CS HW #1

I worked alone throughout the weeks.

$u = e^{-t} \quad v = -\cos(t)$

1) a) $\int_0^{\pi} \sin(t) e^{-t} \, dt \quad ; \quad u' = -e^{-t} \quad v' = \sin(t)$

↳ $\int_0^{\pi} u v' \, dt = uv - \int_0^{\pi} v u' \, dt$

$= \left[-e^{-t} \cos(t) \right]_0^{\pi} - \underbrace{\int_0^{\pi} e^{-t} \cos(t) \, dt}$

$u = e^{-t} \quad v = \sin(t)$

$u' = -e^{-t} \quad v' = \cos(t)$

↳ $- \left[e^{-t} \sin(t) \right]_0^{\pi} - \int_0^{\pi} e^{-t} \sin(t) \, dt$

↳ $2 \int_0^{\pi} \sin(t) e^{-t} \, dt = \underbrace{\left[-e^{-t} \cos t \right]_0^{\pi} + \left[-e^{-t} \sin t \right]_0^{\pi}}$

$\left(\left[-e^{-\pi} \cos(\pi) \right] - \left[-e^{-(0)} \cos(0) \right] \right) + \left(\left[-e^{-\pi} \sin(\pi) \right] - \left[-e^{-(0)} \sin(0) \right] \right)$

↳ $\frac{+1}{e^{\pi}} + 1 + 0 + 0 \to \frac{1}{e^{\pi}} + 1$

↳ $2 \int_0^{\pi} \sin(t) e^{-t} \, dt = \frac{1}{e^{\pi}} + 1$

↳ $\int_0^{\pi} \sin(t) e^{-t} \, dt = \frac{1}{2e^{\pi}} + \frac{1}{2}$

b) $f(x) = \int_0^{x^2} t\cos(\sqrt{t})\,dt$

$u = \sqrt{t} \rightarrow du = \frac{1}{2\sqrt{t}}\,dt \rightarrow du(2\sqrt{t}) = dt$

$\hookrightarrow u^3 = t^{3/2}$

$\hookrightarrow f(x) = \int_0^{x^2} t\cos(u)\cdot 2\sqrt{t}\,du$

$= \int_0^{x^2} 2t^{3/2}\cos(u)\,du$

$= 2\int_0^{x^2} u^3\cos(u)\,du$

IBP $\hookrightarrow$ $u = u^3 \quad v = +\sin(u)$

$u' = 3u^2 \quad v' = \cos(u)$

$\hookrightarrow 2\int_0^{x^2} u^3\cos(u)\,du = 2\left(u^3\sin(u) - 3\underbrace{\int_0^{x^2} u^2\sin(u)\,du}\right)$

IBP

$\hookrightarrow u = u^2 \quad v = -\cos(u)$

$u' = 2u \quad v' = \sin(u)$

$\hookrightarrow -u^2\cos(u) + 2\underbrace{\int_0^{x^2} u\cos(u)\,du}$

IBP

$\hookrightarrow u = u \quad v = \sin u$

$u' = 1 \quad v' = \cos u$

$\rightarrow u\sin u - \int_0^{x^2}\sin(u)\,du$

$\hookrightarrow u\sin(u) + \cos(u)$

Add all parts together

$$\left[2u^3\sin(u) + 6u^2\cos(u) - 12u\sin(u) - 12\cos(u)\right]_0^{x^2}$$

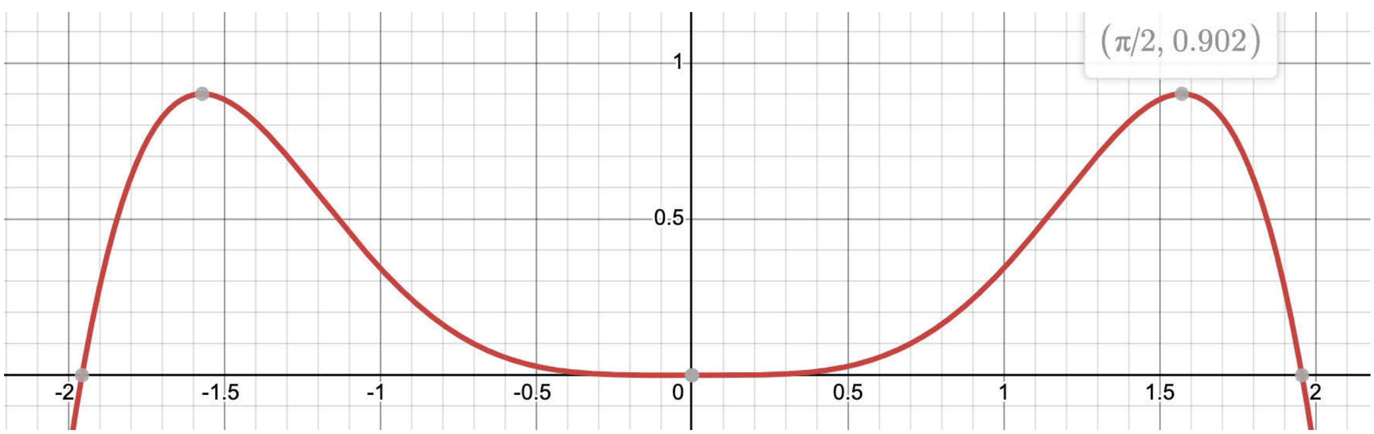
revert u to t or ($u^3 = t^{3/2}$) ($u = \sqrt{t}$)

