FFT Applications - Basic Principles of FFT Usage

DFT/FFT definition

FFT (Fast Fourier Transform) is an algorithm for fast computation of the DFT (Dscrete Fourier Transform), defined as:

$$X(kf_0) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{kf_0}{f_s}n} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, 2, ..., N-1 .$$

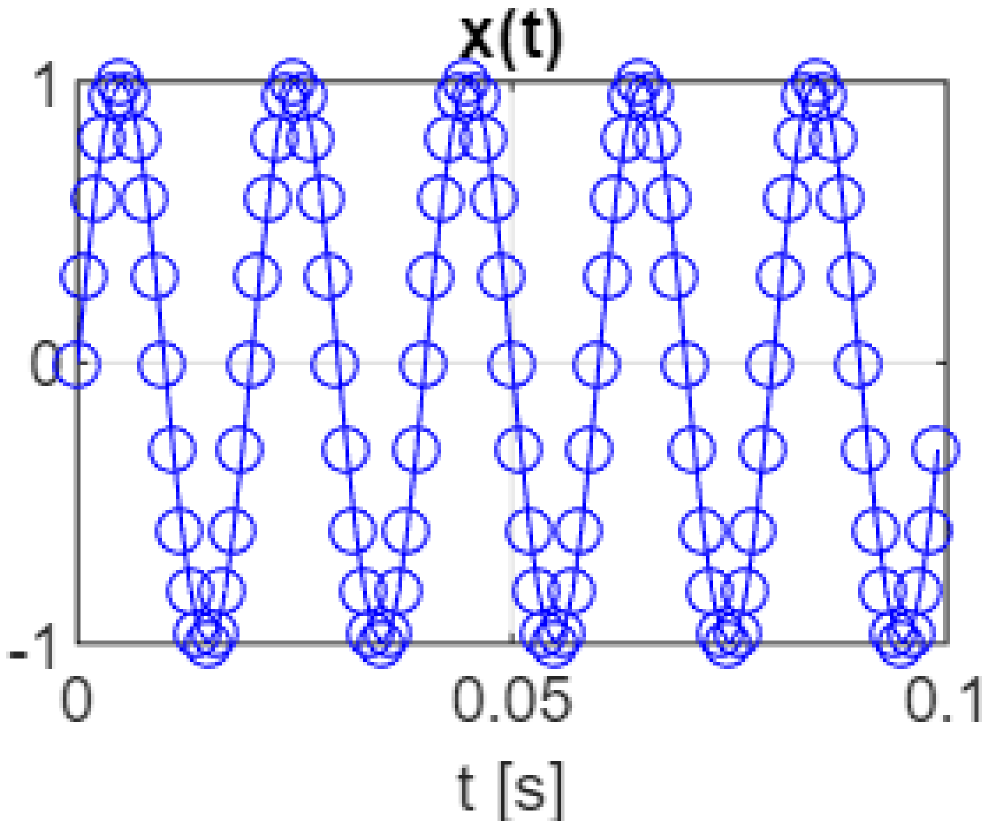
The same is computed but in a faster way.

Analyzed signal

Let's assume that the N-samples long fragment of single sinusoidal signal $x(t) = \sin(2\pi f_x t)$ will be analyzed by us by means of FFT. Run the code for different number of signal samples `N` and different sine repetition frequency `fx` (sine amplitude is constant and equals 1). Observe the signal shape.

```
clear all; close all;         % "washing hands"
fs = 1000;                    % sampling ratio (Hz)
N  = 100;                     % number of signal samples, 100 or 1000
dt=1/fs; t=dt*(0:N-1);        % time scaling
f0 = 1/(N*dt);                % fundamental frequency of N-point DFT = fs/N

% Signal
fx=50; x = sin(2*pi*fx*t);   % signal with frequency fx = 25,50,100,125,200,225 Hz
figure; plot(t,x,'bo-'); xlabel('t [s]'); title('x(t)'); grid;
```



