial x} + \sum_{i=0}^{s} z^i \phi_{(i)}\right]$$

$$\sigma_{xz} = \bar{\mu}\frac{\partial w}{\partial x} + (\bar{\mu} + \eta) \sum_{i=0}^{r\geq 1} i z^{i-1} \psi_{(i)} - \eta \sum_{i=0}^{s} z^i \phi_{(i)} \cdots\cdots\cdots\cdots (18)$$

A simple approximate theory

We construct a simple micropolar beam theory by choosing some special terms in the general power series expansions given in equation (8). By setting r=1 and s=0 in equation (10), we obtain a (1,0) theory with the following reduced approximate displacement and micro-rotation fields

$$u_x(x,z,t) \approx \sum_{i=0}^{r=1} z^i \psi_{(i)}(x,t) = z^0 \psi_{(0)}(x,t) + z^1 \psi_{(1)}(x,t) = \psi_{(0)}(x,t) + z\psi_{(1)}(x,t)$$

$$u_x(x,z,t) \approx \sum_{i=0}^{r=1} z^i \psi_{(i)}(x,t) = u(x,t) + z\psi(x,t) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (19)$$

$$\phi_y(x,z,t) \approx \sum_{i=0}^{s=0} z^i \phi_{(i)}(x,t) = z^0 \phi_{(0)}(x,t) = \phi_{(0)}(x,t)$$

$$\phi_y(x,z,t) = \varphi(x,t) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (20)_1$$

From equation (9) the displacement component u_z is written as it is,

$$u_z(x,z,t) \approx w(x,t) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (20)_2$$

In equations (19), $(20)_1$ and $(20)_2$, u, ψ and φ have been used instead of $\psi_{(0)}$, $\psi_{(1)}$ and $\phi_{(0)}$

Balance equations for the micropolar beam

Now integrating equation $(4)_1$ over the cross section with the aid of equation (6), we obtain,

$$\iint_A \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} + \rho(f_x - \ddot{u}_x)\right] dA = 0$$

$$\iint_A \frac{\partial \sigma_{xx}}{\partial x} dA + \iint_A \frac{\partial \sigma_{zx}}{\partial z} dA + \iint_A [\rho(f_x - \ddot{u}_x)] dA = 0$$

$$\frac{\partial}{\partial x}\left[\iint_A \sigma_{xx} dA\right] + \iint_A \frac{\partial \sigma_{zx}}{\partial z} dA + \rho\left\{\iint_A f_x dA - \iint_A \ddot{u}_x dA\right\} = 0$$

$$\frac{\partial}{\partial x}(AN_{xx}) + \tau + \rho A(\bar{f}_x - \ddot{\bar{u}}_x) = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (21)$$

Now integrating equation (4)₂ over the cross section with the aid of equation (6), we obtain,

$$\iint_A \left[\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + \rho(f_z - \ddot{u}_z)\right] dA = 0$$

$$\iint_A \frac{\partial \sigma_{xz}}{\partial x} dA + \iint_A \frac{\partial \sigma_{zz}}{\partial z} dA + \iint_A [\rho(f_z - \ddot{u}_z)] dA = 0$$

$$\frac{\partial}{\partial x}\left[\iint_A \sigma_{xz} dA\right] + \iint_A \frac{\partial \sigma_{zz}}{\partial z} dA + \rho\left\{\iint_A f_z dA - \iint_A \ddot{u}_z dA\right\} = 0$$

$$\frac{\partial}{\partial x}(AN_{xz}) + p + \rho A(\bar{f}_z - \ddot{\bar{u}}_z) = 0 \cdots\cdots\cdots\cdots (22)$$

Now integrating equation (5) over the cross section with the aid of equation

$$\iint_A \left[\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{zy}}{\partial z} + \sigma_{zx} - \sigma_{xz} + \rho(l_y - J\ddot{\varphi}_y)\right] dA = 0$$

$$\iint_A \frac{\partial m_{xy}}{\partial x} dA + \iint_A \frac{\partial m_{zy}}{\partial z} dA + \iint_A \sigma_{zx} dA - \iint_A \sigma_{xz} dA + \iint_A \rho(l_y - J\ddot{\varphi}_y) dA = 0$$

$$\frac{\partial}{\partial x}\left[\iint_A m_{xy} dA\right] + \iint_A \frac{\partial m_{zy}}{\partial z} dA + \iint_A \sigma_{zx} dA - \iint_A \sigma_{xz} dA + \rho\left\{\iint_A l_y dA - \iint_A J\ddot{\varphi}_y dA\right\} = 0$$

$$\frac{\partial}{\partial x}(AM_{xy}) + m + AN_{zx} - AN_{xz} + \rho(AL_y - AJ\ddot{\bar{\varphi}}_y) = 0$$

$$\frac{\partial}{\partial x}(AM_{xy}) + m + A(N_{zx} - N_{xz}) + \rho A(L_y - J\ddot{\bar{\varphi}}_y) = 0 \cdots\cdots\cdots\cdots (23)$$

(7), we obtain,

To obtain equations (21), (22) and (23) we have used the following definitions,
$\{N_{xx}, N_{xz}, N_{zx}, M_{xy}, \bar{f}_x, \bar{f}_z, L_y, \bar{u}_x, \bar{u}_z, \bar{\phi}_y\} =$
$\frac{1}{A}\iint_A \{\sigma_{xx}, \sigma_{xz}, \sigma_{zx}, m_{xy}, f_x, f_z, l_y, u_x, u_z, \phi_y\} dA \cdots\cdots\cdots$ (24)

From the above considered definition we have $\bar{u}_x = \frac{1}{A}\iint_A u_x dA$, then if we partially double differentiate this on both sides we get $\frac{\partial^2 \bar{u}_x}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[\frac{1}{A}\iint_A u_x dA\right]$

$$\ddot{\bar{u}}_x = \frac{1}{A}\iint_A \frac{\partial^2 u_x}{\partial t^2} dA = \frac{1}{A}\iint_A \ddot{u}_x dA$$

Similarly, $\ddot{\bar{u}}_z = \frac{1}{A}\iint_A \ddot{u}_z dA$

$$\ddot{\bar{\phi}}_y = \frac{1}{A}\iint_A \ddot{\phi}_y dA$$

Now multiplying equation $(4)_1$ with 'z' and integrating the obtained equation on the beam cross section we will get,

$$\iint_A z\left[\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{zx}}{\partial z} + \rho(f_x - \ddot{u}_x)\right] dA = 0$$
$$\iint_A z\frac{\partial\sigma_{xx}}{\partial x} dA + \iint_A z\frac{\partial\sigma_{zx}}{\partial z} dA + \iint_A z\rho(f_x - \ddot{u}_x) dA = 0$$

We know that, $\frac{\partial}{\partial z}(z\sigma_{zx}) = \sigma_{zx} + z\frac{\partial\sigma_{zx}}{\partial z}$, therefore the above equation can be rewritten as,

$$\iint_A z\frac{\partial\sigma_{xx}}{\partial x} dA + \iint_A \frac{\partial}{\partial z}(z\sigma_{zx}) dA - \iint_A \sigma_{zx} dA + \iint_A z\rho(f_x - \ddot{u}_x) dA = 0$$
$$\iint_A \frac{\partial(\sigma_{xx} z)}{\partial x} dA + \iint_A \frac{\partial}{\partial z}(z\sigma_{zx}) dA - \iint_A \sigma_{zx} dA + \rho\left[\iint_A z f_x dA - \iint_A z\ddot{u}_x dA\right] = 0$$
$$\frac{\partial}{\partial x}\left(\iint_A \sigma_{xx} z dA\right) + \iint_A \frac{\partial}{\partial z}(z\sigma_{zx}) dA - \iint_A \sigma_{zx} dA + \rho\left[\iint_A z f_x dA - \iint_A z\ddot{u}_x dA\right] = 0$$
$$\frac{\partial M}{\partial x} + h\bar{\tau} - AN_{zx} + \rho\left(\tilde{f}_x - \ddot{\tilde{u}}_x\right) = 0 \cdots\cdots\cdots\cdots\cdots\cdots (25)$$

To obtain equation (25) we have considered the following definitions,

$$\left\{M, \tilde{f}_x, \tilde{u}_x\right\} = \iint_A \{\sigma_{xx}, f_x, u_x\} z dA$$

and $\bar{\tau}$ is defined as $\bar{\tau} = \frac{1}{h}\left(h^+\tau_{top} - h^-\tau_{bottom}\right)$ ·· (26)

We know that N_{xx} is defined as,

$$N_{xx} = \frac{1}{A}\iint_A \sigma_{xx} dA$$

Using equation $(17)_1$

$$N_{xx} = \frac{1}{A}\iint_A \left[\bar{E}\sum_{i=0}^{r} z^i \frac{\partial \psi_{(i)}}{\partial x}\right] dA$$

For (1,0) micropolar beam theory,

$$N_{xx} = \frac{1}{A}\iint_A \left[\bar{E}\sum_{i=0}^{1} z^i \frac{\partial \psi_{(i)}}{\partial x}\right] dA$$

$$N_{xx} = \frac{1}{A}\iint_A \bar{E}\left[\frac{\partial}{\partial x}\big(\psi_{(0)}(x,t)\big) + \frac{\partial}{\partial x}\big(z\psi_{(1)}(x,t)\big)\right] dA$$

$$N_{xx} = \frac{1}{A}\iint_A \bar{E}\left[\frac{\partial u(x,t)}{\partial x} + z\frac{\partial \psi(x,t)}{\partial x}\right] dA$$

$$N_{xx} = \frac{1}{A}\iint_A \bar{E}\frac{\partial u(x,t)}{\partial x} dA + \frac{1}{A}\iint_A \bar{E}\, z\frac{\partial \psi(x,t)}{\partial x} dA$$

$$N_{xx} = \frac{\bar{E}}{A}\frac{\partial u(x,t)}{\partial x}\iint_A dA + \frac{\bar{E}}{A}\frac{\partial \psi(x,t)}{\partial x}\iint_A z dA \qquad (\because dA = dy.dz)$$

$$N_{xx} = \frac{\bar{E}}{A}\frac{\partial u(x,t)}{\partial x} A + 0 \qquad \left(\because \iint_A z dA = 0\right) \quad (1^{st}\ moment\ of\ area\ is\ zero)$$

$$N_{xx} = \bar{E}\frac{\partial u(x,t)}{\partial x} \cdots\cdots\cdots (27)_1$$

We know that M is defined as,

$$M = \iint_A \sigma_{xx} z dA$$

Using equation $(17)_1$

$$M = \iint_A \left[\bar{E} \sum_{i=0}^{r} z^i \frac{\partial \psi_{(i)}}{\partial x} \right] z dA$$

