6B13 MAT : LINEAR ALGEBRA

UNIT I - VECTOR SPACES (20 Hours)

Introduction, vector spaces, subspaces, Linear Combinations & Systems of Linear Equations (Section 1.1, 1.2, 1.3 of T1)

UNIT II - BASES AND DIMENSION (20 Hours)

Linear dependence & Linear Independence, Bases & Dimensions, Maximal linearly Independent Subsets (Sections 1.5, 1.6, 1.7 of T1)

UNIT III - LINEAR TRANSFORMATIONS, MATRICES (25 Hours)

Linear transformations, Null spaces & ranges (proof of thm 2.3 excluded), The Matrix Representation of a Linear transformation (Sections 2.1, 2.2 of T1) (operation of Linear Transformations & related theorems are excluded)

Introduction, Rank of a Matrix, Elementary transformations of a Matrix, Invariance of rank through elementary transformations, elementary transformations of a matrix do not alter its rank, Multiplication of the elements of a row by a ~~matrix~~ ~~do~~ non-zero number does not alter the rank, Addition to the elements of a row the products by a number of the corresponding elements of a row does not alter the rank, Reduction to normal form (Proof of theorem excluded). Elementary matrices, elementary transformations and elementary matrices, Employment of only row (column) transformations. The rank of a product. A convenient method for computing the inverse of a non singular matrix by elementary row transformations (section 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11, 4.12, 4.13 of T2)

UNIT IV - SYSTEM OF LINEAR EQUATIONS, EIGEN VALUES AND EIGEN VECTORS (25 Hours)

Introduction, System of linear homogenous eq^ns. Null Space & nullity of matrix, Sylvesters law of nullity, Range of a Matrix, Systems of linear non homogenous eq^ns (Sections 6.1, 6.2, 6.3, 6.4, 6.5, 6.6 of T2)

Eigen Values, eigen vectors, properties of eigen values, Cayley-Hamilton theorem (without proof) (Section 2.13, 2.14, 2.15 of T3)

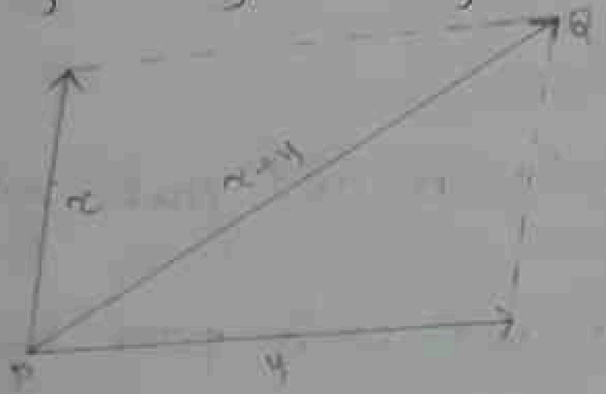
TEXTS

1. SH Friedberg, A.J Insel & L.E Spence, Linear Algebra (4th edition) PH Inc
2. S. Narayan & Mittal, A Text Book of Matrices (Revised edition) S Chand
3. B.S Grewal, Higher Engineering Mathematics (41st edition) Khanna Publishers

VECTOR SPACE

PARALLELOGRAM LAW FOR VECTOR ADDITION

The sum of two vectors x & y that act at the same point P is the vector begining at P that is represented by the diagonal of parallelogram having x & y as adjecent sides.



VECTOR ADDITION

The operation of adding two or more vectors together into a vector sum.

SCALAR MULTIPLICATION

Multiplication of a vector by a scalar (having magnitude)

The algebraic description of vector addition & scalar multiplication for vectors in a plane yield the following prop.:-

1) ∀ vectors x & y, $x+y = y+x$

2) ∀ vectors x, y & z,

$(x+y)+z = x+(y+z)$

3) ∃ a vector denoted by 0 ∋ $x+0 = x$ for each vector x.

4) For each vector x, there is a vector y ∋

$x+y=0$

5) For each vector x, $1x = x$

6) For each pair of R's 'a & b' and each vector x,

$(ab)x = a(bx)$

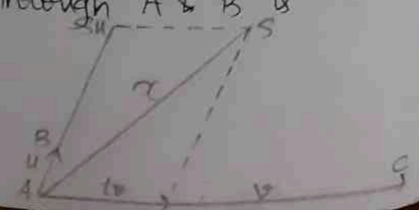
7) For each R 'a' and each pair of vectors x & y, $a(x+y) = ax+ay$

8) For each pair of R's 'a' & 'b' and each vector x, $(a+b)x = ax+bx$.

Example 1

Let A & B be points having coordinates $(-2,0,1)$ & $(4,5,3)$ resp. The end point C of the vector emanating from the origin & having the same direction as the vector begining at A & terminating at B has coordinates $(4,5,3)-(-2,0,1) = (6,5,2)$. Hence the eq^n of the line through A & B is

$x = (-2,0,1) + t(6,5,2)$



Now let A, B & C denote any 3 non-collinear points in space. These points determine a unique plane. Let u & v denote vectors begining at A & ending at B & C resp. Observe that any point in the plane containing A, B & C is the endpoint S of a vector x begining at A & having the form $su+tv$ for some real no.s s & t. The end point of su is the point of intersection of the line through A & B with the line through S ∥^{l} to the line through A & C. A similar procedure locates the endpoint of tv. Moreover, for any real no.s s & t, the vector $su+tv$ lies in the plane containing A, B & C. It follows that an eq^n of the plane containing A, B & C is

$$x = A + su + tv,$$

where s & t are arbitrary real no.s & x denotes an arbitrary point in the plane.

Example 2

Let A, B & C be the points having coordinates $(1,0,2)$, $(-3,-2,4)$ & $(1,8,-5)$ resp. The endpoint of the vector emanating from the origin & having the same length & direction as the vector begining at A & terminating at B is

$$(-3,-2,4) - (1,0,2) = (-4,-2,2)$$

III^{ly}, the endpoint of a vector emanating from the origin and having the same length & direction as the vector begining at A & terminating at C is

$(1,8,-5) - (1,0,2) = (0,8,-7)$

Hence the eqn of the plane containing the 3 given points is

$x = (1,0,2) + s(-4,-2,2) + t(0,8,-7)$

Any mathematical structure possessing the 8 prop. is called a vector space.

VECTOR SPACES

A vector space V over a field F consists of a set on which two operations called addition & scalar multiplication resp. are defined so that for each pair of elements x, y in V there is a unique element $x+y$ in V, and for each 'a' in F and each element x in V there is a unique element ax in V & the following conditions hold.

(VS1) $\forall x, y \in V$, $x+y = y+x$ (Commutativity of addition)

(VS2) $\forall x, y, z \in V$, $(x+y)+z = x+(y+z)$ (associativity of addition)

(VS3) $\exists$ an element in V denoted by 0 s.t. $x+0 = x$ for each $x \in V$

(VS4) For each element $x \in V$ $\exists$ an element $y \in V$ s.t. $x+y=0$

(VS5) For each element $x \in V$, $1x = x$

(VS6) For each pair of elements $a, b \in F$ and each element $x \in V$, $(ab)x = a(bx)$

(VS7) For each element $a \in F$ and each pair of elements $x, y \in V$, $a(x+y) = ax+ay$

(VS8) For each pair of elements $a, b \in F$ and each element $x \in V$, $(a+b)x = ax+bx$

The elements $x+y$ and ax are called the sum of x & y and the product of a & x resp.

The elements of the field F are called scalars and the elements of the vector space V are called vectors.

An object of the form $(a_1, a_2, \dots a_n)$ where the entries $a_1, a_2 \dots a_n$ are elements of a field F, is called an n-tuple with entries from F. The elements $a_1, a_2 \dots a_n$ are called the entries or components of the n-tuples. Two n-tuples $(a_1, a_2 \dots a_n)$ & $(b_1, b_2 \dots b_n)$ with entries from a field F are called equal if $a_i = b_i$ for $i = 1, 2 \dots n$.

POLYNOMIAL

A polynomial with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

n - non-ve integer

a_k - coefficient of x^k in F.

If $f(x)=0$, ie if $a_n = a_{n-1} = \dots = a_0 = 0$, then $f(x)$ is called zero polynomial and its degree is defined to be -1, otherwise, the degree of a polynomial is defined to be the largest exponent of x that appears in the representation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

with a zero coefficient.

Polynomials of degree zero may be written in the form $f(x) = c$ for some non-zero scalar c.

SEQUENCE

Let F be any field. A sequence in F is a function σ from the +ve integer into F. The sequence σ ∋ $\sigma(n) = a_n$ for $n = 1, 2 \dots$ is denoted $\{a_n\}$. Let V consist of all sequences $\{a_n\}$ in F that have only a finite no. of non-zero terms a_n. If $\{a_n\}$ & $\{b_n\}$ are in V & $t \in F$, ·

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \&$$

$$t\{a_n\} = \{ta_n\}$$

with these operations V is a vector space.

EXAMPLE

Let $S = \{(a_1, a_2) : a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in S$ & $c \in R$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$ &
$c(a_1, a_2) = (ca_1, ca_2)$

∵ (VS 1), (VS 2) & (VS 8) fails, S is not a vector space.

For $(a_1, a_2), (b_1, b_2) \in S$ & $c \in S$,

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0) \quad \&$$

$$c(a_1, a_2) = (ca_1, 0)$$

Then S is not a vector space. ∵ (VS 3) (VS 4) & (VS 5) fails.

CANCELLATION LAW FOR VECTOR ADDITION

If x, y & z are vectors in a vector space V ∋ $x + z = y + z$, then $x = y$

PROOF

∃ a vector $v \in V$ ∋ $z + v = 0$ (VS 4)

Thus,

$$x = x + 0 = x + (z + v) = (x + z) + v$$

$$= (y + z) + v = y + (z + v)$$

$$= y + 0 = y$$

[(VS 2) & (VS 3)]

COROLLARY 1

The vector zero described in the definition of vector space is unique [identity element is unique]

PROOF

Let $x \in V$

Suppose that there are two elements 0 & $0'$ ∋

$x + 0 = x$ & $x + 0' = x$

$\alpha + 0 = \alpha + 0'$

using cancellation law,

$0 = 0'$

$\therefore$ the vector zero is unique

OR

Let e, e' be two identities

$e + a = a$

$e' + a = a$

$e + a = e' + a$

cancellation law,

$e = e'$

COROLLARY 2

The vector y described in the definition of vector space is unique [additive inverse is unique]

PROOF

Let $\alpha \in V$

Suppose that there are two elements y & z s

$\alpha + y = 0$ & $\alpha + z = 0$

$\alpha + y = \alpha + z$

By cancellation law

$y = z$

$\therefore$ the vector y is unique

OR

Let a', a'' are two inverses

$a' + a = 0$

$a'' + a = 0$

$a' + a = a'' + a$

$a' = a''$

THEOREM (1.2)

In any vector space V, the following statements are true:

a) $0\alpha = 0$ for each $\alpha \in V$

b) $(-a)\alpha = -(a\alpha) = a(-\alpha)$ for each $a \in F$ & each $\alpha \in V$

c) $a0 = 0$ for each $a \in F$

PROOF

a) Consider $0\alpha + 0\alpha = (0+0)\alpha$ (distributive law)

$= 0\alpha$ — ①

Also we have

$0\alpha + 0 = 0\alpha$ — ② (existance identity)

from ① & ②

$0\alpha + 0\alpha = 0\alpha + 0$

$\therefore$ By cancellation law

$0\alpha = 0$

b) If $a\alpha \in V \ \exists \ -(a\alpha) \in V$ s

$a\alpha + (-(a\alpha)) = 0$ — ③

Now.

$a\alpha + (-a)\alpha = (a+-a)\alpha$ (distributive law)

$= 0\alpha = 0$ — ④

③ & ④ $\Rightarrow a\alpha + (-(a\alpha)) = a\alpha + (-a)\alpha$

By cancellation law

$-(a\alpha) = (-a)\alpha$

Put $a = 1$ in this eqn

$-(1\alpha) = (-1)\alpha$

$-\alpha = (-1)\alpha$

$a(-\alpha) = a[(-1)\alpha]$

$= [a(-1)]\alpha$

$= (-a)\alpha$

$\therefore -(a\alpha) = (-a)\alpha$

$= a(-\alpha)$

c) $a0 + a0 = a(0+0)$

$= a0$ — ⑤

$a0 + 0 = a0$ — ⑥

from ⑤ & ⑥

$a0 + a0 = a0 + 0$

$\therefore$ By cancellation law, $a0 = 0$

REMARK

The set of all n-tuples with entries from a field F is denoted by F^n.

EXAMPLES

1) P.T $\mathbb{R}^2$ over $\mathbb{R}$ is a vector space with the follow. operations

we have $\mathbb{R}^2 = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$

For all $u = (a_1, a_2)$ & $v = (b_1, b_2) \in \mathbb{R}^2$

Define $u + v = (a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2) \in \mathbb{R}^2$

& $au = a(a_1, a_2) = (aa_1, aa_2) \in \mathbb{R}^2$

A) 1) Let $u, v \in \mathbb{R}^2$

Thus $u = (a_1, a_2)$ & $v = (b_1, b_2)$

$u + v = (a_1 + b_1, a_2 + b_2)$

$= (b_1 + a_1, b_2 + a_2)$

$= (b_1, b_2) + (a_1, a_2)$

$= v + u$

$\therefore$ Addition is commutative

2) Let $w = (c_1, c_2) \in \mathbb{R}^2$

$(u+v) + w = [(a_1, a_2) + (b_1, b_2)] + (c_1, c_2)$

$= [a_1+b_1, a_2+b_2] + (c_1, c_2)$

$= [a_1 + b_1 + c_1, a_2 + b_2 + c_2]$

$= (a_1, a_2) [(b_1 + c_1), (b_2 + c_2)]$

$= u + (v + w)$

3) $x \in V,\ x + 0 = x$

$(a_1, a_2) \in R^2$

$(a_1, a_2) + (0,0) = (a_1, a_2)$

$\therefore$ zero element exist in V

4) $x \in V\ \exists\ y \in V,\ x + y = 0$

$(a_1, a_2) \in R^2$

$(a_1, a_2) + (-a_1, -a_2) = (0,0)$

$\therefore$ inverse exist

5) $(a_1, a_2) \in R^2$

$(a_1, a_2)(1,1) = (a_1, a_2)$

6) $(a,b) \in R$ & $x \in R^2$

$(a,b)(c_1, c_2) = (abc_1, abc_2)$

$= a(bc_1, bc_2)$

$= a(b(c_1, c_2))$

7) $a((a_1, a_2) + (b_1, b_2)) = a(a_1 + b_1, a_2 + b_2)$

$= (aa_1 + ab_1, aa_2 + ab_2)$

$= a(a_1, a_2) + a(b_1, b_2)$

8) $(a+b)(a_1, a_2) = (aa_1, aa_2) + (ba_1, ba_2)$

$= a(a_1, a_2) + b(a_1, a_2)$

$= au + bu$

SUBSPACES

A subset W of a V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

In any vector space V, V & {0} are subspaces. The latter is called the zero subspace of V.

THEOREM (1.3

A subset W of a vector space V is a subspace iff the following properties hold.

1) $x + y \in W$ if $x, y \in W$ (W is called closed under addition)

2) $cx \in W$, $c \in F$ & $x \in W$ (W is closed under scalar multiplication)

3) W has a zero vector

4) Each vector in W has a additive inverse in W.

THEOREM

Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

a) $0 \in W$

b) $x+y \in W$ whenever $x \in W$ & $y \in W$

c) $cx \in W$ whenever $c \in F$ & $x \in W$

PROOF

Let V be a vector space and W be a subset of V. Suppose that W is a subspace of V

$\because$ W is a subspace of V, W is a vector space with the operations of additions and scalar multiplication defined on V.

$\therefore$ Condition b) & c) hold

$\because$ W is a subspace of V, for $x \in W$ $\exists$ $0' \in W$ s.t $x + 0' = x$

Also, $x + 0 = x$ for each $x \in V$

$\Rightarrow x + 0' = x + 0$

Then by using cancellation law,

$0' = 0$

$\therefore$ $0 \in W$

$\therefore$ Condition i) hold

Conversely suppose that Condition i), ii), iii) hold

Let $W \subseteq V$

We want to P.T W is a subspace of V

condition ii) & iii) implies W is closed under addition & scalar multiplication

Also condition i) implies W has a zero vector

$\therefore$ to prove W is a subspace of V it is enough to prove that each vector in W has a additive inverse in W.

Let $x \in W$ then choosing $c = -1 \in F$ in Condition iii) we get $(-1)x \in W \Rightarrow -x \in W$

$\therefore$ additive inverse exist in W

$\therefore$ W is a subspace of V

TRANSPOSE

The transpose of A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns.

i.e. $(A^t)_{ij} = A_{ji}$

eg: $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

SYMMETRIC MATRIX

A symmetric matrix is a matrix A s.t.

$$A^t = A$$

A symmetric matrix must be square

EXAMPLES

1) The set W of all symmetric matrices in $M_{n\times n}(F)$ is a subspace of $M_{n\times n}(F)$.

i) The zero matrix is equal to its transpose & hence belongs to W.

ii) If $A \in W$ & $B \in W$, then $A^t = A$ & $B^t = B$. Thus

$(A+B)^t = A^t + B^t = A+B$

Thus $A+B \in W$

iii) If $A \in W$, then $A^t = A$

For any $a \in F$, we've

$(aA)^t = aA^t = aA$

Thus $aA \in W$

Hence, W is a subspace of $M_{n\times n}(F)$ (from thm)

2) Let $C(R)$ denote the set of all continuous real valued functions defined on R. Then $C(R)$ is a subspace of the vector space $F(R,R)$ (i.e. set of all fun. from $R \to R$)

i) The zero fun. of $F(R,R)$ is the constant fun. defined by $f(t) = 0 \ \forall \ t \in R$. $\because$ Constant fun.s are continuous, we've $f \in C(R)$

ii & iii) The sum of 2 conti. fun.s is conti. & the product of a real no. & a conti. fun is conti. So $C(R)$ is closed under addition & scalar multiplication and hence is a subspace of $F(R,R)$ by thm.

