Analog Filters

Theoretical introduction

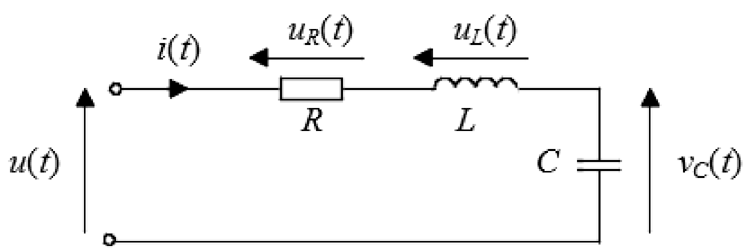
Analog circuits are described by **differential & integral equations**. After performing upon them the **Laplace transform** and the **continuos Fourier transform** (the special case of the Laplace transform for $s = j2\pi f$):

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt, \qquad X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

one obtains, after some math, a **circuit transfer function** $H(s)$ and a **circuit frequency response** $H(f)$, i.e. an information what our circuit will do with input sinusoidal oscillation with frequeny f: $|H(f)|$ - is sinusoid gain while $\angle H(f)$ - sinusoid shift angle.

RLC circuit as an example

Let's discuss t**he RLC circuit** (R-resistor, L-induction coil, C-capacitor), presented in figure below, as an example.



Its output and input voltages are equal:

$$u_{out}(t) = u_C t = \frac{1}{C}\int_0^t i(t)dt$$

$$u_{in}(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt$$

After: 1) calculating the Laplace (Fourier) transform of the above equations, and 2) dividing the second by the first, one obtatains the circuit transfer function $H(s)$ (the circuit frequency response $H(f)$):

$$H(s) = \frac{U_{out}(s)}{U_{out}(s)} = \frac{\frac{1}{sC}}{R + Ls + \frac{1}{sC}} = \frac{1}{LCs^2 + RCs + 1}, \qquad \left(H(f) = \frac{1}{LC(j2\pi f)^2 + RC(j2\pi f) + 1}\right)$$

which is a ratio of two polynomials of variable s (or $j2\pi f$). Circuit response to the Dirac delta function (impulse) $\delta(t)$ is equal (for $t \geq 0$):

$$h(t) = Ae^{-\alpha t}\sin(\omega_1 t), \qquad \omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{R/L}{2\omega_0}, \quad A = \frac{\omega_0}{\sqrt{1-\xi^2}}, \quad \alpha = \xi\omega_0, \quad \omega_1 = \omega_0\sqrt{1-\xi^2}$$

Matlab code given below calculates the frequency response of the RLC circuits and simulate its impulse respone. Change values of R, L, C and observe changes of both characteristics. Change differenet frequency range of observation. Is the impulse response always oscillatory? $H(f)$ is the result of Fourier transform of $h(t)$.

```
clear all; close all;

R = 10;                    % resistance in ohms
L = 2*10^(-3);             % inductance in henrys
C = 5*10^(-6);             % capacitance in farads

w0  = 1/sqrt(L*C);         f0 = w0/(2*pi),    % undumped resonance frequency
```

```
f0 = 1.5915e+03
```

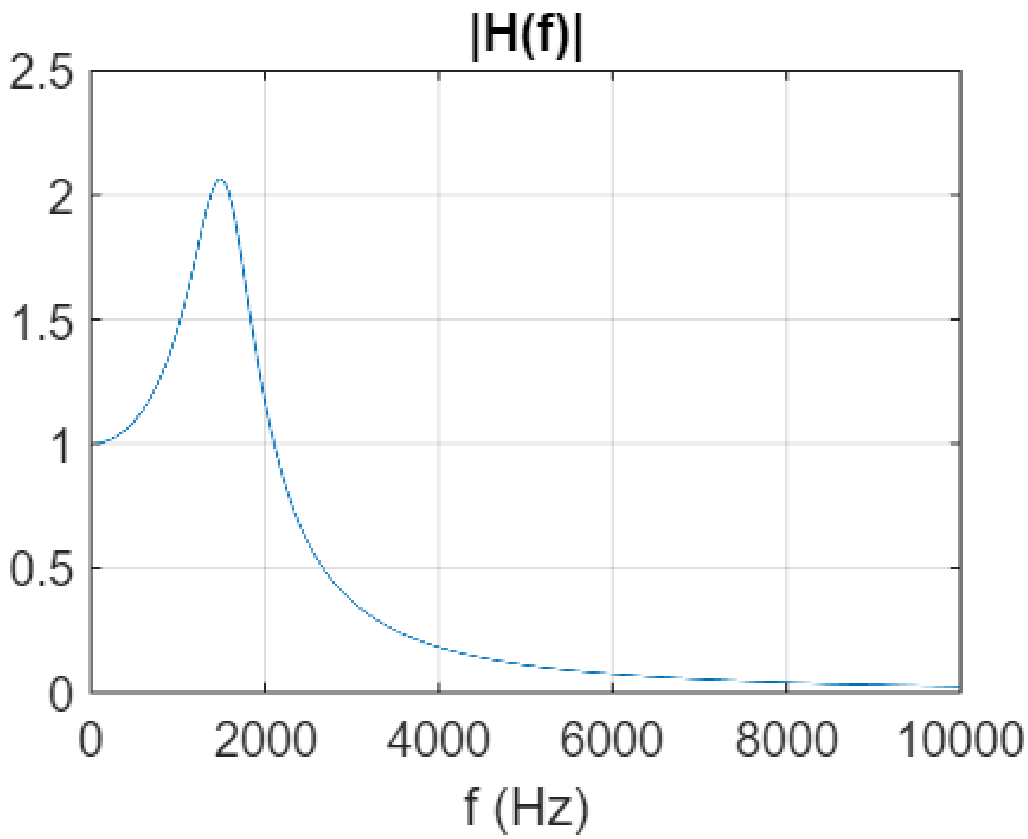
```
ksi = (R/L)/(2*w0),                           % should be smaller than 1
```

```
ksi = 0.2500
```