For (1,0) micropolar beam theory,

$$M = \iint_A \bar{E} \left[\frac{\partial}{\partial x} \left(\psi_{(0)}(x,t) \right) + \frac{\partial}{\partial x} \left(z \psi_{(1)}(x,t) \right) \right] z dA$$

$$M = \iint_A \bar{E} \left[\frac{\partial u(x,t)}{\partial x} + z \frac{\partial \psi(x,t)}{\partial x} \right] z dA$$

$$M = \iint_A \bar{E} \frac{\partial u(x,t)}{\partial x} z dA + \iint_A \bar{E} \frac{\partial \psi(x,t)}{\partial x} z^2 dA$$

$$M = \bar{E} \frac{\partial u(x,t)}{\partial x} \iint_A z dA + \bar{E} \frac{\partial \psi(x,t)}{\partial x} \iint_A z^2 dA$$

$$M = 0 + \bar{E} \frac{\partial \psi(x,t)}{\partial x} I \qquad \left(\because \iint_A z dA = 0 \right) \quad and \quad \left(\because \iint_A z^2 dA = I = Area\ moment\ of\ Inertia \right)$$

$$M = \bar{E} I \frac{\partial \psi(x,t)}{\partial x} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \cdot (27)_2$$

We know that N_{xz} is defined as, $\qquad N_{xz} = \frac{1}{A} \iint_A \sigma_{xz} dA$

Using equation (18) and (1,0) micropolar beam theory we get,

$$N_{xz} = \frac{1}{A} \iint_A \left[\bar{\mu} \frac{\partial w}{\partial x} + (\bar{\mu} + \eta) \sum_{i=0}^{r=1} i z^{i-1} \psi_{(i)} - \eta \sum_{i=0}^{s=0} z^i \phi_{(i)} \right] dA$$

$$N_{xz} = \frac{1}{A} \iint_A \left[\bar{\mu} \frac{\partial w(x,t)}{\partial x} + (\bar{\mu} + \eta) \psi_{(1)}(x,t) - \eta \phi_{(0)}(x,t) \right] dA$$

$$N_{xz} = \frac{1}{A}\iint_A \left[\bar{\mu}\frac{\partial w(x,t)}{\partial x} + (\bar{\mu}+\eta)\psi(x,t) - \eta\varphi(x,t)\right] dA$$
$$N_{xz} = \frac{1}{A}\iint_A \bar{\mu}\frac{\partial w(x,t)}{\partial x} dA + \frac{1}{A}\iint_A (\bar{\mu}+\eta)\psi(x,t) dA - \frac{1}{A}\iint_A \eta\varphi(x,t) dA$$
$$N_{xz} = \frac{\bar{\mu}}{A}\frac{\partial w(x,t)}{\partial x}\iint_A dA + \frac{(\bar{\mu}+\eta)}{A}\psi(x,t)\iint_A dA - \frac{\eta}{A}\varphi(x,t)\iint_A dA$$
$$\boldsymbol{N_{xz} = \bar{\mu}\frac{\partial w(x,t)}{\partial x} + (\bar{\mu}+\eta)\psi(x,t) - \eta\varphi(x,t)} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot (\mathbf{28})$$

We know that N_{zx} is defined as, $N_{zx} = \frac{1}{A}\iint_A \sigma_{zx} dA$

Using equation (17)2 and (1,0) micropolar beam theory we get,

$$N_{zx} = \frac{1}{A}\iint_A \left[(\bar{\mu}+\eta)\frac{\partial w}{\partial x} + \bar{\mu}\sum_{i=0}^{r=1} iz^{i-1}\psi_{(i)} + \eta\sum_{i=0}^{0} z^i\phi_{(i)}\right] dA$$
$$N_{zx} = \frac{1}{A}\iint_A \left[(\bar{\mu}+\eta)\frac{\partial w(x,t)}{\partial x} + \bar{\mu}\psi_{(1)}(x,t) + \eta\phi_{(0)}(x,t)\right] dA$$
$$N_{zx} = \frac{1}{A}\iint_A \left[(\bar{\mu}+\eta)\frac{\partial w(x,t)}{\partial x} + \bar{\mu}\psi(x,t) + \eta\varphi(x,t)\right] dA$$
$$N_{zx} = \frac{1}{A}\iint_A (\bar{\mu}+\eta)\frac{\partial w(x,t)}{\partial x} dA + \frac{1}{A}\iint_A \bar{\mu}\psi(x,t) dA + \frac{1}{A}\iint_A \eta\varphi(x,t) dA$$
$$N_{zx} = \frac{(\bar{\mu}+\eta)}{A}\frac{\partial w(x,t)}{\partial x}\iint_A dA + \frac{\bar{\mu}}{A}\psi(x,t)\iint_A dA + \frac{\eta}{A}\varphi(x,t)\iint_A dA$$
$$\boldsymbol{N_{zx} = (\bar{\mu}+\eta)\frac{\partial w(x,t)}{\partial x} + \bar{\mu}\psi(x,t) + \eta\varphi(x,t)} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot (\mathbf{29})$$

Substituting i=x and j=y in equation (1),

$$\kappa_{xy} = \phi_{y,x}$$
$$\kappa_{xy} = \frac{\partial \phi_y}{\partial x}$$

Using equation (20)1,

$$\boldsymbol{\kappa_{xy}} = \frac{\boldsymbol{\partial\varphi(x,t)}}{\boldsymbol{\partial x}} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \mathbf{(30)_1}$$

M_{xy} is defined as, $\qquad M_{xy} = \frac{1}{A}\iint_A m_{xy} dA$

Using equation (3)$_2$, $\quad m_{xy} = \beta\kappa_{xy} + \gamma\kappa_{yx}$

$$M_{xy} = \frac{1}{A}\iint_A [\beta\kappa_{xy} + \gamma\kappa_{yx}] dA$$

$$M_{xy} = \frac{1}{A}\iint_A [\beta\phi_{y,x} + \gamma\phi_{x,y}] dA$$

$$M_{xy} = \frac{1}{A}\iint_A \beta\phi_{y,x} dA \qquad (\because \phi_{x,y} = 0)$$

$$M_{xy} = \frac{1}{A}\iint_A \beta\frac{\partial\varphi(x,t)}{\partial x} dA$$

$$M_{xy} = \frac{1}{A}\beta\frac{\partial\varphi(x,t)}{\partial x}\iint_A dA$$

$$\boldsymbol{M_{xy}} = \boldsymbol{\beta}\frac{\boldsymbol{\partial\varphi(x,t)}}{\boldsymbol{\partial x}} = \boldsymbol{\beta\kappa_{xy}} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \mathbf{(30)_2}$$

Using equation (17)$_1$ for (1,0) micropolar beam theory we obtain,

$$\sigma_{xx} = \bar{E}\sum_{i=0}^{1} z^i \frac{\partial\psi_{(i)}}{\partial x}$$

$$\sigma_{xx} = \bar{E}\left[\frac{\partial\psi_{(0)}}{\partial x} + z\frac{\partial\psi_{(1)}}{\partial x}\right]$$

$$\sigma_{xx} = \bar{E}\left[\frac{\partial u(x,t)}{\partial x} + z\frac{\partial\psi(x,t)}{\partial x}\right]$$

$$\sigma_{xx} = \bar{E}\frac{\partial u(x,t)}{\partial x} + z\bar{E}\frac{\partial\psi(x,t)}{\partial x}$$

Comparing with equations (27)$_1$ and (27)$_2$, the above equation can be rewritten as,

$$\boldsymbol{\sigma_{xx}} = \boldsymbol{N_{xx}} + \boldsymbol{z}\frac{\boldsymbol{M}}{\boldsymbol{I}} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \mathbf{(31)}$$

The total amount of bending moment at the beam cross section, $\bar{M}$, can be obtained from the following equation,

$$\bar{M} = \iint_A \sigma_{xx} z dA + \iint_A m_{xy} dA$$

From the definitions of M and M_{xy},

$$\boldsymbol{\bar{M} = \iint_A \sigma_{xx} z dA + \iint_A m_{xy} dA = M + AM_{xy}} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \mathbf{(32)}$$

We can improve the micropolar beam theory by choosing more terms in the power series expansions present in equation (10). In fact, a proper micropolar beam theory should yield the following relation forσ_{xz},

$$\sigma_{xz} = \frac{1}{b}\frac{\partial}{\partial x}\left(\frac{MQ}{I}\right) + \sigma_{xz}^{\tau} + \sigma_{xz}^{\phi}$$

$$\sigma_{xz} = \frac{Q}{Ib}\frac{\partial M}{\partial x} + \frac{M}{Ib}\frac{\partial Q}{\partial x} + \frac{MQ}{b}\frac{\partial}{\partial x}\left(\frac{1}{I}\right) + \sigma_{xz}^{\tau} + \sigma_{xz}^{\phi}$$

$$\sigma_{xz} = \frac{VQ}{Ib} + \frac{M}{Ib}\frac{\partial Q}{\partial x} - \frac{MQ}{bI^2}\frac{dI}{dx} + \sigma_{xz}^{\tau} + \sigma_{xz}^{\phi} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (33)$$

where, $V = \frac{\partial M}{\partial x}$ is the shear force in the beam cross section

$Q = \int_z^{h^+} z dA$ is the first moment of area of the cross section

σ_{xz}^{τ} is the effect of shear loading on the lower and upper surfaces of the beam

σ_{xz}^{ϕ} is the part of transverse shear σ_{xz} resulting from the micro-rotation vector ϕ

Governing equations of motion

Let $\bar{u}_x = u$, $\bar{u}_y = v$and $\bar{u}_z = w$, then $\ddot{\bar{u}}_x = \ddot{u}$, $\ddot{\bar{u}}_y = \ddot{v}$and $\ddot{\bar{u}}_z = \ddot{w}$

Now substituting equation $(27)_1$ in equation (21) we get,

$$\frac{\partial}{\partial x}(AN_{xx}) + \tau + \rho A(\bar{f}_x - \ddot{\bar{u}}_x) = 0$$
$$\frac{\partial}{\partial x}\left(A\bar{E}\frac{\partial u(x,t)}{\partial x}\right) + \tau + \rho A(\bar{f}_x - \ddot{\bar{u}}_x) = 0$$
$$\boldsymbol{\frac{\partial}{\partial x}\left(\bar{E}A\frac{\partial u}{\partial x}\right) + \tau + \rho A(\bar{f}_x - \ddot{u}) = 0} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \boldsymbol{(34)}$$