3) Let n be a non-ve integer & let $P_n(F)$ consist of all polynomials in $P(F)$ having degree less than or equal to n. P.T $P_n(F)$ is a subspace of $P(F)$

I) $\because$ the zero polynomial has degree -1, it is in $P_n(F)$.

II) The sum of two polynomials with degree less than or equal to n is a polynomial with degree less than or equal to n. So $P_n(F)$ is closed under addition.

III) The product of a scalar & a polynomial of degree less than or equal to n is a poly. of degree less than or equal to n. $\therefore P_n(F)$ is closed under scalar multiplication.

Hence by thm, $P_n(F)$ is a subspace of $P(F)$.

4) P.T the set of diagonal matrices is a subspace of $M_{n\times n}(F)$

i) Zero matrix is a diagonal matrix because all of its entries are 0.

ii) & iii) If A & B are diagonal $n \times n$ matrices, then whenever $i \neq j$, $(A+B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$ &

$(CA)_{ij} = CA_{ij} = c0 = 0$, for any scalar c.

Hence $A+B$ & CA are diagonal matrices for any scalar c $\therefore$ the set of diagonal matrices is a subspace of $M_{n \times n}(F)$ by thm.

5) P.T the set of $n \times n$ matrices having trace equal to zero is a subspace of $M_{n \times n}(F)$

i) Trace of a zero matrix equal to zero, because all of its entries are 0.

ii) If A, B are $n \times n$ matrices having trace equal to zero. ie $a_{11} + a_{22} + \cdots + a_{nn} = 0$ & $b_{11} + b_{22} + \cdots + b_{nn} = 0$

Then,

$A+B$ is also an $n \times n$ matrix with $tr(A+B) = 0$

$$\because tr(A+B) = (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn})$$
$$= (a_{11}+a_{22}+\cdots+a_{nn}) + (b_{11}+b_{22}+\cdots+b_{nn})$$
$$= 0 + 0 = 0$$

iii) If A be any $n \times n$ matrix with $tr(A) = 0$, then A is also an $n \times n$ matrix with $tr(CA) = 0$, where c is any scalar.

$$\left[tr(CA) = (ca_{11} + ca_{22} + \cdots + ca_{nn}) \right.$$
$$= c(a_{11} + a_{22} + \cdots + a_{nn})$$
$$\left. = c \times 0 = 0 \right]$$

6) Is $M_{m \times n}(\mathbb{R})$ having non-ve entries subspace of $M_{m \times n}(\mathbb{R})$. Justify your ans.

$M_{m \times n}(\mathbb{R})$ having non-ve entries is not a subspace of $M_{m \times n}(\mathbb{R})$ because it is not closed under scalar multiplication (by -ve scalar)

THEOREM (1.4)

Any intersection of subspaces of a vector space V is a subspace of V.

PROOF

Let c be a collection of subspaces of V & let W denote the intersection of the subspaces in c. $\because$ every subspaces contains the zero vector, $0 \in W$. Let $a \in F$ & $x, y \in W$. Then x & y are contained in each subspace in c. Because each subspace in c is closed under addition & scalar multiplication, it follows that $x+y$ & ax are contained in each subspace in c. Hence $x+y$ & ax are also contained in W, so that W is a subspace of V by thm.

6 Is union of subspaces of V a subspace of V. Justify.

The union of subspaces must contain the zero vector & be closed under scalar multiplication, but in general the union of subspaces of V need not be closed under addition.

eg. $W_1 = \{(a,0) \& (0,b) \mid a,b \in R\}$

$W_2 = \{(0,b) \mid b \in R\}$

are subset of R^2

$W_1 \cup W_2 = \{(a,0) \& (0,b) \mid a,b \in R\}$

$(1,0) \& (0,1) \in W_1 \cup W_2$ but

$(1,0) + (0,1) = (1,1) \notin W_1 \cup W_2$

ie $W_1 \cup W_2$ is not closed under addition

$\therefore W_1 \cup W_2$ is not a subspace

NOTE:- The union of two subspaces of V is a subspace of V iff one of the subspaces contains the other.

LINEAR COMBINATIONS AND SYSTEM OF LINEAR EQUATIONS

Let V be a vector space & S a non-empty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if $\exists$ a finite no. of vectors, $u_1, u_2 \ldots u_n$ in S & scalars $a_1, a_2 \ldots a_n$ in F s $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. In this case we also say that v is a linear combination of $u_1, u_2 \ldots u_n$ & call $a_1, a_2 \ldots a_n$ the coefficients of the linear combination

NOTE:- In any vector space V, $0v = 0$ for each $v \in V$. Thus the zero vector is a linear combination of any non-empty subset of V.

EXAMPLE

1) Determine the vector $(2,6,8)$ can be expressed as a linear combination of $u_1 = (1,2,1)$, $u_2 = (-2,-4,-2)$, $u_3 = (0,2,3)$, $u_4 = (2,0,-3)$ & $u_5 = (-3,8,16)$

For this we must determine if there are scalars a_1, a_2, a_3, a_4 & a_5 s

$$(2,6,8) = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5$$

$$= a_1(1,2,1) + a_2(-2,-4,-2) + a_3(0,2,3) + a_4(2,0,-3) + a_5(-3,8,16)$$

$$= (a_1 - 2a_2 + 2a_4 - 3a_5,\ 2a_1 - 4a_2 + 2a_3 + 8a_5,\ a_1 - 2a_2 + 3a_3 - 4a_4 + 16a_5)$$

Hence (2,6,8) can be expressed as a linear combination of u_1, u_2, u_3, u_4 & u_5 iff there is a 5-tuple of scalars $(a_1, a_2, a_3, a_4, a_5)$ satisfying the system of linear equations

$$\left.\begin{aligned} a_1 - 2a_2 + 2a_4 - 3a_5 &= 2 \\ 2a_1 - 4a_2 + 2a_3 + 8a_5 &= 6 \\ a_1 - 2a_2 + 3a_3 - 4a_4 + 16a_5 &= 8 \end{aligned}\right\} \quad (1)$$

NOTE:- To solve system (1), we replace it by another system with the same sol., but which is easier to solve.

Procedure

Uses 3 types of operations to simplify the original system:

1) Interchanging the order of any 2 equ. in the system.

2) Multiplying any equ. in the system by a non-zero constant.

3) Adding a constant multiple of any eq^n to another eq^n in the system.

These operations do not change the set of sol^s to the original system. We use these operations to obtain a system of eq^{n}s that had the follow. properties:-

(1) The 1st non-zero coefficient in each eq^n is one.

(2) If an unknown is the 1st unknown with a non-zero coefficient in some eq^n, then that unknown occurs with a zero coefficient in each of the other eq^{n}s

(3) The 1st unknown with a non-zero coefficient in any eq^n has a larger subscript than the 1st unknown with a non-zero coefficient in any preceding eq^n.

eg.- None of the follow. systems meets these requirements

$$\left.\begin{aligned} x_1 + 3x_2 + x_4 &= 7 \\ 2x_3 - 5x_4 &= -1 \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 + x_5 &= -5 \\ x_3 - 2x_5 &= 9 \\ x_4 + 3x_5 &= 6 \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} x_1 - 2x_3 + x_5 &= 1 \\ x_4 - 6x_5 &= 0 \\ x_2 + 5x_3 - 3x_5 &= 2 \end{aligned}\right\} \quad (4)$$

System (2) does not satisfy prop. (1) because the 1st non-zero coefficient in the 2nd eq^n is 2; System (3) does not satisfy prop. (2) because x_3, the 1st unknown with a non-zero coefficient in the 2nd eq^n, occurs with a non-zero coefficient in the 1st eqn, & system ④ does not satisfy prop (3) because a_2 the 1st unknown with a non-zero coefficient in the 3rd eqn, does not have a longer subscript than a_4, the 1st unknown with a non-zero coefficient in the 2nd eqn.

Once a system with prop (1), (2) & (3) has been obtained, it is easy to solve for some of the unknowns in terms of the others.

In the course of using equations ~~①, (2) & (3)~~ 1), 2) & 3) a system containing an eqn of the form $0 = c$, where c is non-zero, is obtained, then the original system has no sol.s

To solve system ① we replace it by another system with the same sol.s but which is easier to solve

$$\left.\begin{aligned} a_1 - 2a_2 + 2a_4 - 3a_5 &= 2 \\ 2a_3 - 4a_4 + 14a_5 &= 2 \\ 3a_3 - 5a_4 + 19a_5 &= 6 \end{aligned}\right\} ②$$

$$\left.\begin{aligned} a_1 - 2a_2 + 2a_4 - 3a_5 &= 2 \\ a_3 - 2a_4 + 7a_5 &= 1 \\ 3a_3 - 5a_4 + 19a_5 &= 6 \end{aligned}\right\} ③$$

$$\left.\begin{aligned} a_1 - 2a_2 + 2a_4 - 3a_5 &= 2 \\ a_3 - 2a_4 + 7a_5 &= 1 \\ a_4 - 2a_5 &= 3 \end{aligned}\right\} ④$$

$$\left.\begin{aligned} a_1 - 2a_2 + a_5 &= -4 \\ a_3 + 3a_5 &= 7 \\ a_4 - 2a_5 &= 3 \end{aligned}\right\} ⑤$$

ie $a_1 = 2a_2 - a_5 - 4$

$a_3 = -3a_5 + 7$

$a_4 = 2a_5 + 3$

Thus for any choice of scalars a_2 & a_5, a vector of the form

$(a_1, a_2, a_3, a_4, a_5) = (2a_2 - a_5 - 4, a_2, -3a_5 + 7, 2a_5 + 3, a_5)$

is a sol. to system ①. In particular, the vector $(-4, 0, 7, 3, 0)$ obtained by setting $a_2 = 0$ & $a_5 = 0$ is a sol to ①.

$\therefore (2,6,8) = -4u_1 + 0u_2 + 7u_3 + 3u_4 + 0u_5$

So that $(2,6,8)$ is a linear combination of u_1, u_2, u_3, u_4 & u_5.

2) The vectors $(1,1,0)$, $(1,0,1)$ & $(0,1,1)$ generates R^3 since an arbitrary vector (a_1, a_2, a_3) in R^3 is a linear combination of the 3 given vectors. in fact the scalars r, s & t for which

$$r(1,1,0) + s(1,0,1) + t(0,1,1) = (a_1, a_2, a_3)$$

are

$$r = \frac{1}{2}(a_1 + a_2 - a_3)$$

$$s = \frac{1}{2}(a_1 - a_2 + a_3)$$

$$t = \frac{1}{2}(-a_1 + a_2 + a_3)$$

3) The polynomials $x^2 + 3x - 2$, $2x^2 + 5x - 3$ & $-x^2 - 4x + 4$ generate $P_2(R)$ since each of the 3 given polynomials belongs to $P_2(R)$ & each polynomial $ax^2 + bx + c$ in $P_2(R)$ is a linear combination of these 3, namely

$$(-8a+5b+3c)(x^2+3x-2) + (4a-2b-c)(2x^2+5x-3)$$
$$+(-a+b+c)(-x^2-4x+4) = ax^2+bx+c$$

4) The matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \ \& \ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

generate $M_{2\times2}(R)$ since an arbitrary matrix A in $M_{2\times2}(R)$ can be expressed as a linear combination of the 4 given matrices a follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \left(\tfrac{1}{3}a_{11} + \tfrac{1}{3}a_{12} + \tfrac{1}{3}a_{21} - \tfrac{2}{3}a_{22}\right)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$
$$+\left(\tfrac{1}{3}a_{11} + \tfrac{1}{3}a_{12} - \tfrac{2}{3}a_{21} + \tfrac{1}{3}a_{22}\right)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$+\left(\tfrac{1}{3}a_{11} - \tfrac{2}{3}a_{12} + \tfrac{1}{3}a_{21} + \tfrac{1}{3}a_{22}\right)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$+\left(-\tfrac{2}{3}a_{11} + \tfrac{1}{3}a_{12} + \tfrac{1}{3}a_{21} + \tfrac{1}{3}a_{22}\right)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

on the otherhand, the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ \& \ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

do not generate $M_{2\times2}(R)$ because each of these matrices has equal diagonals entries so any linear combination of these matrices has equal diagonal entries. Hence not every 2x2 matrix is a linear combination of these 3 matrices.

Q. Determine whether the vectors emanating from the origin as terminating at the follow. pairs of points are $\parallel^l$.

i) $(3,1,2)$ & $(6,4,2)$ ii) $(-3,1,7)$ & $(9,-3,-21)$

i) $\vec{u} = 3\vec{i} + \vec{j} + 2\vec{k}$ & $\vec{v} = 6\hat{i} + 4\hat{j} + 2\hat{k}$.

In this case $\exists$ no real no. t s.t. $\vec{v} = t\vec{u}$

$\therefore$ these 2 vectors are not $\parallel^l$.

ii) $\vec{u} = -3\hat{i}+\hat{j}+7\hat{k}$ & $\vec{v} = 9\hat{i}-3\hat{j}-21\hat{k}$

In this case $\exists$ a real no. $t \ni \vec{v} = t\vec{u}$

$\therefore$ these 2 vectors are $\parallel$.

Q. Find the eq^n of the lines through the follow. pairs of points in space

i) $(3,-2,4)$ & $(-5,7,1)$

ii) $(2,4,10)$ & $(-3,-6,0)$

i) $\vec{u} = 3\vec{i}-2\vec{j}+4\vec{k}$

$\vec{v} = -5\vec{i}+7\vec{j}+1\vec{k}$

$\vec{v}-\vec{u} = -8\vec{i}+9\vec{j} - 3\vec{k}$

$\therefore x = \vec{u} + t(\vec{v}-\vec{u})$

$\Rightarrow x = 3\vec{i}-2\vec{j}+4\vec{k} + t(-8\vec{i}+9\vec{j}-3\vec{k})$

$\therefore$ the vector eq^n of the line through A & B is

$x(t) = (3-8t)\vec{i} + (-2+9t)\vec{j} + (4-3t)\vec{k}$

ii) $u = (2,4,0)$, $v = (-3,-6,0)$

w.k.t eq^n of the line through A & B is

$x = u + t(\vec{v}-\vec{u})$

$= (2,4,0) + t(-5,-10,0)$

EQUATION OF A PLANE THROUGH 3 NON-COLLINEAR POINTS

Here,

A is a point in space

t & s are variable $(-\infty$ to $\infty)$

$\vec{u} = AB$ & $\vec{v} = AC$

$\therefore eq^n$ of a plane through 3 non-collinear point is,

$$\boxed{x = A + s\vec{u} + t\vec{v}}$$

Q. Find eq^n of the plane containing the points in space $(1,0,2)$, $(-3,-2,4)$ & $(1,8,-5)$ resp

A) $A(1,0,2)$, $B(-3,-2,4)$, $C(1,8,-5)$

$\vec{u} = AB = (-4,-2,2)$

$\vec{v} = AC = (0,8,-7)$

$\therefore eq^n$ of the plane is,

$A + s\vec{u} + t\vec{v} = x$

$\therefore x = (1,0,2) + s(-4,-2,2) + t(0,8,-7)$

EXAMPLE

i) The set of all n-tuples with entries from a field F is denoted by F^n. This set is a vector space over F with the operations of coordinate wise addition & scalar multiplication. i.e if

$\vec{u} = (a_1, a_2, \dots a_n) \in F^n$, $\vec{v} = (b_1, b_2 \dots b_n) \in F^n$ & $c \in F$, then

$\vec{u} + \vec{v} = (a_1+b_1, a_2+b_2, \ldots a_n+b_n)$ &

$c\vec{u} = (ca_1, ca_2 \ldots ca_n)$

→ Also, $\mathbb{R}^3$ is a vector space over R. In this vector space,

$(3,-2,0) + (-1,1,4) = (2,-1,4)$ & $-5(3,-2,0) = (15,10,0)$

$||^{ly}$ $\mathbb{C}^2$ is a vector space over $\mathbb{C}$

ie $(1+i,2) + (2-3i,4i) = (3-2i, 2+4i)$

& $i(1+i,2) = (i-1, 2i)$

2) The set of all m×n matrices with entries from a field F is a vector space, denoted by $M_{m\times n}(F)$, with operations of matrix addition & scalar multiplication

For $A, B \in M_{m\times n}(F)$ & $c \in F$

$(A+B)_{ij} = A_{ij} + B_{ij}$ &

$(cA)_{ij} = cA_{ij}$

for $1 \le i \le m$ & $1 \le j \le m$. For instance,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

&

$$-3\begin{pmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 6 \\ 9 & -6 & -9 \end{pmatrix} \text{ in } M_{2\times 3}(F)$$

3) Let S be any non-empty set & F be any field, & let F(S,F) denote the set of all fun. from S to F. Two functions f & g in F(S,F) are called equal if $f(s) = g(s)$, for each $s \in S$.