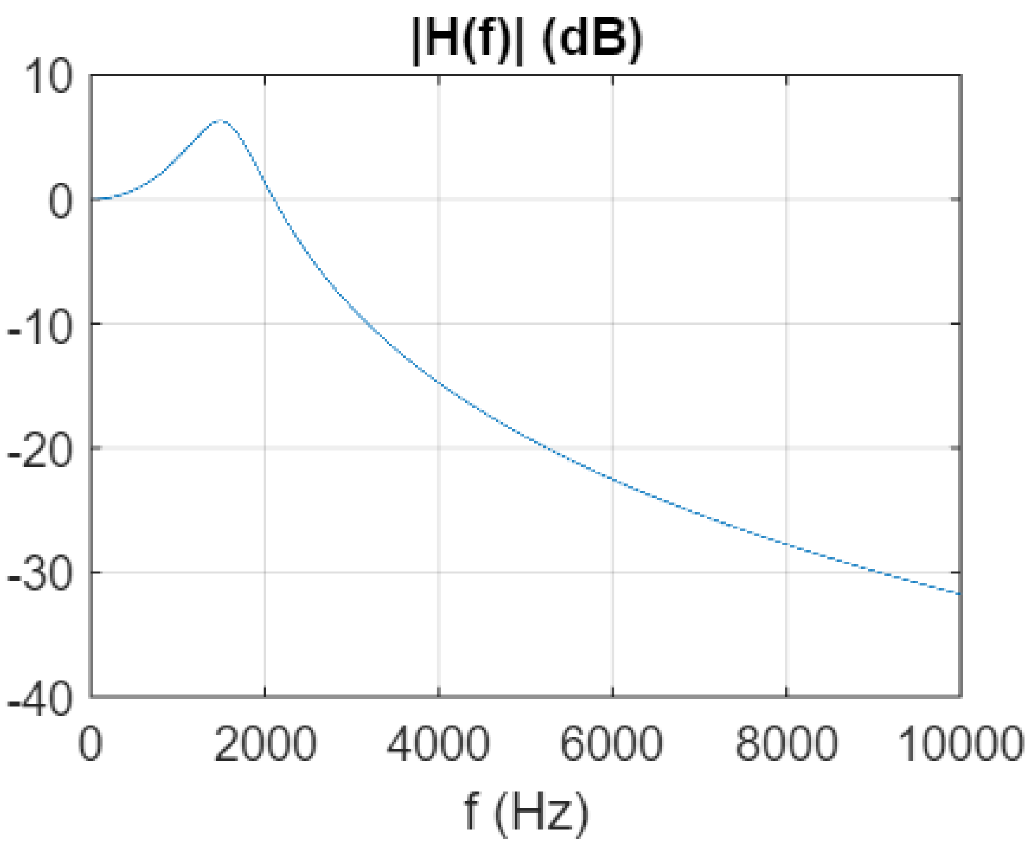
```
w1  = w0*sqrt(1-ksi^2);   f1 = w1/(2*pi),    % damped resonance frequency
```

```
f1 = 1.5410e+03
```

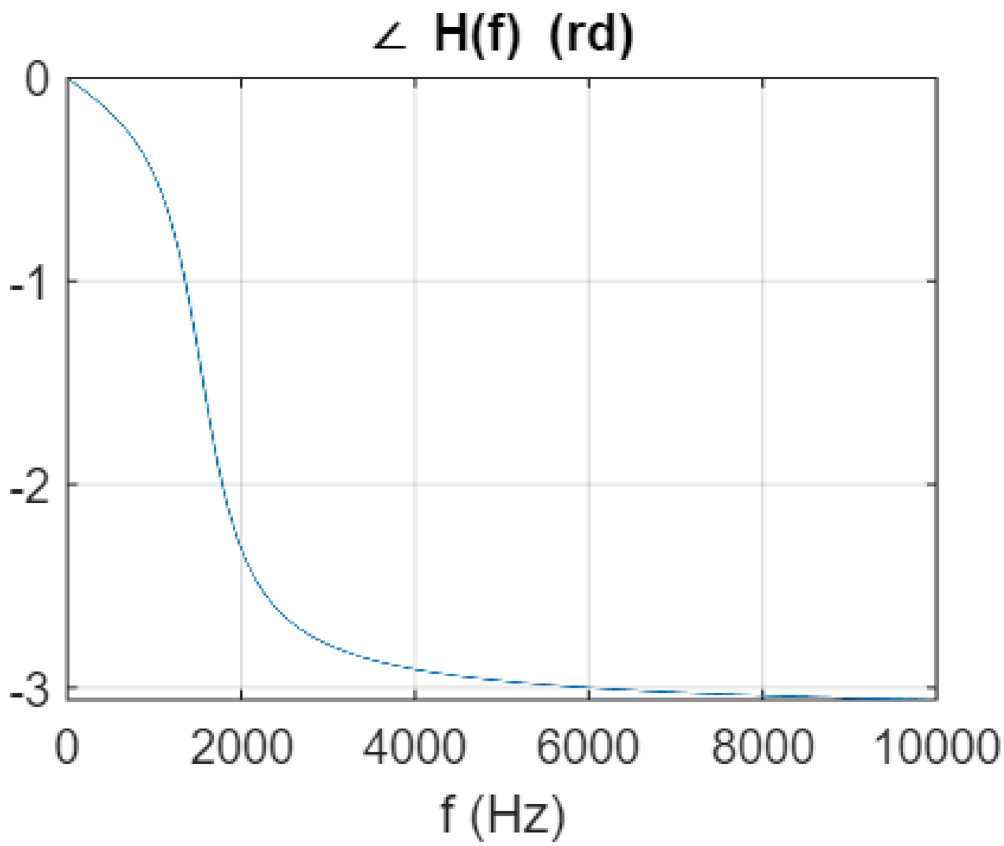
```

b = [ 1 ];                 % coeffs of nominator polynomial
a = [ L*C, R*C, 1 ];       % coeffs of denominator poly (from the highest order)

% Frequency response
  f = 0 : 1 : 10000;               % frequency range of interest
  w = 2*pi*f; s = j*w;             % radial frequency, Laplace transform variable
  H = polyval(b,s)./polyval(a,s); % frequency response H(f) = H(s=j*2*pi*f)
% H = freqs(b,a,2*pi*f);           % Matlab function doing the same
% Figures
  figure; plot(f,abs(H)); xlabel('f (Hz)'); title('|H(f)|'); grid;
```



