Signals #4: DtFT vs. DFT

Discretization of continuous Fourier transform and Fourier series

Continuous Fourier transform (CFT) and Fouries series (FS) are defined as follows for analog signals (non-period and periodic ones, respectively):

(CFT) $X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$,

(FS) $X(kf_0) = \frac{1}{T}\int_0^T x(t)e^{-j2\pi(kf_0)t}dt, \quad k = 0, \pm1, \pm2, \pm3, \ldots,$

After time discretization (f_s - sampling frequency, $dt = 1/f_s$) and assumning having only Nsignal samples ($n = 0, 1, 2, \ldots, N-1$, $t = n \cdot dt$) we obtained discrete versions of CFT and FS, knowns as Distrete time Fourier transform (DtFT) and Discrete Fourier Transform (DFT):

(DtFT) $X(f) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{f}{f_s}n}$

(DFT) $X(kf_0) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi\frac{kf_0}{f_s}n} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, 2, \ldots, N-1.$

As we see equations are almost the same, the only difference is that in DtFT frequency choice is arbitrary while in DFT obligatory (since when the signal is period it can contain only frequencies beeing multiples of its fundamental frequency $f_0 = \frac{1}{T}$ ($T = N \cdot dt$- signal time duration). In this program we will concentrate on basics of frequency analysis using DtFT and DFT.

Signal choice

Signal of our interest:

1. will be sampled with frequency f_s,
2. will have N samples,
3. will consists of two cosines with amplitudes A_1, A_2 and frequencies f_1, f_2:
   $x(t) = A_1\cos(2\pi f_1 t) + A_2\cos(2\pi f_2 t)$.

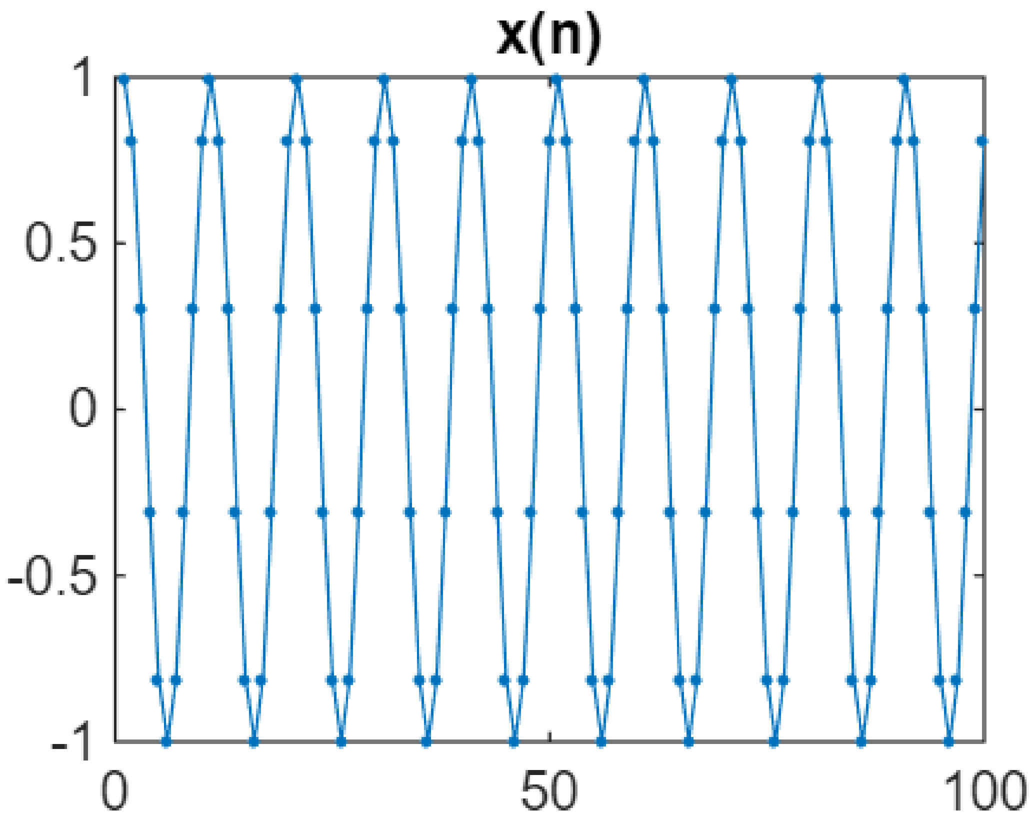
Signal component amplitudes can differ a lot (problem with available **amplitude resolution** of the frequency analysis: how to see amolitude so small?) and signal component frequencies can be very similar (problem with available **frequency resolution** of the frequency analysis: how to distinguish frequencies which lies so close).