The set F(S,F) is a vector space with the operations of addition & scalar multiplication defined for $f, g \in F(S,F)$ & $c \in F$ by,

$(f+g)(s) = f(s) + g(s)$ &

$(cf)(s) = c[f(s)]$

4) Let $S = \{(a_1,a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1,a_2), (b_1,b_2) \in S$ & $c \in \mathbb{R}$, define $(a_1,a_2) + (b_1,b_2) = (a_1+b_1, a_2-b_2)$ and $c(a_1,a_2) = (ca_1, ca_2)$

Here (VS 0) [ie addition & scalar multiplication are closed] hold. But (VS 1), (VS 2) & (VS 8) fail to hold as,

$(1,2) + (2,3) = (3,-1)$ &

$(2,3) + (1,2) = (3,1)$

∴ (VS 2) fail to hold.

Hence, S is not a vector space.

Q Check whether the following sets are form a vector space with the corresp. vector addition & scalar multiplication.

Let $S = \{(a_1, a_2) : a_1 a_2 \in R\}$

for (a_1,a_2) $(b_1,b_2) \in S$, define

$(a_1,a_2) + (b_1,b_2) = (a_1+b_1, a_2-b_2)$ &

$c(a_1,a_2) = (ca_1, ca_2)$

(a_1,a_2) $(b_1,b_2) \in S$

$(a_1,a_2) + (b_1,b_2) = (a_1+b_1, a_2-b_2) \in S$

$c(a_1,a_2) = (ca_1, ca_2) \in S$

i) $(a_1,a_2) + (b_1,b_2) = (a_1+b_1, a_2-b_2)$

$(b_1,b_2) + (a_1,a_2) = (b_1+a_1, b_2-a_2)$

$(a_1,a_2) + (b_1,b_2) \neq (b_1,b_2) + (a_1,a_2)$

is not commutative

S is not a vector space on R

R^3 even R is a vector space

For $u = (\cancel{R}(a_1,a_2,a_3) \mid a_1, a_2, a_3 \in R)$

for $u = (a_1,a_2,a_3) \in R^3$ & $V = (b_1,b_2,b_3) \in R$

$u+v = (a_1+b_1, a_2+b_2, a_3+b_3) \in R^3$

$cu = c(a_1,a_2,a_3)$

$= (ca_1, ca_2, ca_3) \in R^3$

$u+V = (a_1,a_2,a_3) + (b_1,b_2,b_3)$

$= (a_1+b_1, a_2+b_2, a_3+b_3)$

$= (b_1,b_2,b_3) + (a_1,a_2,a_3)$

$= V+u$

$(u+V)+W = (a_1,a_2,a_3) + (b_1,b_2,b_3) + (c_1,c_2,c_3)$

$= (a_1+b_1, a_2+b_2, a_3+b_3) + (c_1,c_2,c_3)$

$= (a_1,a_2,a_3) + (b_1+c_1, b_2+c_2, b_3+c_3)$

$= (a_1,a_2,a_3) + [(b_1,b_2,b_3) + (c_1,c_2,c_3)]$

$= u + (v+w)$

$(a_1,a_2,a_3) + (0,0,0) = (a_1,a_2,a_3)$

$(a_1,a_2,a_3) + (-a_1,-a_2,-a_3) = (0,0,0)$

$(a_1,a_2,a_3)\ (1,1,1) = a_1 \cdot 1,\ a_2 \cdot 1,\ a_3\ 1$

$= (a_1,a_2,a_3)$

$bc(a_1,a_2,a_3) = (bca_1, bca_2, bca_3)$

$= b(ca_1, ca_2, ca_3)$

$= b(c(a_1,a_2,a_3))$

$b((a_1,a_2,a_3) + (b_1,b_2,b_3))$

$= b(a_1+b_1, a_2+b_2, a_3+b_3)$

$= (b(a_1+b_1) + b(a_2+b_2) + b(a_3+b_3))$

$= (ba_1 + bb_1, ba_2 + bb_2, ba_3 + bb_3)$

$= b(a_1, a_2, a_3) + b(b_1, b_2, b_3)$

$(b+c)(a_1,a_2,a_3) = a_1(b+c), a_2(b+c), a_3(b+c)$

$= a_1b + a_1c, a_2b + a_2c, a_3b + a_3c$

$= (a_1b, a_2b, a_3b) + (a_1c, a_2c, a_3c)$

$= b(a_1, a_2, a_3) + c(a_1, a_2, a_3)$

Q F^n over F is a vector space

Let $u, v \in F^n$

$u = (a_1, a_2 \cdots a_n)$

$v = (b_1, b_2 \cdots b_n)$

$u + v = (a_1, a_2 \cdots a_n) + (b_1, b_2 \cdots b_n)$

$= (a_1+b_1, a_2+b_2, \cdots a_n+b_n)$

$= (b_1+a_1, b_2+a_2, \cdots b_n+a_n)$

$= (b_1, b_2, \cdots b_n) + (a_1, a_2 \cdots a_n)$

$= v + u$

let $u, v, w \in F^n$

$(u+v) + w = (a_1, a_2, \cdots a_n) + (b_1, b_2 \cdots b_n) + (c_1, c_2 \cdots c_n)$

$= (a_1+b_1, a_2+b_2, \cdots a_n+b_n)(c_1, c_2 \cdots c_n)$

$= (a_1+b_1+c_1, a_2+b_2+c_2, \cdots a_n+b_n+c_n)$

$= (a_1, a_2, \cdots a_n)(b_1+c_1, b_2+c_2, \cdots b_n+c_n)$

$= u + (v+w)$

$(a_1, a_2, \cdots a_n) + (0, 0, \cdots 0) = (a_1, a_2 \cdots a_n)$

$(a_1, a_2 \cdots a_n) + (-a_1, -a_2, \cdots -a_n) = (0, 0, \cdots 0)$

$c, d \in F$

$(c,d)(a_1, a_2 \cdots a_n) = (cda_1, cda_2, \cdots cda_n)$

$= c(da_1, da_2, \cdots da_n)$

$= c(d(a_1, a_2 \cdots a_n))$

$c \in F$

$c((a_1, a_2, \cdots a_n) + (b_1, b_2, \cdots b_n))$

$= c((a_1+b_1), a_2+b_2, \cdots a_n+b_n))$

$= (ca_1+cb_1, ca_2+cb_2, \cdots, ca_n+cb_n)$

$= (ca_1, ca_2 \cdots ca_n) + (cb_1, cb_2, \cdots cb_n)$

$= c(a_1, a_2 \cdots a_n) + c(b_1, b_2, \cdots b_n)$

$(b+c)(a_1, a_2, \cdots a_n) = (a_1(b+c), a_2(b+c) \cdots a_n(b+c))$

$= (a_1b + a_1c, a_2b + a_2c, \cdots a_nb + a_nc)$

$= (a_1 b, a_2 b, \dots a_n b) + (a_1 c, a_2 c, \dots a_n c)$

$= b(a_1, a_2, \dots a_n) + c(a_1, a_2 \dots a_n)$

Q. S.T the vector $(2,5,7) \in R^3$ can be represented as the linear combination of the set $\{(1,0,0)\ (0,1,0)\ (0,0,1)\}$

$(2,5,7) \in R^3$

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$

$(2,5,7) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$= (a_1, a_2, a_3)$

$\Rightarrow a_1 = 2,\ a_2 = 5,\ a_3 = 7$

$\therefore (2,5,7) = 2(1,0,0) + 5(0,1,0) + 7(0,0,1)$

Q S.T $2x^3 - 2x^2 + 12x - 6$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ & $3x^3 - 5x^2 - 4x - 9$ in $P_3(R)$ Also S.T $3x^3 - 2x^2 + 7x + 8$ is not a linear combination of above.

We want to find scalars a_1 & a_2 S.

$2x^3 - 2x^2 + 12x - 6 = a_1(x^3 - 2x^2 - 5x - 3)$
$+ a_2(3x^3 - 5x^2 - 4x - 9)$

$= a_1 x^3 - 2a_1 x^2 - 5a_1 x - 3a_1 + 3a_2 x^3 - 5a_2 x^2$
$- 4a_2{}^2 - 9a_2$

$= x^3(a_1 + 3a_2) - x^2(2a_1 + 5a_2) - x(5a_1 + 4a_2)$
$- 3(a_1 + 3a_2)$

$= (a_1 + 3a_2)x^3 + (-2a_1 - 5a_2)x^2 + (-5a_1 - 4a_2)x$
$+ (-3a_1 - 9a_2)$

Equating coefficients of x^3, x^2, x & Constant

$a_1 + 3a_2 = 2$ — 1

$-2a_1 - 5a_2 = -2$ — 2

$-5a_1 - 4a_2 = 12$ — 3

$-3a_1 - 9a_2 = -6$ — 4

1 $\times 2 \Rightarrow 2a_1 + 6a_2 = 4$ — 5

5 + 2 $\Rightarrow$ $-2a_1 - 5a_2 = -2$

$a_2 = 2$

$a_2 = 2$ in 1

$\Rightarrow a_1 = 2 - 6 = -4$

$a_1 = -4,\ a_2 = 2$

$\therefore 2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3)$
$+ 2(3x^3 - 5x^2 - 4x - 9)$

ii) $3x^3 - 2x^2 + 7x + 8 = (a_1 + 3a_2)x^3 + (-2a_1 - 5a_2)x^2 + (-5a_1 - 4a_2)x + (-3a_1, -9a_2)$

$a_1 + 3a_2 = 3$ — ⓐ

$-2a_1 - 5a_2 = -2$ — ⓑ

$-5a_1 - 4a_2 = 7$ — ⓒ

$-3a_1 - 9a_2 = 8$ — ⓓ

ⓐ $\times 2 \Rightarrow 2a_1 + 6a_2 = 6$ — ⓔ

$$\begin{array}{r} 2a_1 + 6a_2 = 6 \\ -2a_1 - 5a_2 = -2 \\ \hline a_2 = 4 \end{array}$$

$a_2 = 4$ in ⓐ $\Rightarrow a_1 = 3 - 12 = -9$

Substitute $a_1 = -9$ & $a_2 = 4$ in ⓒ

$\Rightarrow -5\times -9 - 4\times 2 \neq 7$

SPAN

Let S be a non-empty subset of vector space V. The span of S denoted by span(S) is the set consisting of all linear combination of the vectors in S.

$\text{span}(\phi) = 0$

Q. Find the span of the set $S = \{(1,0,0), (0,1,0)\}$

Linear combination of elements of S are in the form $a_1(1,0,0) + a_2(0,1,0) = (a_1, a_2, 0)$

$\therefore \text{span}(S) = (a_1, a_2, 0)$ S

$\{(a_1, a_2, 0) : a_1, a_2 \in F\}$

Q. Find the span of the set $S = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$

Linear combination of elements of S are in the form $a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = a_1 + a_2 + a_3$

$\therefore \text{span}(S) = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in F\}$

THEOREM (1.5)

The span of any subset S of a vector space V is a subspace of V.

Moreover, any subspace of V that contains S, must also contain the span(S)

PROOF

If $S = \phi$

Then,

$\text{span}(\phi) = \{0\}$,

which is a subspace that contained any subspace of V.

Suppose that $S \neq \phi$

Then S contains a vector z, so $0 \cdot z = 0 \in \text{span}(S)$

Let $x, y \in span(S)$

clearly $0 \in span(S)$

Then $\exists$ vectors $u_1, u_2 \dots u_m, v_1, v_2 \dots v_n$ in S

and scalars $a_1, a_2 \dots a_n, b_1 b_2 \dots b_n$ in F

$\ni$ $x = a_1u_1 + a_2u_2 + \dots + a_mu_m$

$y = b_1v_1 + b_2v_2 + \dots + b_nv_n$

Then $x+y = a_1u_1 + a_2u_2 + \dots + a_mu_m + b_1v_1 + b_2v_2 + \dots + b_nv_n$ &

$cx = (ca_1)u_1 + (ca_2)u_2 + \dots + (ca_m)u_m$

are clearly linear combination of the vectors in S.

$\therefore x+y$ & $cx \in span(S)$

span of S is a subspace of V.

Let W denote any subspace of V that contains S

Then we want to S.T W contains span(S) or $span(S) \subseteq W$

Let $w \in span(S)$

Then w has the form $w = c_1w_1 + c_2w_2 + \dots + c_kw_k$ for some vectors $w_1, w_2 \dots w_k$ in S & some scalars $c_1, c_2 \dots c_k$

$\because S$ is a subset of W & $w_1, w_2 \dots w_k \in S$,

$w_1, w_2 \dots w_k \in W$

$\because W$ is a subspace of V

$c_1w_1 + c_2w_2 + \dots + c_kw_k \in W$

$\therefore w \in W$

$\Rightarrow span(S)$ is a subset of W.

($\because w \in span(S)$ then $w \in W$)

DEFINITION

A subset S of a vector space V generates (or spans) V if $span(S) = V$

In this case we say that the vectors of S generates V (or spans)

Q S.T the vectors $(1,1,0)$ $(1,0,1)$ & $(0,1,1)$ generate $\mathbb{R}^3$

Let $a_1, a_2, a_3 \in \mathbb{R}^3$. Suppose that $\exists\, u_1, u_2, u_3 \ni$
$(a_1, a_2, a_3) = u_1(1,1,0) + u_2(1,0,1) + u_3(0,1,1)$

$a_1 = u_1 + u_2$ — ①

$a_2 = u_1 + u_3$ — ②

$a_3 = u_2 + u_3$ — ③

① − ② $\Rightarrow a_1 - a_2 = u_2 - u_3$ — ④

① − ③ $\Rightarrow a_1 - a_3 = u_1 - u_3$ — ⑤

③ + ④ $\Rightarrow a_3 + a_1 - a_2 = 2u_2$

$$u_2 = \frac{a_3 + a_1 - a_2}{2}$$

② + ⑤ $\Rightarrow a_2 + a_1 - a_3 = 2u_1$

$$u_1 = \frac{a_2 + a_1 - a_3}{2}$$

sub value of u in ②

$$a_2 = \frac{a_2 + a_1 - a_3}{2} + u_3$$

$$a_2 = \frac{a_2 + a_1 - a_3 + 2u_3}{2}$$

$$2a_2 = a_2 + a_1 - a_3 + 2u_3$$

$$2a_2 - a_2 - a_1 + a_3 = 2u_3$$

$$u_3 = \frac{a_2 - a_1 + a_3}{2}$$

$$(a_1, a_2, a_3) = \frac{a_2 + a_1 - a_3}{2}(1,1,0) + \left(\frac{a_3 + a_1 - a_2}{2}\right)(1,0,1) + \left(\frac{a_2 - a_1 + a_3}{2}\right)(0,1,1)$$

$\therefore$ The given vectors generate R^3

$\because$ every vector in R^3 can be expressed as the linear combination of $\{(1,1,0)\ (1,0,1)\ (0,1,1)\}$

Q S.T the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ generates $M_{2\times2}(R)$

Let $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times2}(R)$

Suppose that there are scalars p, q, r, s in R s

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = p\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + q\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + r\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + s\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} p & p \\ p & 0 \end{bmatrix} + \begin{bmatrix} q & q \\ 0 & q \end{bmatrix} + \begin{bmatrix} r & 0 \\ r & r \end{bmatrix} + \begin{bmatrix} 0 & s \\ s & s \end{bmatrix}$$

$$= \begin{bmatrix} p+q+r & p+q+s \\ p+r+s & q+r+s \end{bmatrix}$$

$a_{11} = p+q+r$ — ①

$a_{12} = p+q+s$ — ②

$a_{21} = p+r+s$ — ③

$a_{22} = q+r+s$ — ④

$① - ② \Rightarrow a_{11} - a_{12} = r - s \quad — ⑤$

$① - ③ \Rightarrow a_{11} - a_{21} = q - s \quad — ⑥$

$① - ④ \Rightarrow a_{11} - a_{22} = p - s \quad - ⑦$

$⑤ + ⑥ \Rightarrow 2a_{11} - a_{12} - a_{21} = r + q - 2s \quad — ⑧$

$④ - ⑧ \Rightarrow a_{22} - 2a_{11} + a_{12} + a_{21} = 3s$

$$s = \frac{-2a_{11} + a_{12} + a_{21} + a_{21}}{3}$$

Put the value of s in ⑤, ⑥ & ⑦

$$a_{11} - a_{12} = r - \frac{(-2a_{11} + a_{12} + a_{21} + a_{22})}{3}$$

$$(a_{11} - a_{12}) + \frac{(-2a_{11} + a_{12} + a_{21} + a_{22})}{3} = r$$

$$\frac{3a_{11} - 3a_{12} - 2a_{11} + a_{12} + a_{21} + a_{22}}{3} = r$$

$$\frac{a_{11} - 2a_{12} + a_{21} + a_{22}}{3} = r$$

s in ⑥

$$a_{11} - a_{21} = q - \frac{(-2a_{11} + a_{12} + a_{21} + a_{22})}{3}$$

$$3a_{11} - 3a_{21} - 2a_{11} + a_{12} + a_{21} + a_{22} = q$$

$$\frac{a_{11} + a_{12} - 2a_{21} + a_{22}}{3} = q$$

s in ⑦

$$a_{11} - a_{22} = p - \frac{(-2a_{11} + a_{12} + a_{21} + a_{22})}{3}$$

$$3a_{11} - 3a_{22} = p - (-2a_{11} + a_{12} + a_{21} + a_{22})$$

$$3a_{11} - 3a_{22} - 2a_{11} + a_{12} + a_{21} + a_{22} = p$$

$$a_{11} + a_{12} + a_{21} - 2a_{22} = p$$

$$\frac{a_{11} + a_{12} + a_{21} - 2a_{22}}{3}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \frac{a_{11} + a_{12} - 2a_{21} + a_{22}}{3}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$+ \frac{a_{11} - 2a_{12} + a_{21} + a_{22}}{3}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \frac{-2a_{11} + a_{12} + a_{21} + a_{22}}{3}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Q S.T the polynomials $x^2 + 3x - 2$, $2x^2 + 5x - 3$ & $-x^2 - 4x + 4$ spans $P_2(R)$

$$ax^2 + bx + c = a_1(x^2 + 3x - 2) + a_2(2x^2 + 5x - 3) + a_3(-x^2 - 4x + 4)$$

$= a_1x^2 + a_1x - 2a_1 + 2a_2x^2 + 5a_2x - 3a_2 - a_3x^2 - 4a_3x + 4a_3$

$a = a_1 + 2a_2 - a_3 \quad - (1)$

$b = a_1 + 5a_2 - 4a_3 \quad - (2)$

$c = -2a_1 - 3a_2 + 4a_3 \quad - (3)$

$(2) + (3) \Rightarrow b + c = a_1 + 2a_2 \quad - (4)$

$4 \times (1) \Rightarrow 4a = 4a_1 + 8a_2 - 4a_3$

$(1) - (4) \Rightarrow a - b - c = -a_3$

$a_3 = -a + b + c$

Sub. a_3 in (1) & (2)

