Stability Aspects of Scalable delay differential equations

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Introduction

Delay Differential Equations (DDE's):

$$\frac{dx(t)}{dt} = f\big(x(t), x(t-\tau_1), \ldots x(t-\tau_m)\big) \qquad (1)$$

$$\tau_1, \tau_2, \ldots \tau_m \geq 0$$

$$x(t) = \phi(t), \qquad t \leq t_0$$

DDE's are infinite dimensional systems.

$$\frac{dx(t)}{dt} + x(t-\tau) = 0; \tau \geq 0 \qquad (2)$$

Substituting $x(t) = e^{st}$, we get $s + e^{-s\tau} = 0$ ⟵ Infinite roots for s.

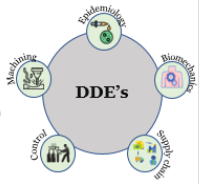
Scalability:

$$x(t) = f(x, y) \qquad (3)$$

$$y(t) = g(x, y) \qquad (4)$$

Scalability in $x(t)$

If $x(t)$ and $y(t)$ are the solutions and $\alpha x(t)$ and $y(t)$ are also solutions for $\alpha \geq 0$



Objective and Problem Statement

Hayes DDE with Hysteresis:

$$\dot{z}(t) = -pz(t) - qz(t-1) - \gamma\theta(t)z(t) \qquad (5)$$

$$\dot{\theta}(t) = \frac{\kappa}{|z(t)| + \epsilon}\{\theta_a + \beta\,\mathrm{sgn}(z(t)\dot{z}(t)) - \theta(t)\}|\dot{z}(t)| \qquad (6)$$

Where

$$\mathrm{sgn}(x) = \frac{x}{|x|}$$

$$p, q \in \mathbb{R}$$

$$\theta_a, \beta, \kappa > 0$$

$$\epsilon \to 0^+$$

$$\gamma > 0$$

γ represents the damping of the hysteresis force $\theta(t)z(t)$.

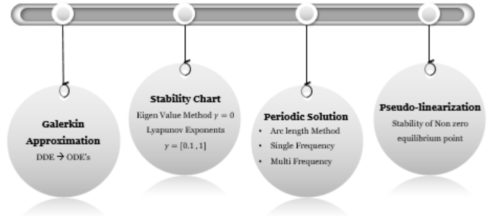
Salient Features :

- Infinite dimensions.
- Non linear and coupled.
- Slope discontinuities at every load reversals.
- Scalable in z(Partially scalable).
  If z is a solution, αz is also solution with $\alpha > 0$

Problem Statement:

Stability analysis of above Hayes equation with Hysteresis for $\gamma = [0 ; 0.1; 1]$.

Methodology



Equilibrium zones: Pseudo-linearization

- Equilibrium conditions

$$\dot{z}(t) = 0, \dot{\theta}(t) = 0 \;\&\; z(t-1) = z(t) \qquad (22)$$

- We get

$$\gamma\theta = -(p+q) \qquad (23)$$

- For the chosen Hysteresis model $\theta \in [0.2, 3.8]$
- The parameter plane containing non zero equilibrium points is $0.2\gamma = -(p+q)$ and $3.8\gamma = -(p+q)$
- Let z_c and θ_c be the non zero equilibrium points, taking small variations

$$\left.\begin{aligned} z(t) &= z_c + \eta(t) \\ \theta(t) &= \theta_c + \phi(t) \end{aligned}\right\} \qquad (24)$$

- Neglecting higher order terms and using scalability

$$\left.\begin{aligned} \dot{\eta}(t) &= q\big(\eta(t) - \eta(t-1)\big) - \gamma\phi(t) \\ \dot{\phi}(t) &= \kappa(\theta_a - \theta_c)|\dot{\eta}(t)| + \kappa\beta\dot{\eta}(t) \end{aligned}\right\}\text{ Non Linear} \qquad (25)$$

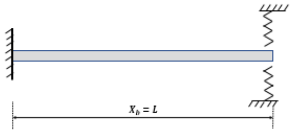
Summary and Conclusions

- Studied the stability charts and boundaries of Hayes DDE with hysteresis.
- Scalability of hysteresis force allowed us to use Lyapunov like exponents.
- Numerical difficulties in computing periodic solution through continuation are resolved using Multi-frequency method.
- Lyapunov like exponents study the exponential growth or lack, but not the stability of individual trajectories.

Future Work

Problem Statement:

Stability Analysis of the Heat Exchanger when the cladding just lost initial contact with the tube. Cladding stiffness is asymmetric



$$EI\frac{\partial^4 W}{\partial X^4} + C\frac{\partial W}{\partial T} + M\frac{\partial^2 W}{\partial T^2} + \delta(X - X_b)\tilde{f}(W) = F(W, \dot{W}, \ddot{W})$$

