Adaptive filters

Basic theory

Adaptive filter is a computation module having two input signals:

1. $d(n)$ - desired/reference one,
2. $x(n)$ - to be adaptively filtered,

and two output signals:

1. $y(n) = \text{adapt}(x(n))$ - filtered $x(n)$,
2. $e(n) = d(n) - y(n)$ - error signal.

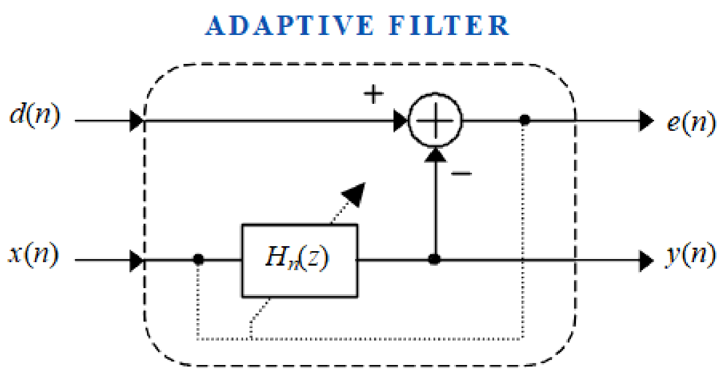


Figure: Block diagram of the adaptive filter

The role of the filter is to make the signal $y(n)$ as much as possible similar to the signal $d(n)$:

$$y(n) \rightarrow d(n)$$

Adaptive changing of filter weights and their frequency response has to ensure minimization of a filter cost function:

1. $J(n) = e^2(n)$ - Normalized Least Mean Squares **(N)LMS**: squared momentum error $e(n)$ of adjustment the signal $y(n)$ to the signal $d(n)$: ,
2. $J(n) = \sum_{k=0}^{+\infty} \lambda^k \cdot e^2(n-k), \;\; \lambda \leq 1$ - Weighted Recursive Least Squares **RLS/WRLS**: weighted sum of the present and previous values of the squared error.

For the FIR LMS filter such values of $h_k(n)$ are searched via adaptation which for every moment n ensure minimization of the following cost function:

$$J(n) = (d(n) - y(n))^2 = \left(d(n) - \sum_{k=0}^{M} h_k(n) x(n-k) \right)^2$$

In order to find them, derivative of the function $J(n)$ in respect to each filter coefficient is calculated:

$$\frac{dJ(n)}{dh_k(n)} = 2 \cdot e(n) \cdot \frac{d}{dh_k(n)}[d(n) - h_0(n) \cdot x(n-0) - \ldots - h_M(n) \cdot x(n-M)],$$

$$\frac{dJ(n)}{dh_k(n)} = -2 * e(n) \cdot x(n-k),$$

and next, during adaptation, filter coefficient values are changed in opposite direction to the function growth, i.e. to its gradient (note minus sign below):

$$h_k(n+1) = h_k(n) - \frac{dJ(n)}{dh_k},$$

$$h_k(n+1) = h_k(n) + 2 \cdot \mu \cdot e(n) \cdot x(n-k)$$

FOR ABIMITIOUS STUDENTS ONLY. Values of optimal filter weights, valid for stationary case, were derived by Wiener:

$$\mathbf{h}_{opt} = \mathbf{R}_{xx}^{-1} \cdot \mathbf{r}_{dx}$$

where:

- $\mathbf{R}_{xx}$ - denotes auto-correlation matrix of the signal $x(n)$,
- $\mathbf{r}_{dx}$ - s a vector of cross-correlation between the signals $d(n)$ and $x(n)$.

Maximum value of adaptation speed coefficient μ is constrained by the following top-bounding equation:

$$0 < \mu < \frac{2}{\lambda_{max}},$$

where λ_{max} is the biggest eigen-value of the matrix $\mathbf{R}_{xx}$ - for bigger values of μ the LMS adaptive filter becomes unstable. The convergence speed is the highest when

$$\lambda_{max}/\lambda_{max} \approx 1.$$

i.e. when $x(n)$ is a pure noise. In RLS adaptive filter the optimal Wiener solution is iteratively solved, i.e. the matrix $\mathbf{R}_{xx}^{-1}$ is initially estimated and adaptively updated.

Examplary usage

In this laboratory we will concentrate only on (N)LMS adaptive filters and consider only two the most typical adaptive filter applications:

- **adaptive correlation / interference canceling (ACC/AIC)** between two signals;
- **linear-prediction-based adaptive signal / telecommunication line enhancement (ASE/ALE)**.