Now substituting equation (28) in equation (22) with the aid of $\bar{\mu} = G - \frac{\eta}{2}$ we get,

$$\frac{\partial}{\partial x}(AN_{xz}) + p + \rho A(\bar{f}_z - \ddot{\bar{u}}_z) = 0$$
$$\frac{\partial}{\partial x}\left\{A\left[\bar{\mu}\frac{\partial w(x,t)}{\partial x} + (\bar{\mu}+\eta)\psi(x,t) - \eta\varphi(x,t)\right]\right\} + p + \rho A(\bar{f}_z - \ddot{\bar{u}}_z) = 0$$
$$\frac{\partial}{\partial x}\left\{A\left[\left(G-\frac{\eta}{2}\right)\frac{\partial w(x,t)}{\partial x} + \left(G+\frac{\eta}{2}\right)\psi(x,t) - \eta\varphi(x,t)\right]\right\} + p + \rho A(\bar{f}_z - \ddot{\bar{u}}_z) = 0$$
$$\boldsymbol{\frac{\partial}{\partial x}\left\{A\left[\left(G-\frac{\eta}{2}\right)\frac{\partial w}{\partial x} + \left(G+\frac{\eta}{2}\right)\psi - \eta\varphi\right]\right\} + p + \rho A(\bar{f}_z - \ddot{w}) = 0} \cdots\cdots\cdots\cdots\cdots\cdots \boldsymbol{(35)}$$

We know the definition,

$$\bar{\phi}_y = \frac{1}{A}\iint_A \phi_y dA$$

Using equation $(20)_1$ in the above definition,

$$\bar{\phi}_y = \frac{1}{A}\iint_A \varphi(x,t)dA$$
$$\bar{\phi}_y = \frac{1}{A}\varphi(x,t)\iint_A dA \qquad (\because dA = dydz)$$
$$\bar{\phi}_y = \varphi$$
$$\therefore \quad \ddot{\bar{\phi}}_y = \ddot{\varphi}$$

Now substituting equations (28), (29) and $(30)_2$ in equation (23) we get,

$$\frac{\partial}{\partial x}\left(AM_{xy}\right) + m + A(N_{zx} - N_{xz}) + \rho A\left(L_y - J\ddot{\bar{\phi}}_y\right) = 0$$

$$\frac{\partial}{\partial x}\left(A\beta\frac{\partial\varphi(x,t)}{\partial x}\right) + m$$
$$+ A\left(\left[(\bar{\mu}+\eta)\frac{\partial w(x,t)}{\partial x} + \bar{\mu}\psi(x,t) + \eta\varphi(x,t)\right]\right.$$
$$\left. - \left[\bar{\mu}\frac{\partial w(x,t)}{\partial x} + (\bar{\mu}+\eta)\psi(x,t) - \eta\varphi(x,t)\right]\right)$$
$$+\rho A\left(L_y - J\ddot{\bar{\phi}}_y\right) = 0$$

$$\frac{\partial}{\partial x}\left(A\beta\frac{\partial\varphi(x,t)}{\partial x}\right) + m + A\left[\eta\frac{\partial w(x,t)}{\partial x} - \eta\psi(x,t) + 2\eta\varphi(x,t)\right]$$
$$+ \rho A\left(L_y - J\ddot{\varphi}\right) = 0 \qquad (\because \bar{\phi}_y = \varphi)$$

$$\frac{\partial}{\partial x}\left(A\beta\frac{\partial\varphi}{\partial x}\right) + m + \eta A\left[\frac{\partial w}{\partial x} - \psi + 2\varphi\right] + \rho A\left(L_y - J\ddot{\varphi}\right)$$
$$= 0 \cdots\cdots\cdots\cdots\cdots\cdots (36)$$

We know the definition,

$$\tilde{u}_x = \iint_A u_x z dA$$

Using equation (19) in the above definition,

$$\tilde{u}_x = \iint_A [u(x,t) + z\psi(x,t)] z dA$$
$$\tilde{u}_x = \iint_A u(x,t) z dA + \iint_A \psi(x,t)\, z^2 dA$$
$$\tilde{u}_x = u(x,t)\iint_A z dA + \psi(x,t)\iint_A z^2 dA$$
$$\tilde{u}_x = 0 + \psi(x,t) I$$
$$\tilde{u}_x = I\psi$$

$$\therefore \quad \ddot{\tilde{u}}_x = I\ddot{\psi}$$

Now substituting equations $(27)_2$ and (29) in equation (25) with the aid of $\bar{\mu} = G - \frac{\eta}{2}$ we get,

$$\frac{\partial M}{\partial x} + h\bar{\tau} - AN_{zx} + \rho(\tilde{f}_x - \ddot{u}_x) = 0$$

$$\frac{\partial}{\partial x}\left[\bar{E}I\frac{\partial\psi(x,t)}{\partial x}\right] + h\bar{\tau} - A\left[(\bar{\mu}+\eta)\frac{\partial w(x,t)}{\partial x} + \bar{\mu}\psi(x,t) + \eta\varphi(x,t)\right] + \rho(\tilde{f}_x - \ddot{u}_x) = 0$$

$$\frac{\partial}{\partial x}\left[\bar{E}I\frac{\partial\psi(x,t)}{\partial x}\right] + h\bar{\tau} - A\left[\left(G+\frac{\eta}{2}\right)\frac{\partial w(x,t)}{\partial x} + \left(G-\frac{\eta}{2}\right)\psi(x,t) + \eta\varphi(x,t)\right] + \rho(\tilde{f}_x - \ddot{u}_x) = 0$$

$$\frac{\partial}{\partial x}\left[\bar{E}I\frac{\partial\psi}{\partial x}\right] + h\bar{\tau} - A\left[\left(G+\frac{\eta}{2}\right)\frac{\partial w}{\partial x} + \left(G-\frac{\eta}{2}\right)\psi + \eta\varphi\right] + \rho(\tilde{f}_x - I\ddot{\psi}) = 0 \cdots\cdots\cdots\cdots (37)$$

State space equations of motion

In case of static problems (all time derivatives will go to zero) and constant material and geometric properties, we can implement the state space method to solve the differential equations (35), (36) and (37).

Let the state variables be,

$$\psi = X_1, \qquad \frac{d\psi}{dx} = X_{2,} \qquad \varphi = X_{3,} \qquad \frac{d\varphi}{dx} = X_{4,} \qquad \frac{dw}{dx} = X_5 \cdots\cdots\cdots\cdots (38)$$

From observation we know that,

$$\boldsymbol{X_1' = X_2} \cdots\cdots\cdots\cdots \boldsymbol{(39)_1}$$

Using the defined state variables in equation (37) we get,

$$\bar{E}I\frac{d^2\psi}{dx^2} + h\bar{\tau} - A\left[\left(G+\frac{\eta}{2}\right)\frac{dw}{dx} + \left(G-\frac{\eta}{2}\right)\psi + \eta\varphi\right] + \rho\tilde{f}_x = 0$$

$$\bar{E}IX_2' + h\bar{\tau} - A\left[\left(G+\frac{\eta}{2}\right)X_5 + \left(G-\frac{\eta}{2}\right)X_1 + \eta X_3\right] + \rho\tilde{f}_x = 0$$

$$X_2' = \frac{1}{\bar{E}I}\left\{A\left[\left(G+\frac{\eta}{2}\right)X_5 + \left(G-\frac{\eta}{2}\right)X_1 + \eta X_3\right] - h\bar{\tau} - \rho\tilde{f}\right\} \cdots\cdots\cdots\cdots (39)_2$$

From observation we know that,

$$\boldsymbol{X_3' = X_4} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \boldsymbol{(40)_1}$$

Using the defined state variables in equation (36) we get,

$$A\beta \frac{d^2\varphi}{dx^2} + m + \eta A\left[\frac{dw}{dx} - \psi + 2\varphi\right] + \rho A L_y = 0$$
$$\beta \frac{d^2\varphi}{dx^2} + \frac{m}{A} + \eta\left[\frac{dw}{dx} - \psi + 2\varphi\right] + \rho L_y = 0$$
$$\beta X_4' + \frac{m}{A} + \eta[X_5 - X_1 + 2X_3] + \rho L_y = 0$$
$$\boldsymbol{X_4' = \frac{1}{\beta}\left\{\eta[X_1 - 2X_3 - X_5] - \frac{m}{A} - \rho L_y\right\}} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \boldsymbol{(40)_2}$$

Using the defined state variables in equation (35) we get,

$$A\left[\left(G - \frac{\eta}{2}\right)\frac{d^2w}{dx^2} + \left(G + \frac{\eta}{2}\right)\frac{d\psi}{dx} - \eta\frac{d\varphi}{dx}\right] + p + \rho A\bar{f}_z = 0$$
$$\left(G - \frac{\eta}{2}\right)\frac{d^2w}{dx^2} + \left(G + \frac{\eta}{2}\right)\frac{d\psi}{dx} - \eta\frac{d\varphi}{dx} + \frac{p}{A} + \rho\bar{f}_z = 0$$
$$\left(G - \frac{\eta}{2}\right)X_5' + \left(G + \frac{\eta}{2}\right)X_2 - \eta X_4 + \frac{p}{A} + \rho\bar{f}_z = 0$$
$$X_5' = \frac{1}{\left(G - \frac{\eta}{2}\right)}\left[\eta X_4 - \left(G + \frac{\eta}{2}\right)X_2 - \frac{p}{A} - \rho\bar{f}_z\right] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (41)$$

The system of equations $(39)_1$, $(39)_2$, $(40)_1$, $(40)_2$, (41) can be written in matrix form as,

$$\begin{Bmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \\ X_5' \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{A\left(G - \eta/_2\right)}{\bar{E}I} & 0 & \frac{A\eta}{\bar{E}I} & 0 & \frac{A\left(G + \eta/_2\right)}{\bar{E}I} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{\eta}{\beta} & 0 & \frac{-2\eta}{\beta} & 0 & \frac{-\eta}{\beta} \\ 0 & \frac{-\left(G + \eta/_2\right)}{G - \eta/_2} & 0 & \frac{\eta}{G - \eta/_2} & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\frac{h\bar{\tau} + \rho\tilde{f}_x}{\bar{E}I} \\ 0 \\ \frac{-1}{\beta}\left(\frac{m}{A} + \rho L_y\right) \\ \frac{-1}{G - \eta/_2}\left[\frac{p}{A} + \rho\tilde{f}_z\right] \end{Bmatrix}$$