$$\tilde{f}(W) = K(1 + a\,sign(W))W$$

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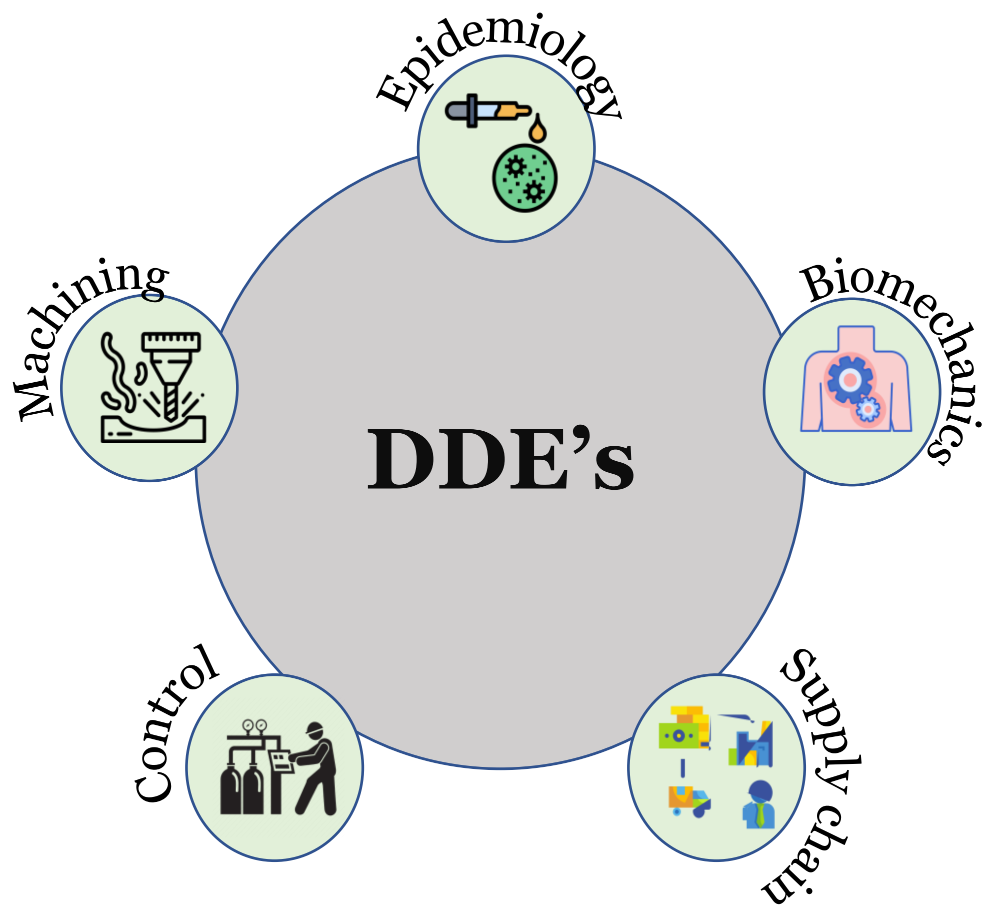

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Methodology

Galerkin Approximation

DDE → ODE's

Stability Chart

Eigen Value Method $\gamma = 0$

Lyapunov Exponents

$\gamma = [0.1\,,1]$

Periodic Solution

- Arc length Method
- Single Frequency
- Multi Frequency

Pseudo-linearization

Stability of Non zero equilibrium point

Methodology: Galerkin Approximation

Galerkin Approximation:

$$z(t-s) = F(t,s) = a_0(t) + a_1(t)\, s + \sum_{k=1}^{N-2} a_{k+1}(t)\sin(k\pi s) \tag{7}$$

Where $a_0(t), \ldots a_{N-1}(t)$ are state variables , N: Number of shape functions.

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial s} = 0 \tag{8}$$

$$z(t) = F(t,0) = a_0(t) \Rightarrow \dot{z}(t) = \dot{a_0}(t) \tag{9}$$

Residual, $R(t,s)$

$$R(t,s) = \int_0^s \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial s}\right) ds \tag{10}$$

Galerkin approximation method

$$\int_0^1 R(t,s)\, s\, ds = 0 \tag{11}$$

$$\int_0^1 R(t,s)\sin(k\pi s)\, ds = 0, k = 1, \ldots, N-2 \tag{12}$$

Methodology: Galerkin Approximation

Considering N=15, we get 16 system of coupled ODE's of the form

$$\frac{d\boldsymbol{a}(t)}{dt} = \boldsymbol{g}(\boldsymbol{a}(t)) \tag{13}$$

Where

$$\mathbf{a}(\mathrm{t}) = [a_0(t), .. a_{14}(t), \theta(t)]'$$

$$\boldsymbol{g}(\boldsymbol{a}(\boldsymbol{t})) = [g_1(\boldsymbol{a}(t)), .. g_{16}(\boldsymbol{a}(\boldsymbol{t}))]'$$

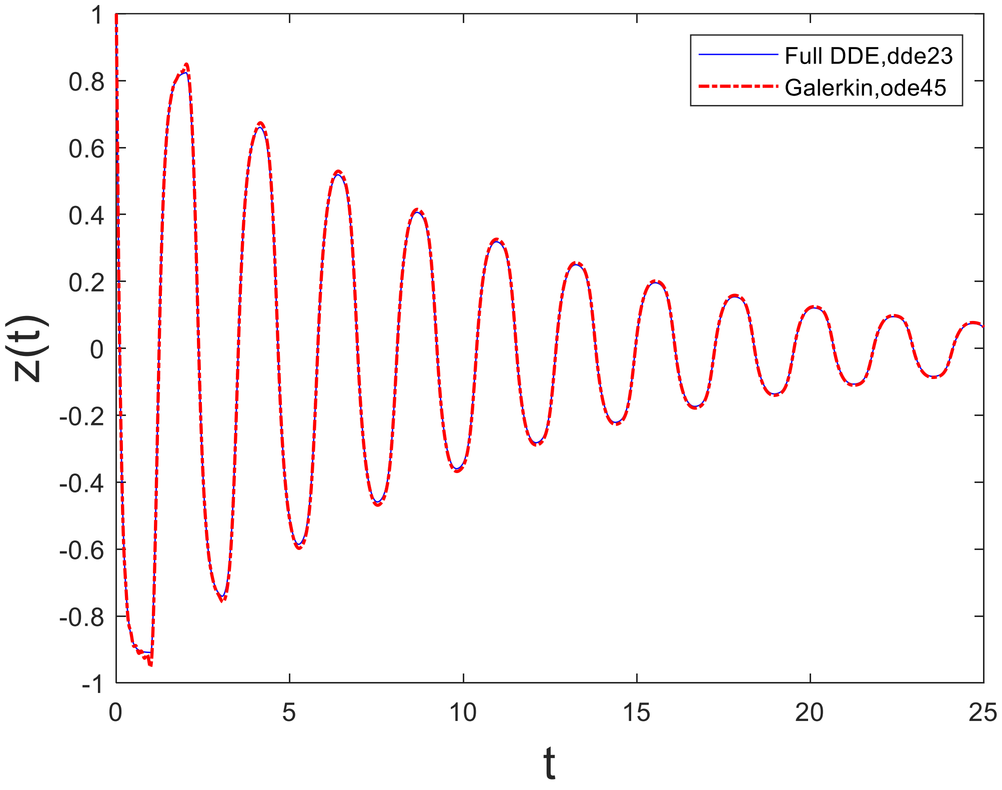
Integrate the above system of ODE's using MATLAB's ode45.