Analyze the Matlab code. Generate different signals and observe their shapes.

```
% clear all; close all;

% Signal
N = 100;                              % number of samples: 100 --> 1000
fs = 1000; dt=1/fs; t=dt*(0:N-1);     % sampling ratio
```

```
fx1 = 100;  Ax1 =1;                 % frequency and amplitude of signal component 1
fx2 = 250;  Ax2 =1;                 % frequency and amplitude of signal component 2
                                    % 250 --> 110, 0 --> 0.001 --> 0.00001
x1 = Ax1*cos(2*pi*fx1*t);           % first component
x2 = Ax2*cos(2*pi*fx2*t);           % fx2 250Hz --> 110Hz, Ax2 1 --> 0.001 --> 0.00001
x = x1; % + x2;                     % x1, x1+x2, 20*log10(0.00001)=-100 dB
plot(x,'.-'); title('x(n)');        % analyzed signal
```

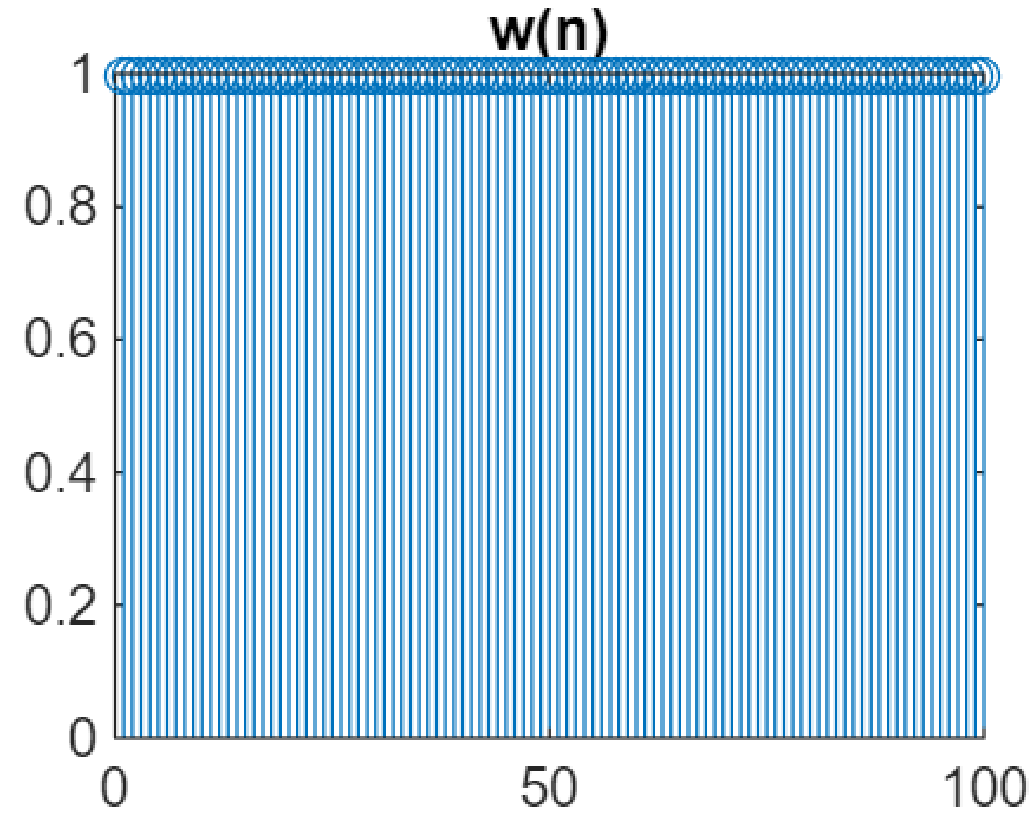


Window choice

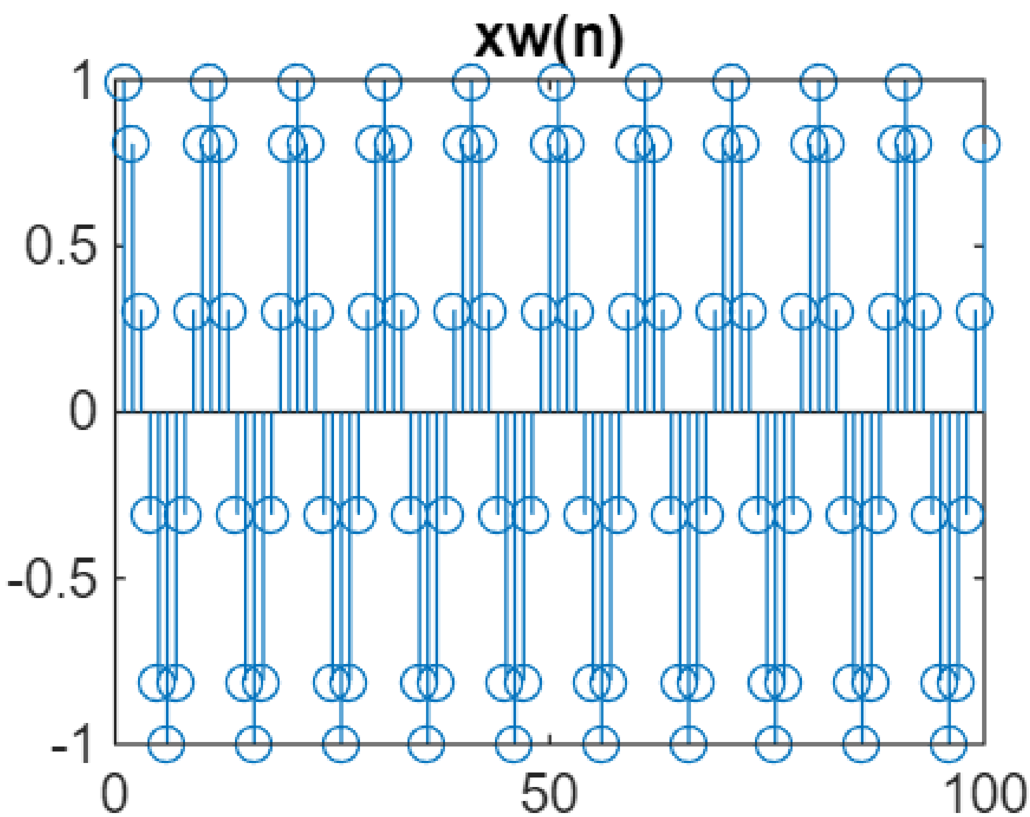
