Hilbert and differentiation FIR filters

Introduction

In this laboratory we learn how to design and use two special FIR digital filters: the Hilbert (H) filter (-90 degrees phase shifter) and differentiation (D) filter. The first of them is widely used for creation of complex-value analytic signals, which simplify amplitude and phase/frequency signal demodulation. The second - in signal processing schemes in which calculation of signal derivative is required, e.g. for frequency demodulation also.

Both filters will be designed by the window method. Their weights will be computed analytically by means of the inverse discrete-time Fourier transform (DtFT):

$$h_{H/D}(k+P) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} H_{H/D}(f) e^{j2\pi(f/f_s)k} df \quad \text{for} \quad -P \le k \le P.$$

and applied to different signals using the digital convolution equation ($x(n) \rightarrow [h_{H/D}(n)] \rightarrow y(n)$):

$$y(n) = \sum_{k=0}^{2P} x(k) h_{H/D}(n-k).$$

Filter output sample $y(n)$ will be computed as the weighted summation of $2P+1$ last input samples $x(n-k),\ k = 0, 1, ..., 2P$.

FIR Hilbert filter

The Hilbert filter is a phase shifter by -90 degrees ($-\pi/2$ radians). Let's apply such phase shift to a pure cosine:

$$\cos(\omega t) \cdot e^{-j\pi/2} = \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) \cdot e^{-j\pi/2} = \frac{e^{j(\omega t - \pi/2)} + e^{-j(\omega t + \pi/2)}}{2}.$$

We see that the Fourier harmonic with positive frequency is shifted by $-\pi/2$, while the harmonic with negative frequency - by $+\pi/2$. In consequence, the frequency response (FR) $H_H(f)$ of the Hilbert transformer/filter, is equal to:

$$\text{for} \quad f > 0: \qquad H_H(f) = e^{-j\pi/2} = -j,$$

$$\text{for} \quad f = 0: \qquad H_H(f) = 0,$$

$$\text{for} \quad f < 0: \qquad H_H(f) = e^{+j\pi/2} = +j.$$

Please note that $H_H(f) = 1$ for all frequencies except zero: $H_H(0) = 0$. Calculating inverse DtFT of the $H_H(f)$, the same way as in laboratory on FIR digital filters, one gets impulse response of the digital FIR Hilbert filter, i.e. equation for the filter weights:

$$h_H(n) = \frac{1 - \cos(\pi n)}{\pi n} = \frac{\sin^2(\pi n/2)}{\pi n/2}, \quad n \neq 0, \qquad h_H(0) = 0.$$

Cut theoretical Hilbert filter impulse response $h_H(n)$ should be shaped (multiplied) by arbitrary chosen window function $w(n)$:

$$h_H^{(w)} = h_H(n) \cdot w(n), \qquad n = -P, ..., -1, 0, 1, ..., P.$$

This operation reduces oscillations present in obtained filter amplitude response, which are caused by taking only a fragment of $h_H(n)$.

Hilbert transformer is used for creation of an analytic complex-value signal $x_a(t)$, associated with a real-value signal $x(t)$:

$$x(t) \quad \rightarrow \quad \text{Hilbert}[x(t)] \quad \rightarrow \quad x_a(t) = x(t) + j \cdot H[x(t)]$$

Spectrum of the analytic version of any real-value signal does not contain negative frequencies, only positive ones - they are amplified 2 times is regard to the original. Let's derive this very important analytic signal feature:

$$X_a(f) = X(f) + j \cdot H[X(f)] = X(f) + j \cdot X(f)H_H(f) = X(f) \cdot (1 + j \cdot H_H(f)),$$

and

for $f < 0$: $\quad X_a(f) = X(f) \cdot (1 + j \cdot (j)) = X(f) \cdot (1 - 1) = 0,$

for $f > 0$: $\quad X_a(f) = X(f) \cdot (1 + j \cdot (-j)) = X(f) \cdot (1 + 1) = 2X(f).$

Therefore the spectrum is not symmetrical around 0 Hz! Explanation of this feature is as follows:

$$x(t) = A(t)\cos(\phi(t)) \quad \rightarrow \quad x_a(t) = A(t)e^{j\phi(t)} = A(t)\cos(\phi(t)) + jA(t)\sin(\phi(t)),$$

in special case:

$$x(t) = A(t)\cos(2\pi ft) \quad \rightarrow \quad x_a(t) = A(t)e^{j2\pi ft}.$$

therefore it a pure complex-value Fourier harmonic with positive frequency f only is obtained (the negative is absent). Amplitude and phase of analytic signal can be extracted very easily from it:

amplitude demodulation: $\quad A(t) = |x_a(t)|,$

phase demodulation: $\quad \phi(t) = \angle x_a(t).$

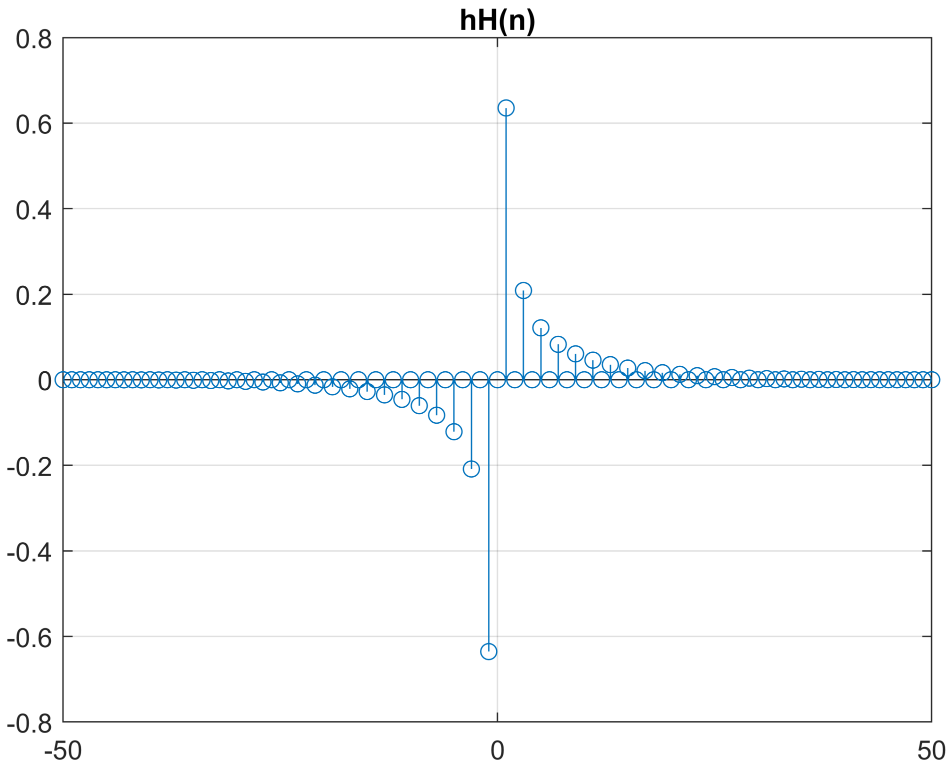
Going further: instantaneous frequency can be found from:

frequency demodulation: $\quad f_{inst} = \dfrac{1}{2\pi}\dfrac{d\phi(t)}{dt}.$

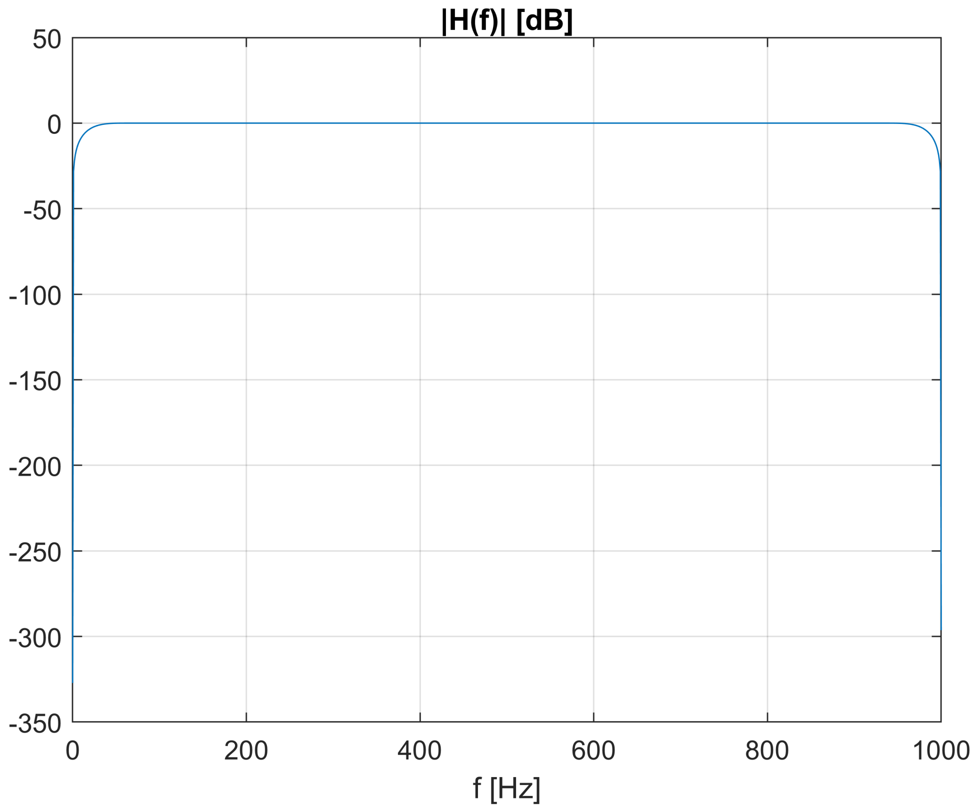
```
clear all; close all;

% Weights h(n) and frequency response H(f) of Hilbert filter (-pi/2 rad phase shifter)

fs = 2000;                                  % sampling frequency
M=50; N=2*M+1; n=-M:M;                      % filter length, indexes of samples
h = (1-cos(pi*n)) ./ (pi*n); h(M+1) = 0;    % filter weights = impulse response
h = h .* kaiser(N,10)';                     % windowing
stem(n,h); title('hH(n)'); grid;            % plot of the Hilbert filter weights
```




```
f = 0 : 1 : fs/2;                                % frequencies of interest
z = exp(-j*2*pi*f/fs);                           % variable z^(-1) of the Z transform
H = polyval(h(N:-1:1),z);                        % value of H(z) polynomial
H = H .* exp(j*2*pi*f/fs*M);                     % correction of time shift by M samples
figure; plot( f, 20*log10( abs(H))); xlabel('f [Hz]'); title('|H(f)| [dB]'); grid;
```

