Gerig, 2/19/2018

Noise Reduction by pixel averaging/filtering

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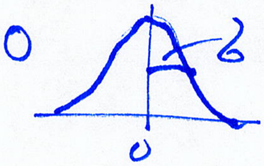
model: $s_i = \underbrace{s + n_i}_{\text{noise-corrupted signal}}$

$n_i = N(0, \sigma^2)$

Gaussian-distributed noise, 0-mean, stdev: σ

$\langle n_i \rangle = 0$



sample mean, averaging:

- take N samples from a pdf
- sum up and divide by N

$$S = \frac{1}{N}\sum_{i=1}^{N} s_i$$

S is random variable, sample mean

$$E(S) = E\left(\frac{1}{N}\sum_{i=1}^{N} s_i\right) = \frac{1}{N}\sum_{i=1}^{N} E(s_i) = \frac{1}{N}\sum s + \underbrace{\frac{1}{N}\sum n_i}_{0 \text{ (zero-mean)}} = s$$

expected value, mean

s is true mean

how close is sample mean from "true" mean?

Variance:

$$E[(S-s)^2] = E\left[\left(\frac{1}{N}\sum n_i\right)^2\right] = \frac{1}{N^2} E\left[\left(\sum n_i\right)^2\right]$$

$$\overset{!}{=} \frac{1}{N^2} E\left[\sum_i \sum_j n_i n_j\right] \overset{\text{linear}}{=} \frac{1}{N^2} \sum_i \sum_j E[n_i n_j]$$

$$= \frac{1}{N^2} \underbrace{\sum_i n_i^2}_{i=j,\ \text{sum of N exp.}} + \frac{1}{N^2} \underbrace{\sum_i \sum_j}_{i \neq j} \underbrace{E[n_i]E[n_j]}_{\emptyset:\ \text{independent, 0-mean}}$$

$$= \frac{1}{N^2} \cancel{N} \cdot E[n^2] = \frac{1}{N} E[n^2] = \frac{1}{N} \sigma^2$$

$$\Rightarrow \text{Variance } \underline{\underline{E[(S-s)^2]}} = \underline{\underline{\frac{1}{N} \sigma^2}} \quad \text{q.e.d.}$$

averaging of N pixels (or N images)
reduces variance σ^2 by $1/N$ $\left(\frac{\sigma^2}{N}\right)$,
standard deviation σ is reduced by $\frac{1}{\sqrt{N}}$.

Example: 3x3 pixels: $\sigma^* = \frac{\sigma}{3}$

11x11 pixels: $\sigma^* = \frac{\sigma}{11}$

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Other formulation, see slides

How close is sample mean (e.g. by averaging N pixels) to true mean?

Assumption: independence of sampling, independence of noise at each pixel

Define new random variable $D = (S - \bar{s})^2 = \left(\frac{1}{N}\sum s_i - \bar{S}\right)^2$

$$\Rightarrow D = \frac{1}{N^2}\left(\sum s_i\right)^2 - \frac{1}{N} 2\bar{s}\sum s_i + \bar{s}^2$$

$$= \frac{1}{N^2}\sum_i s_i \sum_j s_j - \frac{1}{N} 2\bar{s}\sum s_i + \bar{s}^2$$

$$E[D] = \frac{1}{N^2} E\left[\sum_i s_i \sum_j s_j\right] \underbrace{- \frac{1}{N} 2\bar{s} E\Big[\overbrace{\sum s_i}^{N\cdot\bar{s}}\Big]}_{-2\bar{s}^2} + \bar{s}^2$$

$$= \frac{1}{N^2} E\left[\sum s_i \sum s_j\right] - \bar{s}^2$$

$$= \underbrace{\frac{1}{N^2}\sum_i E[s_i^2]}_{i=j} + \underbrace{\frac{1}{N^2}\sum_i\sum_j E[s_i s_j]}_{i\neq j} - \bar{s}^2$$

$$= \frac{1}{N^2} N E[s_i^2] + \frac{1}{N^2}\sum_{i\neq j}\sum E[s_i]E[s_j] - \bar{s}^2$$

$$= \frac{1}{N^2} N E[s^2] + \underbrace{\frac{N(N-1)}{N^2}\bar{s}^2} - \bar{s}^2$$

$$\Rightarrow \boxed{E[D] = \frac{1}{N}E[s^2] - \frac{1}{N}\bar{s}^2 = \frac{1}{N}\left(E[s^2] - \bar{s}^2\right) = \frac{1}{N}\sigma^2}$$

Definition Variance: $\sigma^2 = E\left[(s-\bar{s})^2\right] = E[s^2] - \underbrace{E[s]}_{\bar{s}}{}^2$

s: random sample

$\bar{s}$: true mean

Our result:

$$E[D] = E[(\underset{\text{sample mean}}{S} - \underset{\text{true mean}}{\bar{s}})^2] = \frac{1}{N}\left(E[s^2] - \bar{s}^2\right) = \frac{1}{N}\sigma^2$$

- variance reduced by N
- standard deviation reduced by $\sqrt{N}$

if we average over N pixels
or average N instances of repeated images