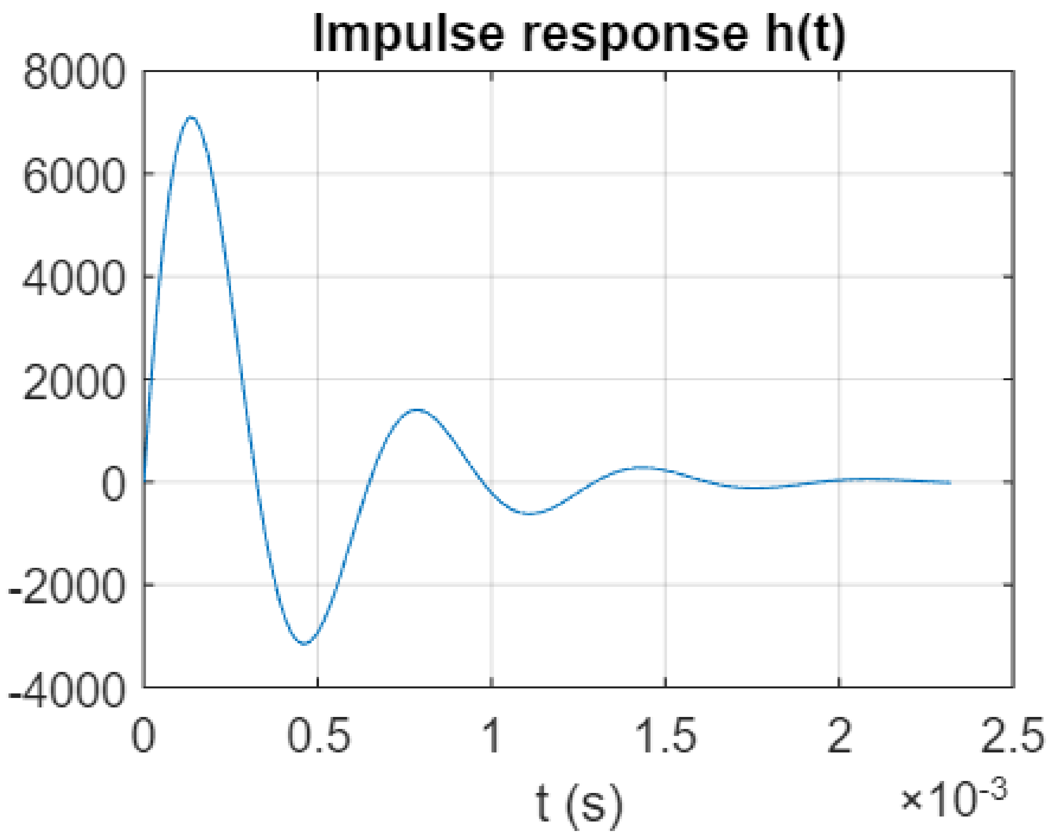

```
figure; plot(f,20*log10(abs(H))); xlabel('f (Hz)'); title('|H(f)| (dB)'); grid;
```



```
figure; plot(f,angle(H)); xlabel('f (Hz)'); title('\angle H(f) (rd)'); grid;
```




```
[h,t] = impulse(tf(b,a));       % calculate circuit impulse response h(t)
plot(t,h); xlabel('t (s)'); title('Impulse response h(t)'); grid;
```



H(s) design by zeros & poles selection

In genaral an analog circuit is defined by differentian input $x(t)$ - output $y(t)$ equation:

$$b_0 x(t) + b_1 \frac{dx(t)}{dt} + \ldots + b_M \frac{dx^M(t)}{dt^M} = y(t) + a_1 \frac{dy(t)}{dt} + \ldots + a_N \frac{dy^N(t)}{dt^N}$$

After performing Laplace transform (LT) of its bothsides and exploiting LT feature $L\left(\frac{dx^m(t)}{dt^m}\right) = s^m X(s)$, one obtains:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s^1 + b_2 s^2 + \ldots + b_M s^M}{1 + a_1 s^1 + a_2 s^2 + \ldots + a_N s^N} = \frac{b_M \cdot (s - z_1)(s - z_2) \cdot \ldots \cdot (s - z_M)}{a_N \cdot (s - p_1)(s - p_2) \cdot \ldots \cdot (s - p_N)}.$$

Instead of "design" of polynomial coefficients $\{b_k, a_k\}$ ensuring required $|H(f)| = |H(s)|_{s=j2\pi f}$, we can choose values of polynomial roots $\{z_k, p_k\}$ what is more easier:

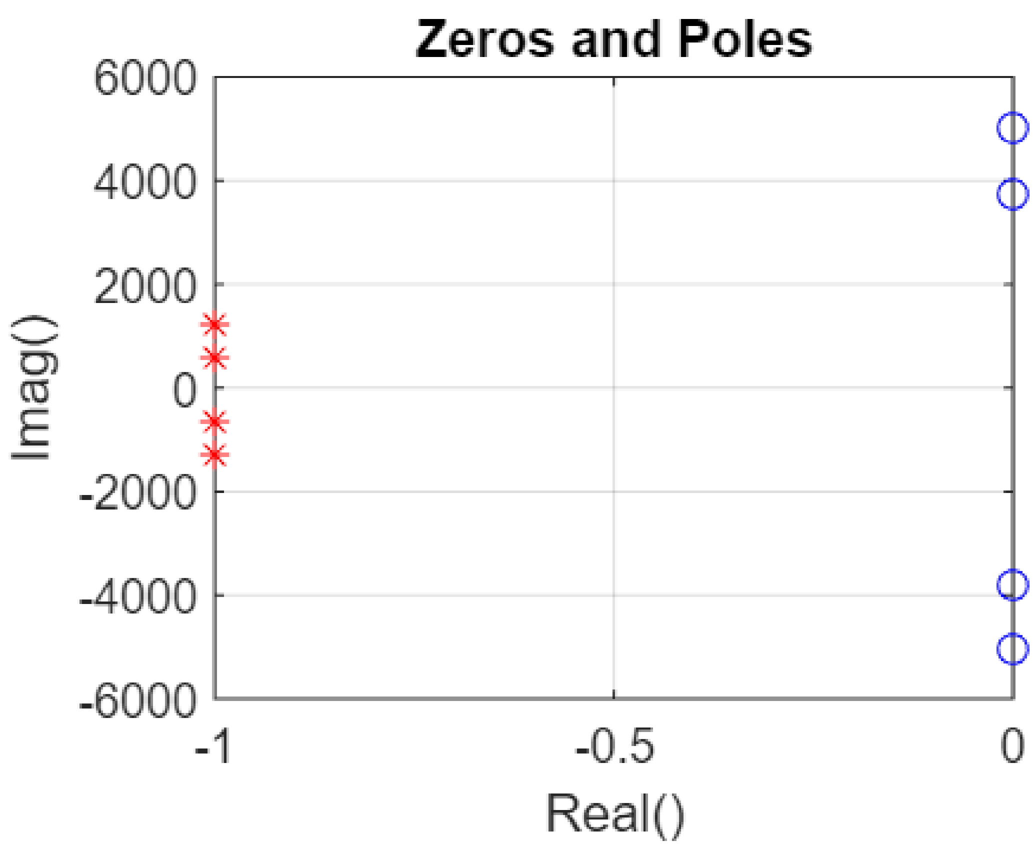
- setting TF zero $z_k = j2\pi f_k$ - removes sinusoid with frequency f_k from circuit output,
- setting TF pole $p_k = \varepsilon - j2\pi f_k$ - amplifies sinusoid with frequency f_k on circuit output (ε denotes small number).

