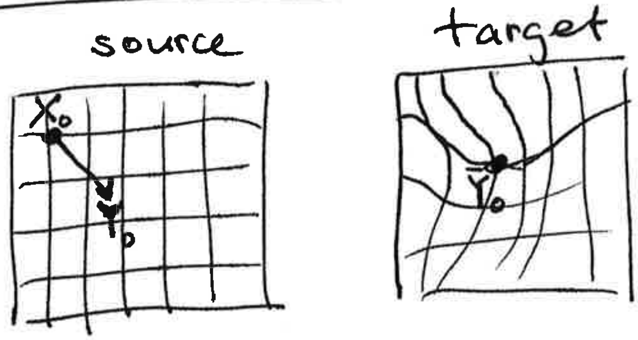
GG 10/20/2014 1

Warping with Radial Basis Functions RBF's



$(\bar{X}_0, \bar{Y}_0)$ pair of landmarks in source and target image

concept: → find ~~transformation~~ so that:

- exact ~~transformation~~ at landmarks
- smooth interpolation in neighborhood
- $\boxed{T(\bar{X}) = \bar{X} + \bar{v}(\bar{X}) = \bar{Y}}$

– radial basis functions:

$$\phi(\|\bar{X}-\bar{Y}\|)$$

possible choices: $\cdot\ \phi(\,) = \exp\left(-\frac{\|\bar{X}-\bar{Y}\|^2}{\sigma^2}\right)$

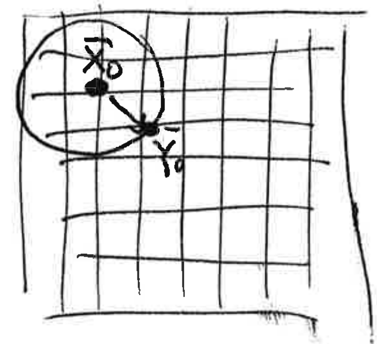
$\cdot\ \phi(\,) = \frac{\sigma^2}{\|\bar{X}-\bar{Y}\|^2 + \sigma^2}$

$\cdot\ \ldots$

important: functions have maximum value for $\bar{X} = \bar{Y}$, and gradually fall off with increasing radial distance

– use radial basis functions centered at landmarks $\bar{X}_0$

- let $\bar{X}_0, \bar{Y}_0$ be a set of landmarks



- $\phi(\ )$ centered at $\bar{X}_0$

$$\rightarrow \quad \phi(\|\bar{X}-\bar{X}_0\|)$$

- $T(\bar{X}) = \bar{X} + \underbrace{\bar{v}(\bar{X})}_{\text{displacement field}}$

$$= \bar{X} + \bar{k}_0 \cdot \phi(\|\bar{X}-\bar{X}_0\|) = \bar{Y}$$

$\bar{k}_0$: weight 2D vector to be determined, called "momentum"

- (x,y)-components: $T^X(\bar{x}) = X^X + k_0^X \cdot \phi(\|\bar{X}-\bar{X}_0\|) = Y^X$

$$T^Y(\bar{x}) = X^Y + k_0^Y \cdot \phi(\|\bar{X}-\bar{X}_0\|) = Y^Y$$

Estimation of $\bar{k}_0$:

$$T(\bar{X}_0) = \bar{Y}_0$$

transformation exactly matches landmark deformation

$$\Rightarrow \quad \bar{X}_0 + \underbrace{\bar{k}_0\, \phi(\bar{X}_0 - \bar{X}_0)}_{\bar{v}(\bar{X}_0)} = \bar{Y}_0$$

$$\Rightarrow \quad \bar{X}_0 + \bar{k}_0 \cdot \underbrace{\phi(0)}_{1} = \bar{Y}_0 \quad \Rightarrow \quad \underline{\underline{\bar{k}_0 = (\bar{Y}_0 - \bar{X}_0)}}$$

$$\underline{\underline{T(\bar{X}) = \bar{X} + \bar{k}_0 \cdot \phi(\|X - X_0\|) = \bar{Y}}}$$