$$\left[2t^{3/2}\sin(\sqrt{t}) + 6t\cos\sqrt{t} - 12\sqrt{t}\sin(\sqrt{t}) - 12\cos(\sqrt{t})\right]_0^{x^2}$$

simplify

$$\left(2x^3\sin(x) + 6x^2\cos(x) - 12x\sin(x) - 12\cos(x) + 12\right)$$

Graph function $x \in (-2, 2)$

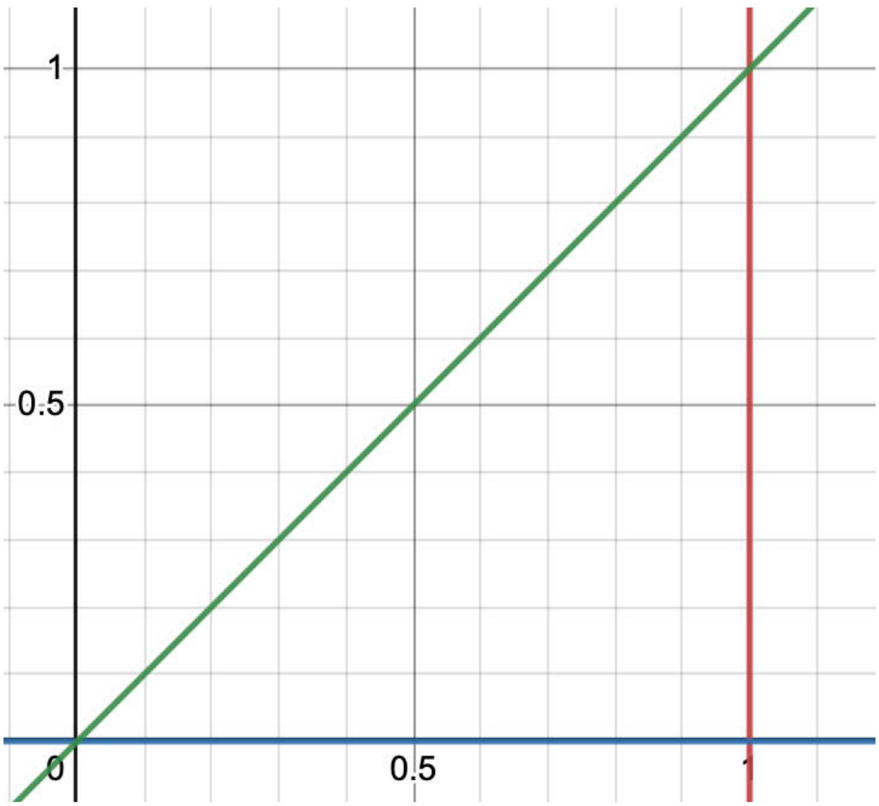


Local maxima: $(-\pi/2, 0.902)$ and $(\pi/2, 0.902)$

Local minima: $(0, 0)$

c) $\iint_R 2x + y \, dA$; R bound by $x = 1$, $y = 0$, $y = x$

Graph of bounds



$0 \leq x \leq y$

$0 \leq y \leq 1$

$$\int_0^1 \int_0^y 2x + y \, dx \, dy$$

↳ $\int_0^y 2x + y \, dx \to 2(y(y)) \to 2y^2$

↳ $\int_0^1 2y^2 \, dy \to \left[\frac{2}{3}y^3\right]_0^1$

↳ $\left(\frac{2}{3}(1)^3\right) - \left(\frac{2}{3}(0)^3\right) = \frac{2}{3}$

2) a) $(\forall n \in \mathbb{N})$ if n is odd, then n^2+4n is odd

proof by contraposition

→ if n^2+4n is even then n is even

$n = 2k$ for some integer k

↳ $(2k)^2 + 4(2k) = 4k^2 + 8k = 4\underbrace{(k^2+2k)}$

We know an even number (4) times an even or odd number will always be even.

↳ By contraposition, this means that if n is odd, then n^2+4n is odd also.