Analyze this code. Run it, observe figures. And one more frequency to reject and one more - to amplify. Try to design an analog filter having flat passing band (linear gain 1 or decibel gain 0 dB) in some frequency interval:

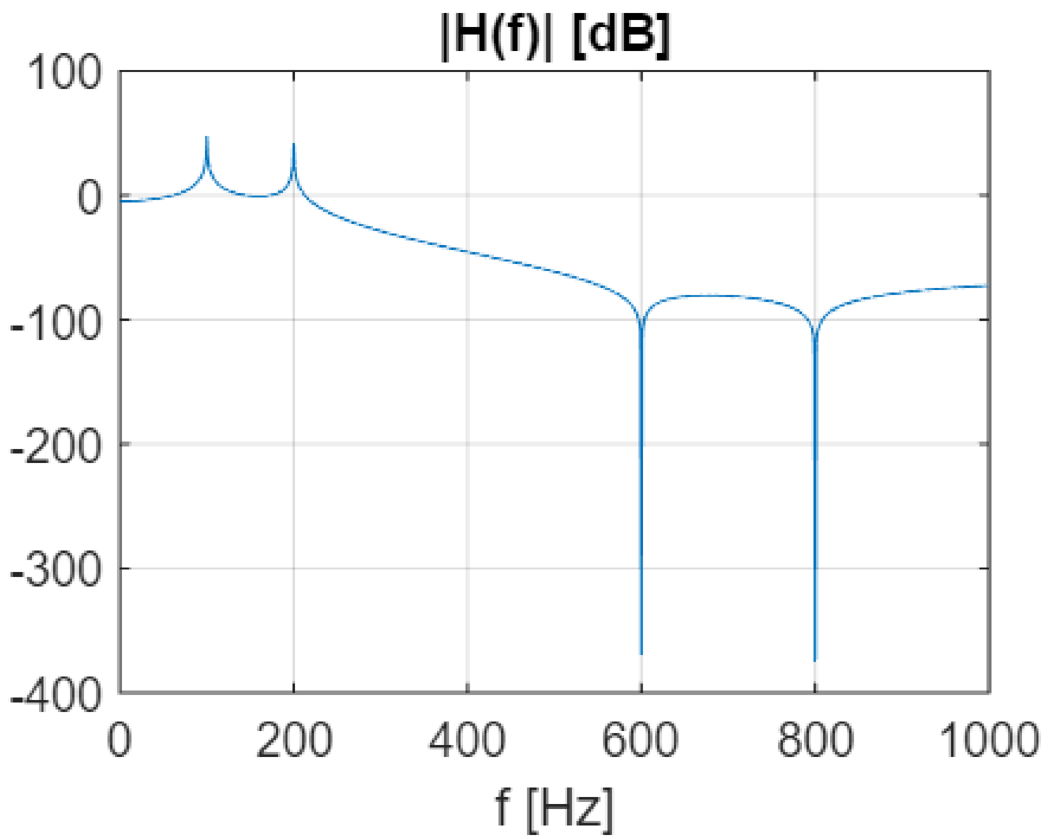
- Low-Pass (LP): $[0, f_{stop}]$ Hz,
- High-Pass (HP): $[f_{start}, +\infty]$ Hz,
- Band-Pass (BP): $[f_{start}, f_{stop}]$ Hz,
- Band-Stop (BS): $[0, f_{stop}]$ and $[f_{stop}, +\infty]$ Hz.

All other frequencies attenuate as much as possible.

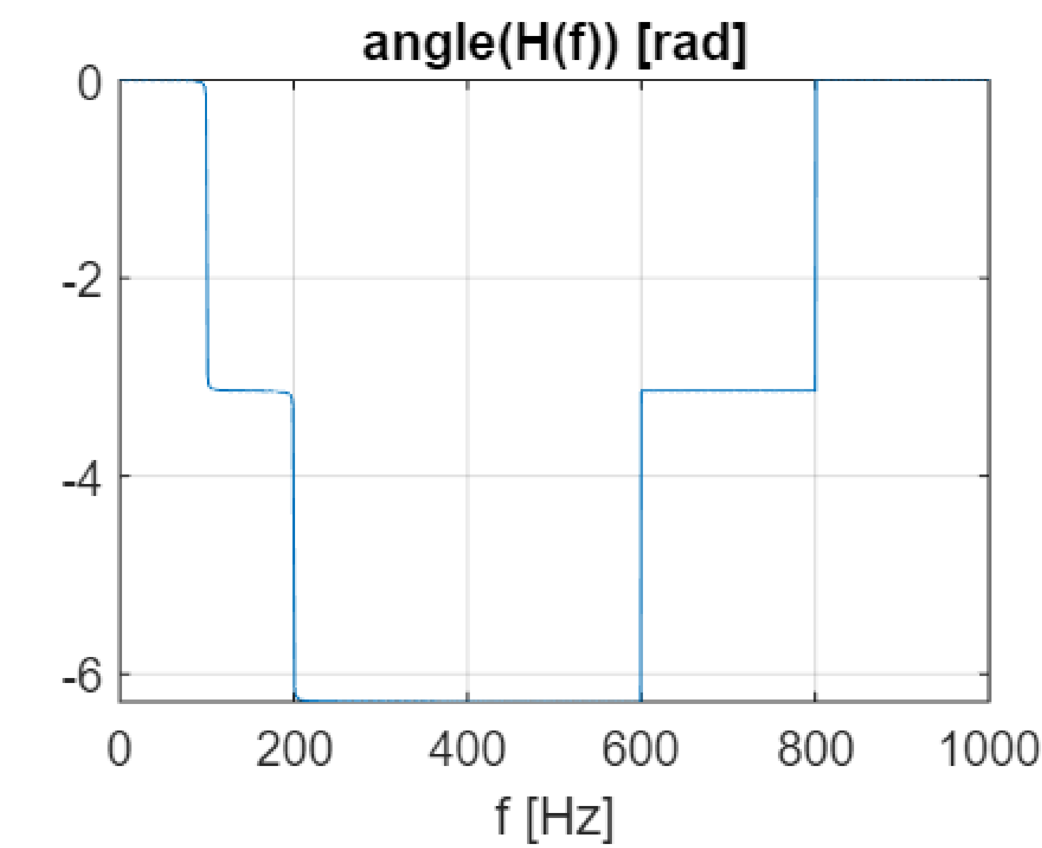
```
% Design of analog circuit/filter transfer function (TF) polynomials
  if(0)  % choosing polynomial coefficients [b,a] - nobody knows what filter will result
     b = [3,2];                       % [ b1, b0 ]
     a = [4,3,2,1];                   % [ a3, a2, a1, a0=1]
     z = roots(b); p = roots(a);      % [b,a] --> [z,p]
  else   % choosing polynomial roots [z,b] - a conscious decision
     gain = 0.001;
     z = j*2*pi*[ 600,800 ];          z = [z conj(z)];    % removing frequencies 600 and 800 Hz
     p = [-1,-1] + j*2*pi*[100,200]; p = [p conj(p)];    % amplifying frequencies 100 and 200 H
     b = gain*poly(z);  a = poly(p);                      % [z,p] --> [b,a]
  end
  figure; plot(real(z),imag(z),'bo', real(p),imag(p),'r*'); grid;
  title('Zeros and Poles'); xlabel('Real()'); ylabel('Imag()');
```




```
% Verification of filter frequency response: amplitude change, phase shift
  f = 0:1:1000;                       % frequencies of interest in hertz
  H = freqs(b,a,2*pi*f);              % frequency response via Matlab function
  figure; plot(f,20*log10(abs(H))); xlabel('f [Hz]'); title('|H(f)| [dB]'); grid;
```




```
figure; plot(f,unwrap(angle(H))); xlabel('f [Hz]'); title('angle(H(f)) [rad]'); grid;
```



