ANOVA

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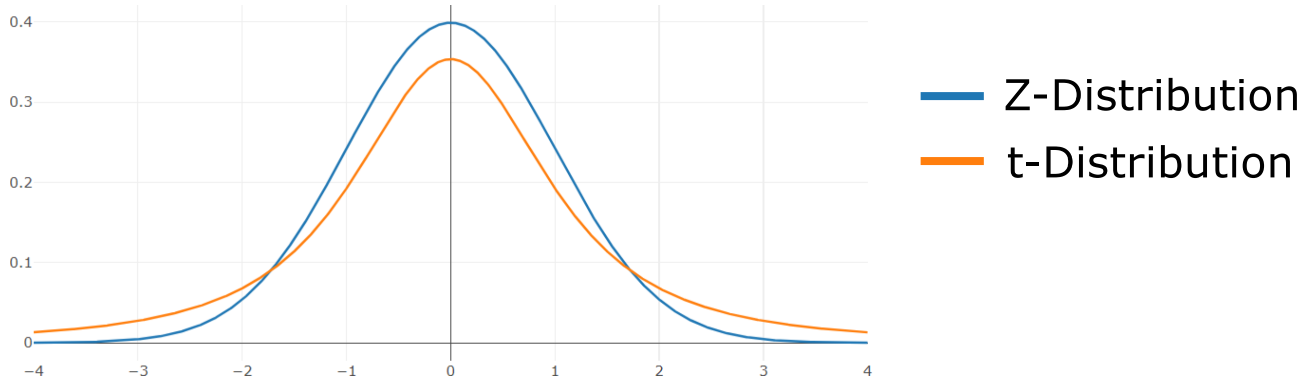
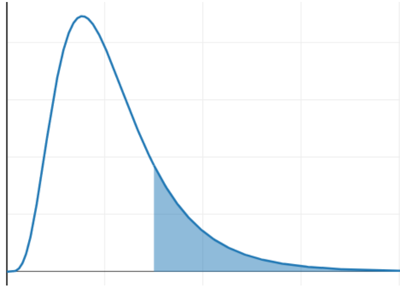
Analysis of Variance

- In the previous section we used Z- and t-Distributions to answer the question "*What is the probability that two samples come from the same population?*"



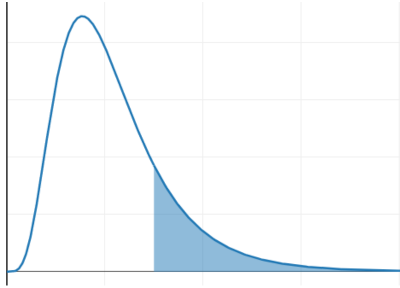
Analysis of Variance

- In this section we introduce a new distribution – the F-Distribution
- Used to answer the question "*What is the probability that two samples come from populations that have the same variance?*"



Analysis of Variance

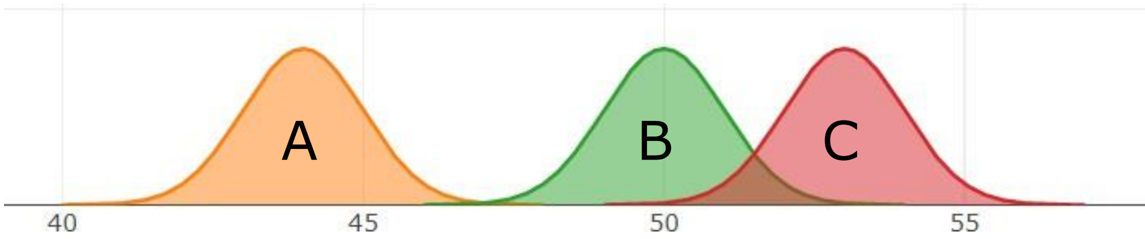
- In this section we introduce a new distribution – the F-Distribution
- Can also answer the question "*What is the probability that* ***three or more*** *samples come from the same population?*"



ANOVA
Analysis of Variance

ANOVA

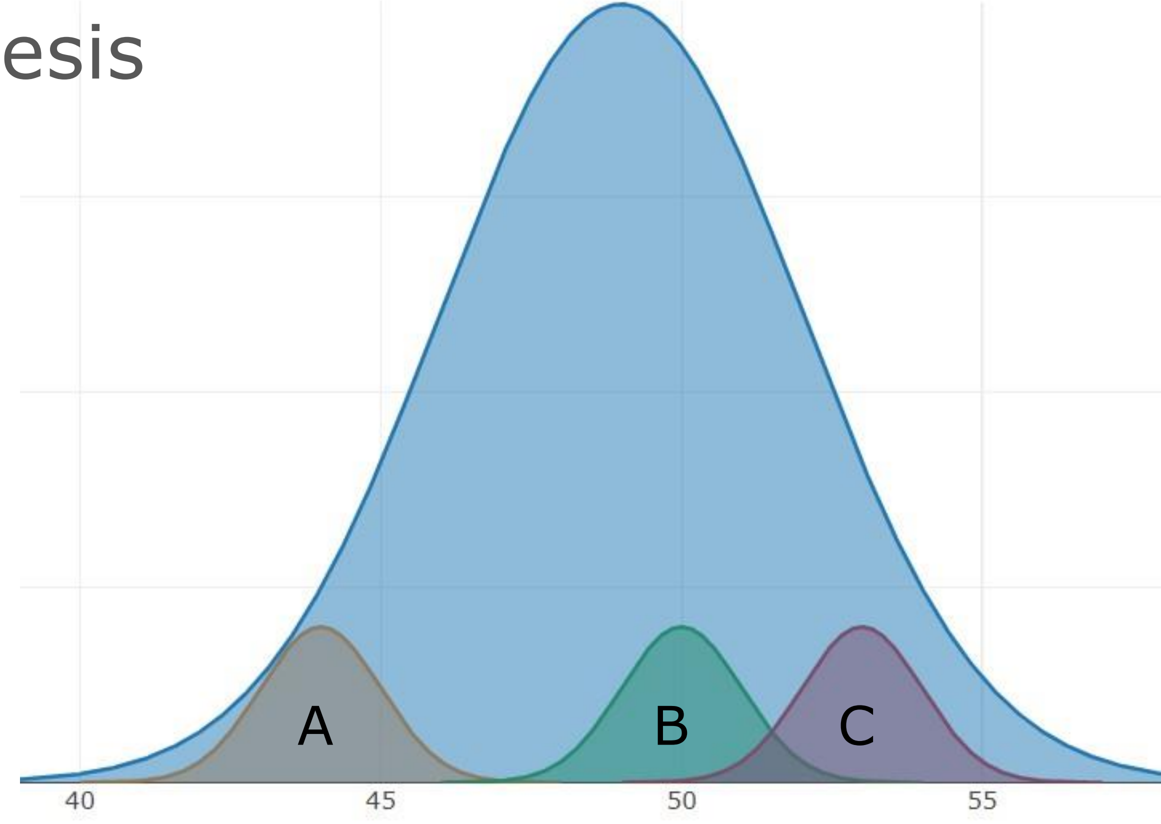
- In the previous section we tested two samples to see if they likely came from the same parent population.
- What if we had three (or more) samples?
- Could we do the same thing?