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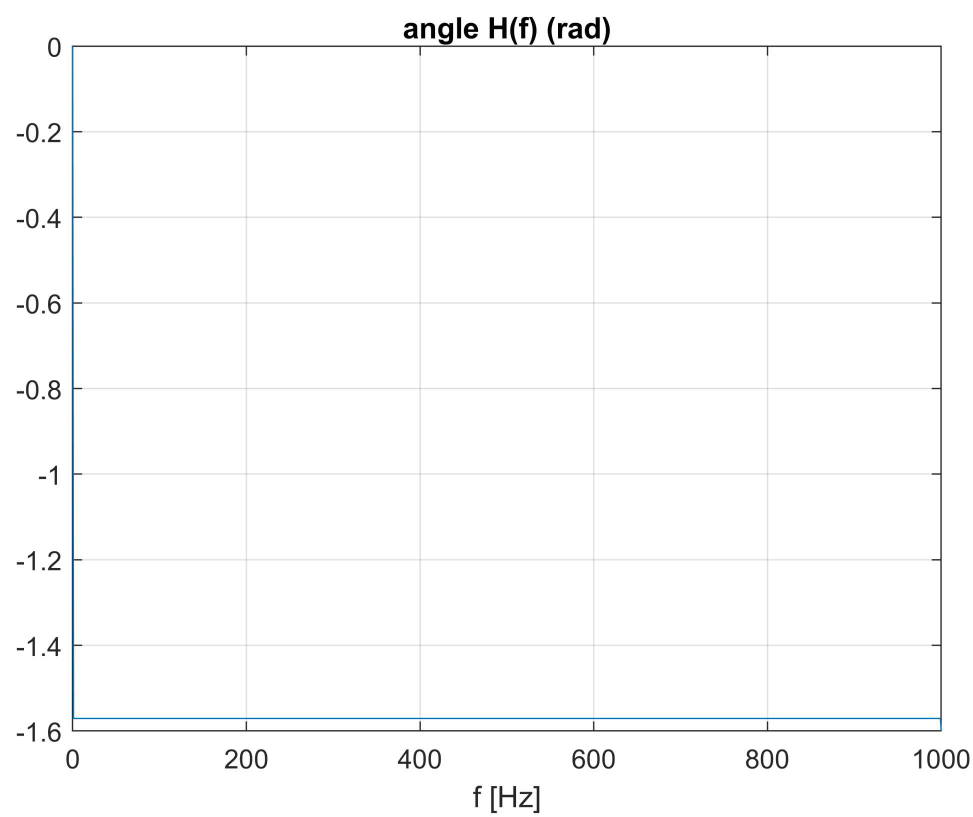
figure; plot( f, unwrap(angle(H)));  xlabel('f [Hz]'); title('angle H(f) (rad)'); grid;
```



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% Example of AM-AM demodulation

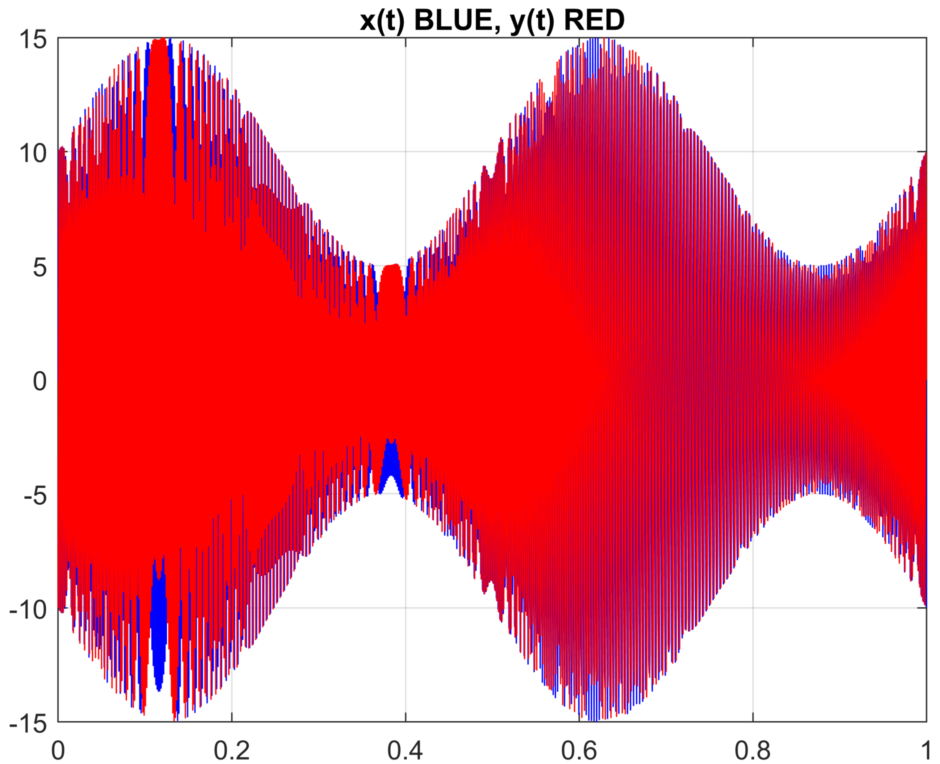
Nx=2000; n=0:Nx-1; dt=1/fs; t=dt*n;                               % number of signal samples
A=10; kA=0.5; fA=2; xA = A*(1 + kA*sin(2*pi*fA*t));               % AM
kF=250; fF=1;       xF = kF*sin(2*pi*fF*t);                       % FM
fc=500;             x = xA .* sin(2*pi*(fc*t + cumsum(xF)*dt));   % modulated carrier

% Analytic signal calculation

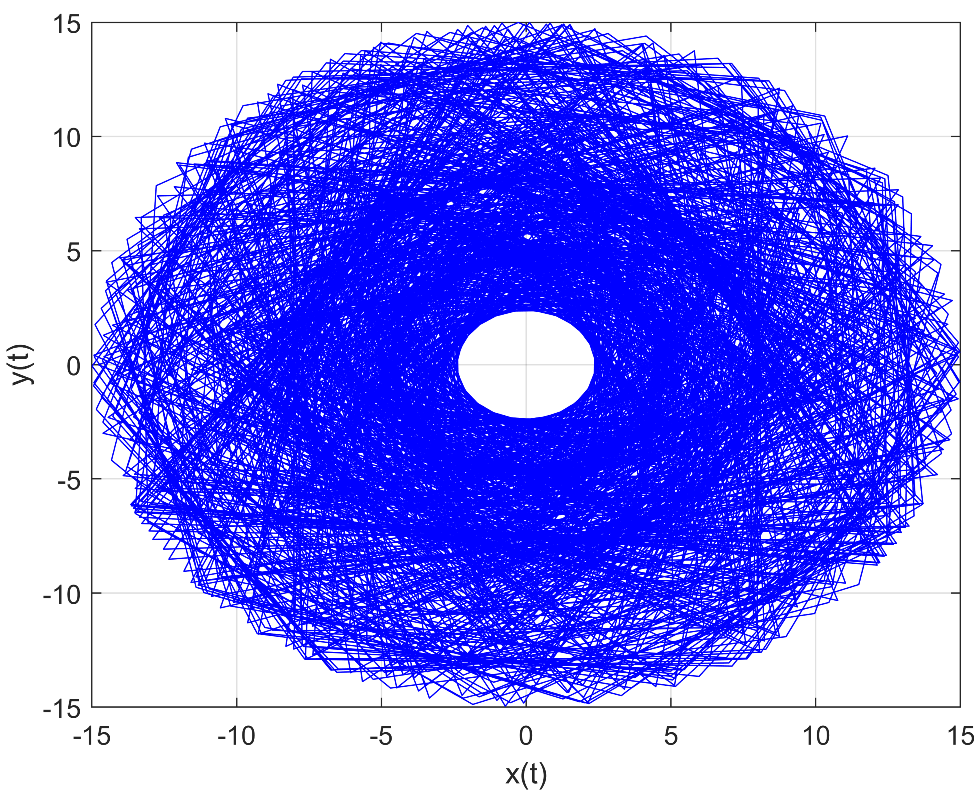
if(1)           % in frequency domain
   X = fft(x);                        % signal FFT, then its modification:
   X(1)=0; X(Nx/2+1)=0;               % # 0 for 0 Hz and fs/2
   X(2:Nx/2)    = -j*X(2:Nx/2);       % # (-j) for positive frequencies
   X(Nx/2+2:Nx) =  j*X(Nx/2+2:Nx);    % # (+j) for negative frequencies
   y = ifft(X);                       % inverse FFT of the modified spectrum