$a = a_1 + 2a_2 - (-a + b + c)$

$a - a + b + c = a_1 + 2a_2$

$b + c = a_1 + 2a_2 \quad - (5)$

$b = a_1 + 5a_2 - 4(-a + b + c)$

$b + 4(-a + b + c) = a_1 + 5a_2$

$b - 4a + 4b + 4c = a_1 + 5a_2$

$-4a + 5b + 4c = a_1 + 5a_2 \quad - (6)$

$3 \times (5) \Rightarrow 3b + 3c = 3a_1 + 6a_2 \quad - (7)$

$(6) - (7) \Rightarrow -a_2 = -4a + 2b + c$

$a_2 = 4a - 2b - c$

Put a_2 in (5)

$a_1 + 8a - 4b - 2c = b + c$

$a_1 = b + c - 8a + 4b + 2c$

$a_1 = -8a + 5b + 3c$

$ax^2 + bx + c = (-8a + 5b + 3c)(x^2 + 3x - 2) + (4a - 2b - c)(2x^2 + 5x - 3) + (-a + b + c)(-x^2 - 4x + 4)$

Q Is $3x^3 - 2x^2 + 7x + 8$ a linear combination of $x^3 - 2x^2 - 5x - 3$ & $3x^3 - 5x^2 - 4x - 9$

$3x^2 - 2x^2 + 7x + 8 = c_1(x^3 - 2x^2 - 5x - 3) + c_2(3x^3 - 5x^2 - 4x - 9)$

$= c_1x^3 - 2c_1x^2 - 5c_1x - 3c_1 + 3c_2x^3 - 5c_2x^2 - 4c_2x - 9c_2$

$= (c_1 + 3c_2)x^3 - (2c_1 + 5c_2)x^2 - (5c_1 + 4c_2)x - 3c_1 - 9c_2$

$c_1 + 3c_2 = 3 \quad - (1)$

$-(2c_1 + 5c_2) = -2 \quad - (2)$

$-(5c_1 + 4c_2) = 7 \quad - (3)$

$-3c_1 - 9c_2 = 8 \quad - (4)$

① × −2 ⇒

$-2c_1 - 6c_2 = -6$

$2c_1 + 5c_2 = 2$

$-c_2 = -4$

$c_2 = 4$

[① × 5] + ③ ⇒

$5c_1 + 15c_2 = 15$

$-5c_1 - 4c_2 = 7$

$11c_2 = 22$

$c_2 = 2$

① × 3 + ④ ⇒

$3c_1 + 9c_2 = 9$

$-3c_1 - 9c_2 = 8$

$0c_2 = 17$, not possible

No sol.

BASES AND DIMENSIONS

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

A subset S of a vector space V is called linearly dependent if there exist a finite no. of distinct vectors $u_1, u_2 \dots u_n$ in S and scalars $a_1, a_2 \dots a_n$ not all zero, such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$$

In this case we also say that the vectors of S are linearly independent.

REMARK

For any vectors $u_1, u_2 \dots u_n$, we've

$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$

if $a_1 = a_2 = \dots = a_n = 0$.

We call this the trivial representation of zero as a linear combination of $u_1, u_2 \dots u_n$

A subset S of a vector space that is not linearly dependent is called linearly independent. We also say that the vectors of S are linearly independent.

A subspace S of a vector space V is said to be linearly independent if there exist a finite no. of

distinct vectors $u_1, u_2 \dots u_n$ and scalars $a_1, a_2 \dots a_n$ such that $a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$
$\Rightarrow a_1 = a_2 = \dots = a_n = 0$.

The following facts about linearly independent sets are true in any vector space.

1. The empty set is linearly independent, for linearly dependent sets must be non-empty.

2. A set consisting of a single non-zero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some non-zero scalar a. Thus,
$(a^{-1}a)u =$
$u = a^{-1}(au)$
$= a^{-1}0 = 0$, which is a contradiction ($\because u$ is a non-zero vector)
$\therefore \{u\}$ is linearly independent

3. A set is linearly independent iff the only representations of 0 as linear combinations of its vectors are trivial representations. This is a useful method for determining whether a finite set is linearly independent.

THEOREM (1.6)

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

PROOF

Given, S_1 is linearly dependent

Then $\exists$ some $u_1, u_2 \dots u_n$ in S_1 & $a_1, a_2 \dots a_n$ in F not all zero s.t.

$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$

$\because S_1 \subseteq S_2$, $u_1, u_2 \dots u_n \in S_2$ & we've

$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$, $a_i \neq 0$ for some i

$\Rightarrow S_2$ is L.D

COROLLARY

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent

PROOF

Let $S_2 = \{u_1, u_2, \dots, u_n\}$ be a L.I set

Then $\exists$ scalars $a_1, a_2 \dots a_n$ in F s.t.

$0 = a_1u_1 + a_2u_2 + \dots + a_nu_n \quad \forall a_i = 0, i = 1, 2 \dots n$

Given, $S_1 \subseteq S_2$

Let $S_1 = \{u_1, u_2, \dots u_k\}$ where $k \leq n$. Consider 2 cases

Case I

If $k = n$

then $S_1 = S_2$, $\because S_2$ is L.I set, S_1 is L.I set

Case II

If $k < n$

Assume that $\exists$ scalars $b_1, b_2 \ldots b_k$ s.

$$0 = b_1u_1 + b_2u_2 + \ldots + b_ku_k \quad — ①$$

① can be written as

$$0 = b_1u_1 + b_2u_2 + \ldots + b_ku_k + 0 \cdot u_{k+1} + \ldots + 0 \cdot u_n \quad — ②$$

② $\Rightarrow$ 0 is the linear combination of elements of S_2

$\because$ S_2 is L.I, all scalars must be zero.

$\therefore$ ① $\Rightarrow b_1 = b_2 = \ldots = b_k = 0$

$\therefore$ ① $\Rightarrow S_1$ is L.I

ANOTHER PROOF

Let $S_1 = \{u_1, u_2, \ldots u_n\}$ be a L.D set

Then $\exists$ scalars $a_1, a_2 \ldots a_n$ not all zero s

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0 \quad — ①$$

Let $S_2 = \{u_1, u_2, \ldots u_n, w_1, w_2, \ldots w_m\}$ $(S_1 \subseteq S_2)$

then ① can be written as

$$a_1u_1 + a_2u_2 + \ldots + a_nw_n + 0 \cdot w_1 + 0 \, w_2 + \ldots + 0 \cdot w_m = 0 \quad — ②$$

from ①, the scalars $a_1, a_2 \ldots a_n$ not all zero

$\therefore$ ② $\Rightarrow S_2 = \{u_1, u_2, \ldots u_n, w_1, w_2, \ldots w_m\}$

is LD.

THEOREM (1.7)

Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent iff $v \in span(S)$

PROOF

If $S \cup \{v\}$ is linearly dependent, then there are vectors $u_1, u_2, \ldots u_n$ in $S \cup \{v\}$ s

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$$

for some non-zero scalars $a_1, a_2, \ldots a_n$.

Because S is linearly independent, one of the u_i's say u_1, equals v.

Thus,

$$a_1v + a_2v_2 + \ldots + a_nv_n = 0 \quad \& \text{ so}$$

$$v = a_1^{-1}(-a_2u_2 - \ldots - a_nu_n) = -(a_1^{-1}a_2)u_2 - \ldots - (a_1^{-1}a_n)u_n$$

$\because$ v is a linear combination of $u_2, \ldots u_n$, which are in S, we've $v \in span(S)$

Conversely, let $v \in span(S)$. Then $\exists$ vectors $v_1, v_2 \ldots v_m$ in S & scalars $b_1, b_2 \ldots b_m$ s

$$v = b_1v_1 + b_2v_2 + \ldots + b_mv_m.$$

Hence,

$$0 = b_1v_1 + b_2v_2 + \ldots + b_mv_m + (-1)v.$$

$v \neq v_i$ for $i = 1, 2 \ldots m$, the coefficients of v in this linear combination is non zero, & so the set $\{v_1, v_2 \ldots v_m, v\}$ is linearly dependent. Therefore $S \cup \{v\}$ is linearly dependent by above thm.

BASIS

A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

THEOREM (1.8)

Let V be a vector space and $\beta = \{u_1, u_2, \ldots u_n\}$ be a subset of V. Then β is a basis for V iff each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, ie, can be expressed in the form

$$v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$$

for unique scalars $a_1, a_2 \ldots a_n$.

PROOF

Let β is a basis for V

Let $v \in V$. Then v can be expressed as a linear combination of vectors of β ($\because \beta$ spans V using defn)

Then $\exists$ scalars $a_1, a_2 \ldots a_n$ $\ni$

$$v = a_1u_1 + a_2u_2 + \ldots + a_nu_n \quad — ①$$

To prove the uniqueness, consider another linear combination of v.

$$v = b_1u_1 + b_2u_2 + \ldots + b_nu_n \quad — ②$$

Comparing ① & ②, we get

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = b_1u_1 + b_2u_2 + \ldots + b_nu_n$$

$$(a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \ldots + (a_n - b_n)u_n = 0$$

$\because \beta$ is linearly independent we can write

$$a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \ldots, a_n - b_n = 0$$

$$a_1 = b_1, \quad a_2 = b_2, \ldots, a_n = b_n$$

$\Rightarrow v$ is uniquely expressed as a linear combination of vectors of β.

Conversely suppose that $v \in V$ can be uniquely expressed as a linear combination of vectors of V.

ie $\exists$ scalars $a_1, a_2, \ldots a_n$ $\ni$

$$v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$$

ie we get every element of V can be expressed as a linear combination of vectors of β

$\therefore$ span $(\beta) = V$

Now to prove β is a basis for V, it is enough to

S.T β is L.I

$\because$ V is a vector space,

$0 \in V$ (additive identity)

Then by our assumption zero can be expressed as a linear combination of members of β

ie $0 = b_1u_1 + b_2u_2 + \cdots + b_nu_n$

Also we've

$0 = 0.u_1 + 0.u_2 + \cdots + 0.u_n$

By our assumption the linear combination is unique

$\Rightarrow b_1 = 0, b_2 = 0, \ldots, b_n = 0$

$\Rightarrow \beta$ is L.I

$\therefore \beta$ is a basis for V

REMARK

This theorem shows that if the vectors $u_1, u_2 \ldots u_n$ form a basis for a vector space V then every vector in V uniquely express in the form $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ for $v \in V$.

THEOREM (1.1)

If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis

PROOF

If $S = \phi$

Then $V = span(S)$

$= span(\phi)$

$= \{0\}$

& ϕ is L.I also ϕ is a subset of S $(\phi \subseteq \phi = S)$

And clearly ϕ is a basis for V

If $S = \{0\}$

$V = span(S)$

$= span(\{0\})$

$= \{0\}$

clearly ϕ is a subset of S, a basis for $V = \{0\}$

Suppose that S contains a non-zero vector u_1

Then set $\{u_1\}$ is a L.I set of S

Continuing if possible choosing vectors

$u_2, u_3 \ldots u_k$ in S S

$\{u_1, u_2, \ldots u_k\}$ is L.I

$\therefore S$ is finite, we must reach a stage at which $\beta = \{u_1, u_2, \dots u_k\}$ is a L.I subset of S & $\beta \cup \{y\}$ is L.D $\forall\ y \in S - \beta$ $(y \in S, y \notin \beta)$

Claim

β is a basis for V

Clearly β is L.I

Now it remains to S.T $\mathrm{span}(\beta) = V$

$\mathrm{span}(\beta) \subseteq V$ & $V \subseteq \mathrm{span}(\beta)$

(By thm, the span of any subset S of a vector space V is a subspace of V)

$\mathrm{span}(\beta) \subseteq V$ — ①

$\therefore$ it is enough to S.T V is a subspace of $\mathrm{span}(\beta)$

Subclaim

S is a subset of $\mathrm{span}(\beta)$

let $v \in S$

Then there are 2 possibilities $v \in \beta$ & $v \notin \beta$

If $v \in \beta$, then $v \in \mathrm{span}(\beta)$ $(\beta \subseteq S)$

If $v \notin \beta$, then by construction $\beta \cup \{v\}$ is L.D

Then by previous thm, $\mathrm{span}(S) \subseteq \mathrm{span}(\beta)$

$\Rightarrow \mathrm{span}(S) = V \subseteq \mathrm{span}(\beta)$ ($\because V$ is generated by S, $\mathrm{span}(S) = V$)

$\Rightarrow V \subseteq \mathrm{span}(\beta)$ — ②

$\therefore$ ① ② $\Rightarrow \mathrm{span}(\beta) = V$

$\Rightarrow \beta$ is a basis for V.

REPLACEMENT THEOREM (Thm 1.10)

Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linear independent subset of V containing exactly m vectors. Then $m \leq n$ and there exist a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ generates V

PROOF

The proof is by mathematical induction on m

Let $m = 0$

Then $L = \phi$

So taking $H = G$ gives the desired result

Now suppose that the theorem is true for some integers $m \geq 0$. We P.T the thm is true for $m+1$.

Let $L = \{v_1, v_2, \ldots v_{m+1}\}$ be a LI subset of V consisting of $m+1$ vectors.

By the corollary of previous thm, $\{v_1, v_2 \ldots v_m\}$ is LI & so we may apply the induction hypothesis to conclude that $m \le n$ & that there is a subset $\{u_1, u_2 \ldots u_{n-m}\}$ of G s $\{v_1, v_2, \ldots v_m\} \cup \{u_1, u_2, \ldots u_{n-m}\}$ generates V.

Thus $\exists$ scalars $a_1, a_2 \ldots a_m, b_1, b_2 \ldots b_{n-m}$ s

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m + b_1u_1 + b_2u_2 + \cdots + b_{n-m}u_{n-m} = v_{m+1} \quad — ①$$

Note that $n-m > 0$ because $n-m=0$, then ① becomes

$a_1v_1 + a_2v_2 + \cdots + a_mv_m = v_{m+1}$

$\Rightarrow a_1v_1 + a_2v_2 + \cdots + a_mv_m - v_{m+1} = 0$

$\therefore \{v_1, v_2, \ldots v_{m+1}\}$ is LD

$\therefore$ Coefficients of $v_{m+1} = 0$

which is a contradiction to the assumption that L is LI.

$\therefore n-m > 0$

Hence $n > m$

$\therefore n \ge m+1$

Also some b_i say b_1 is non-zero. Otherwise we get some contradiction.

Now ① $\Rightarrow b_1u_1 = -a_1v_1 - a_2v_2 - \cdots - a_mv_m + v_{m+1} - b_2u_2 - \cdots - b_{n-m}u_{n-m}$

$\Rightarrow u_1 = -b_1^{-1}a_1v_1 - b_1^{-1}a_2v_2 - \cdots - b_1^{-1}a_mv_m + b_1^{-1}v_{m+1} - b_1^{-1}b_2u_2 - \cdots - b_1^{-1}b_{n-m}u_{n-m}$

take $H = \{u_2, u_3 \ldots u_{n-m}\}$

Then $u_1 \in$ span $(L \cup H)$

Also $v_1, v_2, \ldots v_m, u_2, u_3 \ldots u_{n-m} \in$ span $(L \cup H)$

$\Rightarrow v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot u_{n-m}$

$v_2 = 0 \cdot v_1 + 1 \cdot v_2 + \cdots + 0 \cdot u_{n-m}$

and so on

$\Rightarrow \{v_1, v_2, \ldots v_m, u_1, u_2, \ldots u_{n-m}\} \subseteq$ span $(L \cup H)$ — ②

$\because \{v_1, v_2, \ldots v_m, u_1, u_2, \ldots u_{n-m}\}$ generates V (by assumption)

② $\rightarrow$ span $\{v_1, v_2 \ldots v_m, u_1, u_2 \ldots u_{n-m}\} \subseteq$ span $(L \cup H)$ (using prev thm)

$V \subseteq$ span $(L \cup H)$ — ⓐ

span $(L \cup H) \subseteq V$ — ⓑ

ⓐ & ⓑ $\Rightarrow V =$ span $(L \cup H)$

Also H is a subset of a that containing $n-m-1 = n-(m+1)$ vectors also $L \cup H$ generates V

$\therefore$ the thm is true for $m+1$

$\therefore$ the thm is true for all m.