ANOVA

- Our null hypothesis would look like:

$$H_0: \mu_A = \mu_B = \mu_C$$



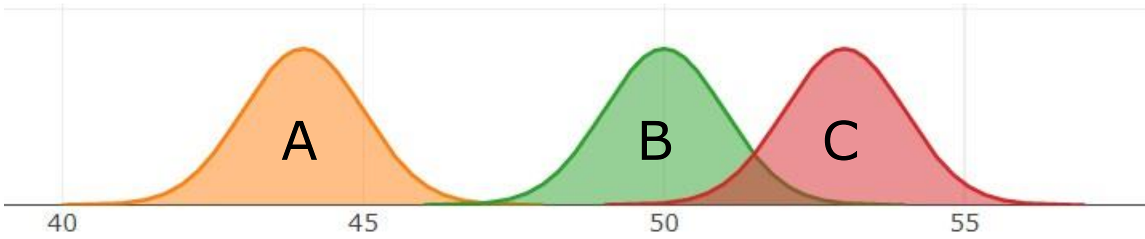
ANOVA

- We *could* test each pair:

$$H_0: \mu_A = \mu_B \qquad \alpha = 0.05$$

$$H_0: \mu_A = \mu_C \qquad \alpha = 0.05$$

$$H_0: \mu_B = \mu_C \qquad \alpha = 0.05$$



ANOVA

- The problem is, our overall confidence drops:

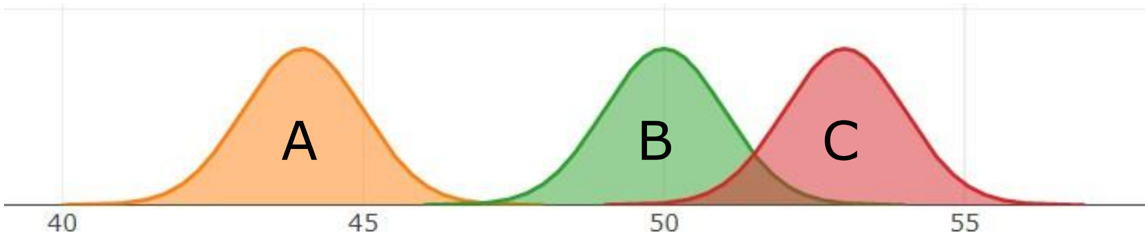
$$H_0: \mu_A = \mu_B \qquad \alpha = 0.05$$

$$H_0: \mu_A = \mu_C \qquad \alpha = 0.05$$

$$H_0: \mu_B = \mu_C \qquad \alpha = 0.05$$

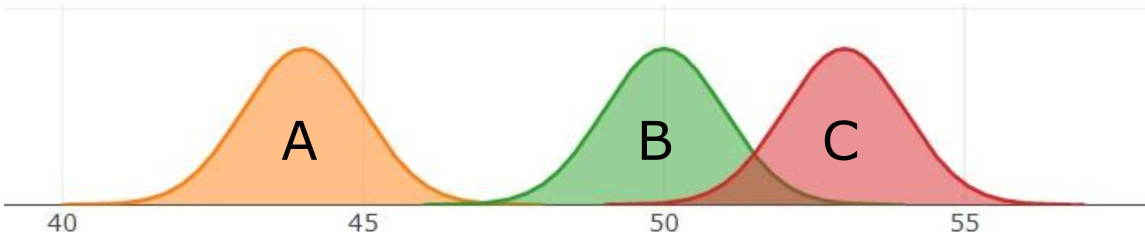
$$.95 \times .95 \times .95 = .857$$

$$85.7\% \ confidence \ level$$



ANOVA

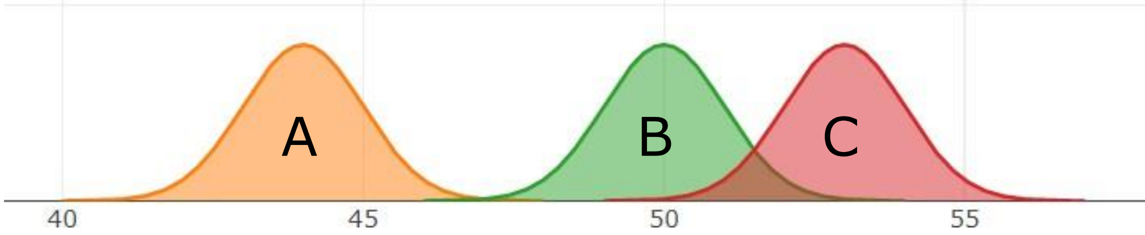
- This is where ANOVA comes in!
- We compute an F value, and compare it to a critical value determined by our degrees of freedom (the number of groups, and the number of items in each group)



ANOVA

Let’s work with some data:

GroupA	GroupB	GroupC
37	62	50
60	27	63
52	69	58
43	64	54
40	43	49
52	54	52
55	44	53
39	31	43
39	49	65
23	57	43



ANOVA

First calculate the sample means

Next calculate the overall mean

	GroupA	GroupB	GroupC
	37	62	50
	60	27	63
	52	69	58
	43	64	54
	40	43	49
	52	54	52
	55	44	53
	39	31	43
	39	49	65
	23	57	43
$\mu_{A,B,C}$	**44**	**50**	**53**
μ_{TOT}	**49**		

ANOVA

ANOVA considers two types of variance:

- Between Groups
  - how far group means stray from the total mean
- Within Groups
  - how far individual values stray from their respective group mean

ANOVA

The F value we're trying to calculate is simply the ratio between these two variances!

$$F = \frac{Variance\ Between\ Groups}{Variance\ Within\ Groups}$$

ANOVA

Recall the equation for variance:

$$s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1} = \frac{SS}{df}$$

Here $\Sigma(x-\bar{x})^2$ is the "sum of squares" SS and $n-1$ is the "degrees of freedom" df

ANOVA

So the formula for the F value becomes:

$$F = \frac{Variance\ Between\ Groups}{Variance\ Within\ Groups} = \frac{\dfrac{SSG}{df_{groups}}}{\dfrac{SSE}{df_{error}}}$$

SSG = Sum of Squares Groups

SSE = Sum of Squares Error

df_{groups} = degrees of freedom (groups)

df_{error} = degrees of freedom (error)

$SSG = 420$

ANOVA

Sum of Squares Groups

$$(\mu_A - \mu_{TOT})^2 = (44 - 49)^2 = 25$$

$$(\mu_B - \mu_{TOT})^2 = (50 - 49)^2 = 1$$

$$(\mu_C - \mu_{TOT})^2 = (53 - 49)^2 = 16$$

$$42$$

Multiply by the number of items in each group:

$$42 \times 10 = 420$$

GroupA	GroupB	GroupC
37	62	50
60	27	63
52	69	58
43	64	54
40	43	49
52	54	52
55	44	53
39	31	43
39	49	65
23	57	43

