Matrix Project

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Question in Geometry :-

Let P be a point on the parabola

$$x^2 + 4y = 0$$

Given that the distance of P from the centre of the circle

$$x^2 + y^2 + 6x + 8 = 0$$

is minimum. Find the equation of the tangent to the parabola at P.

Question in Matrix :-

Let P be a point on the parabola

$$\mathrm{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 4 \end{bmatrix} x = 0.$$

Given that the distance of P from the centre of the circle

$$\mathrm{x}^T x + \begin{bmatrix} 6 \\ 0 \end{bmatrix} x + 8 = 0$$

is minimum. Find the equation of the tangent to the parabola at P.

Solution :-

$$\text{Equation of parabola is : } x^T Vx + 2U^T x + F = 0$$
$$\text{Equation of circle is : } x^T x + 2C_p^T x + E = 0$$

here $V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $C_p = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, $F = 0 and E = 8.$

Let C be the centre of circle, r be the radius of the circle and $x_1 be any point on the given circle.$

$$\implies \quad (x_1 - C)^T (x_1 - C) = r^2$$
$$\implies \quad x_1 x_1^T - 2C^T x + CC^T - r^2 = 0.$$

Comparing this equation with the equation of circle:

$$-2C^T = C_p^T$$
$$\implies \quad C = \frac{-1}{2} C_p = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Solution continued..

The equation of a tangent to given parabola at any point P on the parabola is :

$$(P^T V + U^T)x + P^T U + F = 0.$$

$$\implies \quad (P^T V + U^T)x = -P^T U - F$$

Comparing with the equation of line : $n^T x = \lambda$

$$n^T = P^T V = U^T$$

$$\implies \quad n = (P^T V + U^T)^T = PV^T + U$$

Since P lies on the line $n^T x = \lambda$

$$\implies \quad n^T P = \lambda$$

Then equation of the tangent can be rewritten as

$$n^T x = n^T P \implies n^T (x - P) = 0$$

Solution Approch :

For P to be the point on parabola whose distance is minimun from centre of the circle, **CP** must be normal to the parabola.

$\Longrightarrow$

$$\textbf{CP} \text{ is perpendicular to tangent}$$

Solution continued..

$$\begin{aligned}
&\textbf{CP} \text{ is perpendicular to tangent}\\
\implies\ & \textbf{CP} \perp \textbf{xP}, \textit{where x is any point on the tangent to the}\\
&\text{parabola} \quad \text{at} \quad \text{P}\\
\implies\ & (C-P)^T(x-P) = 0\\
&\text{Since (C-P)}^T(x-P) = 0 \quad \textit{and} \quad n^T(x-P) = 0\\
\implies\ & n = k(C-P), \textit{where is a scaler}.\\
\implies\ & n = K(C-P)\\
&\text{Here K} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\\
\implies\ & PV^T + U = KC - KP \qquad \{n = PV^T + U\}\\
\implies\ & P(V^T + K) = KC - U\\
\implies\ & P = (KC - U)(V^T + K)^{-1}
\end{aligned}$$

Solution continued..

$$\implies \quad P = \left(\left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} -3 \\ 0 \end{bmatrix}\right) - \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\right)^{-1}$$

On solving, we get

$$\mathrm{P} = \begin{bmatrix} \frac{-3k}{k+1} \\ \frac{-2}{k} \end{bmatrix}$$

Since P lies on the parabola, sbstituting P in the equation of parabola :

$$\mathrm{x}^T V x + 2U^T x + F = 0$$

we get $\mathrm{P}^T VP + 2U^T P + F = 0$

$$\implies \left(\begin{bmatrix} \frac{-3k}{k+1} & \frac{-2}{k} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{-3k}{k+1} \\ \frac{-2}{k} \end{bmatrix}\right) + \left(\begin{bmatrix} 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{-3k}{k+1} \\ \frac{-2}{k} \end{bmatrix}\right) + 0 = 0$$

Solution continued..

solving, we get: $\left[\frac{-3k}{k+1}\right]^2 + \left[\frac{-8}{k}\right] = 0$

.

$$\left[\frac{9k^2}{k^2+2k+1}\right] = \left[\frac{8}{k}\right]$$

.

$$9\text{k}^3 - 8k^2 - 16k - 8 = 0$$

.

$$(\text{k-2})(9\text{k}^2 + 10k + 4) = 0$$

.

$$\text{k} = 2$$

Solution continued..

$$\mathrm{P} = \begin{bmatrix} \frac{-3k}{k+1} \\ \frac{-2}{k} \end{bmatrix}, \textit{substituting} \qquad k = 2$$

$$\mathrm{P} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

The equation of a tangent to given parabola at any point P on the parabola is :

$$(\mathrm{P}^T V + U^T)x + P^T U + F = 0.$$

substituting $\qquad \mathrm{P} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$$\left(\begin{bmatrix} -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \end{bmatrix} \right) \mathrm{x} + \begin{bmatrix} -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0$$

$$\implies \qquad \begin{bmatrix} -2 & 2 \end{bmatrix} x - 2 = 0$$

$$\implies \qquad \begin{bmatrix} -1 & 1 \end{bmatrix} x = 1$$

Answer :

The equation of the desired tangent is :-

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$$\begin{bmatrix} -1 & 1 \end{bmatrix} \mathsf{x} = 1$$

Graphical Verification

Plotting the tangent and the normal to this tangent at P, we graphically verify that the normal passes through the center of the circle and hence that P is thepoint on the parabola that is at minimum distance from the center of the circle.

Graph