The above set of equations can be represented as $X' = AX + B$

where, $X = \{X_1\ X_2\ X_3\ X_4\ X_5\}^T$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{A\left(G - \eta/_2\right)}{\bar{E}I} & 0 & \frac{A\eta}{\bar{E}I} & 0 & \frac{A\left(G + \eta/_2\right)}{\bar{E}I} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{\eta}{\beta} & 0 & \frac{-2\eta}{\beta} & 0 & \frac{-\eta}{\beta} \\ 0 & \frac{-\left(G + \eta/_2\right)}{G - \eta/_2} & 0 & \frac{\eta}{G - \eta/_2} & 0 \end{bmatrix} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (42)$$

$$B = \left\{0 \quad -\frac{h\bar{\tau} + \rho\tilde{f}_x}{\bar{E}I} \quad 0 \quad \frac{-1}{\beta}\left(\frac{m}{A} + \rho L_y\right) \quad \frac{-1}{G - \eta/_2}\left[\frac{p}{A} + \rho\tilde{f}_z\right]\right\}^T \cdots\cdots\cdots\cdots\cdots\cdots (43)$$

Eigen values of **A** are as

$$\lambda_i \;=\; 0,0,0,\pm i\xi \qquad\qquad where \qquad \xi = \sqrt{\frac{\eta G(EI \;\;+\; \boldsymbol{A}\beta)}{EI\beta(G-\eta/2)}}$$

Let,

$$\boldsymbol{X} = C_0\boldsymbol{V_0} + C_1\boldsymbol{V_1}x + C_2\boldsymbol{V_2}x^2 + C_3\boldsymbol{V_3}e^{i\xi x} + C_4\boldsymbol{V_4}e^{-i\xi x}$$

$$d\boldsymbol{X}/dx = 0 + C_1\boldsymbol{V_1} + 2C_2\boldsymbol{V_2}x + i\xi C_3\boldsymbol{V_3}e^{i\xi x} - i\xi C_4\boldsymbol{V_4}e^{-i\xi x}$$

$d\boldsymbol{X}/dx = C_1\boldsymbol{V_1} + 2C_2\boldsymbol{V_2}x + i\xi C_3\boldsymbol{V_3}[\cos(\xi x) + isin(\xi x)] - i\xi C_4\boldsymbol{V_4}[cos(\xi x) - isin(\xi x)]$

For homogenous solution of equation, **X'=AX**

$$\begin{aligned} C_1\boldsymbol{V}_1 + 2C_2\boldsymbol{V_2}x + i\xi C_3\boldsymbol{V_3}[\cos(\xi x) + isin(\xi x)] & \\ - i\xi C_4\boldsymbol{V_4}[cos(\xi x) - isin(\xi x)] & \\ = \boldsymbol{A}[C_0\boldsymbol{V_0} + C_1\boldsymbol{V_1}x + C_2\boldsymbol{V}_2x^2 + cos\xi x(C_3\boldsymbol{V_3} + C_4\boldsymbol{V_4}) & \\ + sin\xi x(C_3\boldsymbol{V_3} - C_4\boldsymbol{V_4})] & \end{aligned}$$

By comparing both sides,

$C_1\boldsymbol{V_1} = \mathrm{A}C_0\boldsymbol{V_0}$, $2C_2\boldsymbol{V_2} = \boldsymbol{A}C_1\boldsymbol{V_1} = \boldsymbol{A^2}C_0\boldsymbol{V_0}$, $\boldsymbol{A}C_2\boldsymbol{V_2} = 0$ implies $\boldsymbol{A}^3C_0\boldsymbol{V_0} = 0$

$$i\xi C_3\boldsymbol{V_3} - i\xi C_4\boldsymbol{V_4} = A(C_3\boldsymbol{V_3} + C_4\boldsymbol{V_4})$$

$$-\xi C_3\boldsymbol{V}_3 - \xi C_4\boldsymbol{V}_4 = A(iC_3\boldsymbol{V_3} - iC_4\boldsymbol{V_4})$$

$\rightarrow \boldsymbol{A}C_3\boldsymbol{V_3} = i\xi C_3\boldsymbol{V_3}$ and $\boldsymbol{A}C_4\boldsymbol{V_4} = -i\xi C_4\boldsymbol{V_4}$

$$X_0 = C_0\boldsymbol{V_0} + C_3\boldsymbol{V_3} + C_4\boldsymbol{V_4}$$

From $\boldsymbol{A^3}C_0\boldsymbol{V_0} = 0$

$$\boldsymbol{A^3X_0} = \boldsymbol{\theta}^3C_3\boldsymbol{V_3} + \boldsymbol{A}^3C_4\boldsymbol{V_4}$$

By putting values,

$$\boldsymbol{A^3}X_0 = -i\xi^3C_3\boldsymbol{V_3} + i\xi^3C_4\boldsymbol{V_4}$$

$$-\frac{\boldsymbol{A^3X_0}}{i\xi^3} = C_3\boldsymbol{V_3} - C_4\boldsymbol{V_4} \qquad \text{---(1)}$$

$$\boldsymbol{A^4X_0} = (i\xi)^4 C_3\boldsymbol{V_3} + (-i\xi)^4 C_4\boldsymbol{V_4}$$

$$\frac{\boldsymbol{A^4}X_0}{\xi^4} = C_3\boldsymbol{V_3} + C_4\boldsymbol{V_4} \qquad \text{---(2)}$$

From (1) and (2)

$$C_3\boldsymbol{V_3} = \frac{1}{2}\left[\frac{-\boldsymbol{A^3X_0}}{i\xi^3} + \frac{\boldsymbol{A^4X_0}}{\xi^4}\right]$$

And

$$C_4\boldsymbol{V_4} = \frac{1}{2}\left[\frac{\boldsymbol{A^3X_0}}{i\xi^3} + \frac{\boldsymbol{A^4X_0}}{\xi^4}\right]$$

From before,

$$\boldsymbol{X_0} = C_0\boldsymbol{V_0} + C_3\boldsymbol{V_3} + C_4\boldsymbol{V_4}$$

$$C_0\boldsymbol{V_0} = \boldsymbol{X_0} - \frac{\boldsymbol{A^4X_0}}{\xi^4}$$

$$\therefore C_1\boldsymbol{V_1} = \boldsymbol{AX_0} - \frac{\boldsymbol{A^5X_0}}{\xi^4} = \boldsymbol{A}C_0\boldsymbol{V_0}$$

$$C_2\boldsymbol{V_2} = \frac{1}{2}\boldsymbol{A}^2C_0\boldsymbol{V_0} = \frac{1}{2}\boldsymbol{A^2X_0} - \frac{\boldsymbol{A^6X_0}}{2\xi^4}$$

$$\boldsymbol{A}C_2\boldsymbol{V_2} = 0 \qquad \rightarrow \qquad \boldsymbol{A}\left[\frac{1}{2}\boldsymbol{A^2X_0} - \frac{\boldsymbol{A^6X_0}}{2\xi^4}\right] = 0$$

$$\boldsymbol{A^3X_0} = \frac{\boldsymbol{A^7X_0}}{\xi^4}$$

$$\frac{\boldsymbol{A^5X_0}}{\xi^4} = i\xi C_3\boldsymbol{V_3} - i\xi C_4\boldsymbol{V_4} \qquad \rightarrow \qquad \frac{\boldsymbol{A^5X_0}}{i\xi^5} = C_3\boldsymbol{V_3} - C_4\boldsymbol{V_4}$$

$$-\frac{\boldsymbol{A^3X_0}}{i\xi^3} = \frac{\boldsymbol{A^5X_0}}{i\xi^5} \qquad \rightarrow \quad -\xi\,(\boldsymbol{A^3X_0}) = \frac{\boldsymbol{A^5X_0}}{\xi^4} = \frac{-\boldsymbol{A^3X_0}}{\xi^2}$$

$$\frac{-\boldsymbol{A^5X_0}}{\xi^4} = \frac{\boldsymbol{A^3X_0}}{\xi^2}$$

From(2),

$$\frac{\boldsymbol{A^4X_0}}{\xi^4} = C_3\boldsymbol{V_3} + C_4\boldsymbol{V_4} \qquad \text{----(3)}$$

$$\frac{\boldsymbol{A^5X_0}}{\xi^4} = i\xi C_3\boldsymbol{V_3} - i\xi C_4\boldsymbol{V_4}$$

$$\frac{\boldsymbol{A^6X_0}}{\xi^4} = -\xi^2 C_3\boldsymbol{V_3} - \xi^2 C_4\boldsymbol{V_4}$$

$$\rightarrow \qquad \frac{\boldsymbol{A^6X_0}}{\xi^6} = C_3\boldsymbol{V_3} + C_4\boldsymbol{V_4} \qquad \text{-----(4)}$$

From(3) and (4)

$$\frac{\boldsymbol{A^4X_0}}{\xi^4} = \frac{\boldsymbol{A^6X_0}}{\xi^6} \qquad \therefore \frac{\boldsymbol{A^6X_0}}{\xi^4} = \frac{\boldsymbol{A^4X_0}}{\xi^2}$$

Similarly,

$$C_1\boldsymbol{V_1} = \boldsymbol{AX_0} + \frac{\boldsymbol{A^3X_0}}{\xi^2} \qquad and \qquad C_2\boldsymbol{V_2} = \frac{1}{2}\boldsymbol{A^2X_0} + \frac{\boldsymbol{A^4X_0}}{2\xi^2}$$

$$\therefore \boldsymbol{X_h} = \left[\boldsymbol{X}_0 - \frac{\boldsymbol{A^4X_0}}{\xi^4}\right] + \left[\boldsymbol{AX_0} + \frac{\boldsymbol{A^3X_0}}{\xi^2}\right]x + \left[\frac{1}{2}\boldsymbol{A^2X_0} + \frac{\boldsymbol{A^4X_0}}{2\xi^2}\right]x^2 + \frac{\boldsymbol{A^4X_0}}{\xi^4}\cos\xi x - \frac{\boldsymbol{A^3X_0}}{\xi^3}\sin\xi x$$

$$\boldsymbol{X_h} = \boldsymbol{X_0} + x\boldsymbol{AX_0} + \frac{x^2\boldsymbol{A^2X_0}}{2} + \left[\frac{x}{\xi^2} - \frac{sin\,\xi x}{\xi^3}\right]\boldsymbol{A^3X_0} + [\frac{-1}{\xi^4} + \frac{x^2}{2\xi^2} + \frac{cos\,\xi x}{\xi^4}]\boldsymbol{A^4X_0}$$

$$\boldsymbol{X}_h = \left\{\boldsymbol{I} + x\boldsymbol{A} + \frac{x^2}{2}\boldsymbol{A^2} + \left[\frac{\xi x - sin\,\xi x}{\xi^3}\right]\boldsymbol{A^3} + \frac{1}{\xi^4}\left[cos\,\xi x - 1 + \frac{1}{2}\xi^2x^2\right]\boldsymbol{A^4}\right\}\boldsymbol{X_0}$$

where $\mathbf{X_0}$ is the value of **X** at x = 0 and are determined by the

boundary conditions of the problem.

