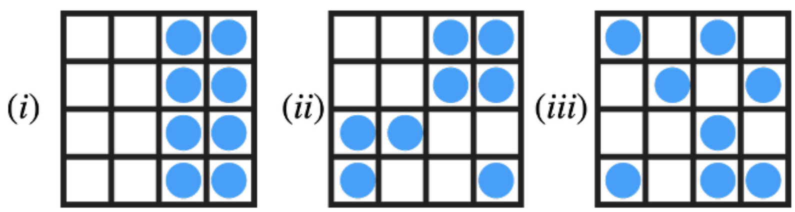
MSEN 640 Fall 2025 – Homework # 5

Instructions: Please answer the problems below either using a computer, tablet, or paper. Save your solutions as a PDF document; remember: if using paper, please **scan** your answers! Remember to **show all of your work** for a full grade.

1.
a. (5 points) How many possible microstates exist for a system containing eight atoms of element A and one atom of element B? Draw all possible microstates for such a system on a three-by-three two-dimensional lattice.
b. (5 points) How many possible microstates exist for a system containing four atoms of element A, two atoms of element B, and three atoms of element C?
c. (5 points) The following lattices show microstates of a 4x4 lattice. Which is the most probable microstate consistent with the macrostate of being 50% occupied. Explain your answer.



2. Consider a small colloidal glass bead of mass m in a container of water held at temperature T. The gravitational potential energy U of the glass particle depends on its height above the bottom of the container z: $U(z) = mgz$, where g is the acceleration due to gravity.

(a) Determine the probability $P(z)$ that the bead is found at a certain height z. Normalize the probability distribution such that $\int_0^\infty P(z)\,\mathrm{d}z = 1$.

(b) Calculate the average height of the bead $\langle z \rangle$. Under what conditions would we expect particles to "sediment" out of a solution?

(c) In the high temperature limit (or $T \to \infty$), where can we expect to find the bead?

(d) Calculate the variance of the bead height $\langle z^2 \rangle - \langle z \rangle^2$.

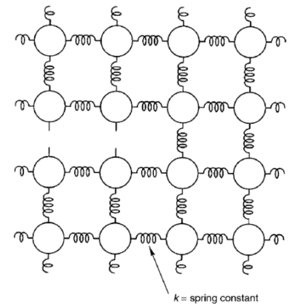
(e) Given what we have learned in part (d), re-interpret your answer to part (c). That is, at any instant in time, where might the bead be found at high temperatures?

(f) Calculate the average energy $\langle U \rangle$

3. One simple model of a crystal consists of a collection of masses and springs. Consider one particular "normal mode" of the crystal, characterized by natural frequency ω. The vibrational energy associated with this normal mode can be quantized such that the n^{th} energy level is given by

$$\epsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

where $n = 0, 1, 2, \ldots$ and $\hbar$ is Planck's constant divided by 2π. This is called the "Einstein model" of a crystalline solid.



(a) Calculate the partition function Z. To simplify, take advantage of the formula for a convergent geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(b) Calculate the entropy, internal energy, and Helmholtz free energy of the crystal. You can express your results in terms of the "Einstein temperature" $\vartheta_E \equiv \hbar\omega/k_B$. (Note that for diamonds, $\vartheta_E \approx 1320\,\mathrm{K}$.)

(c) Calculate the molar heat capacity C_V. Plot C_V/k_B as a function of T/ϑ_E. (This model actually does a good job at capturing the high-temperature behavior of real crystals! The low-temperature requires some corrections, however.)