COROLLARY (1)

Let V be a vector space having a finite basis. Then every basis for V contains the same no. of vectors.

PROOF

Suppose that β is a finite basis for V that contains exactly 'n' vectors, and let γ be any other basis for V. If γ contains more than 'n' vectors, then we can select a subset S of γ containing exactly $n+1$ vectors. $\because$ S is L.I & β generates V, the replacement thm implies that $n+1 \leq n$, a contradiction.

$\therefore \gamma$ is finite & the no. 'm' of vectors in γ satisfies $m \leq n$. Reversing the roles of β & γ, ~~and~~ we obtain $n \leq m$.

Hence $m = n$

$\therefore$ Every basis of V contains the same no. of vectors.

DEFINITIONS

A vector space is called finite-dimensional if it has a basis consisting of a finite no. of vectors.

The unique no. of vectors in each basis for V is called the dimension of V & is denoted by dim(V).

A vector space that is not finite-dimensional is called infinite-dimensional.

EXAMPLES

The vector space $\{0\}$ has dimension zero.

The vector space F^n has dimension n

The vector space $M_{m \times n}(F)$ has dimension mn.

The vector space $P_n(F)$ has dimension $n+1$.

Over the field of complex no.s, the vector space of complex no.s has dimension 1 (A basis is $\{1\}$)

Over the field of real no.s, the vector space of complex no.s has dimension 2 (A basis is $\{1, i\}$)

COROLLARY (2)

Let V be a vector space with dimension n.

a) Any finite generating set of V contains atleast 'n' vectors and a generating set for V that contains exactly 'n' vectors is a basis for V.

b) Any LI subset of V that contains exactly 'n' vectors is a basis for V

c) Every LI subset of V can be extended to a basis for V

PROOF

Let β be a basis for V

a) Let G be a finite generating set for V. By thm some subset H of G is a basis for V. 1st Corollary implies that H contains exactly 'n' vectors

$\therefore$ a subset of G contains 'n' vectors, G must contain at least 'n' vectors. Moreover, if G contains exactly n vectors, then we must have H = G, so that G is a basis for V

b) Let L be a LI subset of V containing exactly 'n' vectors. It follows from the replacement thm that there is a subset H of β containing $n - n = 0$ vectors & $L \cup H$ generate V. Thus $H = \phi$ & L generates V $\because$ L is also LI, L is a basis for V

c) If L is a LI subset of V containing 'm' vectors then the replacement thm asserts that there is a subset H of β containing exactly $n - m$ vectors & $L \cup H$ generates V. Now $L \cup H$ contains at most n vectors, $\therefore$ (a) implies that $L \cup H$ contains exactly 'n' vectors & that $L \cup H$ is a basis for V

COROLLARY

Let V be a vector space having a basis containing exactly 'n' elements. Then any subset of V containing more than 'n' elements is linearly dependent

PROOF

Let V be a vector space having a basis containing exactly 'n' elements and let S be a subset of V containing more than n elements

claim

S is LI

Assume that S is LI

Let H be any subset of S containing exactly 'n' elements ie $H \subset S$

$\therefore$ S is LI, H is LI

Then we've a result. If $\dim(V) = n$, Then any LI subset of V containing exactly n vectors is a basis for V.

Using the result, we can write. H is a basis for V

∴ $H \subset S$, we can choose an element $x \in S$, but $x \notin H$

H is a basis for V, $x \in \text{span}(H)$

ie we get $x \in \text{span}(H)$ & $x \in S$

but $x \notin H$

∵ $x \in S$

Also H is a basis for V, then $\forall x \in V$ can be written as linear combination of members of H

~~H is a basis for V, $x \in \text{span}(H)$~~

~~ie we get $x \in \text{span}(H)$ & $x \in S$~~

And

Then by thm, we've

$H \cup \{x\}$ is LD

Also $H \cup \{x\} \subseteq S$

$H \cup \{x\}$ is LD.

S is LD.

which is a contradiction to our assumption that S is LI

$\Rightarrow$ S is LD

AN OVERVIEW OF DIMENSION AND ITS CONSEQUENCES

1. A basis for a vector space V is a LI subset of V that generates V
2. The no. of elements in a basis is called the dimension of V
3. If the dimension of V is n then every basis for V contains exactly n vectors
4. If $\dim(V) = n$ then any LI subset of V that contains exactly n vectors is a basis for V.
5. Every LI subset of V can be extended to a basis for V

THE DIMENSION OF SUBSPACES

THEOREM (1.11)

Let W be a subspace of a finite-dimensional vector space V. Then W is finite dimensional & $\dim(W) \leq \dim(V)$. Moreover, if

$$\dim(W) = \dim(V),$$

then $V = W$

PROOF

Let $\dim(V) = n$

If $W = \{0\}$ then W is finite dimensional and $\dim(W) = 0 < n$

Otherwise, W contains a non zero vector α_1, so $\{\alpha_1\}$ is a LI set. Continue choosing vectors $\alpha_1, \alpha_2, \ldots \alpha_k$ in W s.t. $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ is LI

$\because$ no LI subset of V can contain more than n vectors, this process must stop at a stage where $k \leq n$ & $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is LI but adjoining any other vectors from W produces a LD set. Previous thm implies that $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ generates W, and hence it is a basis for W.

$\dim(W) = k \leq n$

If $\dim(W) = n$, then a basis for W is a LI subset of V containing n vectors. But 2^{nd} corollary of replacement thm implies that this basis for W is also a basis for V.

So $W = V$.

COROLLARY

If W is a subspace of a finite dimensional vector space V, then any basis for W can be extended to a basis for V

PROOF

Suppose that W be a subspace of a finite dimensional vector space V

Let S be a basis for W

$\because S$ is a LI subset of V then by corollary 4 S can be extended to a basis for V

MAXIMAL LINEARLY INDEPENDENT SUBSETS

Let F be a family of sets. A member M of F is called maximal (w.r.t set inclusion) if M is contained in no member of F other than M itself.

A collection of sets C is called a chain (or nest or tower) if for each pair of sets A & B in C, either $A \subseteq B$ or $B \subseteq A$

MAXIMAL PRINCIPLE

Let $\mathcal{F}$ be a family of sets. If for each chain $\mathcal{C} \subseteq \mathcal{F}$, $\exists$ a member of $\mathcal{F}$ that contains each member of $\mathcal{C}$, then $\mathcal{F}$ contains a maximal member

Let S be a subset of a vector space V. A maximal linearly independent subset of S is a subset B of S satisfying both of the follow. conditions

a) B is linearly independent

b) The only linearly independent subset of S that ~~satisfying both~~ contains B is B itself.

REMARK

A basis B for a vector space V is a maximal linearly independent subset of V, because (B is a basis we've B is linearly & span$(B) = V$)

1) B is linearly independent by defn

2) If $v \in V$ & $v \notin B$. Also span$(B) = V$, then $B \cup \{v\}$ is linearly dependent (by thm)

THEOREM (1.13)

Let S be a L.I subset of a vector space V. There exists a maximal linearly independent subset of V that contains S

PROOF

Let $\mathcal{F}$ denote the family of all linearly indep subsets of a vector space V that contains S

To S.T $\mathcal{F}$ contains a maximal element, it is enough to S.T if $\mathcal{C}$ is a chain in $\mathcal{F}$ then $\exists$ a member U of $\mathcal{F}$ that contains each member of $\mathcal{C}$ (then by maximal priciple $\mathcal{F}$ contains a maximal element).

We claim that U, the union of the members of $\mathcal{C}$ is ~~a~~ required set

Clearly U contains each member ($\because U$ is the union of members of $\mathcal{C}$) of $\mathcal{C}$ & so it is enough to P.T $U \in \mathcal{F}$ (ie to P.T U is a L.I subset of V that contains S)

$\because$ each member of $\mathcal{C}$ is a subset of V containing S ($\because \mathcal{C} \subseteq \mathcal{F}$) we've $S \subseteq U \subseteq V$

We need to S.T U is L.I

Suppose $\exists$ vectors $v_1, v_2 \ldots v_n$ in U & scalars

$a_1, a_2 \ldots a_n \in$

$a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$ — ①

$u_i \in U$ for $i = 0, 1, \ldots n$ $\exists$ a set A_i in $\mathcal{C}$ s

$u_i \in A_i$ (U contains each member of $\mathcal{C}$)

$\because$ $\mathcal{C}$ is a chain, one of these sets say A_k, contains all the others

$\mathcal{C} \subset \mathcal{F}$ & $A_k \in \mathcal{C}$

$\therefore u_i \in A_k$ for $i = 1, 2, \ldots n$

$\therefore A_k$ is L.I

Now $\because v_i \in A_k$ is a linearly independent set

we've

$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

$\Rightarrow U$ is L.I

$\therefore$ we get if $\mathcal{C}$ is a chain in $\mathcal{F}$ then $\exists$ a member U of $\mathcal{F}$ that contains each member of $\mathcal{C}$

Then maximal principle implies that $\mathcal{F}$ has a maximal element

$\therefore \mathcal{F}$ is the family of all linearly independent subsets of V that contains S, we get the maximal element in $\mathcal{F}$ is the maximal L.I subset of V that contains S

COROLLARY

Every vector space has a basis

Let $\mathcal{F}$ be the family of all finite subsets of a infinite set S. Then $\mathcal{F}$ has no maximal element

PROOF

Suppose that $\mathcal{F}$ contains a maximal element M. Let $x \in S$ but $x \notin M$. Then M is a subset of $M \cup \{x\}$ which is a contradiction to our assumption that M is maximal

THEOREM (1.12)

Let V be a vector space & S a subset that generates V. If β is a maximal linearly independent subset of S, then β is a basis for V

PROOF

Let β be a maximal linearly independent subset of S. Because β is linearly independent, it suffices to P.T β generates V. We claim that $S \subseteq \operatorname{span}(\beta)$, for otherwise there exists $a\ v \in S$ $\ni$ $v \notin$ Span (β) $\because$ prev thm implies that $\beta \cup \{v\}$ is linearly independent, we've contradicted the maximality of β $\therefore$ $S \subseteq$ Span (β) Because Span $(S) = V$, it follows from prev thm that Span $(\beta) = V$.

Thus a subset of a vector space is a basis iff it is a maximal linearly independent subset of the vector space.

COROLLARY

A subset β of a vector space V is a basis for V iff β is a maximal linearly independent subset of V

PROOF

Suppose that β is a basis for V
$\Rightarrow$ β is a maximal linearly independent subset of V
Conversely suppose that β is a maximal linearly independent subset of V
Then by above thm, β is a basis for V.

EXAMPLES

Q Consider the set $S = \{(1,3,-4,2), (2,2,-4,0), (1,-3,2,-4), (-1,0,1,0)\}$. S.T S is linearly dependent

A) Consider,

$$a_1(1,3,-4,2) + a_2(2,2,-4,0) + a_3(1,-3,2,-4) + a_4(-1,0,1,0) = 0$$

$$(a_1, 3a_1, -4a_1, 2a_1) + (2a_2, 2a_2, -4a_2, 0) + (a_3, -3a_3, 2a_3, -4a_3) + (-a_4, 0, a_4, 0) = 0$$

$a_1 + 2a_2 + a_3 - a_4 = 0$ — ①

$3a_1 + 2a_2 - 3a_3 = 0$ — ②

$-4a_1 - 4a_2 + 2a_3 + a_4 = 0$ — ③

$2a_1 - 4a_3 = 0$ — ④

$2a_1 = 4a_3$

$a_1 = \frac{4a_3}{2} = 2a_3$

Sub $a_1 = 2a_3$ in ②

$\Rightarrow 6a_3 + 2a_2 - 3a_3 = 0$

$2a_2 = -6a_3 + 3a_3$

$2a_2 = -3a_3$

$a_2 = \frac{-3a_3}{2}$

Sub $a_1 = 2a_3$

$a_2 = -\frac{3a_3}{2}$ in ①

$2a_3 + 2\cdot\frac{-3a_3}{2} + a_3 - a_4 = 0$

$2a_3 - 3a_3 + a_3 - a_4 = 0$

$a_4 = 0$

$(a_1, a_2, a_3, a_4) = (2a_3, -\frac{3}{2}a_3, a_3, 0)$

When $a_3 = 2$

$a_1 = 4$

$a_2 = -\frac{3}{2}a_3 = -\frac{3}{2}\times 2 = -3$

$a_4 = 0$

∴ S is linearly dependent

Q. In $M_{2\times3}(\mathbb{R})$ the set $\left\{ \begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix} \begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix} \begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} \right\}$

A. Consider,

$a_1\begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix} + a_2\begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix} +$

$a_3\begin{bmatrix} -2 & -3 & 1 \\ -1 & -3 & 2 \end{bmatrix} = 0$

Q. The set $S = \{(1,0,0,-1)\ (0,1,0,-1)\ (0,0,1,-1)\ (0,0,0,1)\}$

LI ?

A) Consider.

$a_1(1,0,0,-1) + a_2(0,1,0,-1) + a_3(0,0,1,-1) + a_4(0,0,0,1) = 0$

$(a_1,-a_1) + (a_2,-a_2) + (a_3,-a_3) + (a_4,-a_4) = 0$

$(a_1, a_2, a_3, -a_1 - a_2 - a_3 + a_4) = 0$

$a_1 = 0$

$a_2 = 0$

$a_3 = 0$

$-a_1 - a_2 - a_3 + a_4 = 0$

$\Rightarrow a_4 = 0$

Q. For $k = 0, 1, 2, \dots n$ let $p_k(x) = x^k + x^{k+1} + \dots + x^n$

S.T the set $\{p_0(x), p_1(x), \dots, p_n(x)\}$ is LI ?

A) $p_0(x) = x^0 + x^{0+1} + x^{0+2} + \dots + x^n$

$= 1 + x + x^2 + \dots + x^n$

$P_1(x) = x + x^2 + \cdots + x^n$

$P_2(x) = x^2 + x^3 + \cdots + x^n$

$P_n(x) = x^n$

$a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \cdots + a_nP_n(x) = 0$

$a_0(1 + x + x^2 + \cdots + x^n) + a_1(x + x^2 + \cdots + x^n) +$

$a_2(x^2 + \cdots + x^n) + \cdots + a_n x^n = 0$

$a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_n)x^n = 0$

$a_0 = 0$

$a_0 + a_1 = 0 \Rightarrow a_1 = 0$

$a_0 + a_1 + a_2 = 0$

$\Rightarrow 0 + 0 + a_2 = 0$

$\Rightarrow a_2 = 0$

$a_n = 0$

the given set is LI

6. Determine whether the following sets are LD or LI?

1) $\left\{ \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix} \right\}$ in $M_{2\times2}(\mathbb{R})$

2) $\left\{ \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} \right\}$ in $M_{2\times2}(\mathbb{R})$

3) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(\mathbb{R})$

A) $a_1 \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} + a_2 \begin{bmatrix} -2 & 6 \\ 4 & -8 \end{bmatrix} = 0$

$\begin{bmatrix} a_1 & -3a_1 \\ -2a_1 & 4a_1 \end{bmatrix} + \begin{bmatrix} -2a_2 & 6a_2 \\ 4a_2 & -8a_2 \end{bmatrix} = 0$

$\begin{bmatrix} a_1 - 2a_2 = 0 & -3a_1 + 6a_2 \\ -2a_1 + 4a_2 & 4a_1 - 8a_2 \end{bmatrix} = 0$

$a_1 - 2a_2 = 0$ — ①

$-3a_1 + 6a_2 = 0$ — ②

$-2a_1 + 4a_2 = 0$ — ③

$4a_1 - 8a_2 = 0$ — ④

① $\Rightarrow a_1 = 2a_2$

Put $a_2 = 1$, $a_1 = 2$

$\therefore$ the set is L D

2) $a_1 \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} + a_2 \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$

$$\begin{bmatrix} a_1 & -2a_1 \\ -a_1 & 4a_1 \end{bmatrix} + \begin{bmatrix} -a_2 & a_2 \\ 2a_2 & -4a_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} a_1 - a_2 & -2a_1 + a_2 \\ -a_1 + 2a_2 & 4a_1 - 4a_2 \end{bmatrix} = 0$$

$a_1 - a_2 = 0$ — ①

$-2a_1 + a_2 = 0$ — ②

$-a_1 + 2a_2 = 0$ — ③

$4a_1 - 4a_2 = 0$ — ④

① $\Rightarrow a_1 = a_2$

Sub $a_1 = a_2$ in ①

$-2a_2 + a_2 = 0$

$-a_2 = 0$

$a_2 = 0$

$\Rightarrow a_1 = 0$

$\therefore$ the given set is L I

3) $a_1(x^3 + 2x^2) + a_2(-x^2 + 3x + 1) + a_3(x^3 - x^2 + 2x - 1) = 0$

$a_1x^3 + 2a_1x^2 - a_2x^2 + 3a_2x + a_2 + a_3x^3 - a_3x^2 + 2a_3x - a_3 = 0$

$(a_1 + a_3)x^3 + (2a_1 - a_2 - a_3)x^2 + (3a_2 + 2a_3)x + a_2 - a_3 = 0$

$a_1 + a_3 = 0$

$2a_1 - a_2 - a_3 = 0$

$3a_2 + 2a_3 = 0$

$a_2 - a_3 = 0$ — ④

④ $\Rightarrow a_2 = a_3$

$3a_3 + 2a_3 = 0$

$5a_3 = 0$

$a_3 = 0$

$\Rightarrow a_2 = 0$

① $\Rightarrow a_1 = -a_3$

$a_1 = 0$

$\therefore$ set is L I

4) $\{1, x, x^2, \dots x^n\}$ in $P_n(F)$

$a_0 + a_1x + \dots + a_nx^n = 0$

$\Rightarrow a_i = 0$, $i = 1, 2, \dots n$

$\therefore$ the set is L I

Q If $\{u,v,w\}$ is L.I in V then S.T $\{u+v, u-v, u-2v+w\}$ is L.I

$a_1(u+v) + a_2(u-v) + a_3(u-2v+w) = 0$

$a_1u + a_1v + a_2u - a_2v + a_3u - 2a_3v + a_3w = 0$

$(a_1+a_2+a_3)u + (a_1-a_2-2a_3)v + a_3v = 0$

$\because \{u,v,w\}$ is L.I, the scalars are zero

$a_1 + a_2 + a_3 = 0$ — ①

$a_1 - a_2 - 2a_3 = 0$ — ②

$a_3 = 0$

① & ② $\Rightarrow$

$a_1 + a_2 = 0$ — ③

$a_1 - a_2 = 0$ — ④

④ $\Rightarrow a_1 = a_2$

③ $\Rightarrow a_2 + a_2 = 0$

$2a_2 = 0$

$a_2 = 0$

$\therefore a_2 = a_1 \rightarrow a_1 = 0$

$\therefore$ the set is L.I

In $M_{2\times3}(\mathbb{R})$, the set

$$\left\{ \begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} \right\}$$

is linearly dependent because

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Q S.T $\{(1,0)\ (0,1)\}$ is a basis for $\mathbb{R}^2$. We want to S.T the given set is L.I & generate $\mathbb{R}^2$

A: Let $a_1(1,0) + a_2(0,1) = (0,0)$

$(a_1,0) + a_2(0,1) = (0,0)$

$(a_1, a_2) = (0,0)$

$a_1 = a_2 = 0$

span

$(x,y) \in \mathbb{R}^2$

$(x,y) = a_1(1,0) + a_2(0,1)$

$= (a_1, a_2)$

$a_1 = x$ & $a_2 = y$

$\{(1,0)\ (0,1)\}$ span R^2

$\{(1,0)\ (0,1)\}$ is a basis

Q In F^n let $e_1 = (1,0,0 \dots 0)$

$e_2 = (0,1,0 \dots 0)$

$\vdots$

$e_n = (0,0,0 \dots 1)$,

S.T the set $\{e_1, e_2, \dots e_n\}$ is a basis for F^n.