$\mu_{A,B,C}$	**44**	**50**	**53**
μ_{TOT}	**49**		

ANOVA

$$SSG = 420$$
$$df_{groups} = 2$$

GroupA	GroupB	GroupC
37	62	50
60	27	63
52	69	58
43	64	54
40	43	49
52	54	52
55	44	53
39	31	43
39	49	65
23	57	43
$\mu_{A,B,C}$ **44**	**50**	**53**
μ_{TOT} **49**		

Degrees of Freedom Groups

$$df_{groups} = n_{groups} - 1$$
$$= 3 - 1$$
$$= 2$$

ANOVA

$$SSG = 420$$
$$df_{groups} = 2$$
$$SSE = 3300$$

Sum of Squares Error

$(x_A-\mu_A)^2$	$(x_A-\mu_A)^2$	$(x_B-\mu_B)^2$	$(x_B-\mu_B)^2$	$(x_C-\mu_C)^2$	$(x_C-\mu_C)^2$
49	64	144	16	9	1
256	121	529	36	100	0
64	25	361	361	25	100
1	25	196	1	1	144
16	441	49	49	16	100
	1062		**1742**		**496**

TOTAL	**3300**

$(37-44)^2$
$=(-7)^2$
$=49$

	GroupA	GroupB	GroupC
	37	62	50
	60	27	63
	52	69	58
	43	64	54
	40	43	49
	52	54	52
	55	44	53
	39	31	43
	39	49	65
	23	57	43
$\mu_{A,B,C}$	**44**	**50**	**53**
μ_{TOT}	**49**		

ANOVA

$$SSG = 420$$
$$df_{groups} = 2$$
$$SSE = 3300$$
$$df_{error} = 27$$

GroupA	GroupB	GroupC
37	62	50
60	27	63
52	69	58
43	64	54
40	43	49
52	54	52
55	44	53
39	31	43
39	49	65
23	57	43

	GroupA	GroupB	GroupC
$\mu_{A,B,C}$	**44**	**50**	**53**
μ_{TOT}	**49**		

Degrees of Freedom Error

$$df_{error} = (n_{rows} - 1) * n_{groups}$$
$$= (10 - 1) * 3$$
$$= 27$$

ANOVA

$$SSG = 420$$
$$df_{groups} = 2$$
$$SSE = 3300$$
$$df_{error} = 27$$

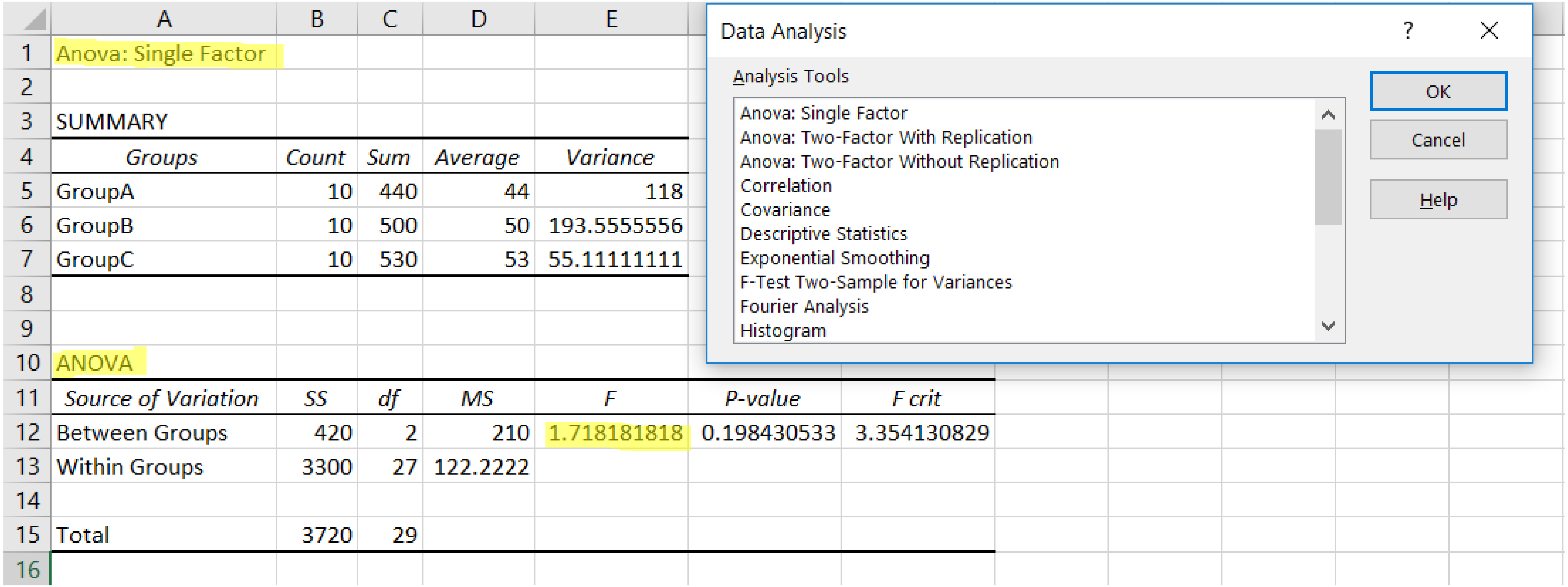
GroupA	GroupB	GroupC
37	62	50
60	27	63
52	69	58
43	64	54
40	43	49
52	54	52
55	44	53
39	31	43
39	49	65
23	57	43

$\mu_{A,B,C}$	**44**	**50**	**53**
μ_{TOT}	**49**		

Plug these into our formula:

$$F = \frac{\frac{SSG}{d\ groups}}{\frac{SSE}{df\,error}} = \frac{\frac{420}{2}}{\frac{3300}{27}} = \frac{210}{122.2\quad 2} = \mathbf{1.718}$$