Example: 2 landmark pairs

$$(\bar{X}_1, \bar{Y}_1) \rightarrow \left[\begin{pmatrix} X_1^x \\ X_1^y \end{pmatrix}, \begin{pmatrix} Y_1^x \\ Y_1^y \end{pmatrix}\right]$$

$$(\bar{X}_2, \bar{Y}_2) \rightarrow \left[\begin{pmatrix} X_2^x \\ X_2^y \end{pmatrix}, \begin{pmatrix} Y_2^x \\ Y_2^y \end{pmatrix}\right]$$

$$T(\bar{X}) = \bar{X} + \sum_{i=1}^{2} \bar{k}_i \, \phi(\|\bar{X} - \bar{X}_i\|)$$

component notation for estimation of "weights" $\bar{k}_i$:

$$\begin{cases} T^x(\bar{X}_1) = X_1^x + k_1^x \phi(\|\bar{X}_1 - \bar{X}_1\|) + k_2^x \phi(\|\bar{X}_1 - \bar{X}_2\|) = Y_1^x \\ T^y(\bar{X}_1) = X_1^y + k_1^y \phi(\|\bar{X}_1 - \bar{X}_1\|) + k_2^y \phi(\|\bar{X}_1 - \bar{X}_2\|) = Y_1^y \end{cases}$$

$$\begin{cases} T^x(\bar{X}_2) = X_2^x + k_1^x \phi(\|\bar{X}_2 - \bar{X}_1\|) + k_2^x \phi(\|\bar{X}_2 - \bar{X}_2\|) = Y_2^x \\ T^y(\bar{X}_2) = X_2^y + k_1^y \phi(\|\bar{X}_2 - \bar{X}_1\|) + k_2^y \phi(\|\bar{X}_2 - \bar{X}_2\|) = Y_2^y \end{cases}$$

let $B = \begin{bmatrix} \phi(\bar{X}_1, \bar{X}_1) & \phi(\bar{X}_1, \bar{X}_2) \\ \phi(\bar{X}_2, \bar{X}_1) & \phi(\bar{X}_2, \bar{X}_2) \end{bmatrix} = \begin{bmatrix} 1 & \phi(\bar{X}_1, \bar{X}_2) \\ \phi(\bar{X}_2, \bar{X}_1) & 1 \end{bmatrix}^*$

$$\begin{pmatrix} X_1^x \\ X_2^x \\ X_1^y \\ X_2^y \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} k_1^x \\ k_2^x \\ k_1^y \\ k_2^y \end{pmatrix} = \begin{pmatrix} Y_1^x \\ Y_2^x \\ Y_1^y \\ Y_2^y \end{pmatrix}$$

$\Rightarrow$ 4 equations to solve for 4 unknowns

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* B is symmetric since $\phi(\|\bar{X}_i - \bar{X}_j\|) = \phi\|\bar{X}_j - \bar{X}_i\|$

After the momenta $(\bar{k}_i)$ are known, the image deformation is finally: $T(\bar{X}) = \bar{X} + \bar{v}(\bar{X}) \Rightarrow$

$$T(\bar{X}) = \bar{X} + \sum_{i=1}^{n} \bar{k}_n \, \phi(\|\bar{X} - \bar{X}_n\|) = \bar{Y}$$

This is a forward transformation and as discussed in the course should rather be implemented as the inverse. However, the transformation may not have an inverse (see document Stanley Durrleman, "lecture_10_18.pdf", page 10 for more details).

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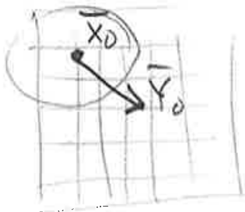
Inverse:

forward: $\bar{Y} = T(\bar{X})$



forward: step through source image $\bar{X}$, calculate target locations $\bar{Y}$, push pixel intensities from $\bar{X}$ to $\bar{Y}$.

backward: $\bar{X} = T^{-1}(\bar{Y})$?

$$\bar{Y} = \bar{X} + \underbrace{\bar{k}_0 \, \phi(\|\bar{X} - \bar{X}_0\|)}_{\bar{v}}$$

$$\bar{X} = \bar{Y} - \bar{v} = \bar{Y} - \bar{k}_0 \, \phi(\|\bar{X} - \bar{X}_0\|)$$

step through target image $\bar{Y}$, calculate $\bar{X}$, but $\bar{X}$ is on both sides

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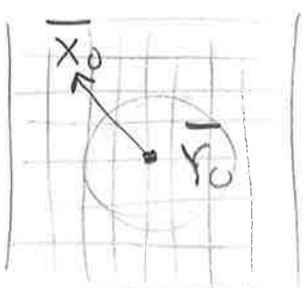
Solution:

There is no inverse of T, but one can calculate a different solution:

$$\bar{Y} = \bar{X} - \bar{k}_0 \, \underbrace{\phi(\|\bar{Y} - \bar{Y}_0\|)}_{\text{kernel centered at } \bar{Y}_0 \,!}$$



verify: for $\bar{Y} = \bar{Y}_0$, we need to get to $\bar{X}_0$ to pull its value:

$$\bar{Y}_0 = \bar{X}_0 - \bar{k}_0 \, \underbrace{\phi(0)}_{1}$$

$$\Rightarrow \bar{k}_0 = \bar{X}_0 - \bar{Y}_0 \quad \checkmark$$