$a_1e_1 + a_2e_2 + \dots + a_ne_n = 0$

Let

$a_1(1,0,0 \dots 0) + a_2(0,1 \dots 0) + \dots + a_n(0,0, \dots 1)$
$= (0,0,0 \dots 0)$

$(a_1, a_2, \dots a_n) = (0,0, \dots 0)$

$\Rightarrow a_1 = a_2 = \dots = a_n = 0$

Span

Let $(b_1, b_2 \dots b_n) \in F^n$

Suppose $\exists$ scalars $c_1, c_2 \dots c_n$ s.

$(b_1, b_2, \dots b_n) = c_1e_1 + c_2e_2 + \dots + c_ne_n$

$= c_1(1,0,0 \dots 0) + c_2(0,1,0 \dots 0) +$
$c_n(0,0, \dots 1)$

$= (c_1, c_2, \dots c_n)$

$b_1 = c_1$

$c_2 = b_2$

$c_n = b_n$

$\therefore \{e_1, e_2 \dots e_n\}$ spans F^n

$\{e_1, e_2 \dots e_n\}$ is a basis

REMARK

The basis $\{e_1, e_2, \dots e_n\} = \{(1,0,0, \dots 0), (0,1,0 \dots 0)$
$(0,0, \dots 0,1)\}$ is called std basis for F^n

The set $\{(1,0)\ (0,1)\}$ is a basis for R^2 by proving

Q S.T in $P_n(F)$, the set $\{1, x, x^2 \dots x^n\}$ is a basis

1) ~~let~~ L.I

$a_0 + a_1x + \dots + a_nx^n = 0$

$a_i = 0, 1, 2 \dots n$

Span

Let $b_0 + b_1x + \dots + b_nx^n \in P_n(F)$

Suppose $\exists$ scalars $c_0, c_1 \dots c_n$ s

$b_0 + b_1x + \dots + b_nx^n = c_0 + c_1x + \dots + c_nx^n$

$\Rightarrow c_0 = b_0$,

$c_1 = b_1, \dots c_n b_n$

REMARK

The basis $\{1, x, x^2, \dots x^n\}$ for $P_n(F)$ is called std basis for $P_n(F)$

Recalling that span $(\phi) = \{0\}$ & ϕ is L.I, we see that ϕ is a basis for the zero vector space

In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is a 1 in the i^{th} row & j^{th} column. Then $\{E^{ij} \; 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_{m\times n}(F)$

Q. S.T $\{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$ is basis for R^4

A) Here R^4 is a vector space with dimension 4

$[R^4 \to \{(1,0,0,0) (0,1,0,0) (0,0,1,0) \underset{\text{std basis}}{(0,0,0,1)}\}]$

clearly the set is L.I

$a_1(1,0,0,-1) + a_2(0,1,0,-1) + a_3(0,0,1,-1) + a_4(0,0,0,1)$

$= (a_1, a_2, a_3, a_4)$

By Corollary, any L.I of R^4 that contains exactly 4 vectors is a basis R^4.

$\therefore$ the given set is a basis

Q. Determine which of the following sets basis for R^3

a) $S_1 = \{(1,0,-1) (2,5,1) (0,-4,3)\}$

b) $S_2 = \{(2,-4,1), (0,3,-1) (6,0,-1)\}$

Here $\dim(R^3) = 3$

We've a result,

Any L.I subset of V that contains exactly 'n' vectors is a basis for V

$\therefore \dim(R^3) = 3$ & the given set containing exactly 3 elements, it is enough to P.T the given set is L.I

$a_1(1,0,-1) + a_2(2,5,1) + a_3(0,-4,3) = (0,0,0)$

$(a_1, 0, -a_1) + (2a_1, 5a_2, a_2) + (0, -4a_3, 3a_3) = (0,0,0)$

$a_1 + 2a_2 = 0 \quad$ — ①

$5a_2 - 4a_3 = 0 \quad$ — ②

$-a_1 + a_2 + 3a_3 = 0 \quad$ — ③

① + ③ $\Rightarrow 3a_2 + 3a_3 = 0 \quad$ — ④

$a_2 + a_3 = 0 \quad$ — ⑤

⑤ in ② $\Rightarrow 5a_2 - 4a_3 = 0$

$a_2 = 0$, $a_3 = 0$

$\therefore a_1 = 0$

$\therefore S_1$ is L.I

b) $a_1(2,-4,1) + a_2(0,3,-1) + a_3(6,0,-1) = (0,0,0,0)$

$= (2a_1, -4a_1, a_1) + (0, 3a_2, -a_2) + 6(a_3, 0, -a_3) = (0,0,0)$

$2a_1 + 6a_3 = 0$ — ①

$-4a_1 + 3a_2 = 0$ — ②

$a_1 - a_2 - a_3 = 0$ — ③

$3 \times ③ = 3a_1 - 3a_2 - 3a_3$ — ④

$② + ④ \Rightarrow -a_1 - 3a_3 = 0$ — ⑤

$⑤ \times -2 = 2a_1 + 6a_3$

it is same as ①

$\therefore$ S_2 is L.D

Q Is $\{(1,4,-6), (1,5,8), (2,1,1), (0,1,0)\}$ is a L.I subset of R^3?

We've

$\dim(R^3) = 3$ & the set contains 4 elements

$\therefore$ given set cannot be a L.I seat of R^3 (Corollary)

$\therefore$ The set is not a basis for R^3

LINEAR TRANSFORMATIONS AND MATRICES

DEFINITION

Let V & W be VS over F
We call a fun $T: V \to W$ a LT from V to W.
if $\forall x, y \in V$ & $c \in F$, we've

1) $T(x+y) = T(x) + T(y)$
2) $T(cx) = cT(x)$

We simply called T linear

PROPERTIES OF LT

1. If T is linear, then $T(0) = 0$

PROOF
Suppose that T is linear
Then
$$T(x+y) = T(x) + T(y)$$
$$T(cx) = cT(x)$$
$$T(0) = T(0x) = 0T(x) \quad [c = 0]$$
$$= 0$$

2. T is linear iff $T(cx+y) = cT(x) + T(y)$ $\forall x, y \in V$ & $c \in F$

PROOF
Suppose that T is linear
Then $T(x+y) = T(x) + T(y)$
$T(cx) = cT(x)$

Consider, $T(cx+y) = T(cx) + T(y)$ by 1)
$= cT(x) + T(y)$ by 2)

Conversely suppose that
$T(cx+y) = cT(x) + T(y)$ — ③
to ST T is linear
Put $c = 1$ in ③
$T(x+y) = T(x) + T(y)$
Put $y = 0$ in ③
$T(cx+0) = cT(x+0)$
$T(cx) = cT(x)$
$\therefore$ T is linear

3 If T is linear, then $T(x-y) = T(x) - T(y)$ $\forall x, y \in V$

PROOF

Suppose that T is linear

Then using 2nd prop. we can write

$T(x+cy) = T(x) + cT(y)$ $\forall x, y \in V$ & $c \in F$

choosing $c = -1$

$T(x-y) = T(x) - T(y)$

4 T is linear iff for $x_1, x_2 \ldots x_n \in V$ & $a_1, a_2 \ldots a_n \in F$, we've

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

PROOF

Suppose that T is linear

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = T\left(a_1 x_1 + \sum_{i=2}^{n} a_i x_i\right)$$

$$= T(a_1 x_1) + T\left(\sum_{i=2}^{n} a_i x_i\right)$$

$$= a_1 T(x_1) + T\left(a_2 x_2 + \sum_{i=3}^{n} a_i x_i\right)$$

$$= a_1 T(x_1) + a_2 T(x_2) + T\left(a_3 x_3 + \sum_{i=4}^{n} a_i x_i\right)$$

$$= a_1 T(x_1) + a_2 T(x_2) + \ldots + a_n T(x_n)$$

$$= \sum_{i=1}^{n} a_i T(x_i)$$

Conversely suppose that

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

take $n = 2$,

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{2} a_i T(x_i)$$

$$T(a_1 x_1 + a_2 x_2) = a_1 T(x_1) + a_2 T(x_2)$$

take $a_2 = 1$

$$T(a_1 x_1 + x_2) = a_1 T(x_1) + T(x_2)$$

Using 2nd prop. T is linear

EXAMPLE

Q. S.T $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a_1, a_2) = (2a_1 + a_2, a_1)$ is linear

A) To prove T is linear it is enough to P.T $T(cx+y) = cT(x) + T(y)$ $\forall x, y \in \mathbb{R}^2$ & $c \in \mathbb{R}$

$\because x, y \in \mathbb{R}^2$

We can take, $x = (b_1, b_2)$, $y = (d_1, d_2)$

$cx+y = c(b_1,b_2)+(d_1,d_2)$

$= (cb_1+d_1, cb_2+d_2)$

$T(cx+y) = T(cb_1+d_1, cb_2+d_2)$

$= (2(cb_1+d_1)+cb_2+d_2, cb_1+d_1)$

$= (2cb_1+2d_1+cb_2+d_2, cb_1+d_1)$

$cT(x)+T(y) = cT(b_1,b_2)+T(d_1,d_2)$

$= c(2b_1+b_2, b_1)+(2d_1+d_2, d_1)$

$= (2b_1c+cb_2+2d_1+d_2, cb_1+d_1)$

$= T(cx+y)$

T is linear

Q. Define $T\ \mathbb{R}^2 \to \mathbb{R}^2$, $T(a_1,a_2) = (a_1,-a_2)$ then T is a LT called Reflection about the x-axis

A) To prove T is linear it is enough to P.T $T(cx+y) = cT(x)+T(y)$ $\forall$ $x,y \in \mathbb{R}^2$ & $c \in F$

$x = (b_1,b_2)$ $\quad y = (d_1,d_2)$

$cx+y = c(b_1,b_2)+(d_1,d_2)$

$= (cb_1,cb_2)+(d_1,d_2)$

$= (cb_1+d_1, (cb_2+d_2))$

$cT(x)+T(y) = cT(b_1,b_2)+T(d_1,d_2)$

$= c(b_1,-b_2)+(d_1,-d_2)$

$= (cb_1,-cb_2)+(d_1,-d_2)$

$= (cb_1+d_1, -(cb_2+d_2))$

$= T(cx+y)$

$\therefore$ T is linear

Q. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1,a_2) = (a_1,0)$. Then T is a LT is called projection on the x-axis

A) $x = (b_1,b_2)$ $\quad y = (d_1,d_2)$

$(cx+y) = c(b_1,b_2)+(d_1,d_2)$

$= (cb_1,cb_2)+(d_1,d_2)$

$= (cb_1+d_1, cb_2+d_2)$

$T(cx+y) = T(cb_1+d_1, cb_2+d_2)$

$= (cb_1+d_1, 0)$

$c(T(x))+T(y) = cT(b_1,b_2)+T(d_1,d_2)$

$= c(b_1,0)+(d_1,0)$

$= (cb_1, 0) + (d_1, 0)$

$= (cb_1 + d_1, 0)$

$= T(cx + y)$

$\therefore T$ is linear

Q S.T $T: M_{m \times n}(F) \to M_{n \times m}(F)$ by $T(A) = A^t$ is linear

$T(cA+B) = (cA+B)^t = cA^t + B^t$

$= cT(A) + T(B)$

Q Let $V = C(\mathbb{R})$ the v.s of conti real valued fun^s on $\mathbb{R}$. let $a, b \in \mathbb{R}$, $a < b$ then S.T $T: V \to \mathbb{R}$ defined by

$T(f) = \int_a^b f(t)\,dt \quad \forall f \in V$ is linear

$T(cf+g) = \int_a^b (cf(t) + g(t))\,dt$

$= \int_a^b cf(t)\,dt + \int_a^b g(t)\,dt$

$= c\int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

$= cT(f) + T(g)$

REMARK

1 For a v.s V over F we define the identity transformation $I: V \to V$ by $I(x) = x \ \forall x \in V$. Then clearly I is linear

PROOF

$I(cx+y) = cx + y$

$= cI(x) + I(y)$

2 For v.s V & W over F, we defined the zero transformation $T_0: V \to W$ by $T_0(x) = 0 \ \forall x \in V$. Then T_0 is linear

$T_0(cx+y) = 0$

$cT_0(x) + T_0(y) = c0 + 0 = 0$

$\therefore T_0$ is linear

DEFINITION

Let V & W be v.s. & let $T: V \to W$ be linear

We define the null space (kernel) of T $N(T)$ to be the set of vectors x in V s.t. $T(x) = 0$

i.e. $N(T) = \{x \in V : T(x) = 0\}$

We define the range or image of T $R(T)$ ~~of~~ to be the subset of W consisting of all images (under T) of vectors in V

ie $R(T) = \{T(x) : x \in V\}$

EXAMPLE

1 Let V & W be V.S., find the null space & range of identity & zero transformations

We've identity transformation $I : V \to V$ defined by $I(x) = x$

$N(I) = \{x \in V : I(x) = 0\}$

$= \{x \in V : x = 0\}$

$= \{0\}$

$R(I) = \{I(x) : x \in V\}$

$= \{x : x \in V\}$

$= V$

Now we've zero transformation is

$T_0 : V \to W$ by $T_0(x) = 0 \ \forall \ x \in V$

$N(T_0) = \{x \in V : T_0(x) = 0\}$

$= V$ [If we take any element from V, then T_0 maps it to 0]

$R(T_0) = \{T_0(x) : x \in V\}$

$= \{0\}$

2 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the L.T defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$ find null space & range of T

verify T is linear

$x = (b_1, b_2, b_3) \qquad y = (d_1, d_2, d_3)$

$cx + y = c(b_1, b_2, b_3) + (d_1, d_2, d_3)$

$= (cb_1, cb_2, cb_3) + (d_1, d_2, d_3)$

$= (cb_1 + d_1, cb_2 + d_2, cb_3 + d_3)$

$T(cx+y) = T(cb_1 + d_1, cb_2 + d_2, cb_3 + d_3)$

$= (cb_1 + d_1 - (cb_2 + d_2), 2cb_3 + 2d_3)$

$= (cb_1 + d_1 - cb_2 - d_2, 2cb_3 + 2d_3)$

$cT(x) + T(y) = cT(b_1, b_2, b_3) + (d_1, d_2, d_3)$

$= c(b_1 - b_2, 2b_3) + (d_1 - d_2, 2d_3)$

$= (cb_1 - cb_2, 2cb_3) + (d_1 - d_2, 2d_3)$

$= (cb_1 - cb_2 + d_1 - d_2, 2cb_3 + 2d_3)$

- Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the L.T. defined by,

$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$N(T) = \{x \in \mathbb{R}^3 : T(x) = 0\}$

$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 : T(a_1, a_2, a_3) = (0,0)\}$

$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 : (a_1 - a_2, 2a_3) = (0,0)\}$

$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - a_2 = 0, 2a_3 = 0\}$

$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_2, a_3 = 0\}$

$= \{0, a_2, 0\}$

$R(T) = \{T(x) : x \in \mathbb{R}^3\} = \mathbb{R}^2$

THEOREM 2:

Let V & W be V.S. & $T: V \to W$ be linear. Then N(T) & R(T) are subspaces of V & W resp.

