Assignment 3 Theoretical Problems

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Reflectance map

1 Special case

(Book problem 2.1). We see a diffuse sphere centered at the origin, L with radius one and albedo ρ, in an orthographic camera, looking down the z-axis. The sphere is illuminated by a distant point light source whose source direction is (0,0,1). There is no other illumination. Show that the shading field (reflectance map) in the camera is:

$$\rho\sqrt{1-x^2-y^2}$$

Solution:
The equation for a sphere with radius one is:

$$z = f(x,y) = \sqrt{1-x^2-y^2}$$

$$p = \frac{\partial f(x,y)}{\partial x} = \frac{\partial\sqrt{1-x^2-y^2}}{\partial x} = \frac{1}{2\sqrt{1-x^2-y^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2-y^2}} \qquad (1)$$

$$q = \frac{\partial f(x,y)}{\partial y} = \frac{\partial\sqrt{1-x^2-y^2}}{\partial y} = \frac{1}{2\sqrt{1-x^2-y^2}} \cdot (-2y) = \frac{-y}{\sqrt{1-x^2-y^2}} \qquad (2)$$

The reflectance map equation for the light source direction (0, 0, 1) is:

$$R(p,q) = \rho\frac{1}{\sqrt{p^2+q^2+1}} \qquad (3)$$

Plug (1), (2) into (3), we have:

$$R(p,q) = \rho\sqrt{1-x^2-y^2}$$

2 General light source direction

Starting from the reflectance map equation for a general light source direction (see slides and book), show that reflectance map isophotes (curves of constant brightness in R(p,q)) represent conic sections, i.e. curves with squared and linear terms in p and q. You may choose an example for a light source direction and plot the result in (p,q)-space for several values of constant brightness.

Solution:
Assuming we are dealing with Lambertian surface, the equation of the reflectance map for a general light source is:

$$R(p,q) = \rho\frac{p \cdot p_L + q \cdot q_L + 1}{\sqrt{p^2+q^2+1}\sqrt{{p_L}^2+{q_L}^2+1}} \qquad (4)$$

where $p = \frac{\partial f(x,y)}{\partial x}$, $q = \frac{\partial f(x,y)}{\partial y}$, $p_L = \frac{L_x}{L_z}$, $q_L = \frac{L_y}{L_z}$.

For one light source direction, the above equation is equivalent to:

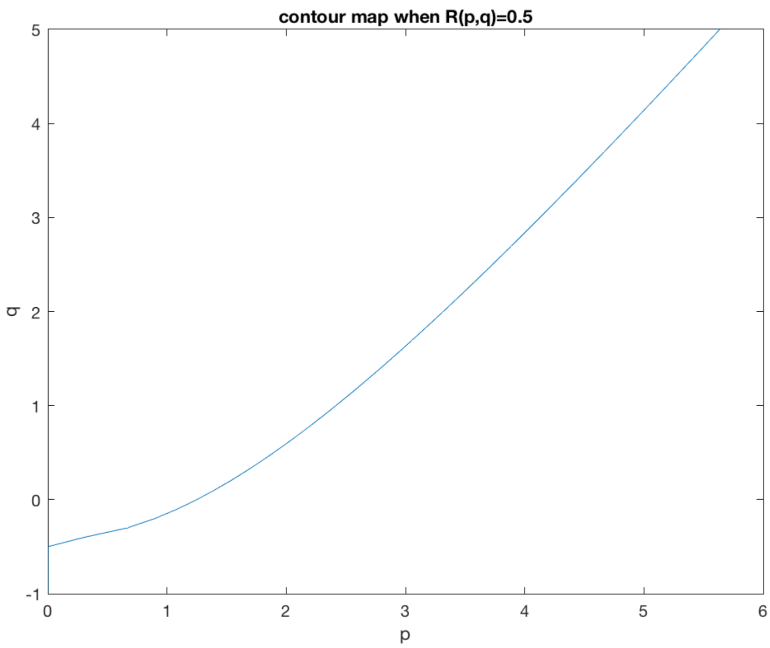
$$R(p,q) = \rho\frac{ap+bq+1}{c\sqrt{p^2+q^2+1}}$$

where a, b, c can be treated as constants for a specific light source.
Let $R(p,q) = 1, \rho = 1$ we have $(ap + bq + 1)^2 = c^2(p^2 + q^2 + 1)$
From this binary quadratic equation we can see that the plot can be curves for some values.
Here we give a simple example for verification:
Suppose the light source direction is (0, 0.6, 0.8) (For simplicity, it is already a normalized vector) and let $R(p,q) = 0.5, \rho = 1$, we have the equation:

$$\frac{[0 \quad 0.6 \quad 0.8]^T \cdot [p \quad q \quad 1]^T}{\sqrt{p^2+q^2+1}} = 0.5$$

It can be derived as: $p = \sqrt{0.44q^2 + 3.84q + 1.56}$

Here is the plot on MATLAB: It can be shown that the isophote is a conic section.



3 Number of images for reconstruction of normals

Following the discussions in the book chapter 2 and slides, explain:
Why is one image with one light source direction not enough?
Getting images with more than one light source directions, justify how many images we would need as a minimum. You can use drawings or equations if you like.

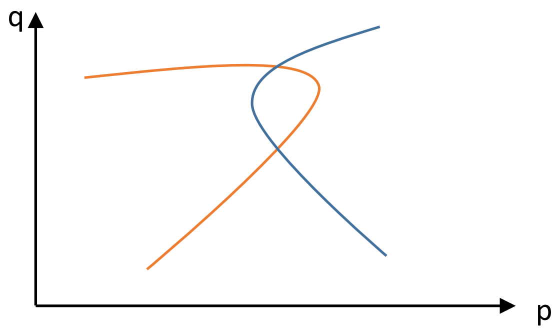
Solution:
From the reflection map we know that given an intensity on the image, there are many (p, q) combinations (i.e. surface orientations), we can hardly determine which surface orientation it has using only one light source direction.

We need at least 3 images to determine the surface normals of a shape.
Given two images from different views, they can be characterized by two independent equations:

$$I_1(x, y) = R_1(p, q) \qquad (5)$$
$$I_2(x, y) = R_2(p, q) \qquad (6)$$

From equation (4) we know that we ρ, p, q are unknown, if we know ρ and if (5) and (6) are linear equations we can determine surface orientation using two equations. However, equation (5) and (6) are nonlinear so that we need a third equation to overdetermine the solution. That is to say, we need three images of different source light directions to determine the surface orientation.

Graphically, we draw two reflectance map (see below) representing two different source light direction of a specific image point under p-q space, we can see that there may exist the situation when two curves intersect at two points. Thus we can't determine from two source light directions.



If we have 3 light source directions, we can determine the unique intensity.