We used the following parameters:

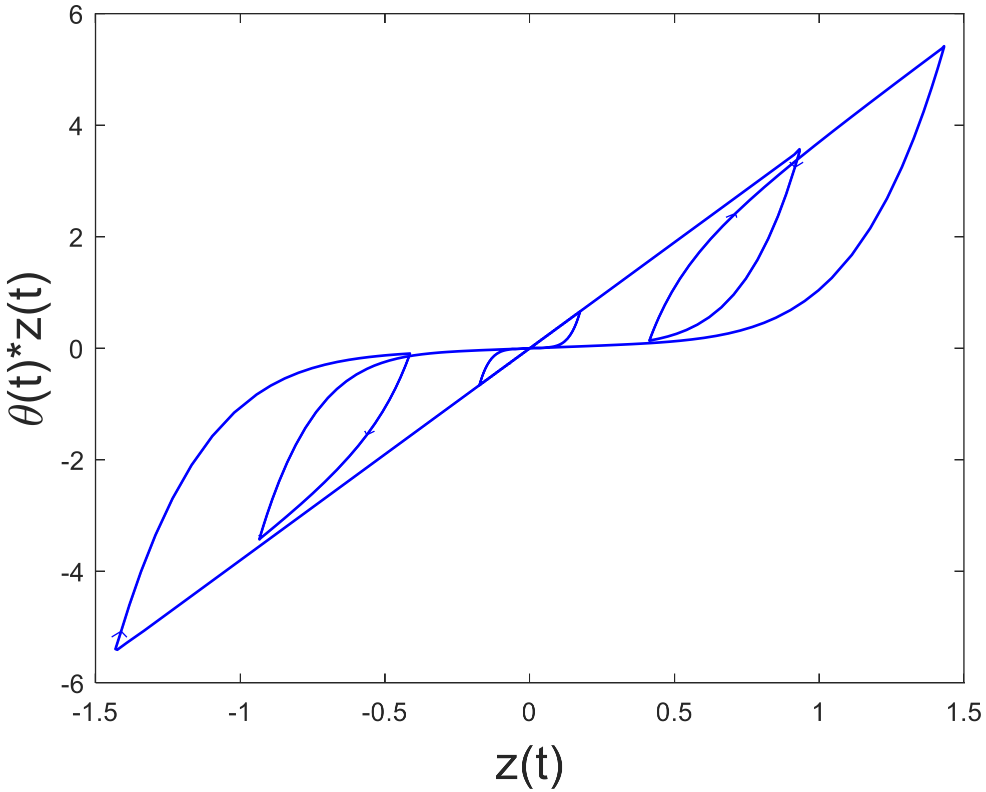
$$p = 5; q = 8; \gamma = 1; \kappa = 4; \theta_a = 2; \beta = 1.8; \epsilon = 10^{-6}$$

ODE: Ordinary differential equations

Methodology: Galerkin Approximation



Comparison of solutions obtained from full DDE solution using MATLAB's dde23 and Galerkin approximation



Response of Hysteresis model.

Methodology: Stability Analysis $\gamma = 0$

$\gamma = 0$

$\dot{z}(t) = -pz(t) - qz(t-1)$

Galerkin approximation
DDE(infinite dimensions)→ system of ODE's(finite dimensions)

$\dot{\boldsymbol{a}} = \boldsymbol{A}\boldsymbol{a}$

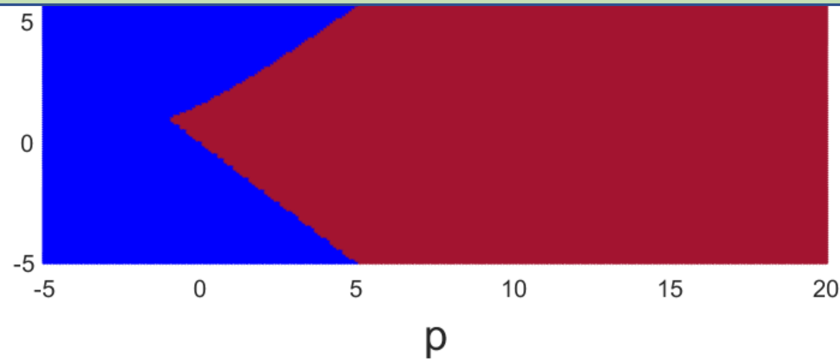
$\lambda_i \leq 0, \forall\, i$

Yes → **Stable**

No → **Unstable**

$$\dot{z}(t) = -pz(t) - qz(t-1) - \gamma\theta(t)z(t) \quad (5)$$

$$\dot{\theta}(t) = \frac{\kappa}{|z(t)| + \epsilon}\{\theta_a + \beta\,\mathrm{sgn}\big(z(t)\dot{z}(t)\big) - \theta(t)\}|\dot{z}(t)| \quad (6)$$



Stability Chart for $\gamma = 0$. Blue is unstable and brown indicates stable regions.