$$\dot{\overline{\boldsymbol{X}}} = \overline{\boldsymbol{A}}\boldsymbol{X} + \overline{\boldsymbol{B}}\overline{\boldsymbol{u}}(\boldsymbol{t})$$

$$\boldsymbol{e}^{-\boldsymbol{A}t}\dot{\boldsymbol{X}}(t) - \boldsymbol{e}^{-\boldsymbol{A}t}\boldsymbol{X}(t) = \frac{d\left(\boldsymbol{e}^{-\boldsymbol{A}t}\boldsymbol{X}(t)\right)}{dt} - \boldsymbol{e}^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

Integrating on both sides,

$$\int_0^t \frac{d}{d\tau}\left(\boldsymbol{e}^{-\boldsymbol{A}\tau}\boldsymbol{X}(\tau)\right) d\tau = \int_0^\tau \boldsymbol{B}\boldsymbol{u}(\tau)\, e^{-\boldsymbol{A}\tau} d\tau$$

$$= e^{-\boldsymbol{A}t}\boldsymbol{X}(t) - e^{-\boldsymbol{A}0}\boldsymbol{X}(0)$$

$$\overline{\boldsymbol{X}}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}(0) + e^{At}\int_0^\tau Bu(\tau)\, e^{-A\tau} d\tau$$

$$\overline{\boldsymbol{X}}(t) = e^{\boldsymbol{A}t}\boldsymbol{X}(0) + \int_0^t \boldsymbol{B}\boldsymbol{u}(\tau)\, e^{\boldsymbol{A}(t-\tau)} d\tau$$

From previous we got,

$$\boldsymbol{X_h} = \left\{\boldsymbol{I} + x\boldsymbol{A} + \frac{x^2\boldsymbol{X_0}}{2}\boldsymbol{A}^2 + \left[\frac{\xi x - \sin\xi x}{\xi^3}\right]\boldsymbol{A}^3 + \frac{1}{\xi^4}\left[\cos\xi x - 1 + \frac{1}{2}\xi^2 x^2\right]\boldsymbol{A}^4\right\}\boldsymbol{X_0}$$

By comparing these two equations,

$$e^{\boldsymbol{A}\boldsymbol{t}} = \boldsymbol{I} + x\boldsymbol{A} + \frac{x^2}{2}\boldsymbol{A}^2 + \left[\frac{\xi x - \sin\xi x}{\xi^3}\right]\boldsymbol{A}^3 + \frac{1}{\xi^4}\left[\cos\xi x - 1 + \frac{1}{2}\xi^2 x^2\right]\boldsymbol{A}^4$$

∴for particular solution,

$x_p = \int_0^x \boldsymbol{B}\boldsymbol{u}(\tau)\, e^{\boldsymbol{A}(x-\tau)} d\tau$ here $u(t) = 1$

$x_p = \left[\int_0^x e^{\boldsymbol{A}(t-\tau)} d\tau\right]\boldsymbol{B}$ ----($\boldsymbol{B}$ $is\ constant$)

=

$$\left\{\int_0^x \left(\boldsymbol{I} + (x-\tau)\boldsymbol{A} + \frac{(x-\tau)^2}{2}\boldsymbol{A^2} + \left[\frac{\xi(x-\tau) - sin\,\xi(x-\tau)}{\xi^3}\right]\boldsymbol{A^3} \right.\right.$$

$$\left.\left. + \frac{1}{\xi^4}\left[cos\,\xi(x-\tau) - 1 + \frac{1}{2}\xi^2(x-\tau)^2\right]\boldsymbol{A^4}\right) d\,\tau\right\}\boldsymbol{B}$$

Let $\quad x - \tau = t \quad \rightarrow \quad -d\tau = dt$

=

$$\int_0^x \left(\boldsymbol{I} + t\boldsymbol{A} + \frac{t^2}{2}\boldsymbol{A}^2 + \left[\frac{\xi t - sin\,\xi t}{\xi^3}\right]\boldsymbol{A^3} + \frac{1}{\xi^4}\left[cos\,\xi t - 1 + \frac{1}{2}\xi^2 t^2\right]\boldsymbol{A^4}\right) dt$$

$$\therefore x_p = \left[\boldsymbol{I}x + \frac{x^2}{2}\boldsymbol{A} + \frac{x^3}{6}\boldsymbol{A^2} + \left\{\frac{x^2}{2\xi^2} - \left(\frac{1 - cos\,\xi x}{\xi^4}\right)\right\}\boldsymbol{A^3} \right.$$

$$\left. + \left\{\frac{sin\,\xi x}{\xi^5} - \frac{x}{\xi^4} + \frac{x^3}{6\xi^4}\right\}\boldsymbol{A^4}\right]\boldsymbol{B}$$

$$\therefore \boldsymbol{X} = \left\{\boldsymbol{I} + x\boldsymbol{A} + \frac{x^2\boldsymbol{X_0}}{2}\boldsymbol{A^2} + \left[\frac{\xi x - sin\,\xi x}{\xi^3}\right]\boldsymbol{A}^3 \right.$$

$$\left. + \frac{1}{\xi^4}\left[cos\,\xi x - 1 + \frac{1}{2}\xi^2 x^2\right]\boldsymbol{A^4}\right\}\boldsymbol{X_0}$$

$$+ \left[\boldsymbol{I}x + \frac{x^2}{2}\boldsymbol{A} + \frac{x^3}{6}\boldsymbol{A^2} + \left\{\frac{x^2}{2\xi^2} - \left(\frac{1 - cos\,\xi x}{\xi^4}\right)\right\}\boldsymbol{A^3} \right.$$

$$\left. + \left\{\frac{sin\,\xi x}{\xi^5} - \frac{x}{\xi^4} + \frac{x^3}{6\xi^4}\right\}\boldsymbol{A^4}\right]\boldsymbol{B}$$

Speed of flexural and longitudinal waves

Here we examine the nature of flexural and longitudinal waves in an infinite length micropolar beam.

We define the functions $\Psi$and Φ as follows,

$$\Psi = \frac{\partial \psi}{\partial x}, \qquad \Phi = \frac{\partial \phi}{\partial x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

Now for harmonic waves, one can write,

$$\{\boldsymbol{w(x,t)} \quad \boldsymbol{\Psi(x,t)} \quad \boldsymbol{\Phi(x,t)}\}^{\boldsymbol{T}} = \{\boldsymbol{W_0} \quad \boldsymbol{\Psi_0} \quad \boldsymbol{\Phi_0}\}^{\boldsymbol{T}} \boldsymbol{e^{i(\omega t+kx)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \boldsymbol{(47)}$$

$$\begin{Bmatrix} w(x,t) \\ \Psi(x,t) \\ \Phi(x,t) \end{Bmatrix} = e^{i(\omega t+kx)} \begin{Bmatrix} W_0 \\ \Psi_0 \\ \Phi_0 \end{Bmatrix}$$

where, ω is the natural frequency of the wave

$k = \frac{2\pi}{\bar{\lambda}}$ is the wave number

$\bar{\lambda}$ is the wave length and

$W_0,\ \Psi_0,\ \Phi_0$ are the constants

Let, $\tau = p = m = \tilde{f}_x = \tilde{f}_z = 0$

Substituting the definitions in equations (46) and (47) into equation (35) we obtain,

$$\frac{\partial}{\partial x}\left\{A\left[\left(G-\frac{\eta}{2}\right)\frac{\partial w}{\partial x} + \left(G+\frac{\eta}{2}\right)\frac{\partial \Psi}{\partial x} - \eta\frac{\partial \Phi}{\partial x}\right]\right\} - \rho A\ddot{w} = 0$$

$$\frac{\partial}{\partial x}\left[\left(G-\frac{\eta}{2}\right)\frac{\partial w}{\partial x} + \left(G+\frac{\eta}{2}\right)\frac{\partial \Psi}{\partial x} - \eta\frac{\partial \Phi}{\partial x}\right] - \rho\ddot{w} = 0$$

$$\left(G-\frac{\eta}{2}\right)\frac{\partial^2 w}{\partial x^2} + \left(G+\frac{\eta}{2}\right)\frac{\partial^2 \Psi}{\partial x^2} - \eta\frac{\partial^2 \Phi}{\partial x^2} - \rho\frac{\partial^2 w}{\partial t^2} = 0$$

$$\left(G-\frac{\eta}{2}\right)W_0 k^2 e^{i(\omega t+kx)} + \left(G+\frac{\eta}{2}\right)\Psi_0 k^2 e^{i(\omega t+kx)} - \eta\Phi_0 k^2 e^{i(\omega t+kx)} - \rho W_0\omega^2 e^{i(\omega t+kx)} = 0$$

$$\left(G-\frac{\eta}{2}\right)W_0k^2+\left(G+\frac{\eta}{2}\right)\Psi_0k^2-\eta\Phi_0k^2-\rho W_0\omega^2=0$$
$$\left[\left(G-\frac{\eta}{2}\right)k^2-\rho\omega^2\right]W_0+\left(G+\frac{\eta}{2}\right)k^2\Psi_0-\eta k^2\Phi_0=0$$