In order to improve amplitude resolution, appropriate **window function** (Hamming, Hanning, Blackman, Kaiser, Chebyshev, ...) should be used which perform smooth tapering the signal shape on its begining and end, and remove possible sharp signal change on its edges.

Add more window functions to the program, observe its shapes as well as shapes of the signal after its different "windowing".

```
% Windowing
w1 = boxcar(N)';                    % rectangular window
w2 = hanning(N)';                   % Hanning window
w3 = chebwin(N,140)';               % Chebyshev window: 80,100,120,140
w = w1; is_w1=1;                    % w1 --> w2, w3 (80,100,120,140)
figure; stem(w); title('w(n)');     % window
```



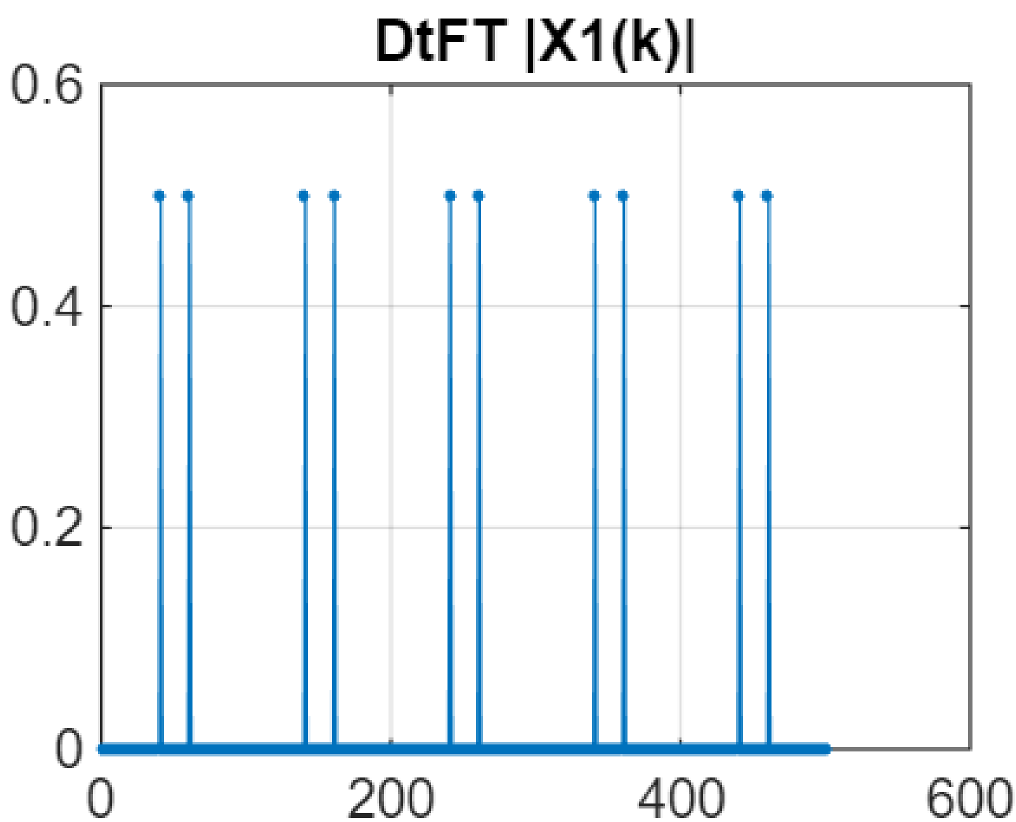

```
x = x .* w;                         % x = x, w, x.*w
figure; stem(x); title('xw(n)');    % windowed signal
```