PROOF

Let 0_V & 0_W denote the zero vector of V & W resp.

CLAIM 1

N(T) is a subspace of V

We've $N(T) = \{x \in V : T(x) = 0_W\}$

1) Existance of identity element in N(T)

$\because T(0_V) = 0_W$, we can write

$0_V \in N(T)$

2) Now let $x, y \in N(T)$ & $c \in F$

Then $T(x+y) = T(x) + T(y)$ $\because$ linear

$= 0_W + 0_W$

$= 0_W$

$\therefore x + y \in N(T)$

$\Rightarrow$ addition is closed in N(T)

~~Next let x, y ∈ N(T) & c ∈ F~~

~~Then T(cx+y) = T(cx) + T(y)~~ *

3) $T(cx) = cT(x)$ $\because$ T is linear

$= c\,0 = 0$

$\Rightarrow cx \in N(T)$

$\therefore$ Scalar multiplication is closed in N(T)

$\Rightarrow$ N(T) is a subspace of V

CLAIM :

$R(T)$ is a subspace of W

we've $R(T) = \{T(x) : x \in V\}$

1) we've $T(0_V) = 0_W$

$\therefore T(0_V) \in R(T)$, we've

$0_W \in R(T)$

2) Now let $x, y \in R(T)$

Then $\exists\ u, v \in V \ni T(u) = x$ & $T(v) = y$

Now $T(u+v) = T(u) + T(v)$ $\because T$ is linear

$= x + y$

$\Rightarrow x + y \in R(T)$

3) let $c \in F$ & $x \in R(T)$

Then $\exists\ u \in V \ni T(u) = x$

$\therefore T(cu) = cT(u)$ $\because T$ is linear

$= cx$

$\Rightarrow cx \in R(T)$

$\Rightarrow R(T)$ is a subspace of W

THEOREM 2.2

Let V & W be v.s & let $T: V \to W$ be linear. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V, then

$R(T) = \text{span}(T(\beta)) = \text{span}\{T(v_1), T(v_2), \dots, T(v_n)\}$

PROOF

Let V & W be v.s & let $T: V \to W$ be linear. Suppose that

$\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V

we've

$R(T) = \{T(x) : x \in V\}$

$\because v_i \in V$ for each i, we can write

$T(v_i) \in R(T)$ for each i

$\because R(T)$ is a subspace of W & the set $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is contained in $R(T)$ by thm

we can write $\text{span}\{T(v_1), T(v_2), \dots, T(v_n)\} \subseteq R(T)$

ie $\text{span}(T(\beta)) \subseteq R(T)$ — ①

Now suppose that $w \in R(T)$

Then $\exists\ v \in V \ni T(v) = w$

$\because \beta$ is a basis for V & $v \in V$, we've

$v = \sum_{i=1}^{n} a_i v_i$ for some $a_1, a_2, \dots, a_n \in F$

$\Rightarrow T(\bar{v}) = T\left(\sum_{i=1}^{n} a_i v_i\right)$

& $w = \sum_{i=1}^{n} a_i T(v_i)$ $\quad \because T$ is linear

$\in \text{span}(T(\beta))$

& $w \in \text{span}(T(\beta))$

$\Rightarrow R(T) \subseteq \text{span}(T(\beta))$ — ②

① & ② $\Rightarrow R(T) = \text{span}(T(\beta))$

DEFINITION

Let V & W be V.S. & let $T: V \to W$ be linear. If $N(T)$ & $R(T)$ are finite dimensional, then we define the nullity of T, denoted by nullity (T) & the rank of T denoted rank (T) to be the dimensions of $N(T)$ & $R(T)$ resp.

i.e. nullity (T) = dim (N(T))

& rank (T) = dim (R(T))

THEOREM 2.3 (DIMENSION THEOREM)

Let V & W be V.S. & let $T: V \to W$ be linear. If V is finite-dimensional, then

nullity (T) + rank (T) = dim (V)

THEOREM 2.4

Let V & W be V.S. & let $T: V \to W$ be linear. Then T is one-to-one iff $N(T) = \{0\}$

PROOF

Suppose that T is one-to-one

We want to P.T $N(T) = \{0\}$

Let $x \in N(T)$

Then by def. of $N(T)$ we've

$T(x) = 0$

But we've $T(0) = 0$

$\Rightarrow T(x) = T(0)$

$\because$ T is one-one, we've $x = 0$

$\therefore N(T) = \{0\}$

Conversely suppose that $N(T) = \{0\}$

To prove T is one-one

Suppose that $T(x) = T(y)$ for some $x, y \in V$

$\Rightarrow T(x) - T(y) = 0$

$\Rightarrow T(x - y) = 0, \quad \because T$ is linear

$\therefore$ By def. of $N(T)$, $x - y \in N(T) = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

$\therefore$ T is one-one

THEOREM 2.5

Let V & W be V.Ss of equal (finite) dimension & let $T: V \to W$ be linear. Then the follow. are equivalent

a) T is one one

b) T is onto

c) $\text{rank}(T) = \dim(V)$

PROOF

$\because$ V & W have equal dimension

$\dim(V) = \dim(W)$

$a \Rightarrow b$

Suppose that T is one one

Then by thm 2.4 $N(T) = \{0\}$

$\therefore \dim(N(T)) = 0$

i.e. $\text{nullity}(T) = 0$

We've from the dimension thm.

$\text{Nullity}(T) + \text{rank}(T) = \dim(V)$

$0 + \text{rank}(T) = \dim(V)$

i.e. $\text{rank}(T) = \dim V$

$\Rightarrow \dim(R(T)) = \dim W$ ($\because \dim V = \dim W$)

$\Rightarrow R(T) = W$ (by thm 1.11)

$\Rightarrow T$ is onto

$b \Rightarrow c$

Suppose that T is onto

$\Rightarrow R(T) = W$

$\Rightarrow \dim(R(T)) = \dim(W)$

i.e. $\text{rank}(T) = \dim W = \dim V$

i.e. $\text{rank}(T) = \dim V$

$c \Rightarrow a$

Suppose that $\text{rank}(T) = \dim(V)$

$\therefore$ we can write $0 + \text{rank}(T) = \dim(V)$ (by dimension thm)

$\therefore \text{nullity}(T) = 0$

i.e. $\dim(N(T)) = 0$

$\Rightarrow N(T) = \{0\}$

Then by thm 2.4, T is one one

THEOREM 2.6

Let V & W be V.Ss over F & suppose that $\{v_1, v_2, \dots v_n\}$ is a basis for V. For $w_1, w_2, \dots w_n$ in W $\exists$ exactly one linear transformation $T: V \to W$ $\ni$ $T(v_i) = w_i$ for $i = 1, 2, \dots n$

Proof

Suppose that V & W be VSs over F & let $\{v_1, v_2, \dots v_n\}$ be a basis for V

Let $x \in V$

$\{v_1, v_2, \dots v_n\}$ is a basis for V, we can write $x = \sum_{i=1}^{n} a_i v_i$, where $a_1, a_2, \dots a_n$ are scalars

Define $T: V \to W$ by $T(x) = \sum_{i=1}^{n} a_i w_i$

1) T is linear

Let $u, v \in V$ & $d \in F$

Then we can write

$u = \sum_{i=1}^{n} b_i v_i$ & $v = \sum_{i=1}^{n} c_i v_i$

where $b_1, b_2, \dots b_n$, $c_1, c_2, \dots c_n$ are scalars

Then $T(u) = \sum_{i=1}^{n} b_i w_i$ & $T(v) = \sum_{i=1}^{n} c_i w_i$

Consider $du + v = d \sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} c_i v_i$

$$= \sum_{i=1}^{n} d b_i v_i + \sum_{i=1}^{n} c_i v_i$$

$$= \sum_{i=1}^{n} (d b_i + c_i) v_i$$

$$T(du+v) = T\left(\sum_{i=1}^{n} (d b_i + c_i) v_i\right)$$

$$= \sum_{i=1}^{n} (d b_i + c_i) w_i$$

$$= \sum_{i=1}^{n} (d b_i w_i + c_i w_i)$$

$$= \sum_{i=1}^{n} d b_i w_i + \sum_{i=1}^{n} c_i w_i$$

$$= d \sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i$$

$$= d\,T(u) + T(v)$$

$\therefore$ T is linear

2) clearly $T(v_i) = w_i$, for $i = 1, 2, \dots n$

3) T is unique

[$x = \sum_{i=1}^{n} a_i v_i$, $T(x) = \sum a_i T(v_i)$

T is linear, also $T(x) = \sum a_i w_i$

$\therefore w_i = T(v_i)$]

Suppose that $\exists$ another LT

$U: V \to W$ defined by $U(v_i) = w_i$ for $i = 1, 2, \dots n$

Then for $x \in V$, we've

$x = \sum_{i=1}^{n} a_i v_i$

$\therefore U(x) = U\left(\sum_{i=1}^{n} a_i v_i\right)$

$= \sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i w_i$

$= T(x)$

$U = T$

Hence the proof

COROLLARY

Let V & W be vector spaces & suppose that V has a finite basis $\{v_1, v_2, \ldots v_n\}$ if $U, T: V \to W$ are linear & $U(v_i) = T(v_i)$ for $i = 1, 2, \ldots n$ then $U = T$

PROOF

Let $x \in V$

$\{v_1, v_2, \ldots v_n\}$ is a basis for V, we can write

$x = \sum_{i=1}^{n} a_i v_i$ where $a_1, a_2, \ldots a_n$ are scalars

Now $U(x) = U\left(\sum_{i=1}^{n} a_i v_i\right)$

$= \sum_{i=1}^{n} a_i U(v_i)$

$= \sum_{i=1}^{n} a_i T(v_i)$ ---- $[\because U(v_i) = T(v_i)]$

$= T\left(\sum_{i=1}^{n} a_i v_i\right)$ $\because$ T is linear

$= T(x)$

Hence $\underline{U = T}$

Q. Define the LT

$T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

$T(f(x)) = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$

find range of T

A) We've,

$\beta = \{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$

Then $R(T) = \text{span}(T(\beta))$

$= \text{span}\{T(1), T(x), T(x^2)\}$ - ① by thm 2.2

To find $T(1)$

$f(x) = 1$

$f(1) = 1, f(2), f(0) = 1$

$T(1) = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

To find x

$f(x) = x$

$f(1) = 1$

$f(2) = 2$

$f(0) = 0$

$T(x) = \begin{bmatrix} 1-2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

To find x^2

$f(x) = x^2$

$f(1) = 1$

$f(2) = 4$

$f(0) = 0$

$T(x) = \begin{bmatrix} 1-4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore (1) \Rightarrow R(T) = \text{Span}\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

Here $\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} = 3\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore T(x)$ & $T(x^2)$ are L.D

6. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be linear & set $T(1,0) = (1,4)$ &

$T(1,1) = (2,5)$

i) find $T(2,3)$

ii) A general expression for T

iii) Is T is one-one

i) Here $\dim(\mathbb{R}^2) = 2$

$a_1(1,0) + a_2(1,1) = (0,0)$

$a_1 + a_2 = 0$

$a_2 = 0$

$\Rightarrow a_1 = 0$

$\therefore \{(1,0),(1,1)\}$ is L.I

The $\dim(\mathbb{R}^2) = 2$ & the above set contain 2 element, then $\{(1,0)\ (1,1)\}$ is a basis for $\mathbb{R}^2$

$\therefore$ any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\{(1,0)\ (1,1)\}$

$(2,3) = a_1(1,0) + a_2(1,1)$

$= a_1 + a_2 = 2$

$a_2 = 3$

$a_1 = -1$

$\therefore (2,3) = -1(1,0) + 3(1,1)$

$T(2,3) = -1\,T(1,0) + 3\,T(1,1)$

$= -1(1,4) + 3(2,5)$

$= (5,11)$

2) Let $(x,y) \in R^2$

$(x,y) = a_1(1,0) + a_2(1,1)$

$a_1 + a_2 = x$

$a_2 = y$

$\Rightarrow a_1 = x - y$

$(x,y) = (x-y)(1,0) + y(1,1)$

$T(x,y) = (x-y)\,T(1,0) + y\,T(1,1)$

$= (x-y)(1,4) + y(2,5)$

$= (x-y, 4x-4y) + (2y, 5y)$

$= (x+y, 4x+y)$

3) To check T is one-one

Suppose that $T(x_1,y_1) = T(x_2,y_2)$

$(x_1+y_1, 4x_1+y_1) = (x_2+y_2, 4x_2+y_2)$

$x_1 + y_1 = x_2 + y_2 \quad — ①$

$4x_1 + y_1 = 4x_2 + y_2 \quad — ②$

② − ①

$3x_1 = 3x_2$

$x_1 = x_2$

① $\Rightarrow y_1 = y_2$

$(x_1,y_1) = (x_2,y_2)$

$\therefore$ T is one one

Q. Check the linearity of 1) $T(a_1,a_2) = (1,a_2)$

2) $T(a_1,a_2) = (a_1,a_2^2)$

3) $T(a_1,a_2) = (a_1+1,a_2)$

4) $T(x,y) = (x,y,0) \quad (R^2 \to R^3)$

5) $T(x,y) = (x, x+y, y) \quad (R^2 \to R^3)$

6) $T(x,y,z) = (x,y,z) \quad (R^3 \to R^3)$

THE MATRIX REPRESENTATION OF A LT

Let V be a finite dimensional v.s. An ordered basis for V with a specific order

ie An ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.

EXAMPLE

Q. In F^3, $\beta = \{e_1, e_2, e_3\}$ is an ordered basis. Also $\gamma = \{e_2, e_1, e_3\}$ is an ordered basis

But $\beta \neq \gamma$ as ordered basis

For the v.s. F^n we call $\{e_1, e_2, \ldots e_n\}$ the std ordered basis for F^n

DEFINITION

Let $\beta = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for a finite dimensional v.s. V

For $x \in V$, let $a_1, a_2, \dots, a_n$ be the unique scalars s.t.

$$x = \sum_{i=1}^{n} a_i u_i$$

we define the coordinate vector of x relative to β denoted by $[x]_\beta$ by

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

eg let $V = P_2(R)$ & let $\beta = \{1, x, x^2\}$ be the std ordered basis for V

If $f(x) = 4 + 6x - 7x^2$, then

$$[f]_\beta = \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}$$

DEFINITION

Suppose that V & W are finite dimensional v.s with ordered basis $\beta = \{v_1, v_2, \dots, v_n\}$ & $\gamma = \{w_1, w_2, \dots, w_m\}$ resp. Let $T: V \to W$ be linear. Then for each j, $1 \le j \le n$, $\exists$ unique scalars $a_{ij} \in F$, $1 \le i \le m$ s.t.

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i, \quad \text{for } 1 \le j \le n$$

we call the $m \times n$ matrix A defined by

$$A_{ij} = a_{ij}$$

the matrix representation of T in the ordered basis β & γ and write $A = [T]_\beta^\gamma$

If $V = W$ & $\beta = \gamma$, then we write

$$A = [T]_\beta$$

DEFINITION

Let V & W be v.s over F we denote the v.s of all L.T from V into W by $\mathcal{L}(V, W)$

In the case that $V = W$ we write $\mathcal{L}(V)$

THEOREM 2.8

Let V & W be finite dimensional v.s with ordered basis β & γ resp & let $T, U: V \to W$ be L.T

Then

a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma \quad \forall$ scalars

Q Let $T: V \to W$ be an invertible LT then ST $T^{-1}: W \to V$ is linear ie inverse LT is linear

Let $y_1, y_2 \in W$ & $c \in F$

T is invertible it is one one & onto

$\therefore T$ is onto $(T: V \to W)$ & $y_1, y_2 \in W$

$\exists\, \alpha_1, \alpha_2 \in V \ni T(\alpha_1) = y_1$ & $T(\alpha_2) = y_2$

$\Rightarrow \alpha_1 = T^{-1}(y_1)$ & $\alpha_2 = T^{-1}(y_2)$

$\because T$ is linear & $\alpha_1, \alpha_2 \in V$

$T(c\alpha_1 + \alpha_2) = cT(\alpha_1) + T(\alpha_2)$

$= cy_1 + y_2$

$T^{-1}(cy_1 + y_2) = c\alpha_1 + \alpha_2$

$= cT^{-1}(y_1) + T^{-1}(y_2)$

$T^{-1}: W \to V$ is linear

EXAMPLE

Q Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the LT by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

find matrix of T

we've

$\beta = \{(1,0)\ (0,1)\}$ &

$\gamma = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$

$\therefore \alpha$

be the std ordered basis for $\mathbb{R}^2$ & $\mathbb{R}^3$ resp

$T(1,0) = (1,0,2)$

$T(0,1) = (3,0,-4)$

$T(1,0) = (1,0,2) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$\left[T(v_j) = \sum_i a_{ij} w_i\right]$

$(1,0,2) = (a_1, a_2, a_3)$

$a_1 = 1,\ a_2 = 0,\ a_3 = 2$

$T(0,1) = (3,0,-4) = b_1(1,0,0) + b_2(0,1,0) + b_3(0,0,1)$

$= (b_1, b_2, b_3)$

$b_1 = 3,\ b_2 = 0,\ b_3 = -4$

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$$

If $\gamma = \{(0,0,1)\ (0,1,0)\ (1,0,0)\}$

Then above eq becomes

$\beta = \{(1,0)\ (0,1)\},\ \gamma' = \{(0,0,1)\ (0,1,0)\ (1,0,0)\}$

$T(1,0) = (1,0,2)$

& $T(0,1) = (3,0,-4)$

$T(1,0) = (1,0,2)$

$= a_1(0,0,1) + a_2(0,1,0) + a_3(1,0,0)$

$(1,0,2) = (a_3, a_2, a_1)$

$a_3 = 1,\ a_2 = 0,\ a_1 = 2$

$T(0,1) = (3,0,-4) = b_1(0,0,1) + b_2(0,1,0) + b_3(1,0,0)$

$= [b_3, b_2, b_1]$

$b_3 = 2,\ b_2 = 0,\ b_1 = -4$

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$$

Q $T: P_3(R) \to P_2(R)$ be L.T by

$f(x) = f'(x)$

find the matrix representation of T

$\beta = \{1, x, x^2, x^3\}$ $\gamma = \{1, x, x^2\}$

$T(1) = 0$

$= 0\cdot 1 + 0x + 0x^2$

$T(x) = 1$

$1\cdot 1 + 1\cdot 0 + 0x^2$

$T(x^2) = 2x$

$= 0\cdot 1 + 2x + 0\cdot x^2$

$f(x^3) = 3x^2$

$= 0\cdot 1 + 0\cdot x + 3x^2$

$$[T]_\beta^\gamma \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Q Let $T: R^2 \to R^3$ & $U: R^2 \to R^3$ be the L.Ts resp defined by

$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

$U(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$

Let β & γ be the std O.B of R^2 & R^3 resp.

