CS 70 HW #9 -

Sunday

worked alone

1) Monty Hall Revenge

a) switching in this scenario will help your odds of getting the car.

probability of picking car with 1st pick = $1/n$

$\downarrow$

The host will always remove a non car option that wasn't picked by player

$\downarrow$

New probability of picking car with switched pick is the chance of not picking the car in the 1st pick times the chance of picking it at this point

$$\hookrightarrow \underbrace{\frac{n-1}{n}}_{\text{chance not pick car w/ 1st pick}} \times \underbrace{\frac{1}{n-1-1}}_{\text{chance picking car after host eliminates bad option}} = \frac{1}{n} \times \frac{n-1}{n-2}$$

$$\underbrace{\frac{1}{n} \times \frac{n-1}{n-2}}_{\text{switch}} \geq \underbrace{\frac{1}{n}}_{\text{nonswitch}}$$

↳ This shows that switching will never hurt your odds

b) Switching will never hurt your odds in this scenario

(Carrying from part a)

probability of picking car with 1st pick = $1/n$

The host will always remove a non car option that wasn't picked by player, this time only $n-2$ doors

New probability of picking car with switching is the chance of not picking the car in the 1st pick times the chance of picking it at this point

$$\hookrightarrow \underbrace{\frac{n-1}{n}}_{\text{chance not pick car w/ 1st pick}} \times \underbrace{\frac{1}{n-1-(n-2)}}_{\text{chance picking car after host eliminates bad option}} = \frac{1}{n} \times \frac{n-1}{1}$$

$$\downarrow$$

$$\underbrace{\frac{1}{n} \times (n-1)}_{\text{switch}} \geq \underbrace{\frac{1}{n}}_{\text{non switch}}$$

↳ This shows that switching will never hurt your odds

c) Again based from a) and b)

probability of picking car with 1st picks $= \dfrac{k}{n}$

↓

The host will always remove a non car option that wasn't picked by player, this time but j doors ($j < n-k$)

↓

New probability of picking car with switch pick is the chance of not picking the car in the 1st pick times the chance of picking it at this point

$$\hookrightarrow \underbrace{\frac{n-k}{n}}_{\text{chance not pick car w/ 1st pick}} \times \underbrace{\frac{k}{n-1-j}}_{\text{chance picking car after host eliminates bad options}}$$

$$\downarrow$$

we want to max k and min j in order for the highest ratio.

$\hookrightarrow$ $k = n-2$

$j < \min(n-k)$

2) Man speaks Truth

a) $H = \frac{1}{3}$ Truth $= \frac{3}{4}$

In other words:

$P[A]$ Man speaks Truth $= \frac{3}{4}$

$P[B]$: Land on heads $= \frac{1}{3}$

$P[\bar{A}] = \frac{1}{4}$

$P[\bar{B}] = \frac{2}{3}$

$\hookrightarrow P[A|B] = \dfrac{P[B|A]\, P(A)}{P(B)}$

$\downarrow$ Bayes theorem

i)

$$\frac{\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{3}\right)\left(\frac{3}{4}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{4}\right)}$$

$$\hookrightarrow \frac{1/4}{1/4 + 2/12} \rightarrow \frac{1}{4} \times \frac{12}{5} = \frac{3}{5}$$

ii)

iii) the probability did change, it went up. Because the man lied is 75%, this increases our confidence that the man isn't lying given that he said he is telling the truth, thus the increase in probability.

b)

Dice $= \frac{1}{6}$ Truth $= \frac{3}{4}$

In other words:

$P[A]$ Man speaks Truth $= \frac{3}{4}$

$P[B]$: Land on 6 $= \frac{1}{6}$

$P[\bar{A}] = \frac{1}{4}$

$P[\bar{B}] = \frac{5}{6}$

$$\hookrightarrow P[A|B] = \frac{P[B|A]\,P[A]}{P[B]}$$

$\downarrow$ Bayes theorem

i)

$$\frac{\left(\frac{1}{6}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{4}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{4}\right)}$$

$$\hookrightarrow \frac{1/8}{1/8 + 5/24} \rightarrow \frac{1/8}{1/3} = 3/8$$

ii) the probability did change, it went up. Because the

iii) man lied is 75%, this increases our confidence that the man isn't lieing given that he said he is telling the truth, thus the increase in probability.