DDE: Delay Differential Equations

ODE: Ordinary Differential Equations

Methodology: Stability Analysis damped

Stability Analysis

$\gamma = 0.1, 1$

Galerkin approximation
DDE(infinite dimensions)→ system of ODE's(finite dimensions)

Compute Lyapunov like exponents l

$l < 0,$

Yes

Stable

No

Unstable

Calculation of Lyapunov like exponents

System of ODE's

$i.c = Z_k^0$ **with** $\|Z_k^0\| = 1$

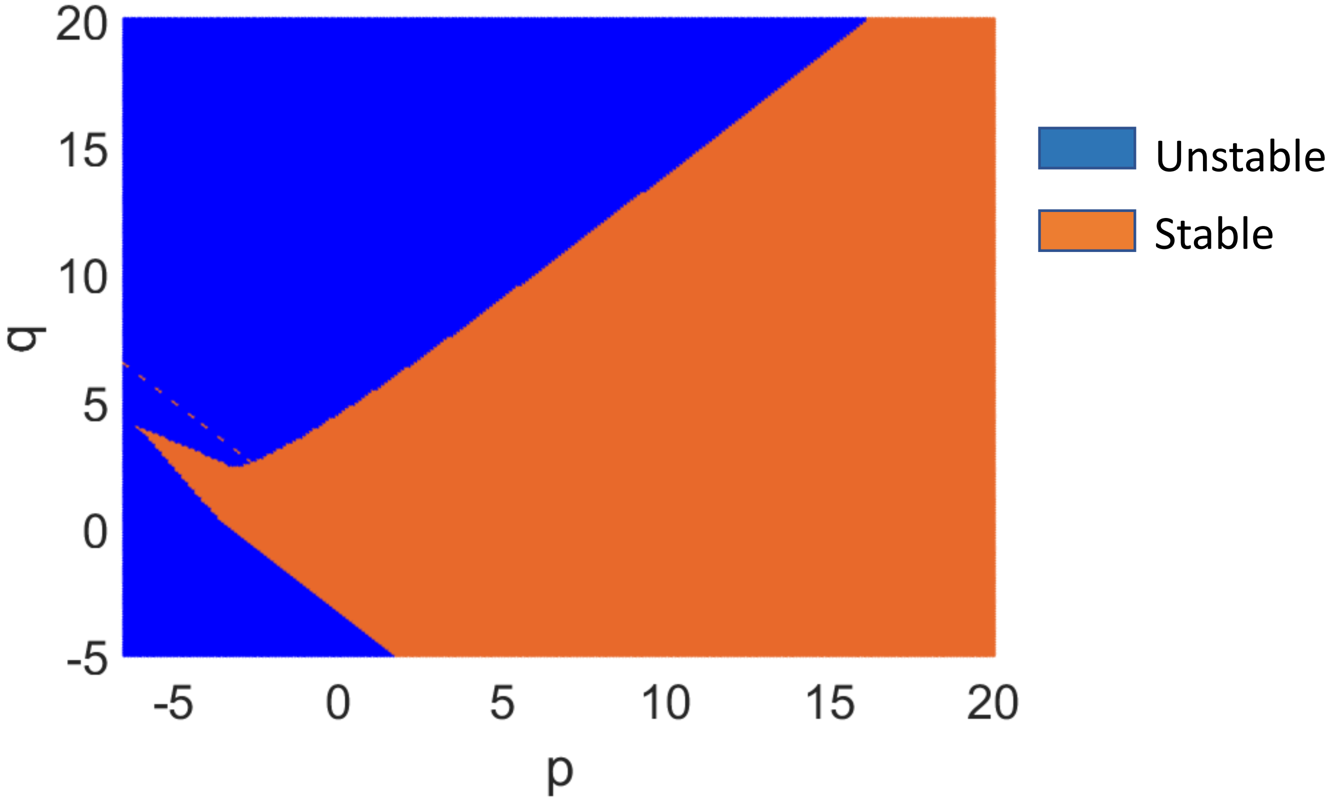
$$Z_k^T = \int_0^T \{System\ of\ ODE's\}$$

Z_{k+1}^0 **= normalize(**Z_k^T**)**

$$l = \lim_{M\to\infty} \frac{1}{M} \sum_{k=0}^{M} ln(\|Z_k^T\|)$$

$k = 1..M$
$M \to \infty$

Methodology: Stability Analysis damped



Stability Chart for $\gamma = 1$. Blue is unstable and orange indicates stable regions.

Methodology: Periodic Solution-Arc length method

Arc length based numerical continuation method :

$$\left.\begin{aligned}\text{Periodicity}\;\left\{\begin{aligned}&a_0(T)-a_0(0)=0,\\&a_1(T)-a_1(0)=0,\\&\quad\vdots\\&a_{N-1}(T)-a_{N-1}(0)=0,\\&\theta(T)-\theta(0)=0\end{aligned}\right.\\ \psi(T)-T=0\qquad\qquad\end{aligned}\right\}\quad(14)$$

Setting RMS value of $a_0(t)=1$; $\dot{\psi}=a_0(t)^2$, $\psi(0)=0$;

For N = 15, Unknown $\boldsymbol{x}=[a_0(0),\dots a_{14}(0),\mathrm{q},\mathrm{T}]'$