Butterworth low-pass prototype filter

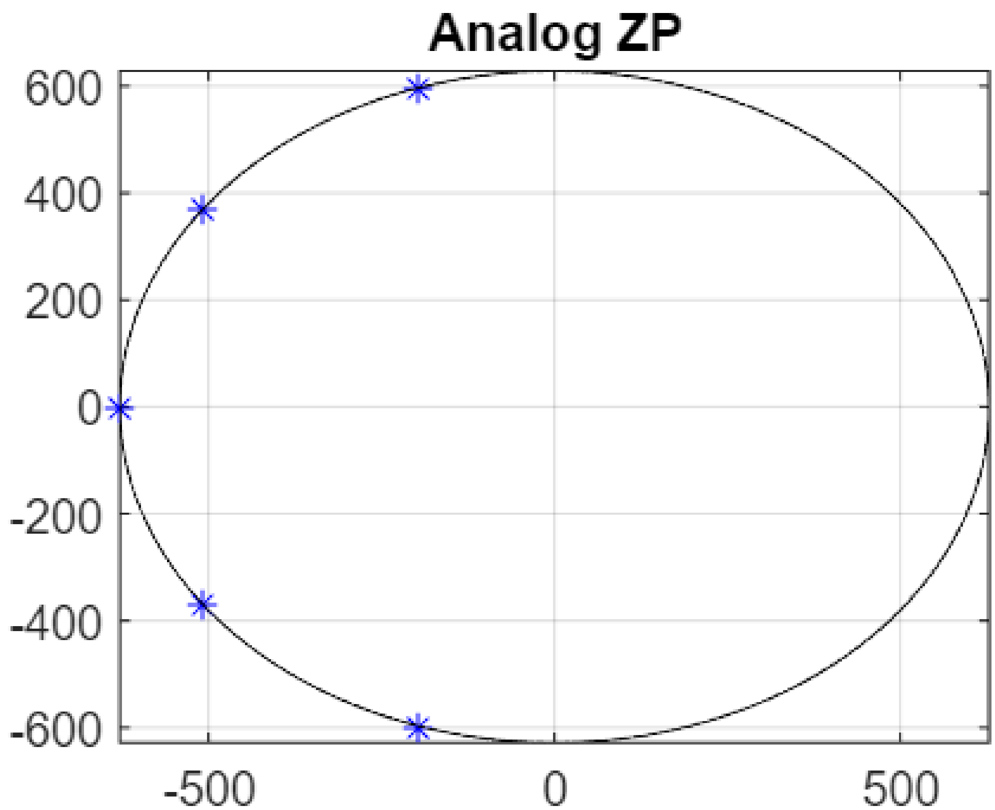
Butterwoth proposed a low-pass $[0, f_{stop}]$ filter having $H(s)$ of the form:

$$H(s) = \frac{(-p_1)(-p_2)...(-p_N)}{(s-p_1)(s-p_2)...(s-p_N)}, \quad p_k = (2\pi f_{stop})e^{j\varphi_k}, \quad \varphi_k = \frac{\pi}{2} + \frac{\Delta\varphi}{2} + \frac{\Delta\varphi}{N}(k-1), \quad \Delta\varphi = \frac{\pi}{N}, \quad k = 1, 2, ..., N.$$

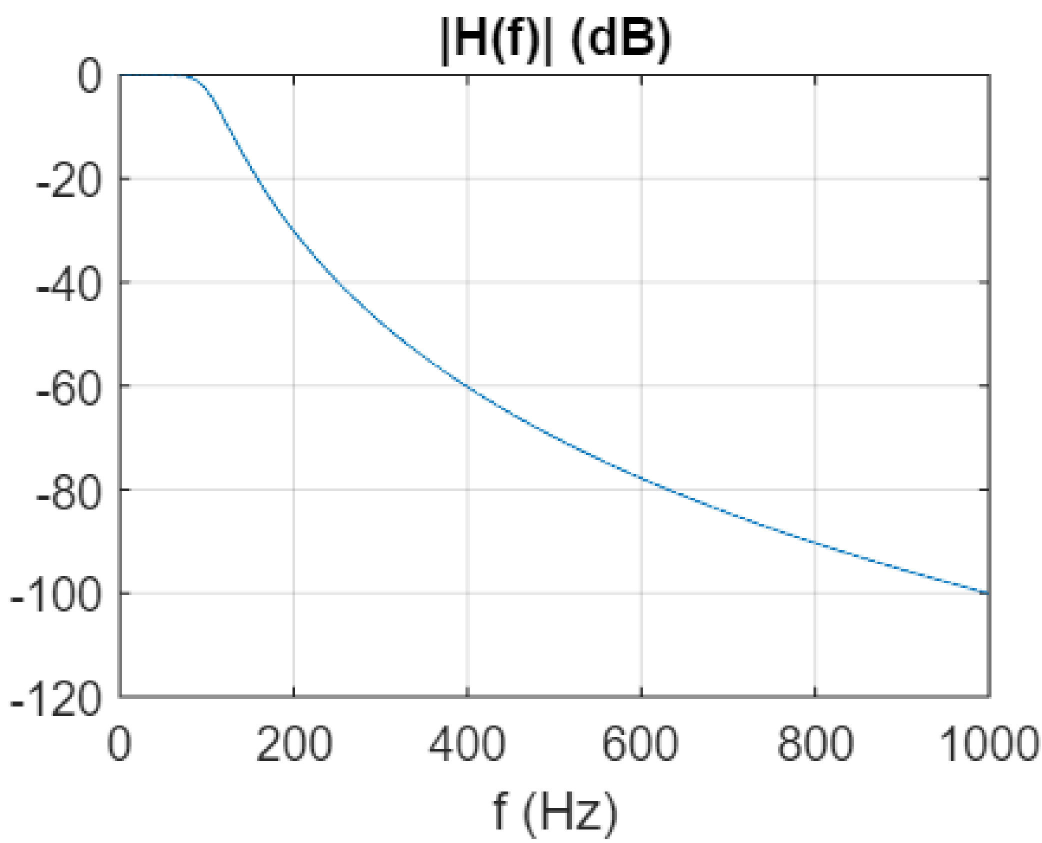
with poles lying on the circle with the radius $R = \omega_{stop} = 2\pi f_{stop}$, but only in the left half-plane of variable s. Value of N decides about the sharpness of the filter cutting edge (decay $-20N$ decibels per increase of frequency by one decade).