Substituting the definitions in equations (46) and (47) into equation (36) we obtain,

$$\frac{\partial}{\partial x}\left(A\beta\frac{\partial^2\Phi}{\partial x^2}\right)+\eta A\left[\frac{\partial w}{\partial x}-\frac{\partial\Psi}{\partial x}+2\frac{\partial\Phi}{\partial x}\right]-\rho AJ\ddot{\varphi}=0$$
$$A\beta\frac{\partial^3\Phi}{\partial x^3}+\eta A\left[\frac{\partial w}{\partial x}-\frac{\partial\Psi}{\partial x}+2\frac{\partial\Phi}{\partial x}\right]-\rho AJ\frac{\partial^2\varphi}{\partial t^2}=0$$
$$\beta\frac{\partial^3\Phi}{\partial x^3}+\eta\left[\frac{\partial w}{\partial x}-\frac{\partial\Psi}{\partial x}+2\frac{\partial\Phi}{\partial x}\right]-\rho J\frac{\partial^2}{\partial t^2}\left(\frac{\partial\Phi}{\partial x}\right)=0$$
$$\beta\Phi_0k^3e^{i(\omega t+kx)}+\eta\left[W_0ke^{i(\omega t+kx)}-\Psi_0ke^{i(\omega t+kx)}+2\Phi_0ke^{i(\omega t+kx)}\right]-\rho J\Phi_0k\omega^2e^{i(\omega t+kx)}=0$$
$$\beta\Phi_0k^3+\eta[W_0k-\Psi_0k+2\Phi_0k]-\rho J\Phi_0k\omega^2=0$$
$$\beta\Phi_0k^2+\eta[W_0-\Psi_0+2\Phi_0]-\rho J\Phi_0\omega^2=0$$
$$\eta W_0-\eta\Psi_0+(2\eta+\beta k^2-\rho J\omega^2)\Phi_0=0$$

Substituting the definitions in equations (46) and (47) into equation (37) we obtain,

$$\frac{\partial}{\partial x}\left[\bar{E}I\frac{\partial^2\Psi}{\partial x^2}\right]-A\left[\left(G+\frac{\eta}{2}\right)\frac{\partial w}{\partial x}+\left(G-\frac{\eta}{2}\right)\frac{\partial\Psi}{\partial x}+\eta\frac{\partial\Phi}{\partial x}\right]-\rho I\ddot{\psi}=0$$
$$\bar{E}I\frac{\partial^3\Psi}{\partial x^3}-A\left[\left(G+\frac{\eta}{2}\right)\frac{\partial w}{\partial x}+\left(G-\frac{\eta}{2}\right)\frac{\partial\Psi}{\partial x}+\eta\frac{\partial\Phi}{\partial x}\right]-\rho I\frac{\partial^2\psi}{\partial t^2}=0$$
$$\bar{E}I\frac{\partial^3\Psi}{\partial x^3}-A\left[\left(G+\frac{\eta}{2}\right)\frac{\partial w}{\partial x}+\left(G-\frac{\eta}{2}\right)\frac{\partial\Psi}{\partial x}+\eta\frac{\partial\Phi}{\partial x}\right]-\rho I\frac{\partial^2}{\partial t^2}\left(\frac{\partial\Psi}{\partial x}\right)=0$$
$$\bar{E}I\Psi_0k^3e^{i(\omega t+kx)}-A\left[\left(G+\frac{\eta}{2}\right)W_0ke^{i(\omega t+kx)}+\left(G-\frac{\eta}{2}\right)\Psi_0ke^{i(\omega t+kx)}+\eta\Phi_0ke^{i(\omega t+kx)}\right]-\rho I\Psi_0k\omega^2e^{i(\omega t+kx)}=0$$
$$\bar{E}I\Psi_0k^3-A\left[\left(G+\frac{\eta}{2}\right)W_0k+\left(G-\frac{\eta}{2}\right)\Psi_0k+\eta\Phi_0k\right]-\rho I\Psi_0k\omega^2=0$$
$$\bar{E}I\Psi_0k^2-A\left[\left(G+\frac{\eta}{2}\right)W_0+\left(G-\frac{\eta}{2}\right)\Psi_0+\eta\Phi_0\right]-\rho I\Psi_0\omega^2=0$$
$$-A\left(G+\frac{\eta}{2}\right)W_0-\left[\rho I\omega^2+A\left(G-\frac{\eta}{2}\right)-\bar{E}Ik^2\right]\Psi_0-A\eta\Phi_0=0$$

The above equations can be written in matrix form as,

$$\begin{bmatrix} \left(G-\frac{\eta}{2}\right)k^2-\rho\omega^2 & \left(G+\frac{\eta}{2}\right)k^2 & -\eta k^2 \\ -A\left(G+\frac{\eta}{2}\right) & -\left[\rho I\omega^2+A\left(G-\frac{\eta}{2}\right)-\bar{E}Ik^2\right] & -A\eta \\ \eta & -\eta & 2\eta+\beta k^2-\rho J\omega^2 \end{bmatrix} \begin{Bmatrix} W_0 \\ \Psi_0 \\ \Phi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The above matrix form can be represented as, $H\ \{W_0 \quad \Psi_0 \quad \Phi_0\}^T = \{0 \quad 0 \quad 0\}^T \cdots\cdots\cdots (48)$

where,

$$H = \begin{bmatrix} \left(G-\frac{\eta}{2}\right)k^2-\rho\omega^2 & \left(G+\frac{\eta}{2}\right)k^2 & -\eta k^2 \\ -A\left(G+\frac{\eta}{2}\right) & -\left[\rho I\omega^2+A\left(G-\frac{\eta}{2}\right)-\bar{E}Ik^2\right] & -A\eta \\ \eta & -\eta & 2\eta+\beta k^2-\rho J\omega^2 \end{bmatrix}$$

For existence of a non-trivial solution (W0,Φ0,Ψ0), it is necessary for the determinant of the coefficient matrix H to vanish. This leads to an algebraic equation of six degree for the wave speed.

C = ω/k. If we neglect rotary inertia effect (ρI → 0), we get the following biquadratic equation

$$\alpha_1 C^4 - 2\alpha_2 C^2 + \alpha_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

where using Λ = 2hk, the following definitions have been used

$$\alpha_1 = \frac{\bar{E}IJ\rho^2}{2Ah(G+\eta/2)}\Lambda^4 + 2hJ\rho^2\Lambda^2$$

$$\alpha_2 = \frac{\bar{E}I\rho[J(G-\eta/2)+\beta]}{4Ah(G+\eta/2)}\Lambda^4 - \frac{\rho h[A(G+\eta/2)(J\eta-\beta)+\bar{E}\ln]}{A(G+\eta/2)}\Lambda^2 - \frac{4\rho\eta h^3(G+2\eta/3)}{G+\eta/2}$$

$$\alpha_3 = \frac{\bar{E}I\beta(G-\eta/2)}{2Ah(G+\eta/2)}\Lambda^4 - \frac{2h\eta(\bar{E}I+A\beta)}{A}\Lambda^2 - \frac{24h^3\eta^2(G+\eta/6)}{G+\eta/2}$$

The roots of the equation (48) are given by,

$$C_{1,2}^2(\Lambda) = \frac{\alpha_2 \mp \sqrt{\alpha_2^2 - \alpha_1\alpha_3}}{\alpha_1}$$

which shows that the waves corresponding to the above wave velocities are dispersive, since they depend on the wave frequencies. When the micro-inertia J vanishes, we have $\alpha 1 = 0$ and hence

$C^2(\Lambda) = \alpha 3/(2\alpha 2)$.

Assuming that J, or equivalently $\alpha 1\alpha 3/\alpha_2^2$ is small, by using the first order Taylor series approximation, we can approximate Eqs.(48) by,

$C_1^2(\Lambda) = \frac{\alpha_3}{\alpha_2}$ $\qquad$ $C_2^2(\Lambda) = \frac{2\alpha_2}{\alpha_1} - \frac{\alpha_3}{2\alpha_2}$

The flexural wave speeds C1(Λ) and C2(Λ) can be plotted against the non-dimensional parameter Λ.

Cantilever Beam

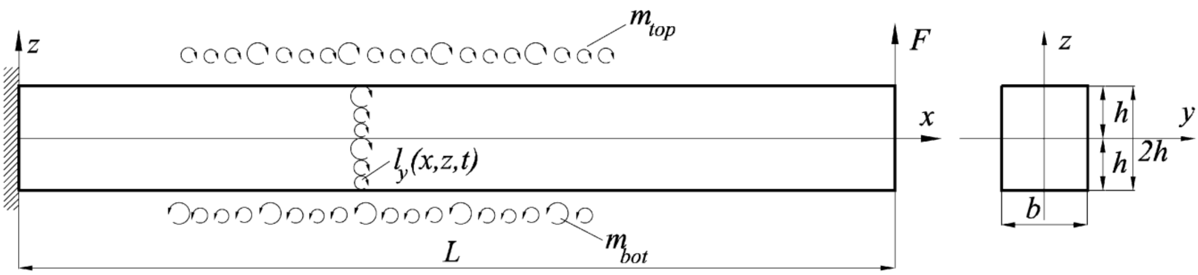


Fig. 3. Geometry and loading of the cantilever beam.

Consider a beam with following geometric and material properties: L = 1 m, h = 0.1 m, b = 0.05 m, E = 20 GPa, ν = 0.3, η = G/20, β = G/5000 (magnitude) and ρ = 2000 kg/m3. Let the beam be fixed at one end and subjected to a transverse concentrated load F in the positive z-direction at the other end. Other loading parameters are $L_y = \iint_A l_y dA/A$ and m = $m_{top} - m_{bottom}$ distributed along the beam as shown in the previous slide.