   xa = x + j*y;                      % creation of analytic signal
   xa = hilbert( x );                 % the same: analytic signal calculated by Matlab
else                 % in time domain
   y = filter(h,1,x);                 % Hilbert filtering, i.e. convolution of x(n) with hH(n)
   y = y(2*M+1 : Nx  );               %
   x = x(M+1   : Nx-M);               % removing transient states and synchronization
   t = t(M+1   : Nx-M);               %
   xa = x + j*y;                      % analytic signal
end

xAest = abs( xa );                                           % AM (amplitude) demodulation
ang = unwrap( angle( xa ) );                                 % PM (phase)     demodulation
xFest = (1/(2*pi)) * (ang(3:end)-ang(1:end-2)) / (2*dt);    % FM (frequency) demodulation

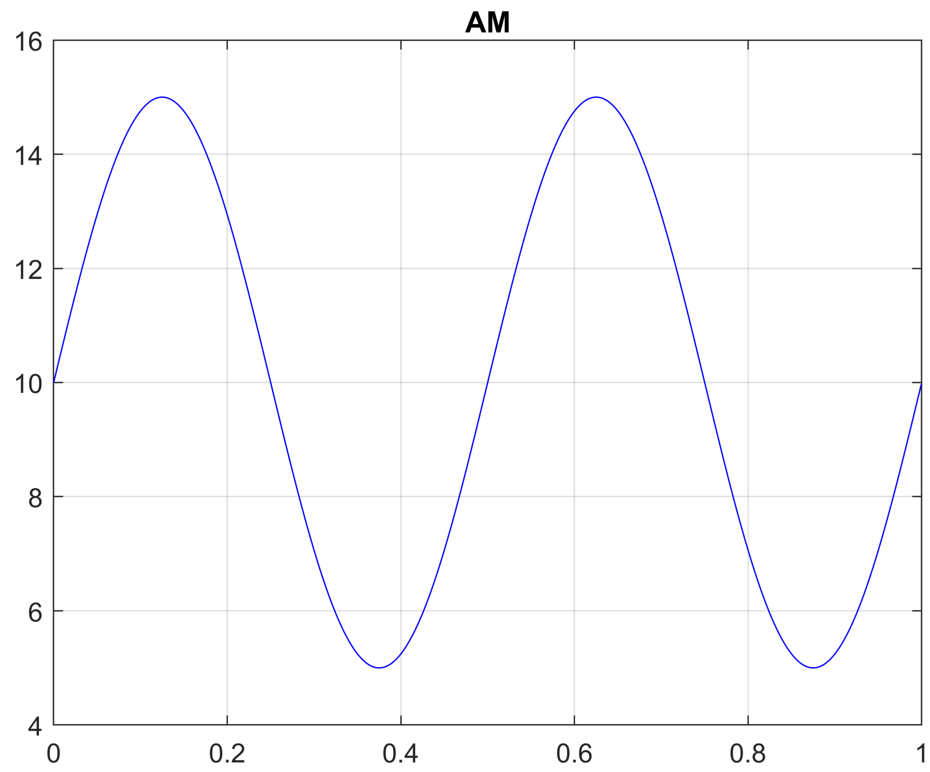
figure; plot(t,x,'b',t,y,'r'); grid; title('x(t) BLUE, y(t) RED');
```




```
figure; plot(x,y,'b'); grid; xlabel('x(t)'); ylabel('y(t)');
```



```
figure; plot(t,xA,'r-',t,xAest,'b-'); title('AM'); grid;
```