DtFT and DFT computation

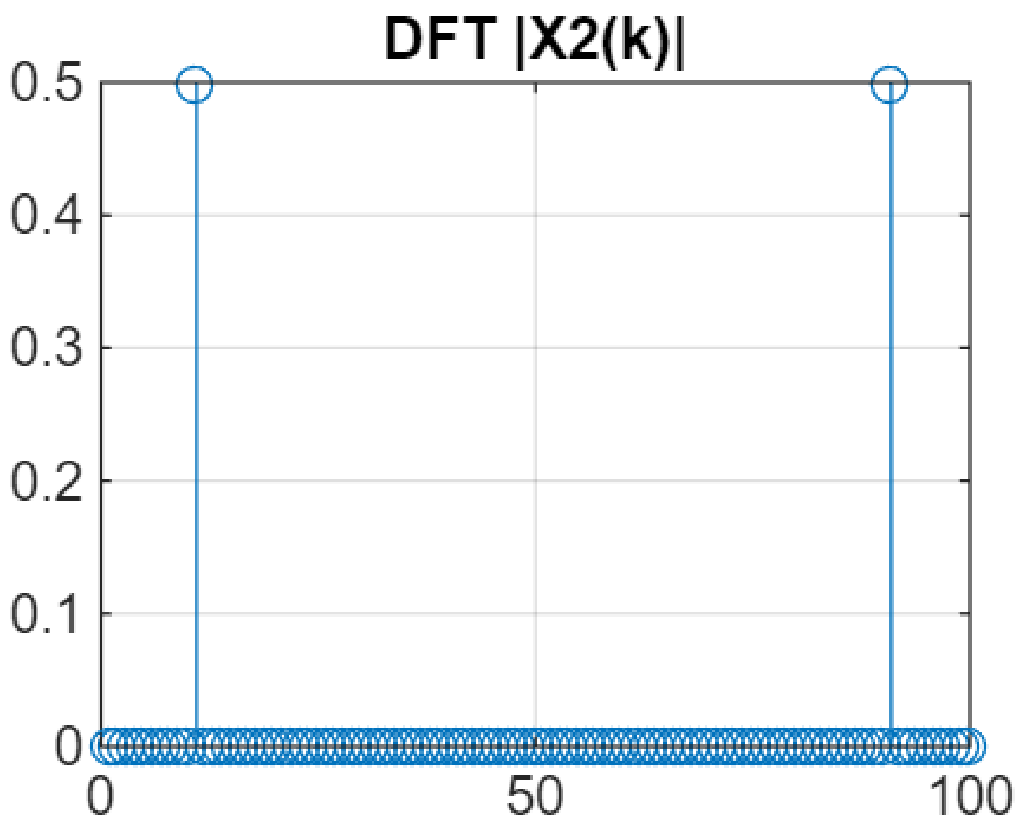
In the code below DtFT and DFT are computed according to their definitions, given above. Change `df,fmax` of DtFT and compare obtained shape of the DtFT spectrum with the DFT spectrum, which should be the same all the time. As you see the DtFT spectrum is periodical and it is sufficient to compute DtFT only in the interval/range $\left[-\frac{f_s}{2}, +\frac{f_s}{2}\right)$ or $[0, f_s)$. Setting `df` to small value we can sample the DtFT spectrum more dense than in DFT (always with the step $f_0 = f_s/N$) . Change values of f_s, N and observe these feature.

```
% DtFT - already discussed (blue line)
df = 10;                        % sampling step:  10 --> 1
fmax = 2.5*fs;                  % sampling range: 2.5 --> 0.5
f1=-fmax:df:fmax;               % frequency range
for k = 1 : length(f1)
    X1(k) = sum( x .* exp(-j*2*pi* (f1(k)/fs) *( 0:N-1) ) ) / N;
end
if(is_w1==1) X1 = N*X1/sum(w); end % scaling for any window different from rectangular
figure; plot(abs(X1),'.-'); title('DtFT |X1(k)|'); grid;  % showing result
```




```
% DFT - later in this chapter (red circles)
% k=0:N-1; n=0:N-1; F = exp(-j*2*pi*(k'*n)); X = (1/N)*F*x;
f0 = fs/N; f2 = f0*(0:N-1);         % DFT freq step = f0 = 1/(N*dt)
for k = 1:N
    X2(k) = sum( x .* exp(-j*2*pi/N* (k-1) *(0:N-1) ) )/ N;       % the same
  % X2(k) = sum( x .* exp(-j*2*pi/N* (f2(k)/fs) *(0:N-1) ) )/ N;  % the same
```