Add calculation and plotting $|H(f)|$ to this program.

```
% Low-pass and high-pass analog Butterworth filters
N = 5;                                  % number of TF poles
f0 = 100;                               % cut-off frequency of low-pass filter
alpha = pi/N;                           % angle of one ``piece of cake''
beta  = pi/2 + alpha/2 + alpha*(0:N-1); % angles of poles
R = 2*pi*f0;                            % circle radius
p = R*exp(j*beta);                      % poles placed on circle, left half-plane
z = []; gain = prod(-p);                % LOW-PASS: TF zeros are not used, gain
%z = zeros(1,N); gain = 1;              % HIGH-PASS: N TF zeros in zero, gain
b = gain*poly(z); a=poly(p);            % [z,p] --> [b,a]
b = real(b);      a=real(a);            %
H = freqs(b,a,2*pi*f);                  % Matlab function
fi=0:pi/1000:2*pi; c=R*cos(fi); s=R*sin(fi);
figure; plot(real(z),imag(z),'ro',real(p),imag(p),'b*',c,s,'k-'); grid; title('Analog ZP');
```




```
figure; plot(f,20*log10(abs(H))); xlabel('f (Hz)'); title('|H(f)| (dB)'); grid;
```



Other Butterworth filters via frequency transformations

Procedure for obtaining other Butterworth filter types, e.g. high-pass, band-pass and band-stop, is as follows:

1. design a low-pass Butterworth filter for $\omega_{stop} = 1$ choosing sufficiently high value of N (decides about filter frequency sharpness),
2. transform it to other filter type using known substitution $s = \text{function}(s')$.

In fact Chebyshev and Cauer filters (topic of our next section) have been also originally defined as low-pass, and has to be transformed in order to obtaine other types.

Let's look deeper into a Low-Pass to Band-Pass transformation of the Butterwort filter. In the LP2BP frequency transformation the s-variable exchange equation is as follows:

$$s = \frac{s'^2 + \omega_{center}^2}{\Delta\omega \cdot s'}, \qquad \omega_{center} = \sqrt{\omega_{start} \cdot \omega_{stop}}, \qquad \Delta\omega = \omega_{stop} - \omega_{start}.$$

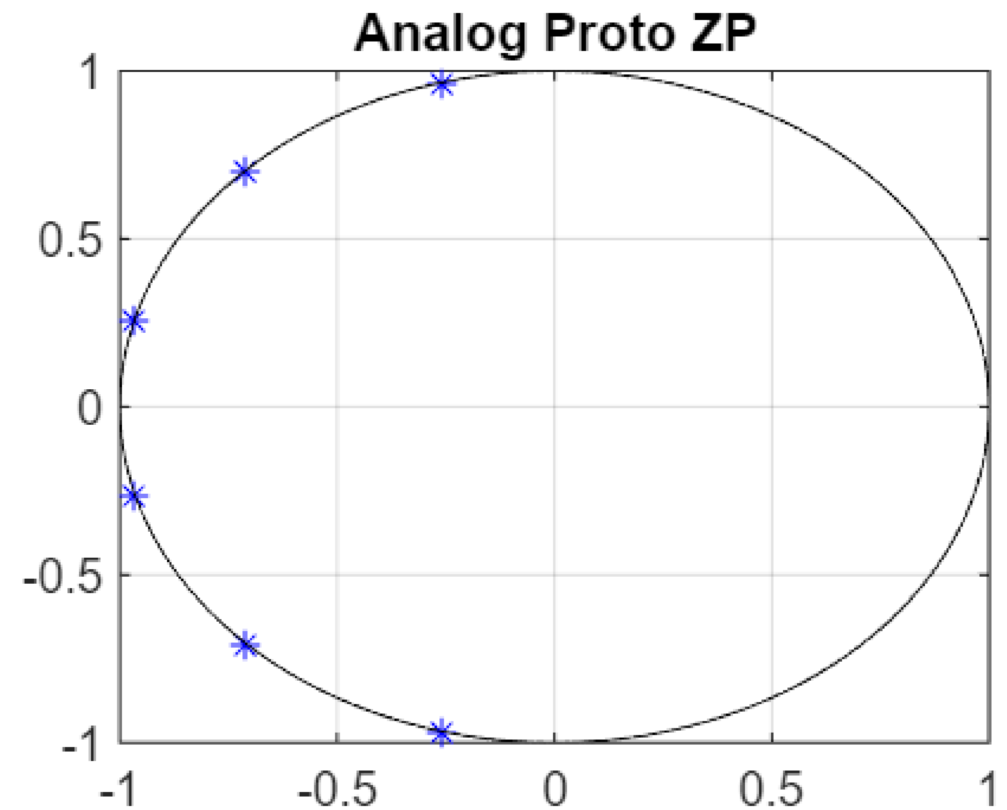
As a result each term $(s - z_k)$ (or $(s - p_k)$) of low-pass $H(s)$ should be replaced by the term:

$$\frac{(s')^2 - z_k \Delta\omega \cdot (s') + \omega_0^2}{\Delta\omega \cdot (s')},$$

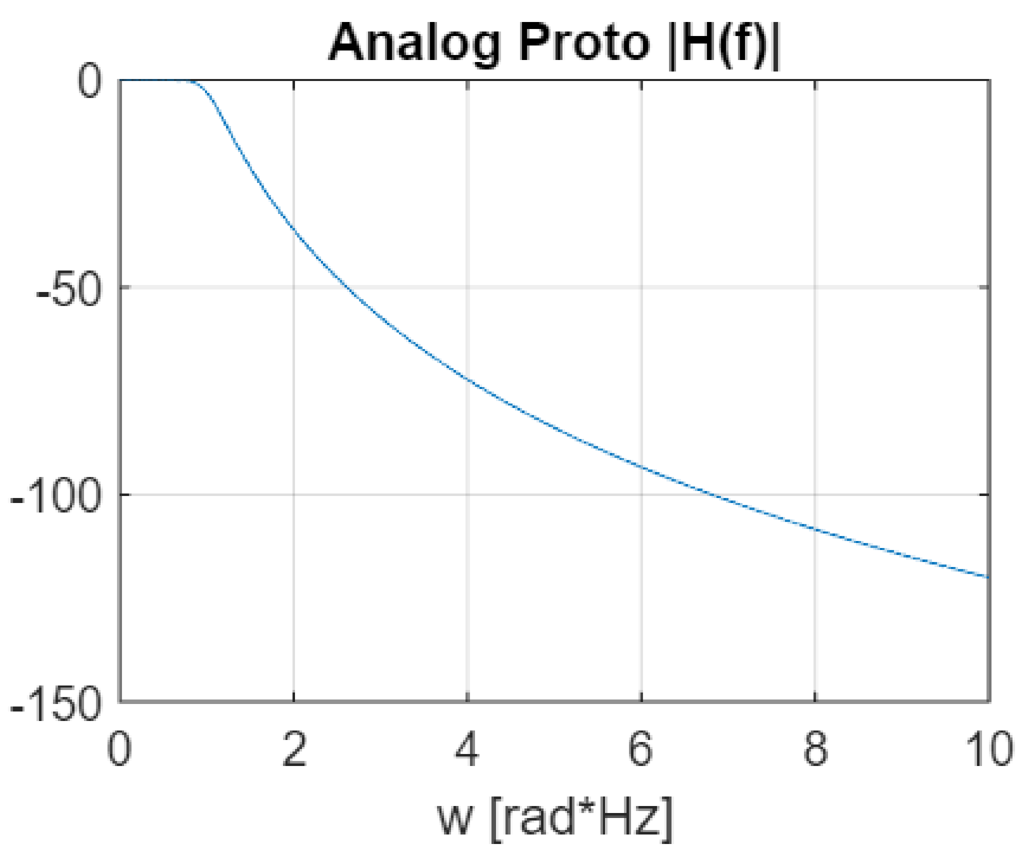
giving in a result ratio of two polynomials with completely different coefficients.