ANOVA with Excel Data Analysis



Anova: Single Factor

SUMMARY

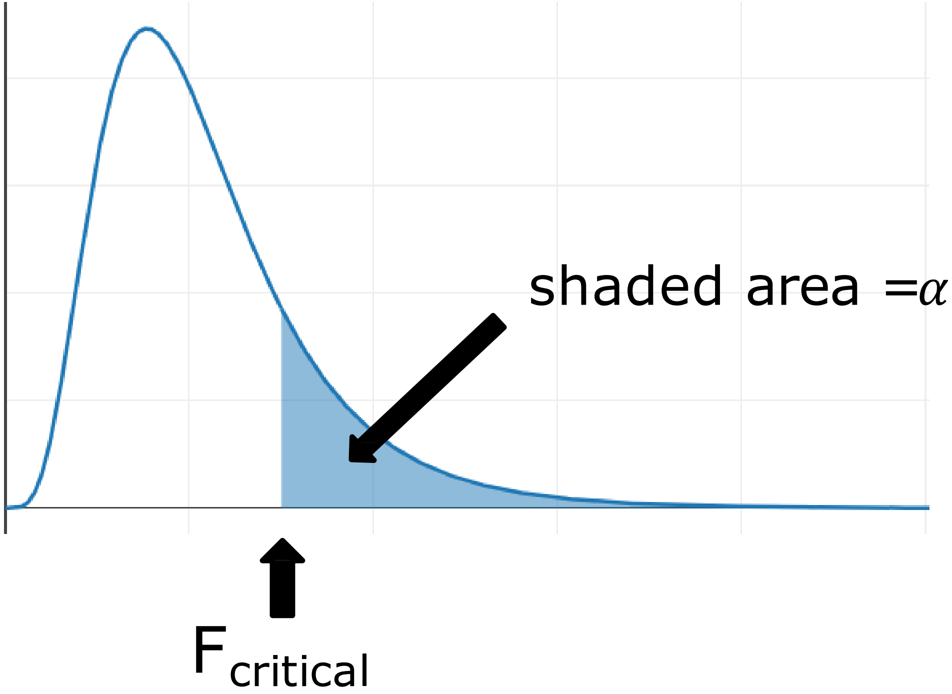
Groups	Count	Sum	Average	Variance
GroupA	10	440	44	118
GroupB	10	500	50	193.5555556
GroupC	10	530	53	55.11111111

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Between Groups	420	2	210	1.718181818	0.198430533	3.354130829
Within Groups	3300	27	122.2222			
Total	3720	29				

F Distribution

F-Distribution



F-Distribution

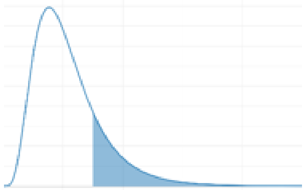
Look up our critical value from an F-table

use a table set for
95% confidence

find numerator df

find denominator df

critical value =3.35



		F-Table Upper Tail Area of 0.05				
		Numerator df				
		1	2	3	4	5
denominator df	25	4.24	3.39	2.99	2.76	2.60
	26	4.23	3.37	2.98	2.74	2.59
	27	4.21	3.35	2.96	2.73	2.57
	28	4.20	3.34	2.95	2.71	2.56
	29	4.18	3.33	2.93	2.70	2.55
	30	4.17	3.32	2.92	2.69	2.53

F-Scores in MS Excel

- In Microsoft Excel, the following function returns an F-score:

α	df1	df2	Formula	Output Value
0.05	2	27	=FINV(A2,B2,C2)	3.3541308285292

F-Scores in Python

```
>>> from scipy import stats
>>> stats.f.ppf(1-.05,dfn=2,dfd=27)
3.3541308285291986
```

ANOVA

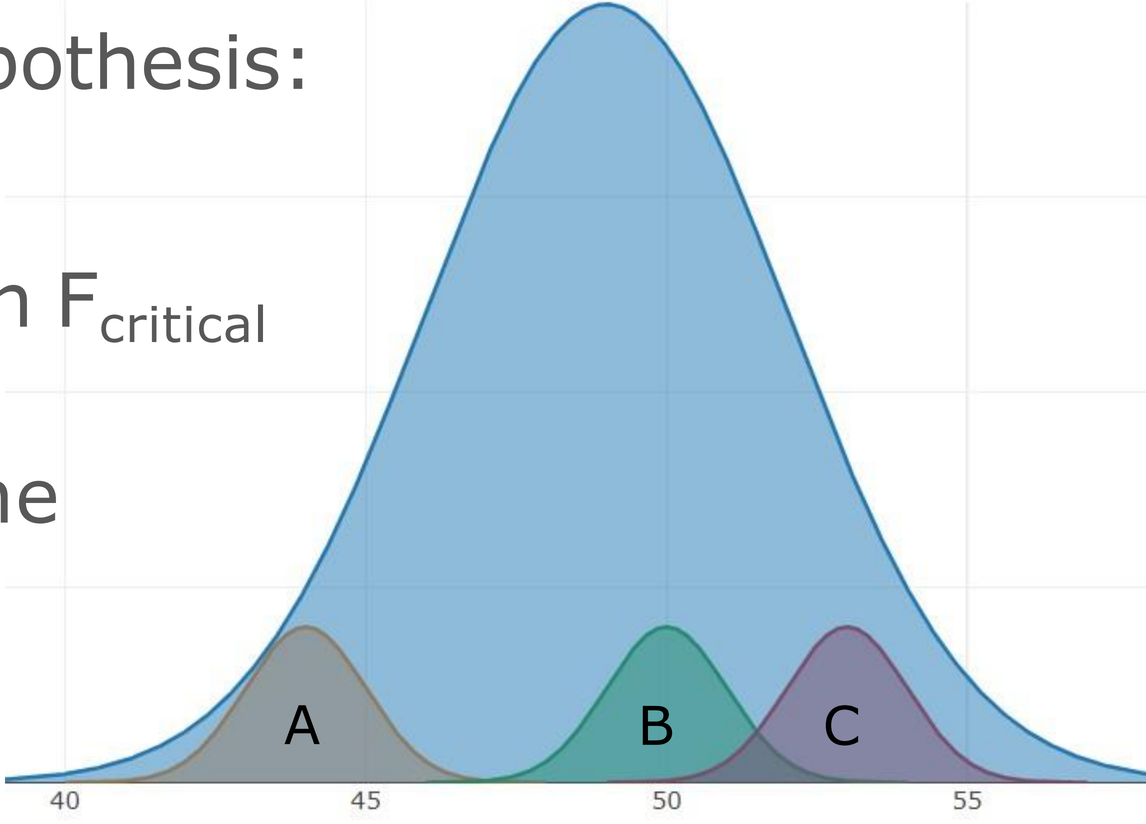
Recall our null hypothesis:

$$H_0: \mu_A = \mu_B = \mu_C$$

Since F is less than $F_{critical}$

$$1.718 < 3.354$$

we fail to reject the null hypothesis!



ANOVA Exercise #1

- In an effort to receive faster payment of invoices, a company introduces two discount plans
- One set of customers is given a 2% discount if they pay their invoice early
- Another set is offered a 1% discount
- A third set is not offered any incentive

ANOVA Exercise #1

- The results are as follows:
- Using ANOVA, can we say that the offers result in faster payments?