3) Mario's Coins

a) No, it is dependent. Since we are picking coins without replacement, and that all the coins have a different probability of Heads this tells me that the 2nd coin we pick will be dependent on the 1st coin we pick

b) No it isn't independent. Y_1 happens to give info on which coin is picked which directly influences Y_2.

c) $\frac{1}{4}$ $\qquad$ $\frac{1}{2}$ $\qquad$ $\frac{3}{4}$

1st $\qquad$ 2nd $\qquad$ 3rd

$\rightarrow$ $P[\text{3rd heads} \mid \text{1st 2 heads}]$

$$= \frac{P[\text{All 3 heads}]}{P[\text{1st 2 heads}]}$$

$$= \frac{\frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)}{\sum_{\text{coins}} P[\text{1st 2 heads} \mid \text{coin}]\,P[\text{coin}]}$$

$$= \frac{\frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)}{\frac{1}{6}\left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)}$$

$\rightarrow$ $\frac{3}{32} \times \frac{48}{11} = 9/22$

4) Symmetric Marbles

a) They are not independent. Since A_1 and A_2 are drawn without replacement, this means that A_2 is dependent on the outcome of A_1, whether its red or blue. Thus, they aren't independent.

b) probability that Rachel wins = Rowlan wins ; $p = p$

↳ $1 - 2p$ = total

↳ $\binom{8}{4}$ ways to draw 4 marbles and $\binom{4}{2}\binom{4}{2}$ for 2 red 2 blue

↳ $\dfrac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \dfrac{18}{35} \rightarrow 1 - 2p = \dfrac{18}{35}$

↳ $p = 17/70$

c) $\left(\frac{1}{2}\right)\left(\frac{3}{7}\right)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) = \dfrac{1}{60}$

1st 2nd 3rd 4th

chance of RRRR

d) By symmetry $P[3rd] = P[1st]$

↳ since 8 red marbles out of 12 marbles

total ↳ $2/3$

e) we know 15 red marbles among 16 drawn

↳ $\frac{15}{16}$ by symmetry

f) Only way to lose is if all blue

↳ $\frac{4}{12} \times \frac{3}{11} \times \frac{8}{10} \times \frac{2}{9} = \frac{8}{495}$

↳ chance that 3rd marble red is $\frac{8}{12}$

↳ $\frac{8}{495} \times \frac{3}{2} = \frac{4}{165}$ chance that she loses since 3rd marble red

↳ complement $= \frac{161}{165}$ chance that she wins since 3rd marble red

5) Cookie Jar

X = # cookies in 1 jar

Y = # cookies in other jar

→ After every step, (X, Y) will either become $(X-1, Y)$ or $(X, Y-1)$ with equal chances of either happening

↓

symmetric random walk in the plane starting at $(X, Y) = (n, n)$

↳ suppose walk ends at $(k, 0)$, this tells me the previous position was $(k, 1)$.

To go from $(n, n) \to (k, 1)$ needs

$(n-k) + (n-1) = 2n-k+1$ steps

of probability $\frac{1}{2}$ each.

↳ Basal Probability: $\left(\frac{1}{2}\right)^{2n-k+1}$

↓

$$\binom{2n-k}{n}$$

* Because of symmetry, we know odds of box 1 or box 2 being empty in the game.

↳ Adding the $\frac{1}{2}$ chance of final draw to $(k,0)$,

we get the following

$$P[(k,0)] = \binom{2n-k}{n} \frac{1}{2^{k-2n}}$$

where $k \in \{ (0, \ldots n) \}$

6) Testing Model Planes

a) Amelia performs n trials each w/ independent probability of success p

$$X_1 \sim \text{Binom}(n,p)$$

$$\Downarrow$$

$$P[X=k] = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } 0 \le k \le n$$

b) Each model plane independently has probability of p^2 of surviving both days

$$X_2 \sim \text{Binom}(n,p^2)$$

$$\Downarrow$$

$$P[X=k] = \binom{n}{k} p^{2k} (1-p^2)^{n-k} \text{ for } 0 \le k \le n$$

c) By extending previous parts, we see plane w/ probability of p^t of surviving t days

$$X_t \sim \text{Binom}(n,p^t)$$

$$\Downarrow$$

$$P[X=k] = \binom{n}{k} p^{tk} (1-p^t)^{n-k} \text{ for } 0 \le k \le n$$

d) We consider the complement

↳ probability that no model remain after t days

By previous parts, $P[X_t = 0] = (1-p^t)^n$

probability at least $1-(1-p^t)^n$

e) NO, not independent

↳ $P[A_1 \cap B_1]$ chance that first 3 models crash $= (1-p)^3$. However, since $P[A_1]P[B_1] = (1-p)^4$, we can say that there was dependence involved

f) $P[A_1 \cap B_1]$ chance that 1st, 2nd model crash. $(1-p)^2 p^{n-2}$. $P[A_1]$ chance that 1st crashes and exactly 1 $n-1$ model more crashes, so $P[A_2] = (1-p)\binom{n-1}{1}(1-p)p^{n-2}$

↳ from this, $n = \frac{1}{1-p} + 1$

a) No, not independent

↳ If $X_1 = 0$ and $X_2 = 1$, then the situation wouldn't be possible because there can't be 1 flower at day 1 and then there be two flowers at day 2.. This tells us that the 2 must not be independent