Analyze the code.

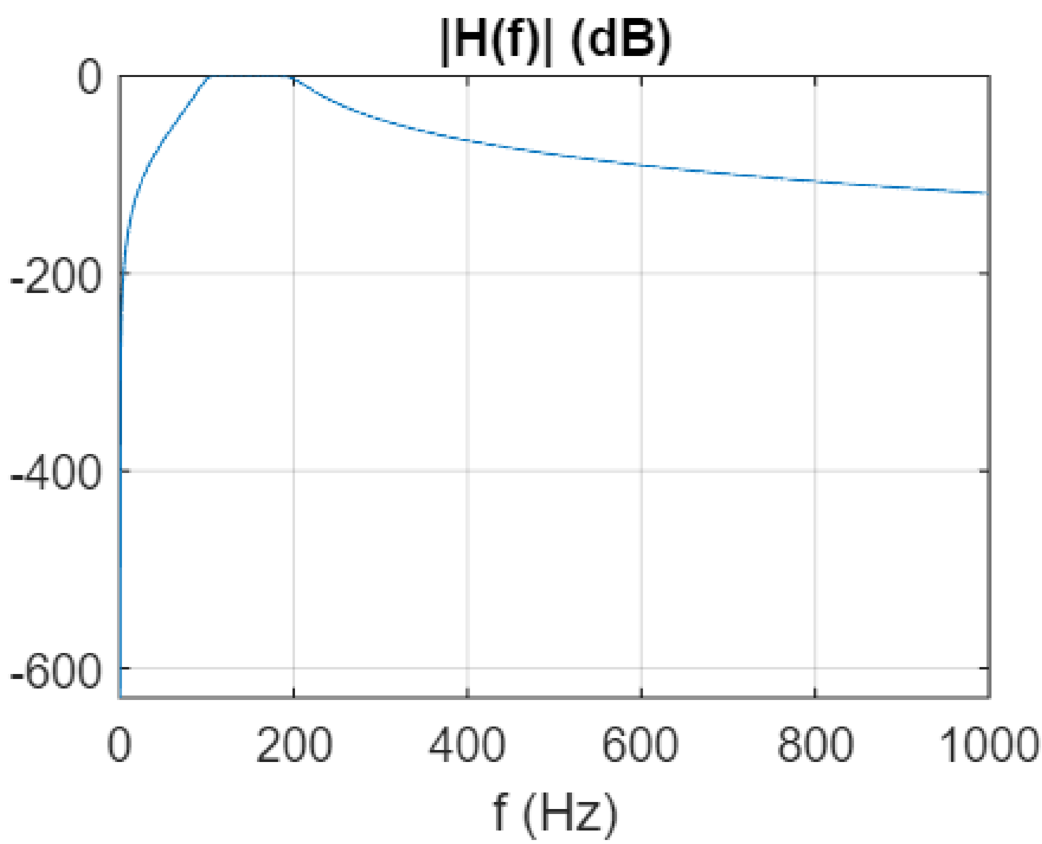
```
% Requirements
  N = 6;                % number of poles in analog prototype
  f0 = 100;             % for LowPass and HighPass
  f1 = 100; f2=200;     % for BandPass and BandStop
% Design of analog LowPass prototype filters for w0 = 1
  w = 0 : 0.01 : 10;                % radial frequencies of interest
  [z,p,gain] = buttap(N);           % analog Butterwoth prototype
  b = gain*poly(z); a = poly(p);    % [z,p,gain] --> [b,a]
  H = freqs(b,a,w);                 % filter frequency response using Matlab
  fi=0:pi/1000:2*pi; c=cos(fi); s=sin(fi);
  figure; plot(real(z),imag(z),'ro',real(p),imag(p),'b*',c,s,'k-'); grid; title('Analog Proto Z
```




```
figure; plot(w,20*log10(abs(H))); grid; xlabel('w [rad*Hz]'); title('Analog Proto |H(f)|');
```



```
% Frequency transformation: Normalized (w0=1) LowPass --> DifferentPass
% Functions xx2yy from Signal Processing Toolbox (xx,yy = lp,hp,bp,bs)
% [b,a] = lp2lp(b,a,2*pi*f0);                          % LowPass to LowPass
% [b,a] = lp2hp(b,a,2*pi*f0);                          % LowPass to HighPass
  [b,a] = lp2bp(b,a,2*pi*sqrt(f1*f2),2*pi*(f2-f1));   % LowPass to BandPass
% [b,a] = lp2bs(b,a,2*pi*sqrt(f1*f2),2*pi*(f2-f1));   % LowPass to BandStop
  H = freqs(b,a,2*pi*f);                               % Matlab function
  figure; plot(f,20*log10(abs(H))); xlabel('f (Hz)'); title('|H(f)| (dB)'); grid;
```