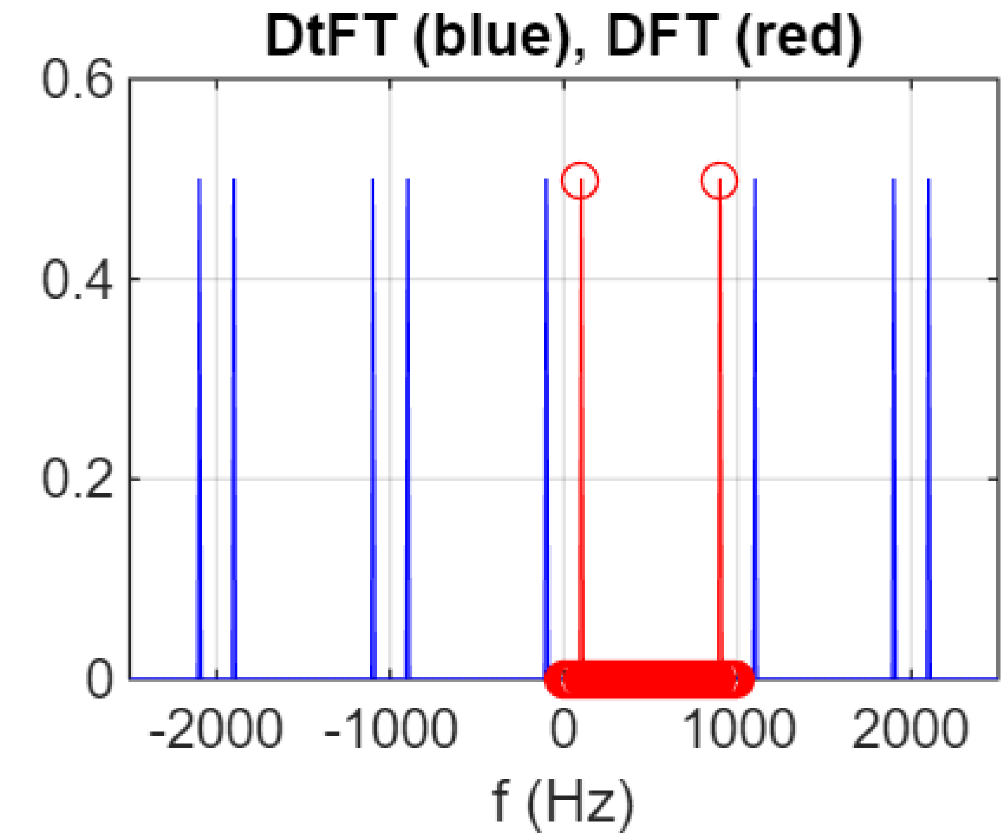
```
end
if(is_w1==1) X2 = N*X2/sum(w); end % scaling for any window different from rectangular
figure; stem(abs(X2)); title('DFT |X2(k)|'); grid;  % showing result
```