2% disc	1% disc	no disc
11	21	14
16	15	11
9	23	18
14	10	16
10	16	21

ANOVA Exercise #1

1. Calculate the means

	2% disc	1% disc	no disc
	11	21	14
	16	15	11
	9	23	18
	14	10	16
	10	16	21
$\mu_{2,1,0}$	**12**	**17**	**16**
μ_{TOT}	**15**		

$SSG = 70$

ANOVA Exercise #1

2. Find Sum of Squares Groups

$(\mu_2 - \mu_{TOT})^2 = (12 - 15)^2 = 9$

$(\mu_1 - \mu_{TOT})^2 = (17 - 15)^2 = 4$

$(\mu_0 - \mu_{TOT})^2 = (16 - 15)^2 = 1$

14

Multiply by the number of items in each group:

$14 \times 5 = 70$

	2% disc	1% disc	no disc
	11	21	14
	16	15	11
	9	23	18
	14	10	16
	10	16	21
$\mu_{2,1,0}$	**12**	**17**	**16**
μ_{TOT}	**15**		

ANOVA Exercise #1

$$SSG = 70$$
$$df_{groups} = 2$$

3. Degrees of Freedom Groups

$$df_{groups} = n_{groups} - 1$$
$$= 3 - 1$$
$$= 2$$

	2% disc	1% disc	no disc
	11	21	14
	16	15	11
	9	23	18
	14	10	16
	10	16	21
$\mu_{2,1,0}$	**12**	**17**	**16**
μ_{TOT}	**15**		

ANOVA Exercise #1

$SSG = 70$
$df_{groups} = 2$
$SSE = 198$

4. Sum of Squares Error

$(x_2-\mu_2)^2$	$(x_1-\mu_1)^2$	$(x_0-\mu_0)^2$
1	16	4
16	4	25
9	36	4
4	49	0
4	1	25
34	**106**	**58**
	TOTAL	**198**

	2% disc	1% disc	no disc
	11	21	14
	16	15	11
	9	23	18
	14	10	16
	10	16	21
$\mu_{2,1,0}$	**12**	**17**	**16**
μ_{TOT}	**15**		

ANOVA Exercise #1

$SSG = 70$
$df_{groups} = 2$
$SSE = 198$
$df_{error} = 12$

5. Degrees of Freedom Error

$$df_{error} = (n_{rows} - 1) * n_{groups}$$
$$= (5 - 1) * 3$$
$$= 12$$

	2% disc	1% disc	no disc
	11	21	14
	16	15	11
	9	23	18
	14	10	16
	10	16	21
$\mu_{2,1,0}$	**12**	**17**	**16**
μ_{TOT}	**15**		

ANOVA Exercise #1

$SSG = 70$
$df_{groups} = 2$
$SSE = 198$
$df_{error} = 12$

2% disc	1% disc	no disc
11	21	14
16	15	11
9	23	18
14	10	16
10	16	21

$\mu_{2,1,0}$	**12**	**17**	**16**
μ_{TOT}	**15**		

6. Calculate F value:

$$F = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}} = \frac{\frac{70}{2}}{\frac{198}{12}} = \frac{35}{16.5} = \mathbf{2.121}$$

7. Look up $F_{critical}$: **3.885**

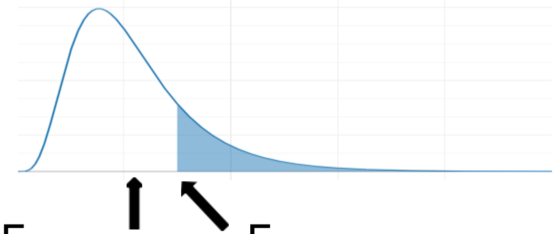
ANOVA Exercise #1

$SSG = 70$
$df_{groups} = 2$
$SSE = 198$
$df_{error} = 12$

Since F falls to the left of $F_{critical}$

$2.121 < 3.885$

we fail to reject the null hypothesis!

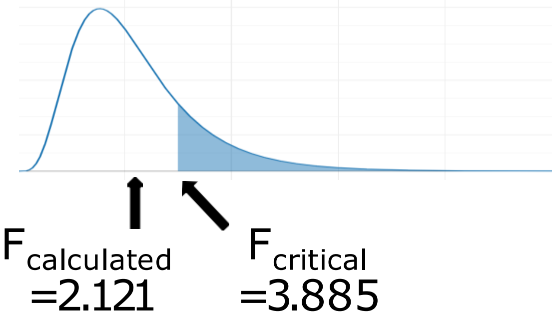


$F_{calculated}$ $=2.121$

$F_{critical}$ $=3.885$

ANOVA Exercise #1

We don't have enough to support the idea that our offers changed the average number of days that customers took to pay their invoices!



Two-Way ANOVA

One-Way vs Two-Way ANOVA

- In the previous examples we used one-way ANOVA to test one independent variable.
- For the invoice problem, the independent variable was the incentive offered.
- The dependent variable was the time it took to receive payment.

One-Way vs Two-Way ANOVA

- Two-Way ANOVA lets us test two independent variables at the same time
- For the invoice example, we might also consider the amount due
- We would have 3 invoices for $50, 3 for $100, etc. and offer different incentives at each dollar amount.

One-Way vs Two-Way ANOVA

- The resulting data might look like this:

	2% disc	1% disc	no disc
$50	16	23	21
$100	14	21	16
$150	11	16	18
$200	10	15	14
$250	9	10	11

- Here, each row or dollar amount is called a block.
- Essentially, we want to isolate and remove any variance contributed by the blocks, to better understand the variance in the groups.

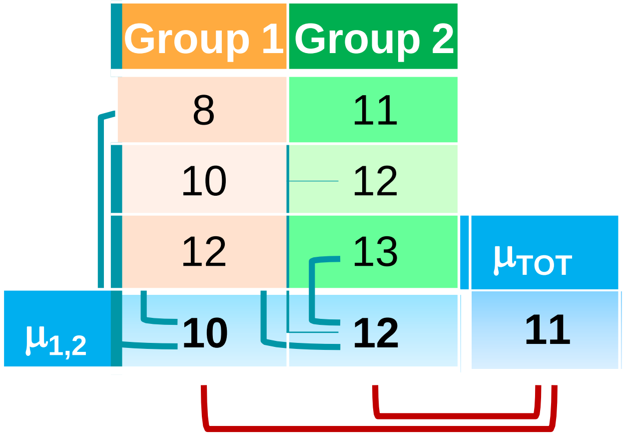
One-Way vs Two-Way ANOVA

- So how do we do that?

	2% disc	1% disc	no disc
$50	16	23	21
$100	14	21	16
$150	11	16	18
$200	10	15	14
$250	9	10	11

Two-Way ANOVA

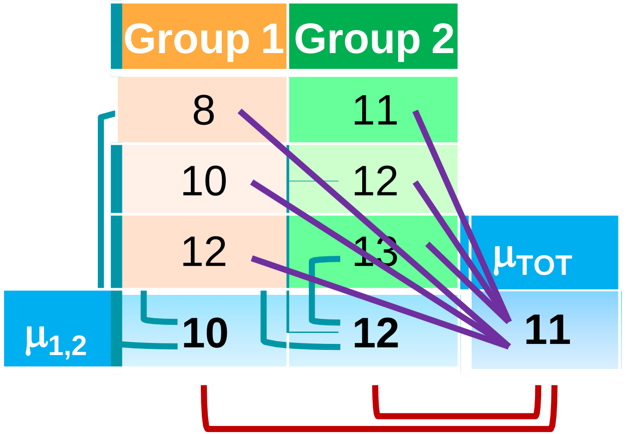
- The goal of ANOVA is to separate different aspects of the total variance.
- In the previous examples we had only Sum of Squares Groups (SSG) and Sum of Squares Error (SSE)
  - » between groups
  - » within groups