p is the independent variable

$\boldsymbol{x}=[a_0(0),\dots a_{14}(0),\mathrm{q},\mathrm{T}]'$

$\boldsymbol{y}=[\boldsymbol{x},p]'$

Newton Raphson method

$\boldsymbol{x_0},\boldsymbol{x_1}$ **for** $p_0;p_1=p_0+\delta$

$\boldsymbol{y_0}=[\boldsymbol{x_0};\boldsymbol{p_0}]$

$\boldsymbol{y_1}=[\boldsymbol{x_1};p_1]$

$s=\|\boldsymbol{y_1}-\boldsymbol{y_2}\|$

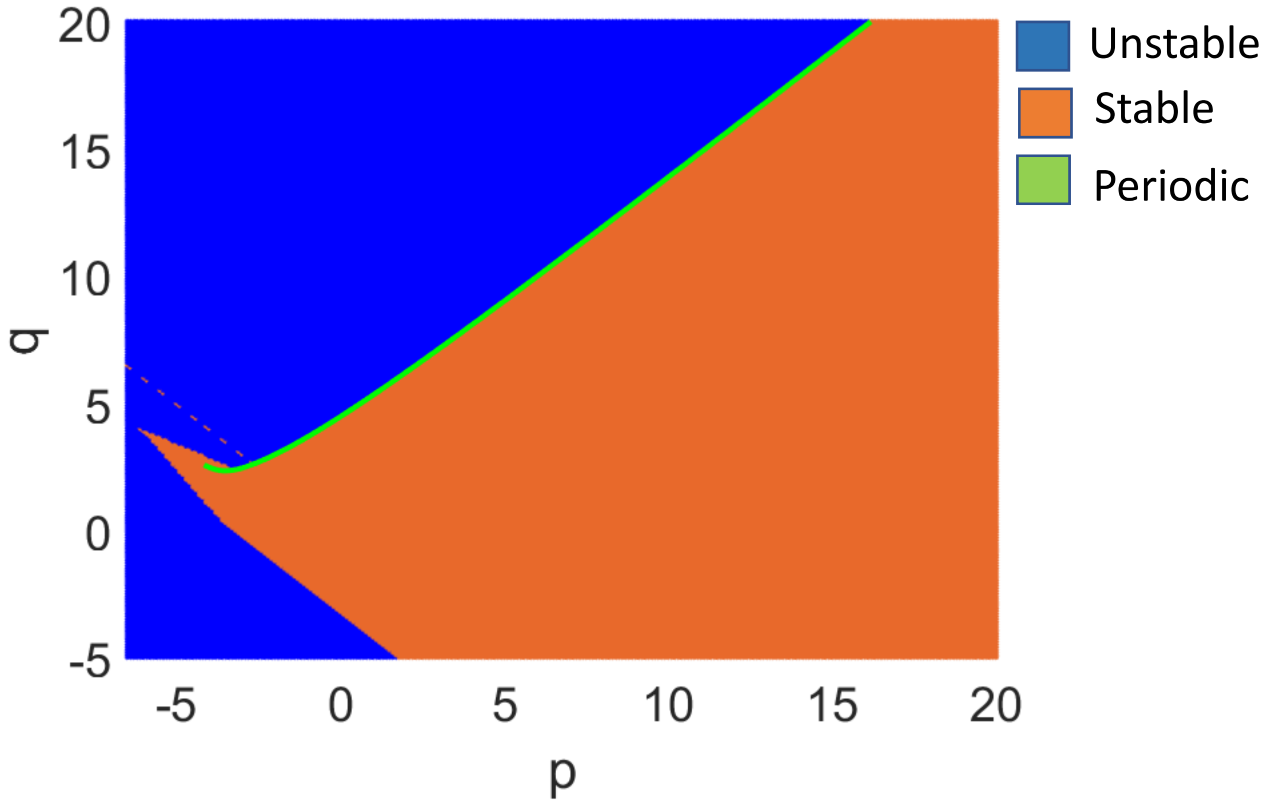
$\boldsymbol{y_c}=\boldsymbol{2}*\boldsymbol{y_1}-\boldsymbol{y_0}$

$\|\boldsymbol{y}-\boldsymbol{y_1}\|-\boldsymbol{s}=\boldsymbol{0}$

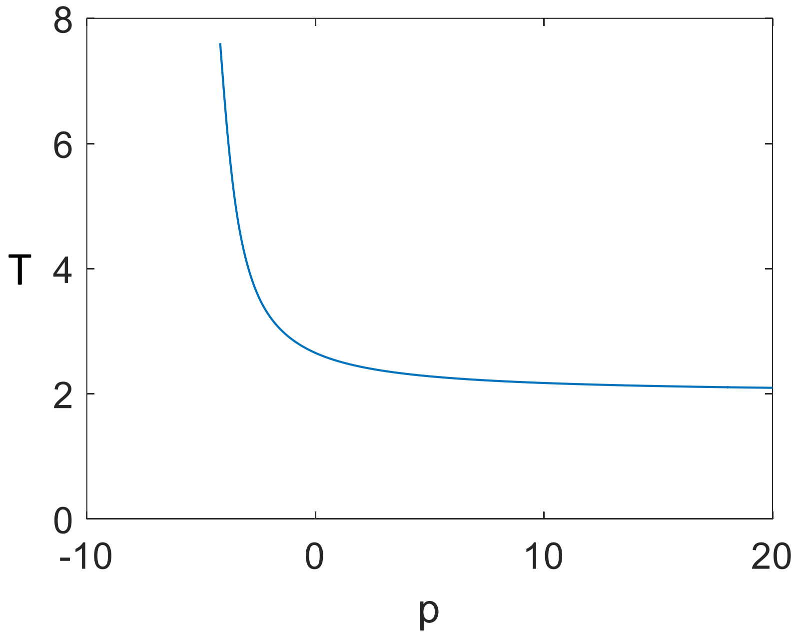
Newton Raphson Method $\boldsymbol{y_2}$

$\boldsymbol{y_1}=\boldsymbol{y_2},\boldsymbol{y_0}=\boldsymbol{y_1}$

Methodology: Periodic Solution-Arc length method



Stability Chart for $\gamma = 1$. Blue is unstable and orange indicates stable regions. Periodic solution obtained using Arc length denoted by green curve