ACC/AIC. The filter is given a disturbed signal $d(n) = s(n) + \varepsilon_1(n)$ and non-perfect, reference pattern of the disturbance $x(n) = \varepsilon_2(n)$. It tries to fit the possessed disturbance pattern $\varepsilon_2(n)$ to the disturbance $\varepsilon_1(n)$, present in $d(n)$, by adaptive filtering. and subtract it:

$$e(n) = d(n) - y(n) = s(n) + \varepsilon_1(n) - \text{filter}(\varepsilon_2 n)) = s(n) + \left(\varepsilon_1(n) - \widehat{\varepsilon}_1(n)\right)$$

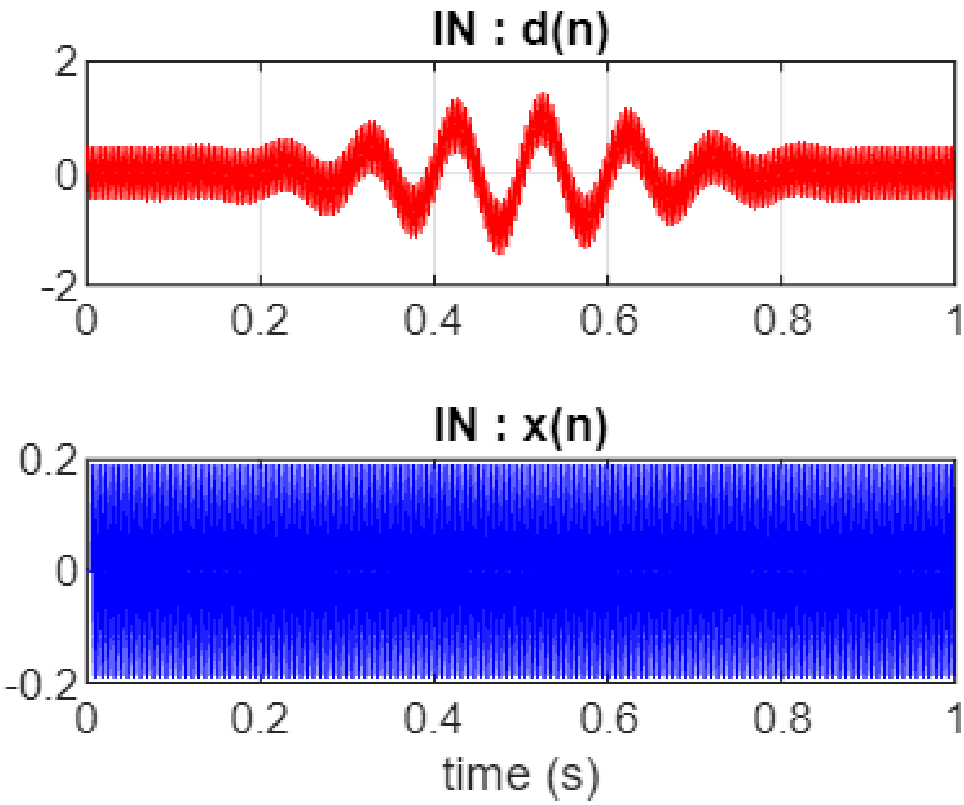
For example, speech of airplane pilot is embedded in motor whir but non-ideal pattern of the whir, captured by the second microphone is available (amplified/attenuated and delayed/stepped-up). The wir reference is adaptively fitted to the wir disturbing a pilot and subtracted from the currupted pilot speech.

ASE/ALE. Signal $x(n) = d(n-1)$, i.e. it is a delayed version of $d(n)$. Therefore, the adaptive filter, working as linear predictor, tries to predict next sample of the signal $d(n)$. It is not possible for noisy components of $d(n)$. And $y(n)$ becomes a denoised version of $d(n)$.

