Assignment 1: Theoretical Problems

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Problem 1: Pinhole Camera

a) "A straight line in the world space is projected onto a straight line at the image plane". Prove by geometric consideration (qualitative explanation via reasoning). Assume perspective projection.

Solution:

Imagine every point on the straight line in the word space emits a light ray through the pinhole and is projected onto the image plane. These rays are coplanar so the intersection of this plane and the image plane forms the projected line, which must be a straight line.

b) Show that, in the pinhole camera model, three collinear points in 3-D space are imaged into three collinear points on the image plane (show via a formal solution).

Solution:

Suppose we have two points in the world space: $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$.

The line these two points form can be represented as:

$$x(t) = x_1 + (x_2 - x_1)t$$
$$y(t) = y_1 + (y_2 - y_1)t$$
$$z(t) = z_1 + (z_2 - z_1)t$$

When t = 0, it represents p_1; when t = 1, it represents p_2.

Now choose the third point $p_3(x_3, y_3, z_3)$ on this line.

Let t = 0.5, we have $p_3(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$.

According to the perspective projection equations,

$$x' = f'\frac{x}{z}$$

$$y' = f'\frac{y}{z}$$

Suppose $f' = 1$, we can transform 3 points onto the image plane:

$$p_1'(\frac{x_1}{z_1}, \frac{y_1}{z_1}),\ p_2'(\frac{x_2}{z_2}, \frac{y_2}{z_2}),\ p_3'(\frac{x_1+x_2}{z_1+z_2}, \frac{y_1+y_2}{z_1+z_2}).$$

The straight line with p_1' and p_2' on the image plane can be represented as:

$$x'(t) = \frac{x_1}{z_1} + \left(\frac{x_2}{z_2} - \frac{x_1}{z_1}\right)t$$

$$y'(t) = \frac{y_1}{z_1} + \left(\frac{y_2}{z_2} - \frac{y_1}{z_1}\right)t$$

p_3' must be on this straight line because of the linearity in parameter t.

Thus three points on the image points are still collinear.

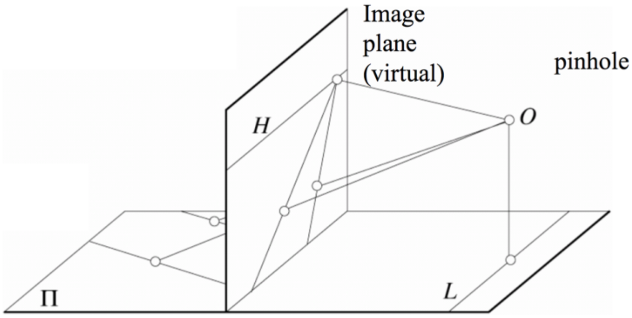
Problem 2: Perspective Projection

See Fig. 1.4 from textbook on page 6 (pdf handouts) for reference.

a) Prove geometrically that the projections of two parallel lines lying in some plane Π appear to converge on a horizon line H formed by the intersection of the image plane with the plane parallel to Π and passing through the pinhole.

Solution:

From the below figure captured from the slides we can see that two parallel lines converge on line H formed by the intersection of the virtual image plane and the plane parallel to Π.



b) Prove the same result algebraically using the perspective projection equation. You can assume for simplicity that the plane Π is orthogonal to the image plane (as you might see in an image of railway tracks, e.g.).

Solution:

Define the equation of the plane Π as y = y_0, the line equation on the plane Π can be written as ax + bz = c, where a, b, c are parameters of the line equation. Based on the perspective projection equation, we have:

$$\begin{cases} x' = f'\dfrac{x}{z} \\ y' = f'\dfrac{y}{z} \end{cases} \Rightarrow \begin{cases} x' = f'\dfrac{c-bz}{az} = f'(\dfrac{c}{az} - \dfrac{b}{a}) \\ y' = f'\dfrac{y_0}{z} \end{cases}$$

When the lines are very far from the camera, which means $z = -\infty$, the above equations are further derived as:

$$\begin{cases} x' = -f'\dfrac{b}{a} \\ y' = 0 \end{cases}$$

This means that all parallel lines with slop $-\frac{b}{a}$ will converge on the point $(-f'\frac{b}{a}, 0)$ on the image plane, namely on the x axis of the image plane. This x axis is the horizontal line H in (a).

Problem 3: Coordinates of Optical Center

Let O denote the homogeneous coordinate vector of the optical center of a camera in some reference frame, and let M denote the corresponding perspective projection matrix. Show that MO = 0. (Hint: Think about the coordinates of the optical center in the world coordinate system, use the notion of transformations between world and camera, and plug this into the projection equation.)

Solution:

Let $O = (O_x, O_y, O_z, 1)^T$ be the coordinate in the world frame, $O_c = (O_{cx}, O_{cy}, O_{cz}, 1)^T$ be the coordinate in the camera frame. We know that $O_c = (0, 0,0,1)^T$ because it's the origin of the camera frame. Define the transformation matrix from the world origin to the camera's optical center as $A = \begin{pmatrix} R^T & t \\ 0^T & 1 \end{pmatrix}$, we know that $M = KA = K \begin{pmatrix} R^T & t \\ 0^T & 1 \end{pmatrix}$, where K is the transformation from camera frame to image frame. Since $AO = O_c, O = A^{-1}O_c$.

Plug $O = A^{-1}O_c$ and $M = KA$ into MO:

$$MO = KAA^{-1}O_c = KO_c = K(0, 0,0,1)^T = 0$$