Arc length method: Plot of P vs T

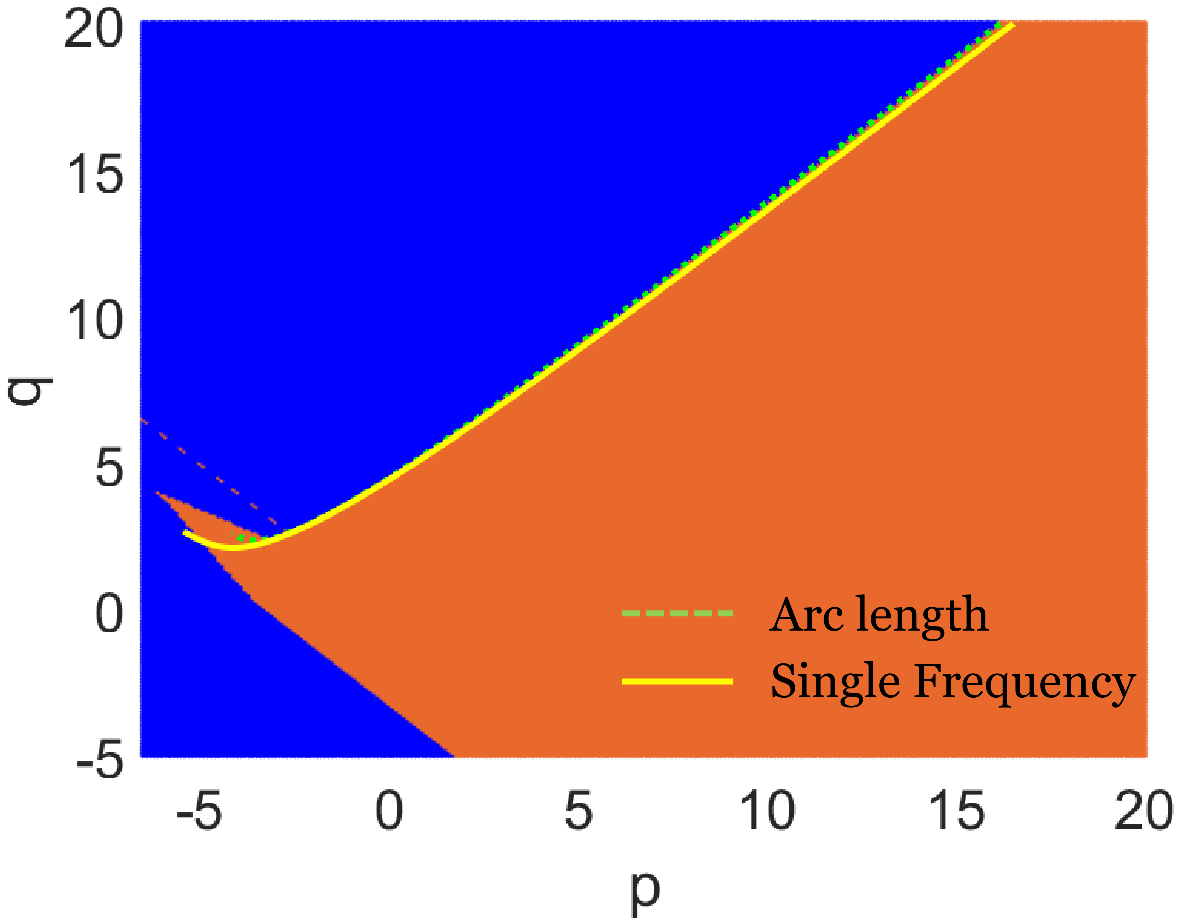
Methodology: Periodic Solution – Single Frequency

- Assume $z(t) = \sin(\omega t)$ (15)
- By fitting the solution using hysteresis model

$$\theta(t)z(t) \approx \alpha_0 \sin(\omega t) + \alpha_1 \cos(\omega t) \quad (16)$$

- Substituting the above two into eq.(5) and eq.(6)
- Set the coefficients of cos and sin to zero

$$\left.\begin{aligned} p + q\,cos(\omega) + \gamma\alpha_0 &= 0 \\ \omega\ - q\sin(\omega) + \gamma\alpha_1 &= 0 \end{aligned}\right\} \quad (17)$$



Periodic solution obtained using Single frequency Harmonic Balance method denoted by yellow curve

Methodology: Periodic Solution – Multi Frequency method

- Define distorted time scale

$$\tau = \omega t + \sum_{k=1}^{n} [A_k \sin(k\omega t) + B_k(\cos(k\omega t) - 1)] \tag{18}$$