Two-Way ANOVA

- These two variances SSG and SSE add up to our total variance

Sum of Squares Total (SST)



Sum of Squares Groups (SSG) » between groups

and Sum of Squares Error (SSE) » within groups

Two-Way ANOVA

- Now we'll look at variance between rows, or blocks

	Group 1	Group 2	
Block A	8	11	
Block B	10	12	
Block C	12	13	μ_{TOT}
$\mu_{1,2}$	**10**	**12**	**11**

Sum of Squares Groups (SSG) » between groups

and Sum of Squares Error (SSE) » within groups

Two-Way ANOVA

- First calculate the block means

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

- Then calculate the
  Sum of Squares Blocks (SSB) » between blocks
  Sum of Squares Groups (SSG) » between groups
  and Sum of Squares Error (SSE) » within groups

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

- ANOVA still considers the relationship between the SSG and the SSE

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

Sum of Squares Blocks (SSB) » between blocks

Sum of Squares Groups (SSG) » between groups

and Sum of Squares Error (SSE) » within groups

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

- By calculating the SSB, we remove some of the variance in SSE

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

Sum of Squares Blocks (SSB) » between blocks

Sum of Squares Groups (SSG) » between groups

and Sum of Squares Error (SSE) » within groups

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

Sum of Squares Groups (SSG)

$$(\mu_1 - \mu_{TOT})^2 = (10 - 11)^2 = 1$$

$$(\mu_2 - \mu_{TOT})^2 = (12 - 11)^2 = 1$$

$$\overline{}$$

$$2$$

multiply by the number of items in each group: $2 \times 3 = 6$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

$$\boxed{SSG = 6}$$

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

Sum of Squares Blocks (SSB)

$$(\mu_A - \mu_{TOT})^2 = (9.5 - 11)^2 = 2.25$$

$$(\mu_B - \mu \qquad)^2 = (11 - 11)^2 = 0$$

$$(\mu_C - \mu_{TOT})^2 = (12.5 - 11)^2 = \underline{2.25}$$

$$4.5$$

multiply by the number of items in each block: $4.5 \times 2 = 9$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	10	12	11

$SSG = 6$

$SSB = 9$

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

Sum of Squares Total (SST)

$$(8-11)^2+(11-11)^2+$$
$$(10-11)^2+(12-11)^2+$$
$$(12-11)^2+(13-11)^2=16$$

no need to multiply since every item is represented

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

$SSG = 6$
$SSB = 9$
$SST = 16$

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

Sum of Squares Error (SSE)

$$SSE = SST - SSG - SSB$$

$$= 16 - 6 - 9 = 1$$

no need to multiply since we're working with totals already

$SSG = 6$
$SSB = 9$
$SST = 16$
$SSE = 1$

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

So how do we calculate F?

Degrees of Freedom Groups is unchanged:

$$df_{groups} = n_{groups} - 1$$
$$= 2 - 1$$
$$= 1$$

$SSG = 6$
$SSB = 9$
$SST = 16$
$SSE = 1$
$df_{groups} = 1$

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

So how do we calculate F?

Degrees of Freedom Error has changed:

$$df_{error} = (n_{blocks} - 1)(n_{groups} - 1)$$
$$= (3 - 1)(2 - 1)$$
$$= 2$$

$SSG = 6$
$SSB = 9$
$SST = 16$
$SSE = 1$
$df_{groups} = 1$
$df_{error} = 2$

Two-Way ANOVA

So how do we calculate F?

$$F = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}} = \frac{\frac{6}{1}}{\frac{1}{2}} = \mathbf{12}$$

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

$SSG = 6$
$SSB = 9$
$SST = 16$
$SSE = 1$
$df_{groups} = 1$
$df_{error} = 2$

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

$F_{groups} = \mathbf{12}$ feels like a high value.

However, in a two-way ANOVA, $F_{critical}$ is found for groups and blocks separately!

$SSG = 6$
$SSB = 9$
$SST = 16$
$SSE = 1$
$df_{groups} = 1$
$df_{error} = 2$

Two-Way ANOVA

$$F = \frac{Var.BetweenGroups}{Var.WithinGroups} = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}}$$

$F_{groups} = \mathbf{12}$ feels like a high value.

For groups, with 1df in the numerator and 2 df in the denominator,

$$F_{critical} = \mathbf{18.5}$$

	Group 1	Group 2	$\mu_{A,B,C}$
Block A	8	11	9.5
Block B	10	12	11
Block C	12	13	12.5
$\mu_{1,2}$	**10**	**12**	**11**

$SSG = 6$
$SSB = 9$
$SST = 16$
$SSE = 1$
$df_{groups} = 1$
$df_{error} = 2$

ANOVA Exercise #2

- Let’s go back to the invoice problem, and add a new independent variable
- Here each block represents an invoice amount
- The dependent variable is still days elapsed until payment

	2% disc	1% disc	no disc
$50	16	23	21
$100	14	21	16
$150	11	16	18
$200	10	15	14
$250	9	10	11

ANOVA Exercise #2

1. Calculate the group means, the block means, and the total mean

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	12	17	16	**15**

ANOVA Exercise #2

2. Sum of Squares Groups

$$(\mu_2 - \mu_{TOT})^2 = (12 - 15)^2 = 9$$

$$(\mu_1 - \mu_{TOT})^2 = (17 - 15)^2 = 4$$

$$(\mu_0 - \mu_{TOT})^2 = (16 - 15)^2 = 1$$

14

Multiply by the number of items in each group:

$$14 \times 5 = 70$$

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	12	17	16	**15**

$SSG = 70$

ANOVA Exercise #2

3. Degrees of Freedom Groups

$$df_{groups} = n_{groups} - 1$$

$$= 3 - 1$$

$$= 2$$

	2% disc	1% disc	no disc	μ_{block}
\$50	16	23	21	20
\$100	14	21	16	17
\$150	11	16	18	15
\$200	10	15	14	13
\$250	9	10	11	10
μ_{col}	12	17	16	**15**

$SSG = 70$ $df_{groups} = 2$

ANOVA Exercise #2

4. Sum of Squares Blocks

$$(\mu_{50} - \mu_{TOT})^2 = (20 - 15)^2 = 25$$

$$(\mu_{100} - \mu_{TOT})^2 = (17 - 15)^2 = 4$$

$$(\mu_{200} - \mu_{TOT})^2 = (15 - 15)^2 = 0$$

$$(\mu_{200} - \mu_{TOT})^2 = (13 - 15)^2 = 4$$

$$(\mu_{250} - \mu_{TOT})^2 = (10 - 15)^2 = 25$$

58

$$58 \times 3 = 174$$

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	**12**	**17**	**16**	**15**