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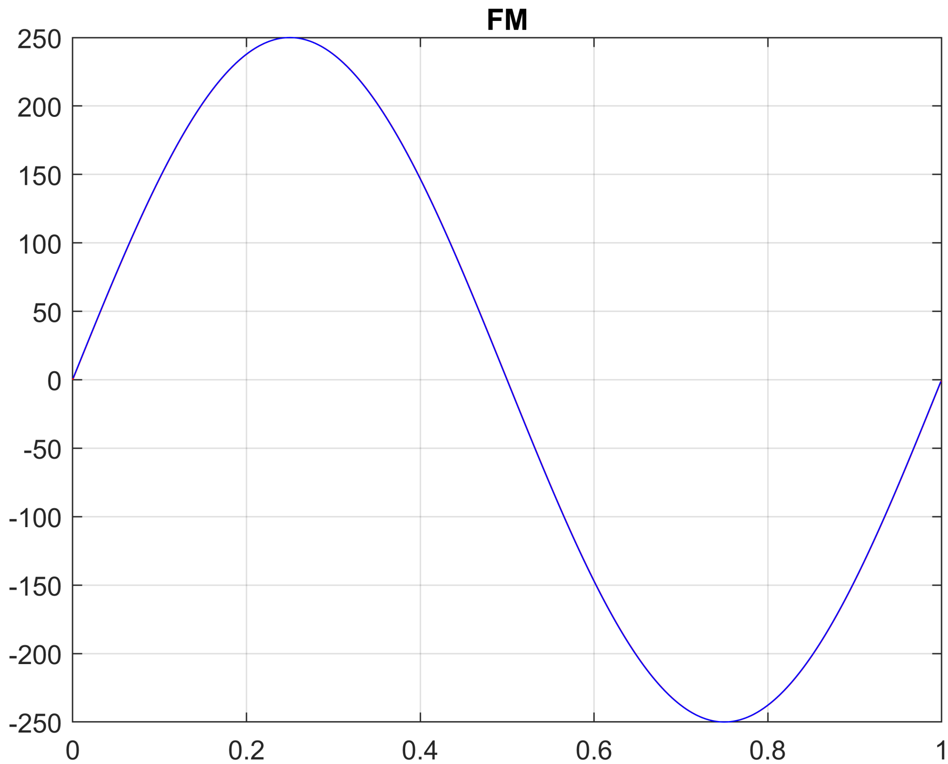
figure; plot(t,xF,'r-',t(2:Nx-1),xFest-fc,'b-'); title('FM'); grid;
```



FIR differentiation filter

Due to continuous Fourier transform features (namely, CFT of the signal derivative is equal to CFT $X(f)$ of the original signal itself multiplied by $j\omega$:

$$F\left(\frac{dx(t)}{dt}\right) = j2\pi f \cdot X(f)$$

frequency response of an analog differentiation filter is equal:

$$H_D(f) = j2\pi f$$

When we calculate inverse DtFT of it, we obtain theoretical impulse response (weights) of the digital FIR differentiation filter equal to:

$$h_D(n) = \frac{\cos(\pi n)}{n}, \ \ h(0) = 0.$$

In order to remove oscillations from the filter frequency response, i.e. DtFT of $h_D(n)$, the weights $h_D(n)$ should be *softly* shaped/tapered on both edges, i.e. multiplied by a appropriately chosen window function:

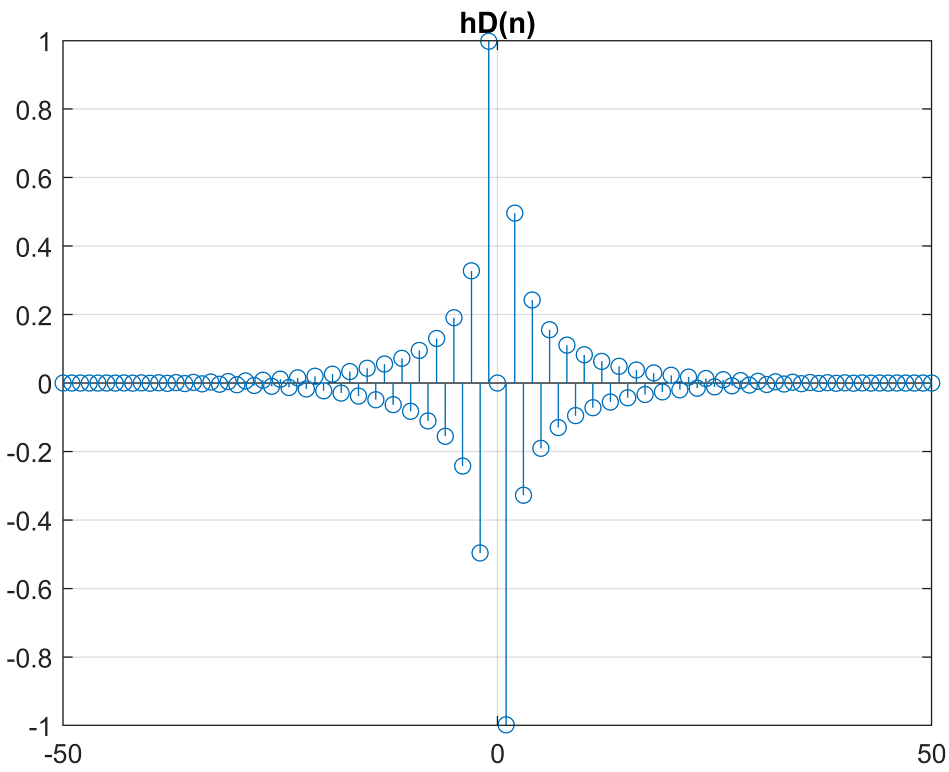
$$h_D^{(w)} = h_D(n) \cdot w(n), \qquad n = -P, ..., -1, 0, 1, ..., P.$$

The simplest differentiation filter weights used in numerical analysis are as follows:

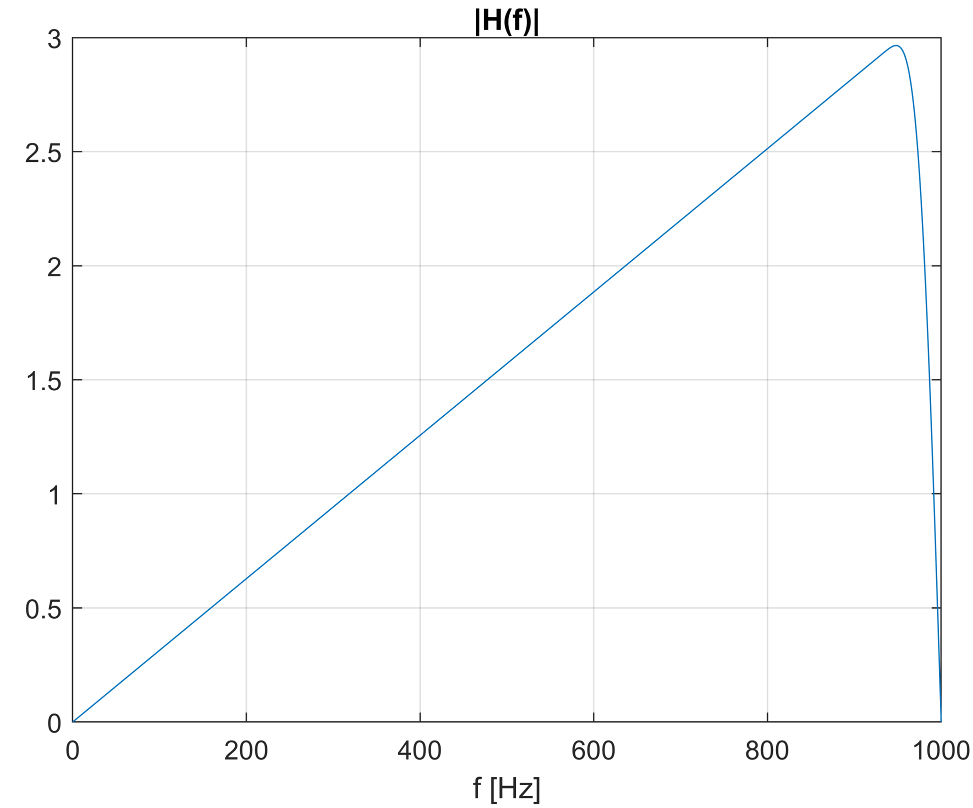
$$h_{D2} = \frac{1}{dt}[-1, 1], \qquad h_{D3} = \frac{1}{2 \cdot dt}[-1, 0, 1], \qquad h_{D5} = \frac{1}{12 \cdot dt}[1, -8, 0, 8, -1].$$