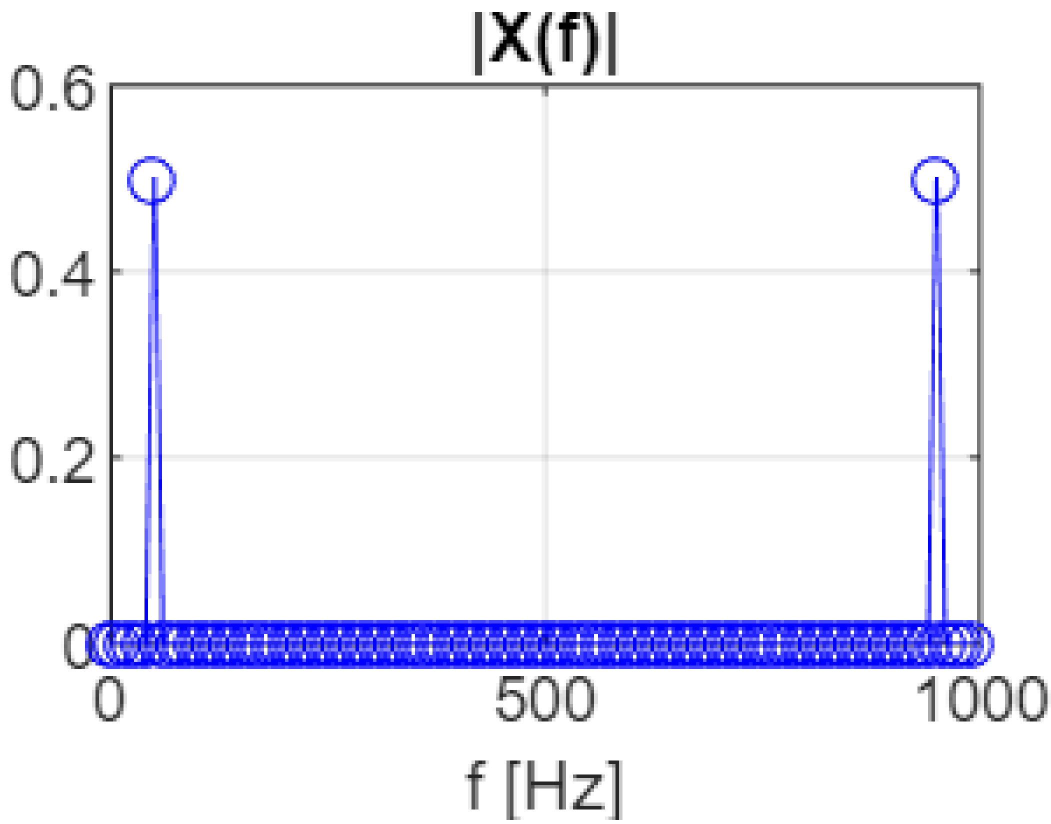
Typical FFT call

The FFT algorithm is widely implemented and available world-wide, also in Matlab - but without division by N inside the procedure. Therefore this division should be performed after the function call. Run first two fragments of the Matlab code: now observe signals and absolute value $|X(kf_0)|$ of their FFT spectra. Observe that FFT peaks are located close to the signal frequency. Two peaks are seen because:

$$\sin(2\pi f_x t) = \frac{e^{j2\pi f_x t} - e^{-j2\pi f_x t}}{2j}.$$

The first peak is for positive frequency f_x and the second one for negative frequency $-f_x$ (in fact for its first copy, i.e. $f_s - f_x$).

```
% FFT spectrum
X = fft(x)/N;                    % FFT unscaled, for positive and negative frequencies
f = fs/N*(0:N-1);                % frequency axis: df=f0=1/(N*dt)=fs/N
figure; plot(f,abs(X),'bo-'); xlabel('f [Hz]'); title('|X(f)|'); grid;
```



Better FFT call

In the simplest FFT usage rectangular window is used for signal fragment shaping and the spectrum is divided by N (summation result of N window samples.

The following "tricks" should be used for improving spectral analysis based on FFT.

1. We remember from laboratory on DFT & DtFT that **application of another window function** can improve amplitude resolution of the DFT spectrum (i.e. increase side-lobes attenuation of the window spectrum) by the cost of decreasing the frequency DFT resolution (caused by widening the main-lobe of the window spectrum). To counteract this effect, the value of N should be increased before signal windowing (i.e. longer signal fragment should be transformed).
2. Since FFT specrum of a real signal is **conjugate symmetric**, it is better to show only the first half of it (for `k=1:N/2+1`).
3. Since number of frequency lags/bins in the output DFT/FFT spectrum in equal to number of input signals samples, what caused denser spectrum sampling (its **interpolation**), we can can artificially append some zeros to the analyzed signal end. This will not change the resultant spectrum shape, only cause its denser sampling.

Analyze the below code. Add more window functions and test their usage. Change side-lobes attenuation `Asl` in the Chebyshev window. Apply different values of `K` (spectrum interpolation order - resulting from number of appended zeros).

```
% Better FFT spectrum: 1) with window usage (better amplitude resulution), 2) with zeros append
% (better spectrum shape due to its interpolation), 3) showing only half of the spectrum scaled
K = 20;                          % interpolation order
w1 = rectwin(N);                 % retangular window
w2 = chebwin(N,100);             % Chebyshev window: Asl=60,80,100,120
w = w1;                          % window choice: w1, w2, ...
x = x.*w';                       % signal windowing
X  = fft(x,N);                   % without appended zeros at signal end
Xz = fft(x,K*N);                 % with these zeros; the same: Xz = fft([x,zeros(1,(K-1)*N)])/sum(w
fz = fs/(K*N)*(0:K*N-1);         % new frequency axis for signal with zeros
figure                           % DFT spectrum inspection
n = 1:N/2+1; nz = 1:K*N/2+1; % only first HALF of the spectrum but scaled by 2
plot(f(n),  20*log10(2*abs(X(n))  /sum(w)),'bo-',...
     fz(nz),20*log10(2*abs(Xz(nz))/sum(w)),'r.-','MarkerFaceColor','b');
xlabel('f (Hz)'); xlabel('[dB]'); title('Zoomed DFT via FFT [dB]'); grid;
```