- Assume $z(\tau) = \sin(\tau)$ (19)
- Hysteresis response

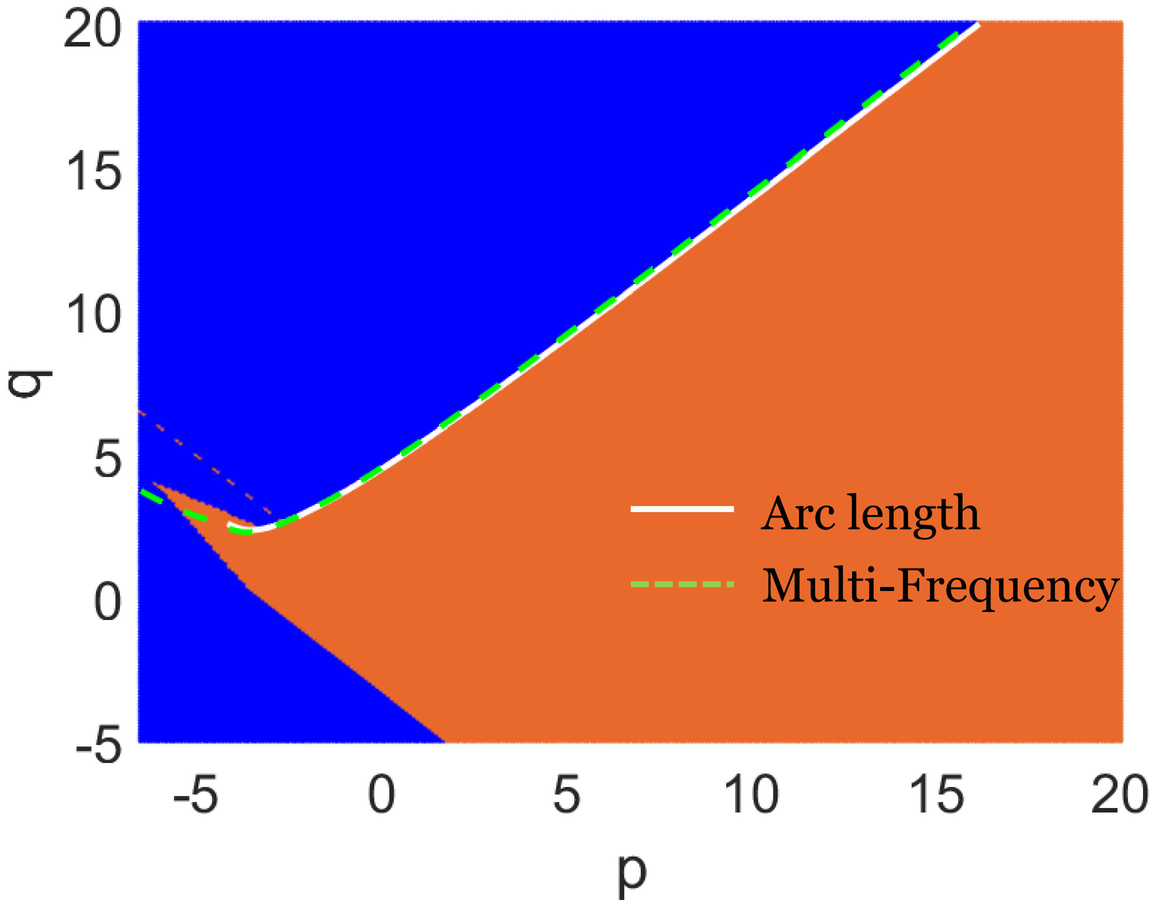
$$\theta(\tau)z(\tau) \approx \alpha_0 \sin(\tau) + \alpha_1 \cos(\tau) + \alpha_2 \sin(3\tau) + \alpha_3 \cos(3\tau) \tag{20}$$

- Numerically fitting we get $\alpha_0 = 3.125, \alpha_1 = 0.7638, \alpha_2 = -0.5625, \alpha_3 = -0.3829$
- Define Residual $R(p, q, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{\omega}, \boldsymbol{t})$
- Minimizing R in least square sense, we get 2n+2 equations.

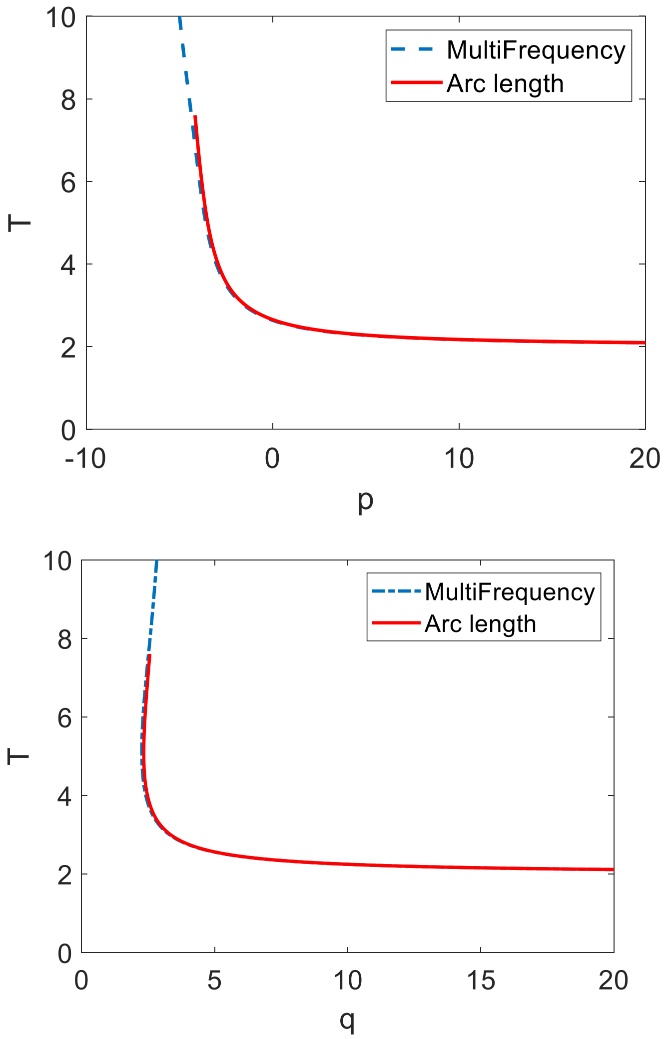
$$\frac{\partial}{\partial p}\int_0^{\frac{2\pi}{\omega}} R^2\, dt = 0, \frac{\partial}{\partial q}\int_0^{\frac{2\pi}{\omega}} R^2\, dt = 0, \frac{\partial}{\partial A_i}\int_0^{\frac{2\pi}{\omega}} R^2\, dt = 0, \frac{\partial}{\partial B_i}\int_0^{\frac{2\pi}{\omega}} R^2\, dt = 0 \tag{21}$$

- Keeping ω as a independent variable , we can solve for p, q, A_i, B_i

Methodology: Periodic Solution – Multi Frequency method



Periodic solution obtained using Multi- frequency method denoted by dotted green curve