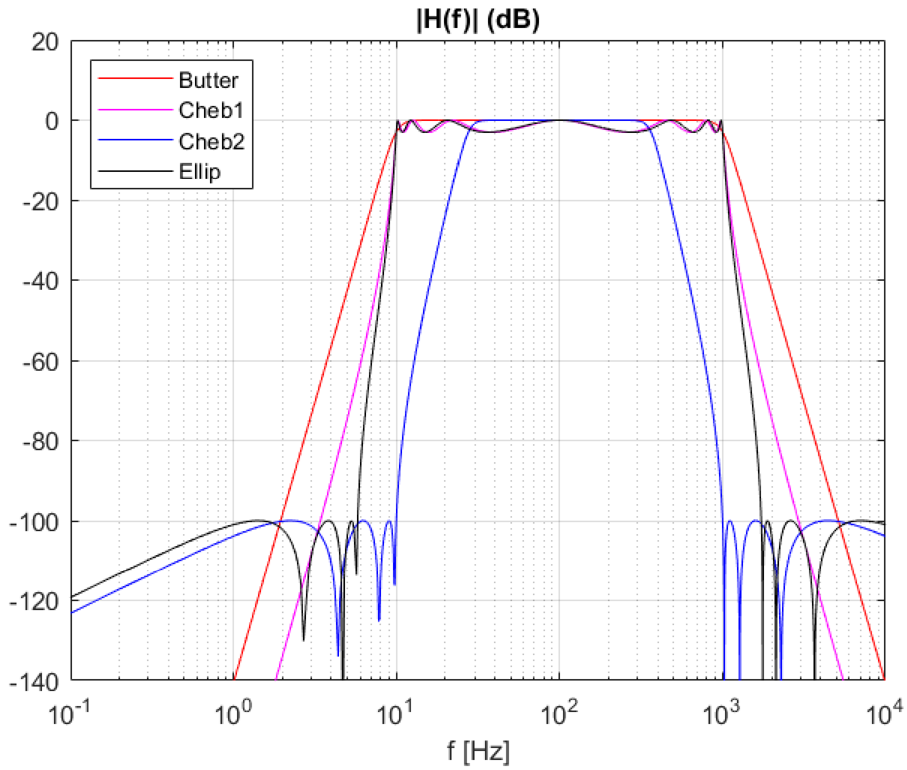


Chebyshev and Cauer (elliptic) filters

Chebyshev I/II and Cauer (elliptic) filters are alternatives to Butterwoth filters. Magnitude frequency response of the Butterworth filter does not have oscillations in the passband (it is flat) and stopband (it is continuously decaying). This is achieved at the price of low filter sharpness. Other filters allows oscillations:

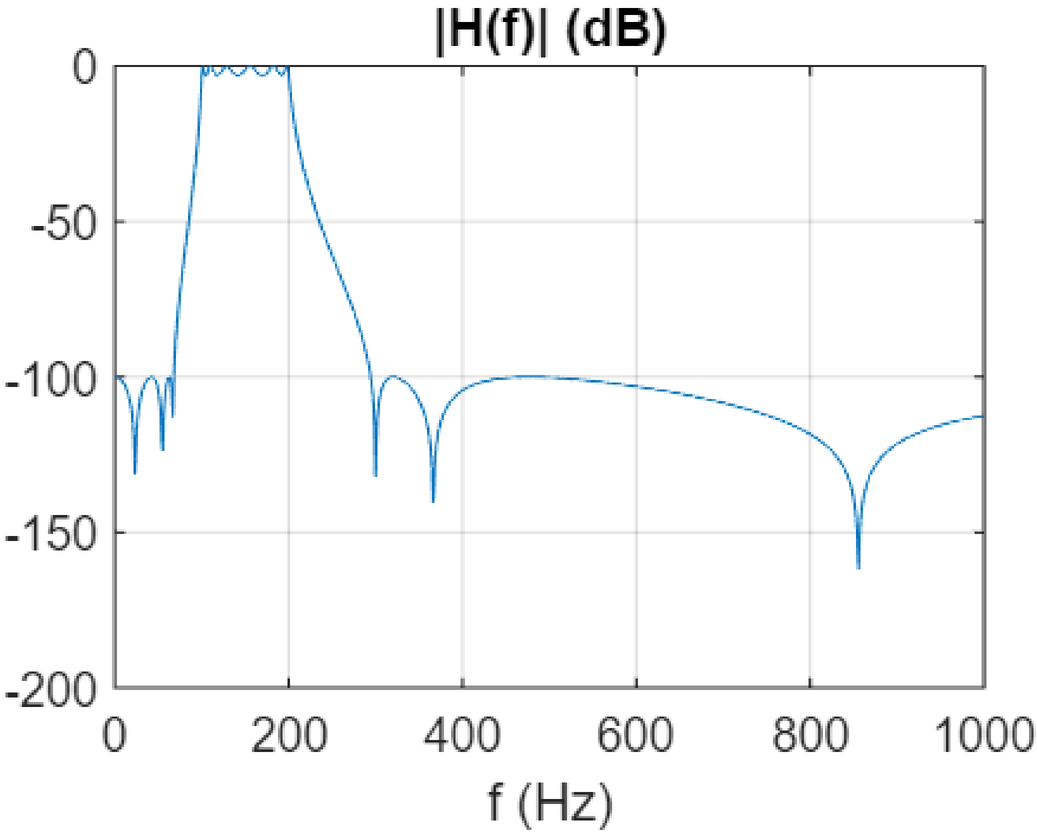
- Chebyshev I - in the passband,
- Chebyshev Ii - in the stopband,
- Cauer (elliptic - both, in the passband and stopband.

Filters with bigger oscillations are sharper. Now we will design analog filter coefficients `[b,a]` using the following Matlab functions: `butter(), cheby1(), cheby2(), ellip()`. Plots of their frequency responses $|H(f)|$ are compred in figure below. Note oscillations and sharpness. Note different behaviour of Chebyshev II filter (during design beging of stopband, not end of passband is specified).



Analyze, run, observe, modify. Try to obtained similar plots as above.

```
N = 6;                    % number of transfer function poles
f0=10;  f1=100;  f2=200; % frequencies in Hz [1/s]
Rp = 3; Rs = 100;         % allowed ripples in dB in passband and stopband
% [b,a] = butter(N, 2*pi*f0, 'low','s');                    % Butt  LowPass
% [b,a] = butter(N, 2*pi*f0, 'high','s');                   % Butt  HighPass
% [b,a] = butter(N, 2*pi*[f1,f2], 'stop', 's');             % Butt  BandStop
% [b,a] = butter(N, 2*pi*[f1,f2], 'bandpass', 's');         % Butt  BandPass
% [b,a] = cheby1(N, Rp, 2*pi*[f1,f2], 'bandpass', 's');     % Cheb1 BandPass
% [b,a] = cheby2(N, Rs, 2*pi*[f1,f2], 'bandpass', 's');     % Cheb2 BandPass
  [b,a] =  ellip(N, Rp, Rs,2*pi*[f1,f2], 'bandpass', 's');  % Ellip BandPass
H = freqs(b,a,2*pi*f);                         % Matlab function
figure; plot(f,20*log10(abs(H))); xlabel('f (Hz)'); title('|H(f)| (dB)'); grid;
```


|H(f)| (dB)
0
-50
-100
-150
-200
0
200
400
600
800
1000
f (Hz)