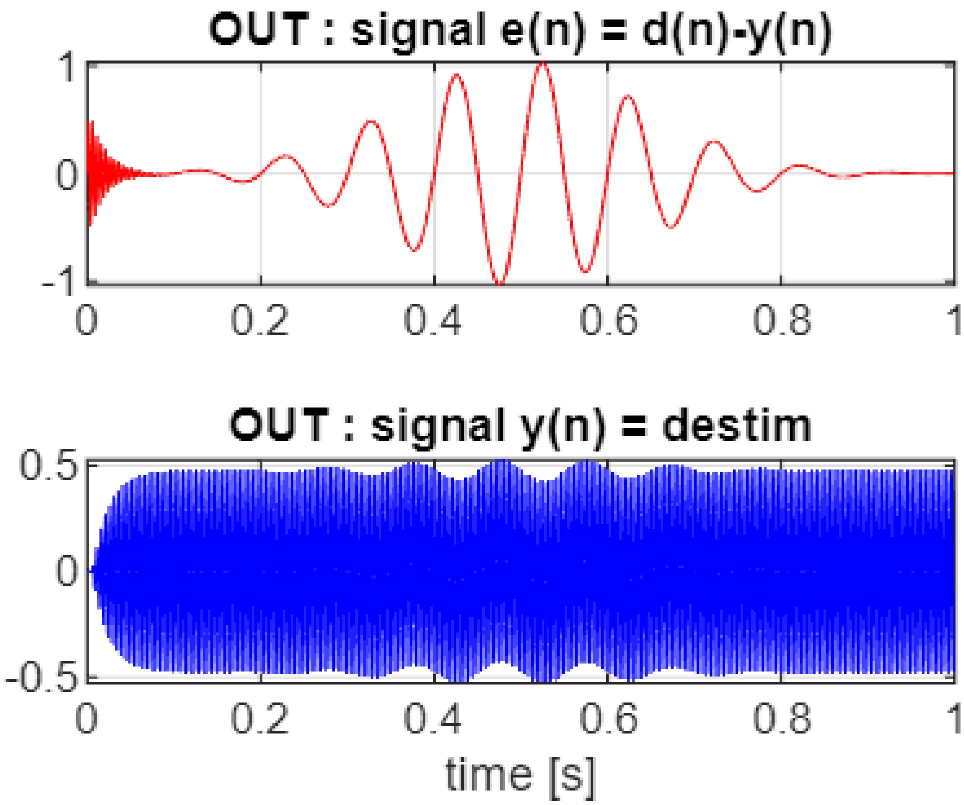
Analyze the Matlab program presented below. It demonstrates adaptive FIR LMS filter usage in two main scenarios, briefly characterized above.

Run the program, observe figures. First set `test=1`, then `test=2`. Try to find better values of `N` (filter length) and `mi` (adaptation speed coefficient). "Better" means: giving faster adaptation and stronger interference rejection. Use LMS and normalized LMS (NLMS) adaptive filter. Record your own speech. Add to it: 1) sinusoid (case 1), and 2) noise (case 2). Check how the filter is removing these disturbances. Try to select optimal values of filter parameters.