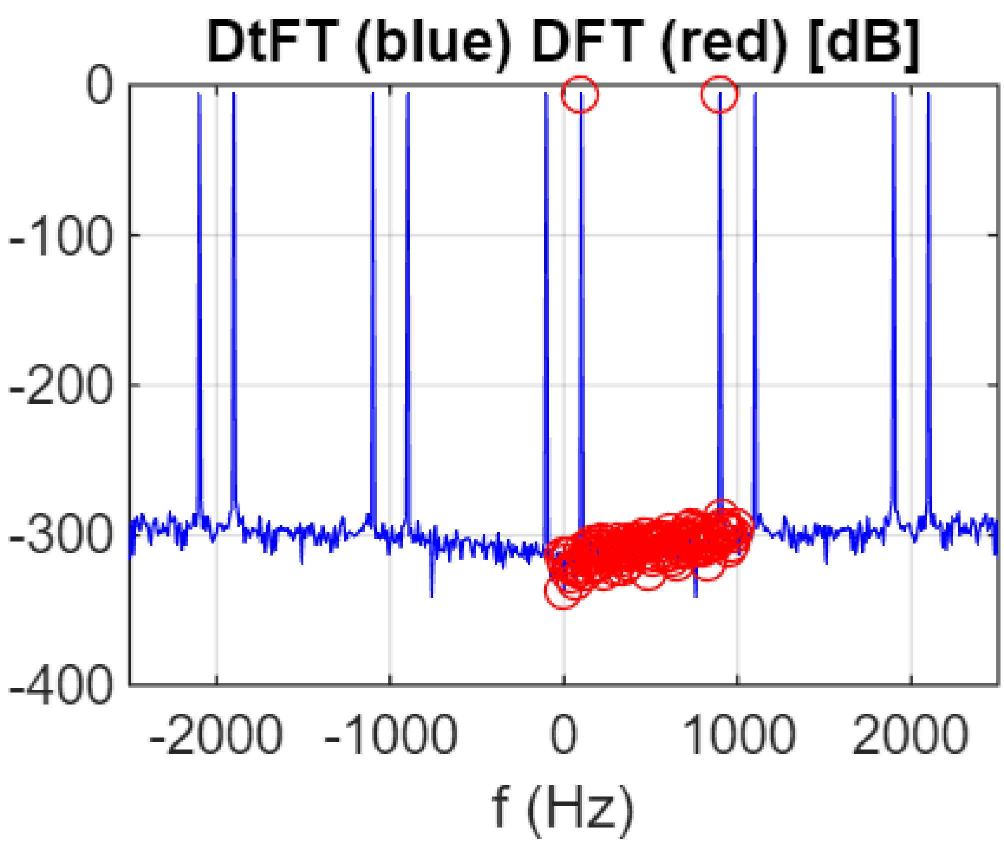
Detailed comparison of DtFT and DFT spectra

These figures can help us with better understanding the difference between the DtFT (dense frequency sampling but extensive computing) and DFT (for fixed values of f_s, N frequency sampling is constant but very fast algorithms exist for the DFT computation).

```
% Figures
figure; plot(f1,abs(X1),'b-',f2,abs(X2),'ro-');
xlabel('f (Hz)'); title('DtFT (blue), DFT (red)'); grid;
```




```
figure; plot(f1,20*log10(abs(X1)),'b-',f2,20*log10(abs(X2)),'ro');
xlabel('f (Hz)'); title('DtFT (blue) DFT (red) [dB]'); grid;
```



1. Set `df=1; fmax=fs/2, x = w.` Observe spectrum of different windows alone. Note different width of their spectral main-lobe around 0Hz, and different attentuation level of spectral side-lobes.
2. Set `x=x1; w=w1;` Observe spectrum of a pure strong cosine computed with rectangular window. Can you read cosine amplitude and frequency from the figure?
3. Set `x=x1+x2; fx2=250; Ax2=1; w=w1;` Do you see the second component? I think that yes.
4. Let 's make the second component weaker. Set `x=x1+x2; fx2=250; Ax2=0.001; w=w1;` Do you see the second component? I think that do not.
5. Let's take better window with higher attentuation of spectral side-lobes. Now set `w=w3; (with 120).` At present you should see the second component.
6. Let's shft the second component in frequency closer to the first one. Set `x=x1+x2; fx2=110; Ax2=0.001; w=w3; (with 120).` Do you see the second component? Not very good?
7. Let's take more signal samples. Set `N=1000.` Is the spectrum sharper?