By assuming u(0) = u(L) = 0, from Eq. (34) we obtain u(x) = 0. The boundary conditions for the micropolar beam are as follows

@x = 0: ψ = φ = w = 0 ⇒ $X_1 = X_3 = X_5 = 0$

@x = L:

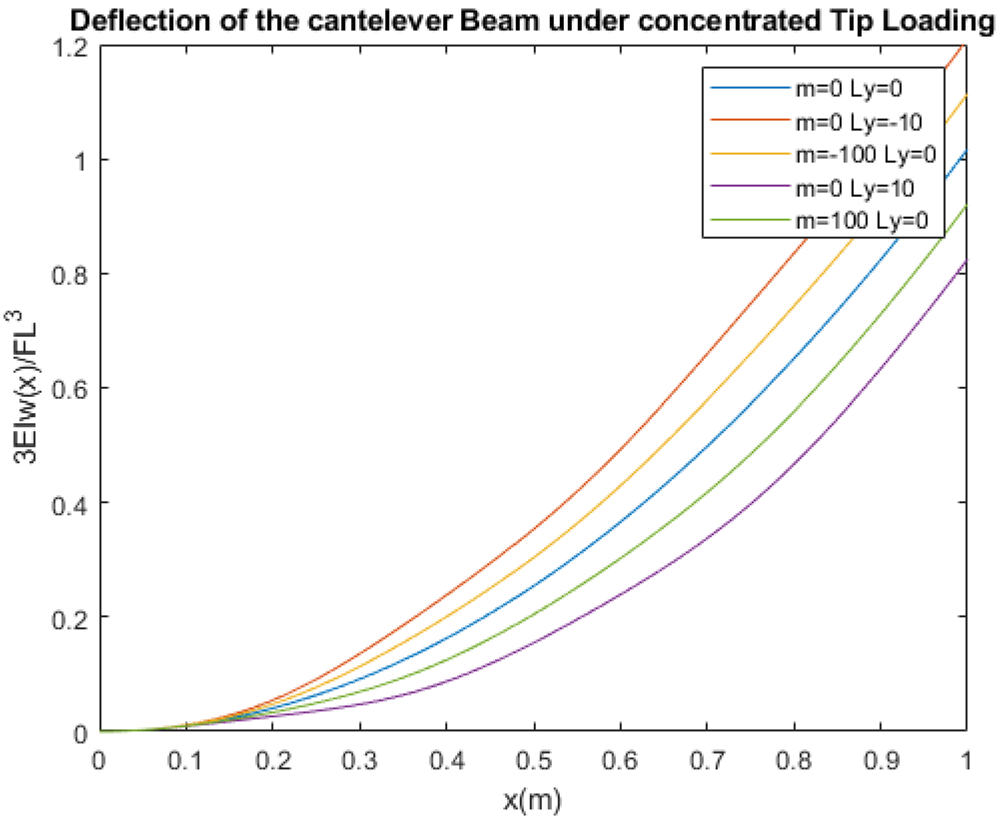
$$\begin{cases} b\int_{-h}^{h} \tau_{xz}\, dz = F \;\Rightarrow\; 2bh[\mu(X_1 + X_6) + \eta(X_1 - X_3)] = F \\ M = m_{xy} = 0 \;\Rightarrow\; X_2 = X_4 = 0. \end{cases}$$

MATLAB CODE AND RESULTS

```
x = linspace(0,1,1000);
F=1000;
E = 20*10^9;
b=0.05;h=0.1;
I = 8*b*h^3/12;
```

Deflection

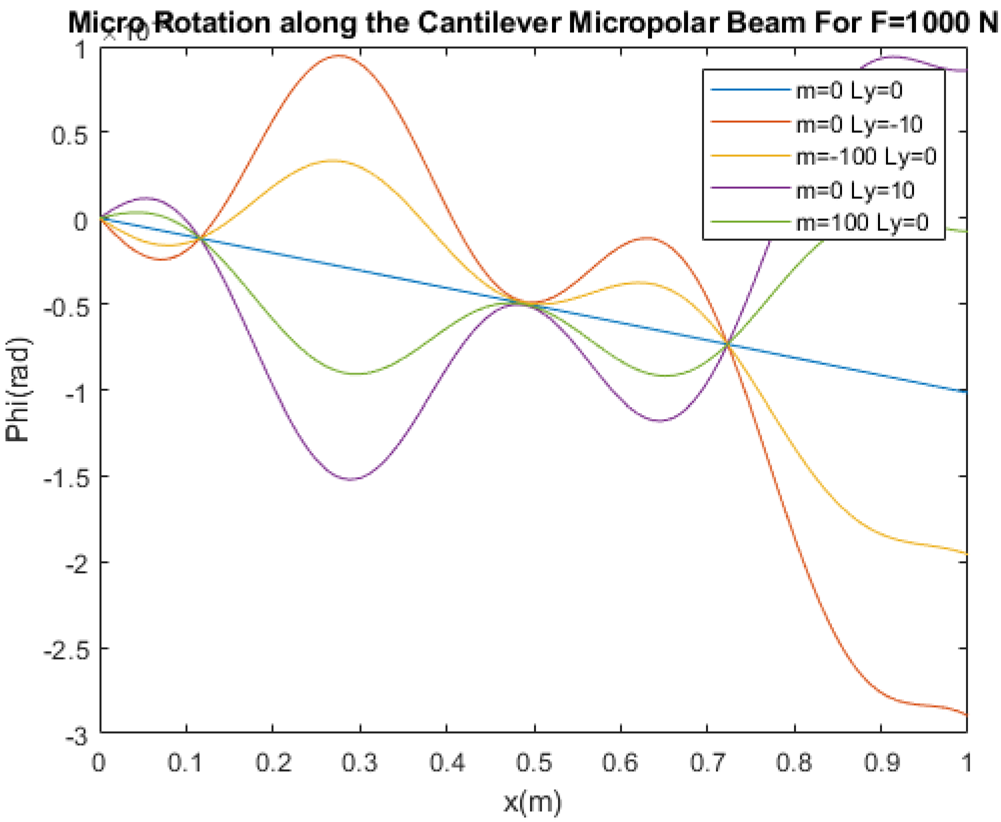
```
m=0;L__y=0;
w1= (-2.409870830*10^9*x.^2-2.814743626*10^7*cos(16.19652482*x))*(4.476877960*10^(-
16)*m+8.953755900*10^(-15)*L__y-2.034587998*10^(-17)*F)+(-
7.530043055*10^8*x.^2+938247.8753*cos(16.19652482*x))*(5.737335600*10^(-
16)*m+1.147467000*10^(-14)*L__y-6.103763994*10^(-16)*F)+(-6.500000002*10^(-18)*m-
1.300000000*10^(-16)*L__y)*(5.629487251*10^9*x-1.606580554*10^11*x.^3-
3.475737736*10^8*sin(16.19652482*x))+1.206295941*10^(-8)*m+2.412591888*10^(-7)*L__y;
w1 =w1*3*E*I/F;
m=0;L__y=-10;
w2= (-2.409870830*10^9*x.^2-2.814743626*10^7*cos(16.19652482*x))*(4.476877960*10^(-
16)*m+8.953755900*10^(-15)*L__y-2.034587998*10^(-17)*F)+(-
7.530043055*10^8*x.^2+938247.8753*cos(16.19652482*x))*(5.737335600*10^(-
16)*m+1.147467000*10^(-14)*L__y-6.103763994*10^(-16)*F)+(-6.500000002*10^(-18)*m-
1.300000000*10^(-16)*L__y)*(5.629487251*10^9*x-1.606580554*10^11*x.^3-
3.475737736*10^8*sin(16.19652482*x))+1.206295941*10^(-8)*m+2.412591888*10^(-7)*L__y;
w2 =w2*3*E*I/F;
m=-100; L__y=0;
w3= (-2.409870830*10^9*x.^2-2.814743626*10^7*cos(16.19652482*x))*(4.476877960*10^(-
16)*m+8.953755900*10^(-15)*L__y-2.034587998*10^(-17)*F)+(-
7.530043055*10^8*x.^2+938247.8753*cos(16.19652482*x))*(5.737335600*10^(-
16)*m+1.147467000*10^(-14)*L__y-6.103763994*10^(-16)*F)+(-6.500000002*10^(-18)*m-
1.300000000*10^(-16)*L__y)*(5.629487251*10^9*x-1.606580554*10^11*x.^3-
3.475737736*10^8*sin(16.19652482*x))+1.206295941*10^(-8)*m+2.412591888*10^(-7)*L__y;
w3 =w3*3*E*I/F;
m=0;L__y=10;
w4= (-2.409870830*10^9*x.^2-2.814743626*10^7*cos(16.19652482*x))*(4.476877960*10^(-
16)*m+8.953755900*10^(-15)*L__y-2.034587998*10^(-17)*F)+(-
7.530043055*10^8*x.^2+938247.8753*cos(16.19652482*x))*(5.737335600*10^(-
16)*m+1.147467000*10^(-14)*L__y-6.103763994*10^(-16)*F)+(-6.500000002*10^(-18)*m-
1.300000000*10^(-16)*L__y)*(5.629487251*10^9*x-1.606580554*10^11*x.^3-
3.475737736*10^8*sin(16.19652482*x))+1.206295941*10^(-8)*m+2.412591888*10^(-7)*L__y;
w4 =w4*3*E*I/F;
m=100; L__y=0;
w5= (-2.409870830*10^9*x.^2-2.814743626*10^7*cos(16.19652482*x))*(4.476877960*10^(-
16)*m+8.953755900*10^(-15)*L__y-2.034587998*10^(-17)*F)+(-
7.530043055*10^8*x.^2+938247.8753*cos(16.19652482*x))*(5.737335600*10^(-
16)*m+1.147467000*10^(-14)*L__y-6.103763994*10^(-16)*F)+(-6.500000002*10^(-18)*m-
1.300000000*10^(-16)*L__y)*(5.629487251*10^9*x-1.606580554*10^11*x.^3-
3.475737736*10^8*sin(16.19652482*x))+1.206295941*10^(-8)*m+2.412591888*10^(-7)*L__y;
w5 =w5*3*E*I/F;
figure
plot(x,w1,x,w2,x,w3,x,w4,x,w5)
legend('m=0 Ly=0','m=0 Ly=-10','m=-100 Ly=0','m=0 Ly=10','m=100 Ly=0')
title('Deflection of the cantelever Beam under concentrated Tip Loading')
axis([0 1 0 1.2])
xlabel('x(m)')
ylabel('3EIw(x)/FL^3')
```