$SSG = 70$

$SSB = 174$

$df_{groups} = 2$

ANOVA Exercise #2

5. Sum of Squares Total

$(x_2-\mu_{tot})^2$	$(x_1-\mu_{tot})^2$	$(x_0-\mu_{tot})^2$
1	64	36
1	36	1
16	1	9
25	0	1
36	25	16
79	**126**	**63**

TOTAL	**268**

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	**12**	**17**	**16**	**15**

$SSG = 70$

$SSB = 174$

$SST = 268$

$df_{groups} = 2$

ANOVA Exercise #2

6. Sum of Squares Error

$$SSE = SST - SSG - SSB$$

$$= 268 - 70 - 174 = 24$$

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	**12**	**17**	**16**	**15**

$SSG = 70$

$SSB = 174$

$SST = 268$

$SSE = 24$

$df_{groups} = 2$

ANOVA Exercise #2

7. Degrees of Freedom Error

$$df_{error} = (n_{blocks} - 1)(n_{groups} - 1)$$
$$= (5 - 1)(3 - 1)$$
$$= 8$$

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	**12**	**17**	**16**	**15**

$SSG = 70$
$SSB = 174$
$SST = 268$
$SSE = 24$

$df_{groups} = 2$
$df_{error} = 8$

ANOVA Exercise #2

8. Calculate F

$$F = \frac{\frac{SSG}{df_{groups}}}{\frac{SSE}{df_{error}}} = \frac{\frac{70}{2}}{\frac{24}{8}} = \frac{35}{3} = \mathbf{11.67}$$

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	**12**	**17**	**16**	**15**

$SSG = 70$

$SSB = 174$

$SST = 268$

$SSE = 24$

$df_{groups} = 2$

$df_{error} = 8$

F=11.67

ANOVA Exercise #2

9. Find $F_{critical}$

$$\alpha = 0.05$$

$$df_{numerator} = 2$$

$$df_{denominator} = 8$$

$$F_{critical} = \mathbf{4.46}$$

	2% disc	1% disc	no disc	μ_{block}
$50	16	23	21	20
$100	14	21	16	17
$150	11	16	18	15
$200	10	15	14	13
$250	9	10	11	10
μ_{col}	12	17	16	15

$SSG = 70$

$SSB = 174$

$SST = 268$

$SSE = 24$

$df_{groups} = 2$

$df_{error} = 8$

$F = 11.67$

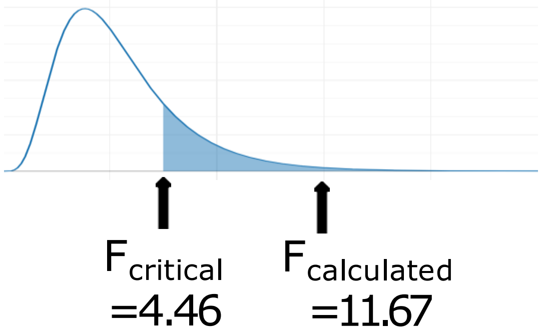
$F_{critical} = 4.46$

ANOVA Exercise #1

Since F falls to the right of $F_{critical}$

$4.46 < 11.67$

we reject the null hypothesis!



$SSG = 70$

$SSB = 174$

$SST = 268$

$SSE = 24$

$df_{groups} = 2$

$df_{error} = 8$

$F = 11.67$

$F_{critical} = 4.46$

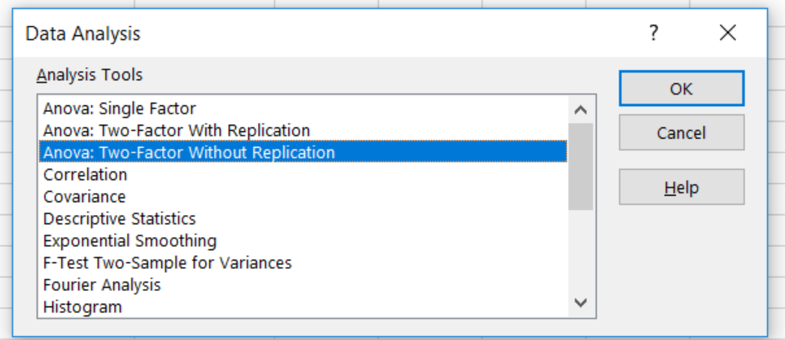
2-way ANOVA in Excel

Anova: Two-Factor Without Replication

SUMMARY	Count	Sum	Average	Variance
Row 1	3	60	20	13
Row 2	3	51	17	13
Row 3	3	45	15	13
Row 4	3	39	13	7
Row 5	3	30	10	1
Column 1	5	60	12	8.5
Column 2	5	85	17	26.5
Column 3	5	80	16	14.5

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Rows	174	4	43.5	14.5	0.000974668	3.837853355
Columns	70	2	35	11.666667	0.004249458	4.458970108
Error	24	8	3			
Total	268	14				



Two-Way ANOVA with Replication

Without vs With Replication

without replication

	GroupA	GroupB	GroupC
Block1	16	23	21
Block2	14	21	16
Block3	11	16	18
Block4	10	15	14
Block5	9	10	11
Block6	8	8	10

with replication

	GroupA	GroupB	GroupC
Block1	16	23	21
	14	21	16
	11	16	18
Block2	10	15	14
	9	10	11
	8	8	10

Samples have multiple values
Samples have a mean value

Two-Way ANOVA with Replication

- Introduces the concept of sample means and sample variance
- Introduces the concept of interactions

Two-Way ANOVA with Replication

- As with our previous 2-way ANOVA, we consider two independent variables organized into groups and blocks
- We sample every block/group combination
- With replication, block/group samples have multiple measurements

Two-Way ANOVA with Replication

- Consider an experiment that measures the height of plants
- We apply three types of fertilizer A, B & C – these are our Groups
- Plants are kept at two temperatures (warm & cold) – these are our Blocks
- We assign 3 plants to each sample

Two-Way ANOVA

- First calculate the mean for each 3-item sample
- Calculate column means
- Calculate block means
- Calculate the overall mean

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	

	A	B	C	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

Two-Way ANOVA

- As before, calculate the Sum of Squares Blocks

$$(16-15)^2+(14-15)^2=2$$

$$\times\, 9\ items\ per\ block = \mathbf{18}$$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	

Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$

Two-Way ANOVA

- As before, calculate the Sum of Squares Columns

$$(15 - 15)^2 + (16 - 15)^2 + (14 - 15)^2 = 2$$

$$\times\, 6\ items\ per\ column = \mathbf{12}$$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	

	A	B	C	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

SSB = 18 SSC = 12

Two-Way ANOVA

- As before, calculate the Degrees of Freedom Columns

$$df_{columns} = (3-1) = 2$$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	

Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $SSC = 12$ $df_{columns} = 2$

Two-Way ANOVA

- We have a new statistic: SS Interactions
- For each sample mean, subtract the matching block and column means, add back the overall mean, square the result

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	

	A	B	C	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $SSC = 12$ $df_{columns} = 2$