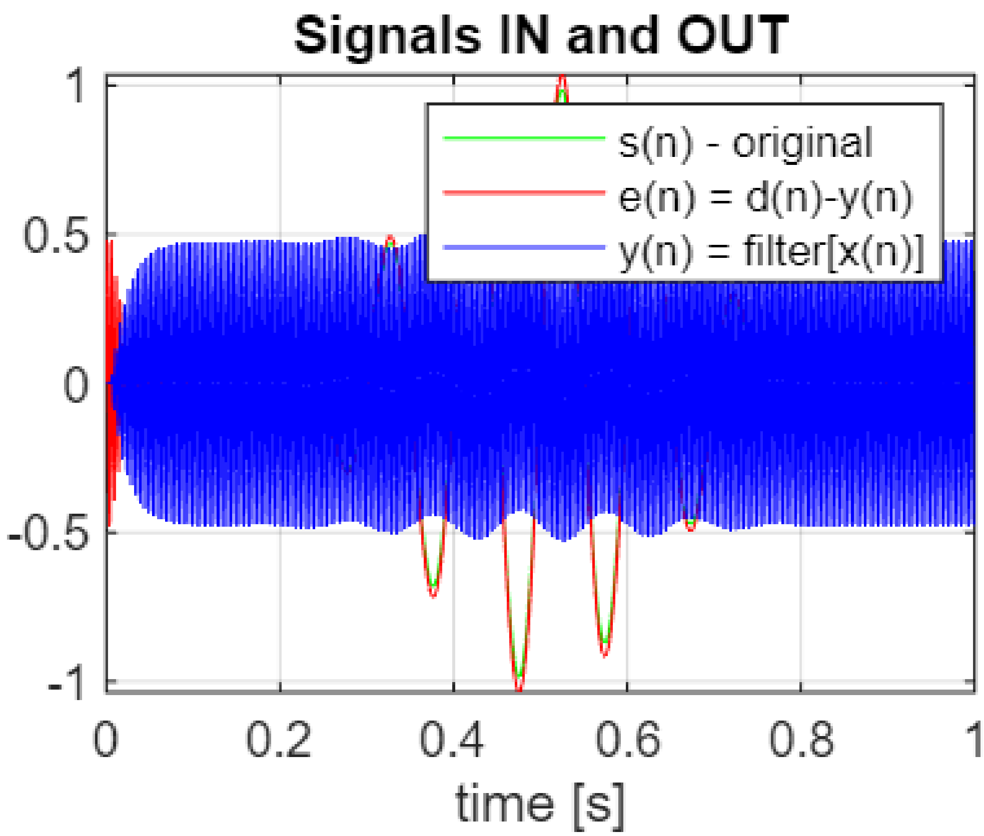
```
clear all; close all;

test=1;      % two adaptive scenarios: 1=ACC/AIC, 2=ASE/ALE

% ##### Input signals and values of adaptive filter parameters #####

fpr = 1000;                                   % sampling frequency
Nx = fpr;                                     % number of samples, 1 secund
dt = 1/fpr; t = 0 : dt :(Nx-1)*dt;           % time
f = 0 : fpr/1000 : fpr/2;                     % frequency

if(test==1) % Scenario #1 ACC/AIC - adaptive correlation/interference cancelling
   M = 50;                                    % number of filter weights
   mi = 0.1;                                  % adaptation speed ( 0<mi<1)
   s = sin(2*pi*10*t).*exp(-25*(t-0.5).^2);  % signal: sine*gaussoid, ECG or speech
   z = sin(2*pi*200*t);                       % interference: harmonic, engine wirr
   d = s + 0.5*z;                             % signal + scaled interference
   x = 0.2*[zeros(1,5) z(1:end-5)];          % delayed and scaled interference "copy"
else  % Scenario #2 ALE/ASE - adaptive line/signal enhancement (denoising)
   M = 10;                                    % number of filter weights
   mi = 0.0025;                               % adaptation speed ( 0<mi<1)
   s = sin(2*pi*10*t);                        % signal: sine, ECG or speech
   z = 0.3*rand(1,Nx);                        % disturbing noise
   d = s + z;                                 % signal disturbed by noise
   x = [ 0, d(1:end-1)];                      % delayed "copy" of the disturbed signal
end
        figure;
        subplot(211); plot(t,d,'r'); grid; title('IN : d(n)');
        subplot(212); plot(t,x,'b'); grid; title('IN : x(n)'); xlabel('time (s)');
```




```
% ###### Adaptive filtering ######

bx=zeros(M,1);            % initialization: buffer fo input signal x(n) samples
h  = zeros(M,1);          % initialization: filter weights
y  = zeros(1,Nx);         % empty output, signal y(n)
e  = zeros(1,Nx);         % empty output, signal e(n)
for n = 1 : length(x)               % main loop
  % n                               % loop index
    bx = [ x(n); bx(1:M-1) ];       % putting new sample of x(n) into the buffer
    y(n) = h' * bx;                 % filtering x(n), i.e. estimation of d(n)
    e(n) = d(n) - y(n);             % estimation error
    h = h + ( 2*mi * e(n) * bx );             % LMS  - filter weights adaptation
  % h = h + mi/(0.0001+bx'*bx) * e(n) * bx;  % NLMS - filter weights adaptation
        if(0) % Observation of filter weights change and filter amplitude response change
            subplot(211); stem(h); xlabel('n'); title('h(n)'); grid;
            subplot(212); plot(f,abs(freqz(h,1,f,fpr))); xlabel('f (Hz)');
            title('|H(f)|'); grid; pause
        end
end

% ###### Figures ###### - output signals from adaptive filter

        figure;
        subplot(211); plot(t,e,'r'); grid; title('OUT : signal e(n) = d(n)-y(n)');
        subplot(212); plot(t,y,'b'); grid; title('OUT : signal y(n) = destim');
        xlabel('time [s]');
```




```
figure; subplot(111); plot(t,s,'g',t,e,'r',t,y,'b');
grid; xlabel('time [s]'); title('Signals IN and OUT');
legend('s(n) - original','e(n) = d(n)-y(n)','y(n) = filter[x(n)]');
```


Signals IN and OUT
s(n) - original
e(n) = d(n)-y(n)
y(n) = filter[x(n)]
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
time [s]