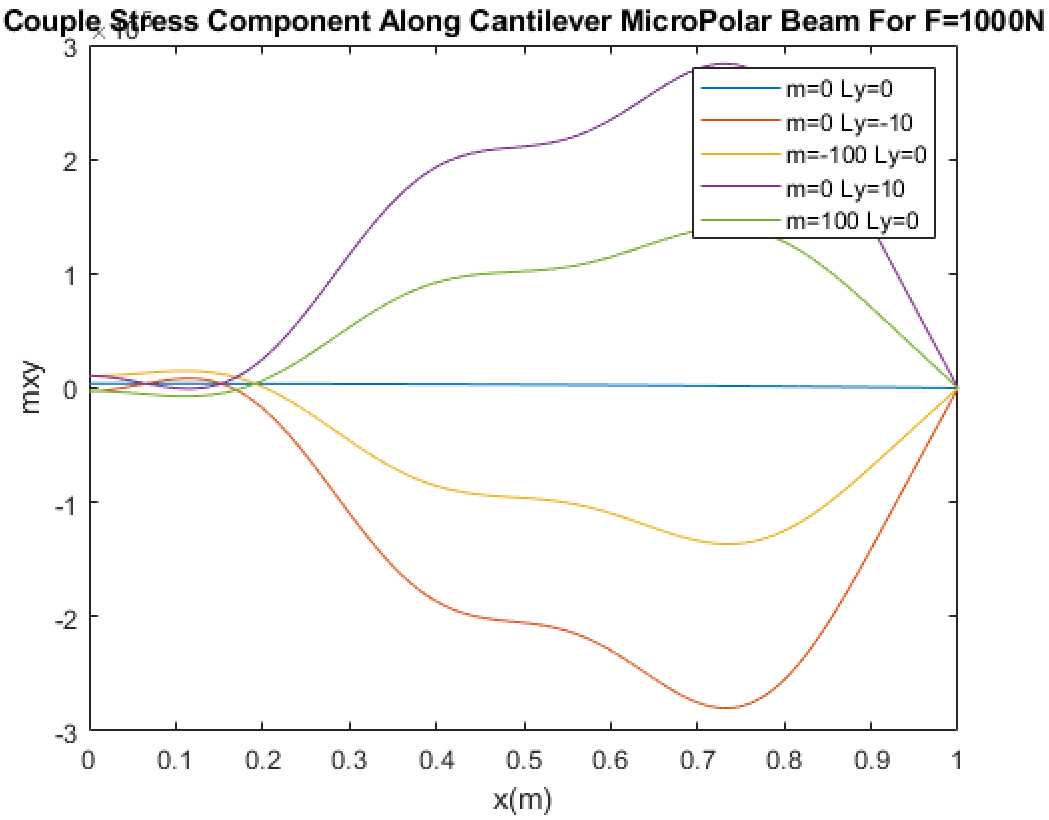


PHI

```
m=0;L__y=0;
phi1 = (1.954887218*x-.1206979423*sin(16.19652482*x))*(9.562226000*10^(-7)*m+0.19124450e-
4*L__y-1.017293999*10^-6*F)+(-.954887218*x+.1206979423*sin(16.19652482*x))*(0.223843898e-
4*m+0.447687795e-3*L__y-1.017293999*10^-6*F)+(-.4774436091*x.^2+0.7452088870e-2-
0.7452088870e-2*cos(16.19652482*x))*(-0.6500000002e-4*m-0.1300000000e-2*L__y);
m=0;L__y=-10;
phi2 = (1.954887218*x-.1206979423*sin(16.19652482*x))*(9.562226000*10^(-7)*m+0.19124450e-
4*L__y-1.017293999*10^-6*F)+(-.954887218*x+.1206979423*sin(16.19652482*x))*(0.223843898e-
4*m+0.447687795e-3*L__y-1.017293999*10^-6*F)+(-.4774436091*x.^2+0.7452088870e-2-
0.7452088870e-2*cos(16.19652482*x))*(-0.6500000002e-4*m-0.1300000000e-2*L__y);
m=-100; L__y=0;
phi3 = (1.954887218*x-.1206979423*sin(16.19652482*x))*(9.562226000*10^(-7)*m+0.19124450e-
4*L__y-1.017293999*10^-6*F)+(-.954887218*x+.1206979423*sin(16.19652482*x))*(0.223843898e-
4*m+0.447687795e-3*L__y-1.017293999*10^-6*F)+(-.4774436091*x.^2+0.7452088870e-2-
0.7452088870e-2*cos(16.19652482*x))*(-0.6500000002e-4*m-0.1300000000e-2*L__y);
m=0;L__y=10;
phi4 = (1.954887218*x-.1206979423*sin(16.19652482*x))*(9.562226000*10^(-7)*m+0.19124450e-
4*L__y-1.017293999*10^-6*F)+(-.954887218*x+.1206979423*sin(16.19652482*x))*(0.223843898e-
4*m+0.447687795e-3*L__y-1.017293999*10^-6*F)+(-.4774436091*x.^2+0.7452088870e-2-
0.7452088870e-2*cos(16.19652482*x))*(-0.6500000002e-4*m-0.1300000000e-2*L__y);
m=100; L__y=0;
phi5 = (1.954887218*x-.1206979423*sin(16.19652482*x))*(9.562226000*10^(-7)*m+0.19124450e-
4*L__y-1.017293999*10^-6*F)+(-.954887218*x+.1206979423*sin(16.19652482*x))*(0.223843898e-
4*m+0.447687795e-3*L__y-1.017293999*10^-6*F)+(-.4774436091*x.^2+0.7452088870e-2-
0.7452088870e-2*cos(16.19652482*x))*(-0.6500000002e-4*m-0.1300000000e-2*L__y);figure
plot(x,phi1,x,phi2,x,phi3,x,phi4,x,phi5)
legend('m=0 Ly=0','m=0 Ly=-10','m=-100 Ly=0','m=0 Ly=10','m=100 Ly=0')
title('Micro Rotation along the Cantilever Micropolar Beam For F=1000 N')
```

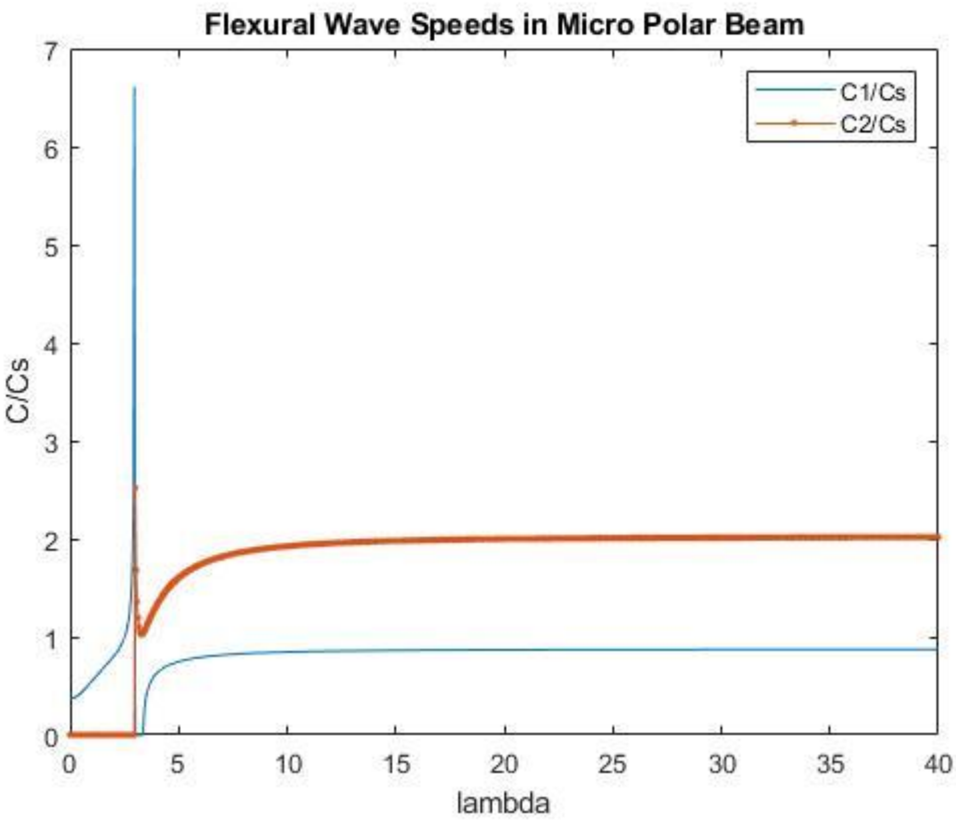
```
xlabel('x(m)')
ylabel('Phi(rad)')

%%mxy
%m=0;L__y=0;
mxy1 = -3912.665314*x.^2+3912.665315;
%m=0;L__y=-10;
mxy2= -1.694497853*10^6*x.^2-25778.24631*cos(16.19652482*x)+22795.22432-
58195.48874*x+1.709401710*10^6*x.^3+4827.917693*sin(16.19652482*x);
%m=-100; L__y=0;
mxy3= -849205.2568*x.^2-12889.12313*cos(16.19652482*x)+23353.94242+854700.8551*x.^3-
39097.74437*x+2413.958846*sin(16.19652482*x);
%m=0;L__y=10;
mxy4= 1.686672522*10^6*x.^2+25778.24630*cos(16.19652482*x)-14969.89368+58195.48874*x-
1.709401710*10^6*x.^3-4827.917693*sin(16.19652482*x);
% m=100; L__y=0;
mxy5= 841379.9261*x.^2+12889.12312*cos(16.19652482*x)-15528.61178-
854700.8551*x.^3+39097.74437*x-2413.958846*sin(16.19652482*x);
figure
plot(x,mxy1,x,mxy2,x,mxy3,x,mxy4,x,mxy5)
legend('m=0 Ly=0','m=0 Ly=-10','m=-100 Ly=0','m=0 Ly=10','m=100 Ly=0')
title('Couple Stress Component Along Cantilever MicroPolar Beam For F=1000N')
xlabel('x(m)')
ylabel('mxy')
```





Published with MATLAB® R2018a

For flex speed




```
clear all
clc
lambda = linspace(0,40,1000);
format long
C1= 0.5036452488e-3*(((4.878048777*10^14*lambda.^4-5.246548322*10^15*lambda.^2-
3.492567470*10^15)./(1.617886178*10^8*lambda.^4-6.931207008*10^8*lambda.^2-
6.454033771*10^9)).^0.5);
C2= 0.5036452488e-3*((7.884615384*10^8*(8.089430889*10^7*lambda.^4-
3.465603504*10^8*lambda.^2-
3.227016886*10^9)./(lambda.^2.*(3.333333332*10^9*lambda.^2+1.576923077*10^10))-
(4.878048777*10^14*lambda.^4-5.246548322*10^15*lambda.^2-
3.492567470*10^15)./(1.617886178*10^8*lambda.^4-6.931207008*10^8*lambda.^2-
6.454033771*10^9)).^0.5);

plot(lambda,C1,lambda,C2,'.-')
xlabel('lambda')
ylabel('C/Cs')
title('Flexural Wave Speeds in Micro Polar Beam')
legend('C1/Cs','C2/Cs')
```

Published with MATLAB® R2018a

Conclusion

In this project, we have analysed the deformation of elastic beams using theory of linear micropolar elasticity based on micropolar continuum mechanics. A simple approximation is done by using power series expansion for the axial displacement and micro-rotation fields.

We have calculated the deflection of cantilever beam using micropolar beam theory. The result which we gat is near to the Euler-Bernoulli and Timoshenko theory. This theory predicts a constant distribution of transverse shear σ_{xz} across the height of the beam which is similar to the Timoshenko beam. The nature of flexural and longitudinal waves in the infinite length micropolar beam has been investigated.