Two-Way ANOVA

$$(13-16-15+15)^2+$$
$$(19-16-16+15)^2+$$
$$(16-16-14+15)^2+$$
$$(17-14-15+15)^2+$$
$$(13-14-16+15)^2+$$
$$(12-14-14+15)^2=28$$

$$\times 3\ \textit{items per sample} = \mathbf{84}$$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	

	A	B	C	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $\quad SSC = 12$ $\quad df_{columns} = 2$

$SSI = 84$

Two-Way ANOVA

- Calculate the Sum of Squares Total

4	36	9	
1	16	0	
9	4	0	
1	1	0	
9	16	4	
4	1	49	**164**

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	

Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $SSC = 12$ $df_{columns} = 2$

$SSI = 84$

$SST = 164$

Two-Way ANOVA

- Calculate the Sum of Squares Error by subtracting the other values from the SST:

$164 - 18 - 12 - 84 = \mathbf{50}$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	16
	14	19	15	
	12	17	15	
Cold	16	14	15	14
	18	11	13	
	17	14	8	

	A	B	C	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $SSC = 12$ $df_{columns} = 2$

$SSI = 84$ $SSE = 50$

$SST = 164$

Two-Way ANOVA

- Degrees of Freedom Error

$$blocks \times columns \times (items - 1)$$

$$= 2 \times 3 \times (3 - 1) = \mathbf{12}$$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	16
	14	19	15	
	12	17	15	
Cold	16	14	15	14
	18	11	13	
	17	14	8	

Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $\quad SSC = 12$ $\quad df_{columns} = 2$

$SSI = 84$ $\quad SSE = 50$ $\quad df_{error} = 12$

$SST = 164$

Two-Way ANOVA

- Calculate F

$$F = \frac{\dfrac{SSC}{df_{columns}}}{\dfrac{SSE}{df_{error}}} = \frac{\dfrac{12}{2}}{\dfrac{50}{12}} = 1.44$$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	**16**
	14	19	15	
	12	17	15	
Cold	16	14	15	**14**
	18	11	13	
	17	14	8	

	A	B	C	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $SSC = 12$ $df_{columns} = 2$

$SSI = 84$ $SSE = 50$ $df_{error} = 12$

$SST = 164$

Two-Way ANOVA

$$F = \mathbf{1.44}$$

- Look up $F_{critical}$

$$F_{(0.05,2,12)} = \mathbf{3.885}$$

Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

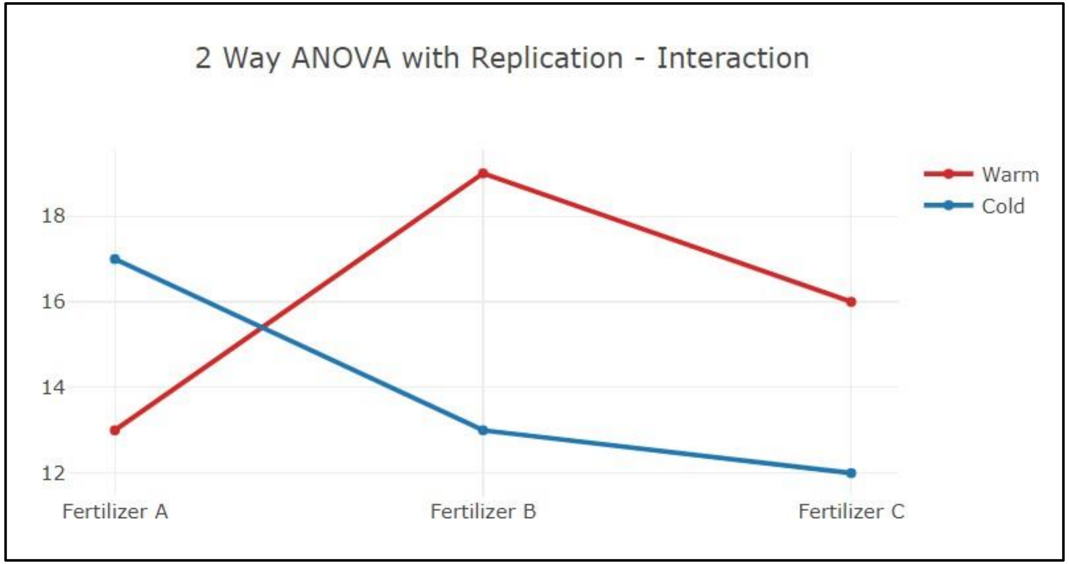
$SSB = 18$ $SSC = 12$ $df_{columns} = 2$

$SSI = 84$ $SSE = 50$ $df_{error} = 12$

$SST = 164$

Two-Way ANOVA

- A look at Interaction:



Fertilizer:	A	B	C	Block Means
Warm	13	21	18	
	14	19	15	**16**
	12	17	15	
Cold	16	14	15	
	18	11	13	**14**
	17	14	8	
Sample Means	13	19	16	
	17	13	12	
Column Means	**15**	**16**	**14**	**15**

$SSB = 18$ $SSC = 12$ $df_{columns} = 2$

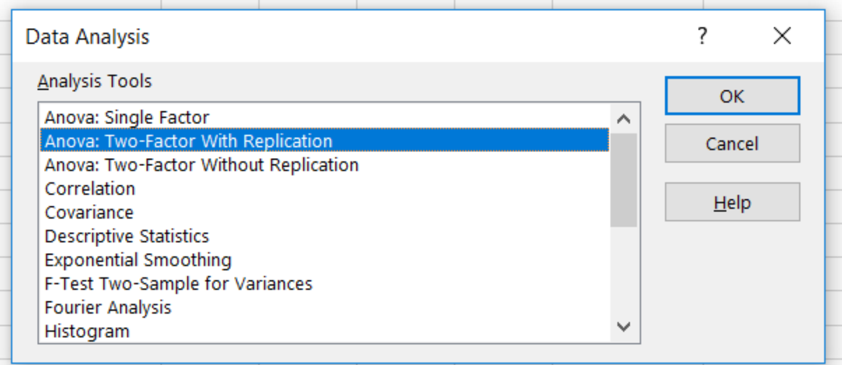
$SSI = 84$ $SSE = 50$ $df_{error} = 12$

$SST = 164$

2-way with Replication in Excel

Anova: Two-Factor With Replication

SUMMARY	Fertilizer A	Fertilizer B	Fertilizer C	Total
WARM				
Count	3	3	3	9
Sum	39	57	48	144
Average	13	19	16	16
Variance	1	4	3	8.75
COLD				
Count	3	3	3	9
Sum	51	39	36	126
Average	17	13	12	14
Variance	1	3	13	9.5
Total				
Count	6	6	6	
Sum	90	96	84	
Average	15	16	14	
Variance	5.6	13.6	11.2	



ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Sample	18	1	18	4.32	0.059785686	4.747225347
Columns	12	2	6	1.44	0.275086887	3.885293835
Interaction	84	2	42	10.08	0.002698928	3.885293835
Within	50	12	4.16667			