```
clear all;

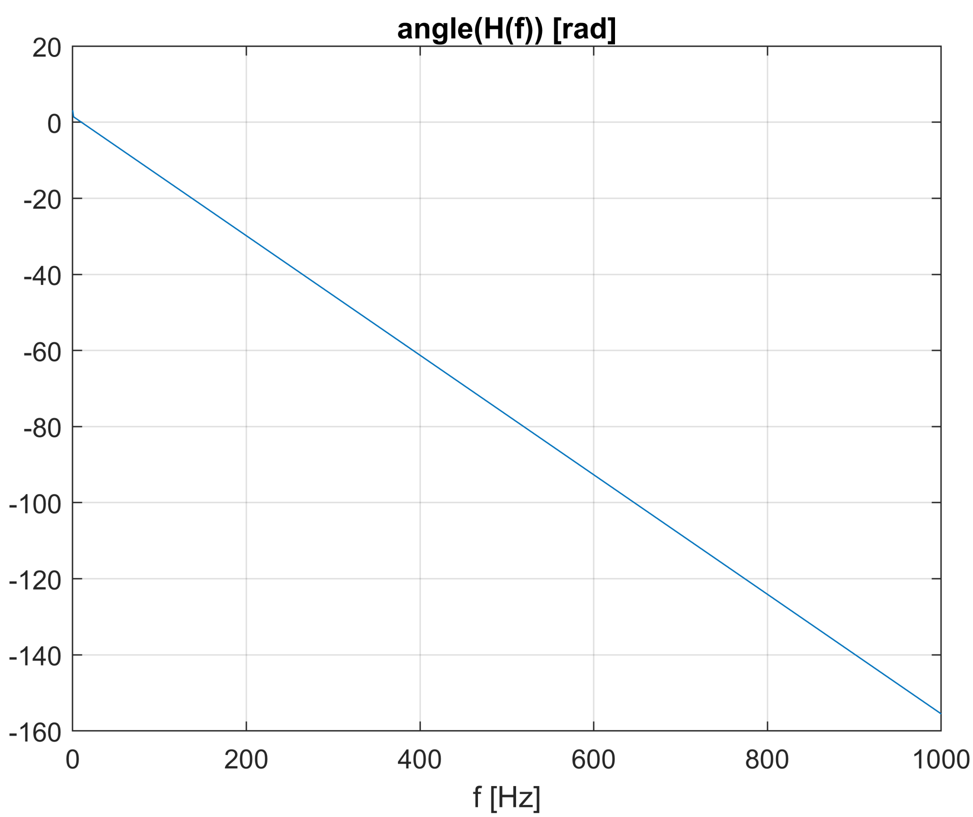
% Differentiation filter weights
M=50; N=2*M+1; n=-M:M; h=cos(pi*n)./n; h(M+1)=0;
h = h .* kaiser(N,10)';
%h = 1/12 * [-1, 8, 0, -8, 1]; M=2; N=2*M+1; n = -M:M; % 1/2*[-1 0 1]
stem(n,h); title('hD(n)'); grid;
```



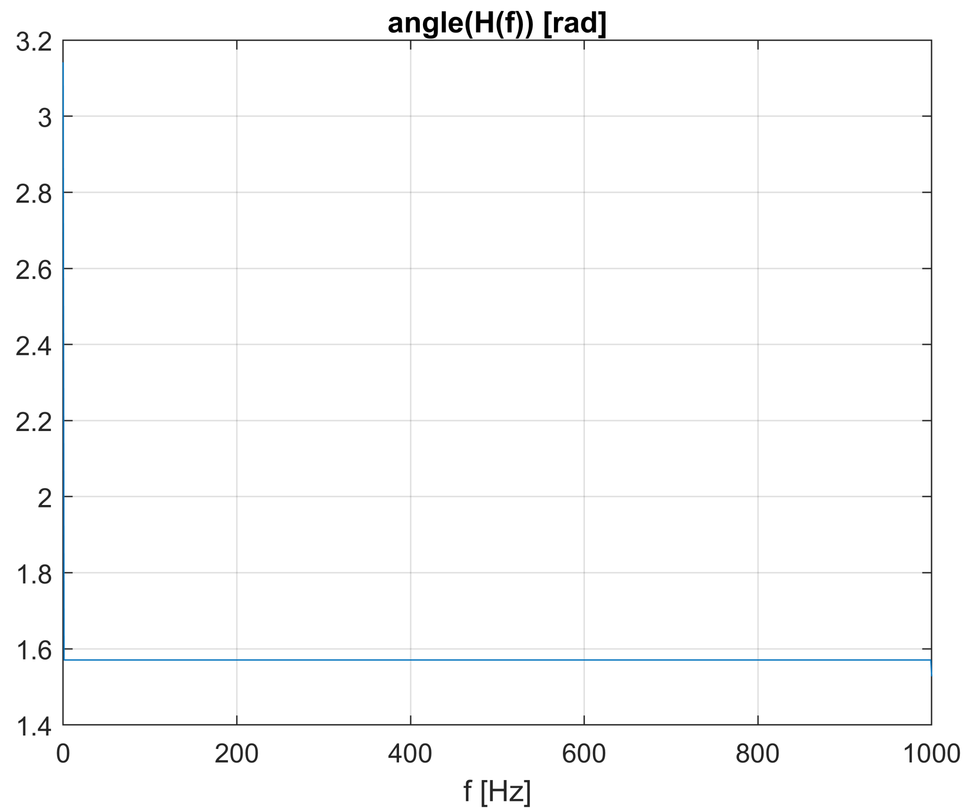