P.T $[T+U] = [T]_\beta^\gamma + [U]_\beta^\gamma$

$\beta = \{(1,0)\ (0,1)\}$

$\gamma = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$

$T(1,0) = (1,0,2)$

$= 1(1,0,0) + 0(0,1,0) + 2(0,0,1)$

$T(0,1) = (3,0,-4)$

$= 3(1,0,0) + 0(0,1,0) + -4(0,0,1)$

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$$

$U(1,0) = (1,2,3)$

$= 1(1,0,0) + 2(0,1,0) + 3(0,0,1)$

$U(0,1) = (-1,0,2)$

$= -1(1,0,0) + 0(0,1,0) + 2(0,0,1)$

$$[U]_\beta^\gamma = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$RHS = [T]_\beta^\gamma + [U]_\beta^\gamma = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$$

$(T+U)(a_1,a_2) = T(a_1,a_2) + U(a_1,a_2)$

$= (a_1+3a_2, 0, 2a_1-4a_2) + (a_1-a_2, 2a_1, 3a_1+2a_2)$

$= (2a_1+2a_2, 2a_1, 5a_1-2a_2)$

$(T+U)(1,0) = (2,2,5)$

$= 2(1,0,0) + 2(0,1,0) + 5(0,0,1)$

$(T+U)(0,1) = (2,0,-2)$

$= 2(1,0,0) + 0(0,1,0) + -2(0,0,1)$

$$LHS = [T+U]_\beta^\gamma = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$$

LHS = RHS

Let $T: P_2(R) \to P_2(R)$ both L.T defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt$ is T one one ? Is T onto?

$\dim(P_2(R)) = 3$

Rank (T) + Nullity $(T) = \dim(V)$

$\{1, x, x^2\}$ is a basis for $P_2(R)$

$\therefore \{T(1), T(x), T(x^2)\}$ span $R(T)$

$T(1) = 2\times 0 + \int_0^x 3\,dt = 3t\Big]_0^x = 3x$

$f(x) = 1$

$T(x) = 2\times 1 + \int_0^x 3t\,dt = 2 + \frac{3}{2}t^2\Big]_0^x$

$= 2 + \frac{3}{2}x^2$

$T(x^2) = 2\times 2x + \int_0^x 3t^2 dt = 4x + x^3\Big]_0^x$

$= 4x + x^3$

$f(x) = x^2$

Q Let $T: R^2 \to R^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$

Let β be the std O.B for R^2 &

$\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$

a) Compute $[T]_\beta^\gamma$

b) If $\alpha = \{(1,2), (2,3)\}$

Compute $[T]_\alpha^\gamma$

a) $\beta = \{(1,0)\ (0,1)\}$

$\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$

$T(1,0) = (1-0, 1, 2\cdot 1 + 0)$

$= (1,1,2)$

$(1,1,2) = c_1(1,1,0) + c_2(0,1,1) + c_3(2,2,3)$

$= (c_1, c_1, 0) + (0, c_2, c_2) + (2c_3, 2c_3, 3c_3)$

$c_1 + 2c_3 = 1$

$c_1 + 2c_2 + 2c_3 = 1$

$c_2 + 3c_3 = 2$

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \end{array}\right] \underset{\sim}{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$\sim R_3 \to R_3 - R_2 \left[\begin{array}{ccc:c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

$c_1 + 2c_3 = 1$ — ①

$c_2 = 0$ — ②

$3c_3 = 2 \Rightarrow c_3 = 2/3$ — ③

$\therefore ① \to c_1 + 2\,\frac{2}{3} = 1$

$c_1 + \frac{4}{3} = 1$

$c_1 = 1 - \frac{4}{3}$

$c_1 = -\frac{1}{3}$

ie $T(1,0) = (1,1,2)$

$= \frac{-1}{3}(1,1,0) + 0\,(0,1,1) + \frac{2}{3}(2,2,3)$

$T(0,1) = (0-1, 0, 2\,0+1)$

$= (-1, 0, 1)$

$(-1,0,1) = c_1(1,1,0) + c_2(0,1,1) + c_3(2,2,3)$

$c_1 + 2c_3 = -1$

$c_1 + c_2 + 2c_3 = 0$

$c_2 + 3c_3 = 1$

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right] \underset{\sim}{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$$\sim R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

$$\sim R_3 \to \frac{1}{3}R_3 \quad \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/3 \end{array}\right]$$

$$\sim R_1 \to R_1 - 2R_3 \quad \left[\begin{array}{ccc:c} 1 & 0 & 0 & -5/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/3 \end{array}\right]$$

$c_1 = -5/3$

$c_2 = 1$

$c_3 = 1/3$

ie, $T(0,1) = (-1,0,1)$

$T(0,1) = -5/3\,(1,1,0) + 1\,(0,1,1) + \frac{1}{3}(2,2,3)$

$$[T]_{\beta}^{\gamma} = \begin{bmatrix} -\frac{1}{3} & -\frac{5}{3} \\ 0 & 1 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

b) $\alpha = \{(1,2), (2,3)\}$

$\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$

$T(1,2) = (1-2, 1, 2\cdot1+2)$

$= (-1,1,4)$

$(-1,1,4) = c_1(1,1,0) + c_2(0,1,1) + c_3(2,2,3)$

$c_1 + 2c_3 = -1$

$c_1 + c_2 + 2c_3 = 1$

$c_2 + 3c_3 = 4$

$$\left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 4 \end{array}\right] \underset{\sim}{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 3 & 4 \end{array}\right]$$

$$\sim R_3 \to R_3 - R_2 \quad \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

$$\sim R_3 \to \frac{1}{3}R_3 \quad \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$$

$$\sim R_1 \to R_1 - 2R_3 \quad \left[\begin{array}{ccc:c} 1 & 0 & 0 & -7/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2/3 \end{array}\right]$$

$c_1 = -7/3$

$c_2 = 2$

$c_3 = 2/3$

$T(1,2) = (-1,1,4)$

$= -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$

$T(2,3) = (2-3, 2, 2\cdot2+3)$

$= (-1,2,7)$

$(-1,2,7) = c_1(1,1,0) + c_2(0,1,1) + c_3(2,2,3)$

$c_1 + 2c_3 = -1$

$c_1 + c_2 + 2c_3 = 2$

$c_2 + 3c_3 = 7$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 7 \end{bmatrix} \underset{\sim}{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 7 \end{array}\right]$$

$$\sim R_3 \to R_3 - R_2 \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$$\sim R_3 \to \frac{1}{3} R_3 \left[\begin{array}{ccc:c} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4/3 \end{array}\right]$$

$$R_1 \to R_1 - 2R_3 \left[\begin{array}{ccc:c} 1 & 0 & 0 & -11/3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4/3 \end{array}\right]$$

$c_1 = -11/3$

$c_2 = 3$

$c_3 = 4/3$

$$T(2,3) = (-1,2,7)$$

$$= \frac{-11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$$

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} -\frac{7}{3} & \frac{-11}{3} \\ 2 & 3 \\ -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

SYSTEM OF LINEAR EQUATIONS, EIGEN VALUES AND EIGEN VECTORS

EIGEN VALUES

If A is any square matrix of order n, we can form the matrix $A-\lambda I$, where I is the n^{th} order unit matrix. The determinant of this matrix equated to zero,

i.e. $$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the characteristic eq^n of A. On expanding the determinant, the characteristic eq^n takes the form

$$(-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \cdots + k_n = 0$$

where k's are expressible in terms of the elements a_{ij}. The roots of this eq^n are called the eigen values or latent roots or characteristic roots of the matrix A.

EIGEN VECTORS

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

then the L.T $Y = AX$ carries the column vector X into the column vector Y by means of the square matrix A. It is required to find such vectors which transform into themselves or to a scalar multiple of themselves.

~~Let X be such a vector into the column vector by means of the square matrix A~~

Let X be such a vector which transforms into λX by means of the transformation $Y = AX$

Then

$$\lambda X = AX$$

$$AX - \lambda IX = 0$$

$$[A - \lambda I]X = 0 \quad \text{--- (1)}$$

This matrix eq^n represents n homogeneous linear eq^s

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} \quad (2)$$

which will have a non-trivial sol. only if the coefficient matrix is singular.

ie if $|A-\lambda I|=0$

This is called the characteristic eqn of the transformation & is same as the characteristic eqn of the matrix A. It has n roots & corresp to each root, ① or ② will have a non-zero sol.

$X=[x_1, x_2, \ldots, x_n]'$, which is known as the eigen vector or latent vector.

NOTE

Corresponding to 'n' distinct eigen values, we get n independent eigen vectors. But when 2 or more eigen values are equal, it may or may not be possible to get L I eigen vectors corresp to the repeated roots.

If X_i is a sol for a eigen value λ_i then it follows from ① that cX_i, is also a sol., where c is arbitrary constant. Thus the eigen vector corresp to a eigen value is not unique but may be any one of the vectors cX_i.

Q Find the eigen values & eigen vectors of the matrix

$$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

A) The characteristic eqn is $[A-\lambda I]=0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}=0$$

$$\lambda^2-7\lambda+6=0$$

$$(\lambda-6)(\lambda-1)=0$$

$$\therefore \lambda=6,1$$

Thus the eigen values are 6 & 1

If x_1, x_2 be the components of an eigen vector corresp to the eigen value λ, then

$$[A-\lambda I]X=\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix}=0$$

when $\lambda=6$

we've

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix}=0$$

$$-x_1+4x_2=0$$

$$x_1-4x_2=0$$

$$\Rightarrow x_1=4x_2$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

The eigen vector corresp. to eigen value is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

when $\lambda = 1$

We've

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 + 4x_2 = 0$$

$$x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The eigen vector corresp. to eigen value is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Q Find the eigen values & eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

The C E eqn is

$|A-\lambda I| = 0$, ie

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

Thus the eigen values of A are 2, 3, 5

If x_1, x_2, x_3 be the components of an eigen vector corresp to the eigen value λ, we've

$$[A-\lambda I]\, X = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Putting $\lambda = 2$, we've

$x_1 + x_2 + 4x_3 = 0$

$6x_3 = 0$, $3x_3 = 0$

ie $x + y = 0$ & $z = 0$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0} = k_1$$

Hence eigen vector corresp to $\lambda = 2$ is $k_1(1, -1, 0)$

Putting $\lambda = 3$, we've

$$x_2 + 4x_3 = 0$$
$$-x_2 + 6x_3 = 0$$
$$2x_3 = 0$$
$$x_2 = 0$$
$$x_3 = 0$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0} = k_2$$

Hence the eigen vector corresp to $\lambda = 3$ is $k_2(1, 0, 0)$

IIIly the eigen vector corresp to $\lambda = 5$ is $k_3(3, 2, 1)$

PROPERTIES OF EIGEN VALUES

1 Any square matrix A & its transpose A' have the same eigen values

we've

$$(A-\lambda I)' = A' - \lambda I' = A' - \lambda I$$
$$|(A-\lambda I)'| = |A' - \lambda I|$$
$$|A - \lambda I| = |A' - \lambda I| \qquad [\because |B'| = |B|]$$

$\therefore |A - \lambda I| = 0$ iff $|A' - \lambda I| = 0$

ie λ is an eigen value of A iff it is an eigen value of A'.

2. The eigen values of a Δlar matrix are just the diagonal elements of the matrix

Let
$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & & \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$
be a Δlar matrix of order n.

Then
$$|A-\lambda I| = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

$\therefore$ roots of $|A-\lambda I| = 0$ are $\lambda = a_{11}, a_{22}, \cdots a_{nn}$.

Hence the eigen values of A are the diagonal elements of A. ie, $a_{11}, a_{22}, \cdots a_{nn}$.

CORROLARY:- The eigen values of a diagonal matrix are just the diagonal elements of the matrix.

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3. The eigen values of an idempotent matrix are either zero or unity.

Let A be an idempotent matrix so that $A^2 = A$. If λ be an eigen value of A, then $\exists$ a non-zero vector X s.t.

$$AX = \lambda X \quad -①$$

$\therefore A(AX) = A(\lambda X)$

ie $A^2X = \lambda(AX)$

$$AX = \lambda(\lambda X) \qquad [\because A^2 = A \ \& \ AX = \lambda X]$$
$$= \lambda^2 X \quad -②$$

from ① & ② we get.

$\lambda^2 X = \lambda X$

or $(\lambda^2 - \lambda)X = 0$

$\lambda^2 - \lambda = 0$ whence $\lambda = 0$ or 1

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4. The sum of the eigen values of a matrix is its trace. ie the sum of the elements of the principal diagonal.

Consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad -①$$

so that

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \cdots \quad -②$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then

$$|A-\lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

$$= -\lambda^3 + \lambda^2(\lambda_1+\lambda_2+\lambda_3) - \cdots \quad — ③$$

Equating RHS of ② & ③ & Comparing Coefficients of λ^2, we get

$$\lambda_1+\lambda_2+\lambda_3 = a_{11}+a_{22}+a_{33}$$

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5. The product of the eigen values of a matrix A is equal to its determinant.

Putting $\lambda = 0$ in ③ we get the result.

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6. If λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}. If X be the eigen vector corresp. to λ, then $AX = \lambda X$.

Premultiplying both sides by A^{-1}, we get

$$A^{-1}AX = A^{-1}\lambda X$$

ie $IX = \lambda A^{-1}X$ or

$$X = \lambda(A^{-1}X)$$

ie $A^{-1}X = (1/\lambda)X$

This being of the same form as ① shows that $\frac{1}{\lambda}$ is an eigen value of the inverse matrix A^{-1}.

7. If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

w.k.t if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1} (Prop 5) $\because$ A is an orthogonal matrix, A^{-1} is same as its transpose A'

$\therefore \frac{1}{\lambda}$ is an eigen value of A'

But the matrices A & A' have the same eigen values, $\because$ the determinant $|A-\lambda I|$ & $|A'-\lambda I|$ are the same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

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8. If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m \ldots \lambda_n^m$ (m being a +ve integer).

Let A_i be the eigen value of A & X_i, the corresp eigen vector. Then

$$AX_i = \lambda_i X_i \quad — ①$$

We've $A^2X_i = A(AX_i) = A(\lambda_i X_i) = \lambda_i(AX_i) = \lambda(\lambda_i X_i) = \lambda_i^2 X_i$

||ly $A^3X_i = \lambda_i^3 X_i$

In general, $A^m X_i = \lambda_i^m X_i$, which is of the same form as ①.

Hence λ^m is an eigen value of A^m

The corresp. eigen vector is the same X_i.

CAYLEY-HAMILTON THEOREM

Every square matrix satisfies its own characteristic eqn; ie if the C.E for the n^{th} order square matrix A is

$$|A-\lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + \dots + k_n = 0$$

then, $$(-1)^n A^n + k_1 A^{n-1} + \dots + k_n = 0$$

Q Verify Cayley-Hamilton thm for the matrix A, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ & find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

A) The C E ~~of~~ of A is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda - 5 = 0 \quad — ①$$

By Cayley-Hamilton thm, A must satisfy its characteristic eqn ① so that

$$A^2 - 4A - 5I = 0 \quad — ②$$

Now,

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 0$$

This verifies the thm.

Multiplying ② by A^{-1}, we get

$$A - 4I - 5A^{-1} - 0$$

$$A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now dividing the polynomial

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$$

by the poly. $\lambda^2 - 4\lambda - 5$,

we obtain

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 - \lambda - 10I = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$
$$= \lambda + 5$$

Hence

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$$

which is a linear polynomial in A.

Q. Find the CE of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$ & hence find its inverse

A. The CE is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$

ie $\lambda^3 - 20\lambda + 8 = 0$

By Cayley-Hamilton thm,

$A^3 - 20A + 8I = 0$

whence $A^{-1} = \frac{5}{2} I - \frac{1}{8} A^2$

$$= \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Q. Find the CE of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ & hence compute A^{-1}. Also find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

A) The C.E of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

According to CH thm, we've

$A^3 - 5A^2 + 7A - 3I = 0$ — ①

multiplying ① by A^{-1}, we get

$A^2 - 5A + 7I - 3A^{-1} = 0$

$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$ — ②

But

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Hence from ②

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= A^2 + A + I$ $\quad (\because A^3 - 5A^2 + 7A - 3I = 0)$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$