Equilibrium zones: Pseudo-linearization

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- For the chosen Hysteresis model $\theta \in [0.2, 3.8]$
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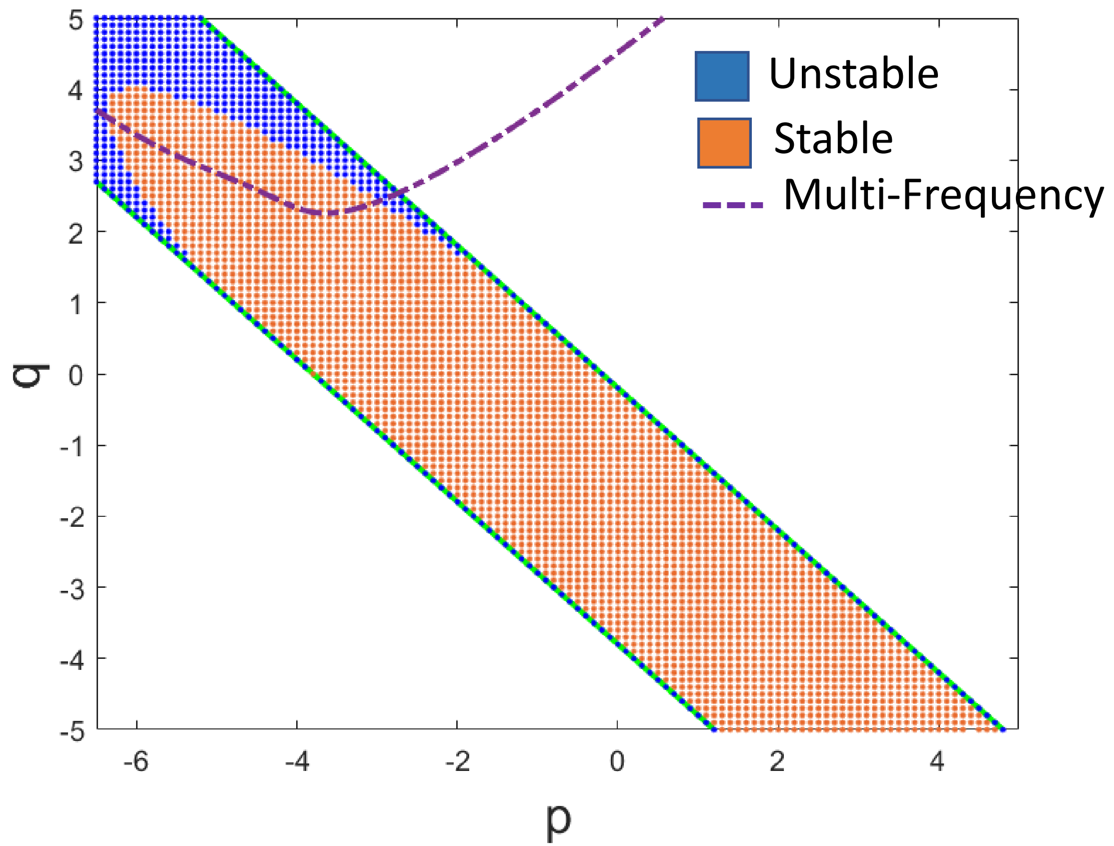

Periodic solution obtained using pseudo-linearization. Parametric zone with non zero equilibrium points are represented using green lines

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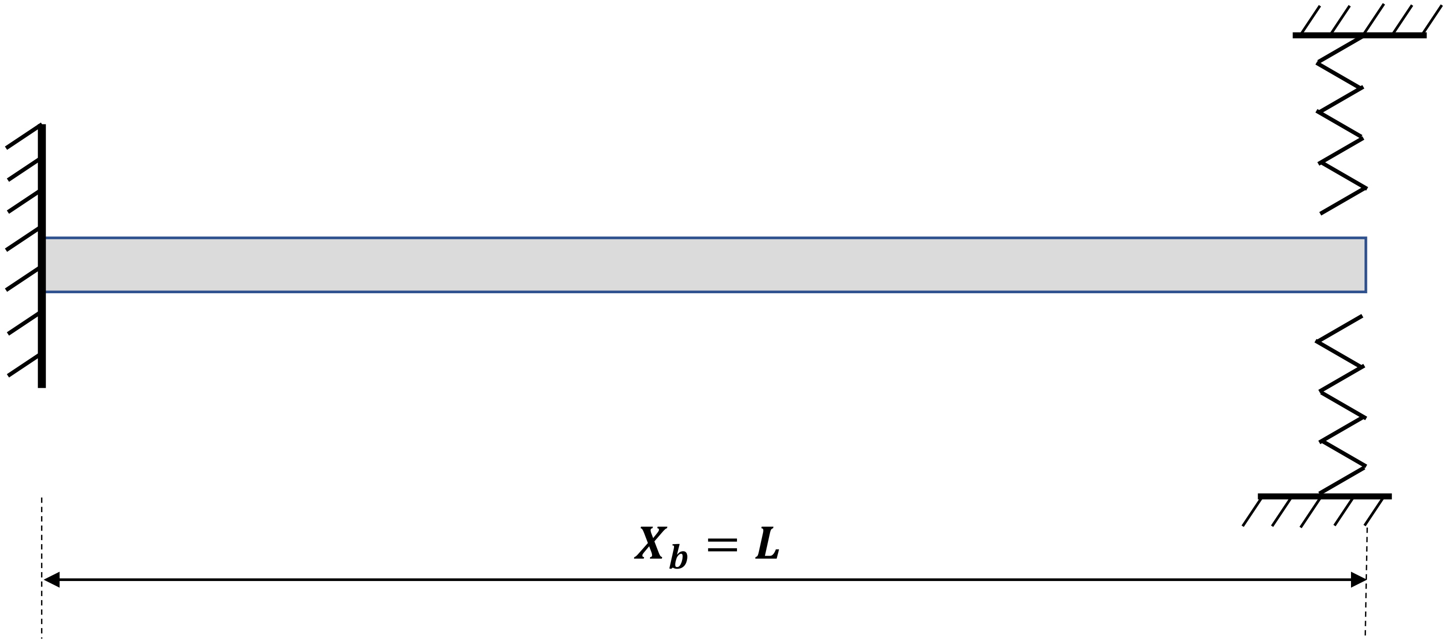

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Acknowledgement

I thank my supervisor Dr. C.P. Vyasarayani for guiding me through difficulties in the problem, and my lab mate Balaji, Junaid for their suggestions.