```
% Verification of filter responses: amplitude, phase, impulse, step
fs = 2000; f = 0:1:fs/2;          % sampling frequency [Hz], frequencies of interest
z = exp(-j*2*pi*f/fs);            % Z transform variable z^(-1)
H = polyval(h(end:-1:1),z);       % value of H(z) polynomial
% H = freqz(h,1,f,fs);            % all-in-one Matlab function
figure; plot(f,abs(H)); xlabel('f [Hz]'); title('|H(f)|'); grid;
```



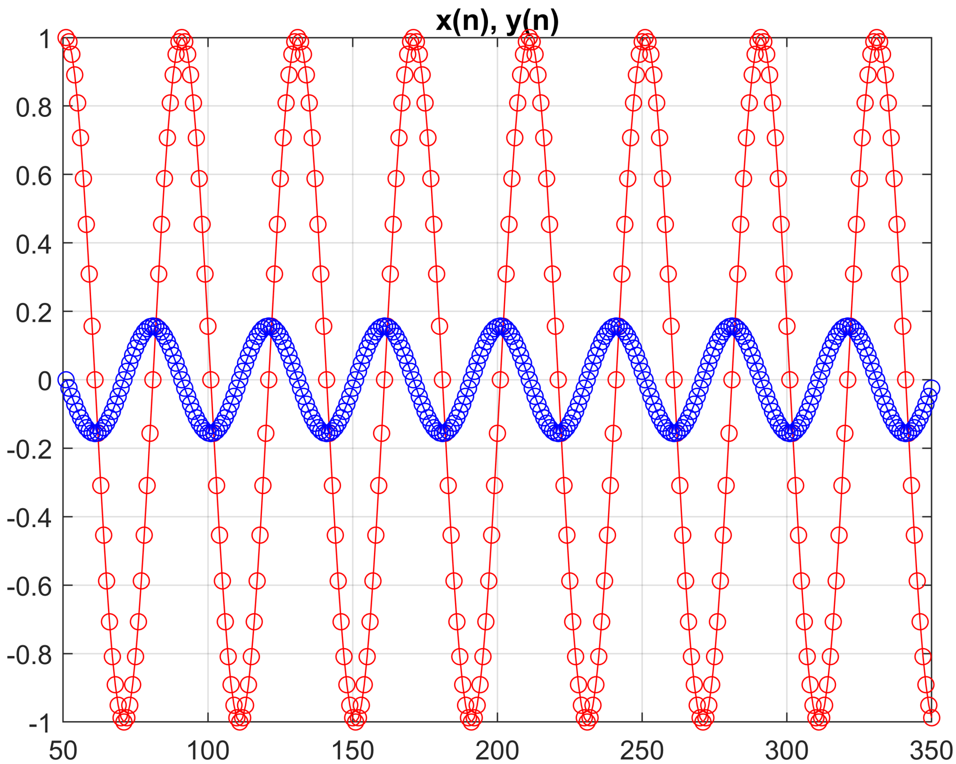

```
figure; plot(f,unwrap(angle(H))); xlabel('f [Hz]'); title('angle(H(f)) [rad]'); grid;
```



```
figure; plot(f,angle(exp(j*2*pi*f/fs*M).*H)); xlabel('f [Hz]'); title('angle(H(f)) [rad]'); gri
```




```
% Filtering - signal differentiation
Nx=400; n=0:Nx-1; dt=1/fs; t=dt*n;
fx=50; x=sin(2*pi*fx*t);
y=filter(h,1,x);
nx=M+1:Nx-M; ny=2*M+1:Nx;
figure; plot(nx,x(nx),'ro-',nx,y(ny),'bo-'); title('x(n), y(n)'); grid;
```


x(n), y(n)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
50
100
150
200
250
300
350