↳ True by contraposition

b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ $b \leq 4$

False

counterexample $b = 11$ $a = 4$

↳ a condition fulfilled but b condition fails

c) $(\forall r \in \mathbb{R})$ if r^2 is irrational then r is irrational

Proof by contraposition

→ if r is rational, then r^2 is rational

↳ $r = \frac{p}{q}$ by knowledge of rational numbers
p and q are integers and $q \neq 0$

↳ $r^2 = \frac{p^2}{q^2}$; since $q \neq 0$ → $q^2 \neq 0$

↳ r^2 can be written as a ratio of two integers
in this case p^2 and q^2 when $q^2 \neq 0$

↳ True by contraposition

d) $(\forall n \in \mathbb{Z}^+)\ 5n^3 > n!$

False

counter example $n = 50$

$\hookrightarrow\ 5(50)^3 = 625{,}000$

$(50)! = 3.04 \times 10^{64}$

$\hookrightarrow\ 3.04 \times 10^{64} >>> 625{,}000$

3) theorem: product of a non-zero rational number and an irrational number is irrational

↳ $a \cdot b = c$; where
a is rational and $\neq 0$
b is irrational
c is irrational
and $c = a \cdot b$

Proof by contradiction

$c = \frac{x}{y}$ where both x and y are integers and $y \neq 0$

↓

$a \cdot b = \frac{x}{y}$; b is rational so can be substituting where h and i are integers and $i \neq 0$.

↓

$a \cdot \left(\frac{h}{i}\right) = \frac{x}{y}$

↳ $a \cdot h = x$ → h and x are integers

↳ $i = y$

↳ a is rational ; $a = x/h$

↳ contradiction

↳ By contradiction, the theorem is true

4 a) $p > 3$ be prime, p is of form $3k+1$ for some
Proof by contradiction $3k-1$ k

Every p will be an odd number since $p > 3$
and the only even prime number is 2.

↳ Let x represent an integer where $x > 7$

→ $\frac{x}{3}$ will have a remainder of either 1, 2 or 0

↳ remainder 0: $x = 3k + 0$
remainder 1: $x = 3k + 1$
remainder 2: $x = 3k + 2$

However for remainder $= 2$; it can alternatively be written as
$x = 3k - 1$ since $x > 3$ and this means we can
write $+2$ as -1.

→ p is in the form of $\begin{bmatrix} 3k+1 \\ 3k-1 \end{bmatrix}$, so by contradiction

↳ p is in the form of $3k$ since that is
the only possible option by the remaining
↓
This means that p will always have a to divide
by which is 3.

↳ By contradiction, the theorem is True

b) Theorem: 5 only number that can be both prime a

Proof by contradiction

in unit a, we proved p is a prime number > 3

↳ Assume $y > 5$ and prime number

Assume $x + 2 = y$ and x, z is a prime numbers
$x + 2 = z$ $x + 2 2 z$

→ Since x, y, z are prime numbers and $x + 2 = y$, we

consult unit a and see that $x = 3k \pm 1$

↳ 1) $3k + 1 = x$

$(3k + 1) + 2 = y \rightarrow 3k + 3 = y \rightarrow 3(k + 1) = y$

↳ y can be divided by 3, making it not prime

a contradiction

2) $3k - 1 = x$

$(3k - 1) + 2 = y \rightarrow 3k + 1 = y$

$(3k + 1) + 2 = z \rightarrow 3k + 3 = z \rightarrow 3(k + 1) = z$

↳ z can be divided by 3, making it not prime

a contradiction

↳ Since both case are contradictory, the

proof of contradiction shows the theorem

is true

5) (2n+1) airports; n positive integer

given
a) distance between any 2 airports all different
b) every airport has 1 airplane depart and lands closest airport

Proof by induction

BC (n=1): 2(1)+1 → 3 airports
↳ by given a, we know that all airports are different distances away from each other. Distance a, b and c where a < b < c when a is distance between airports 1 and 2, b distance between airport 2 and 3 and c distance between airport 3 and 1.
↳ by given b, we know that each airport will send airplanes and send them to the closest airport. Since two airports are not gonna send across distance c, we find that 2 airplanes land on one airport, 1 lands on another airport and 0 land on the other airport
↳ BC is true

IH: Assume that there will at least be 1 airport that doesn't have any airplane destined for it

IS: we apply (k+1) into our current formula

→ 2(k+1)+1 → 2k+2 → 2(k+1) airports

→ Shown in base case, there will be the same number of distances of airports as the number of airports. This means that there will be 2 airports with the highest distance between them known by given a. With given h, we know the airplanes, each from an airport will travel to the closest airport. Since there will be a distance that is the highest between 2 airports, there will be 1 airport with 2 airplanes and 1 airport with 0 airplanes with every other airport having 1 airplane.

↳ thus by induction, we proved k+1

theorem was correct

↳ Thus, True by Induction

6) for positive int $n \geq 1$ prove $\frac{1}{3^1} + \frac{1}{3^2} \dots + \frac{1}{3^n} < \frac{1}{2}$

Proof by induction

BC $(n=1)$: $\frac{1}{3^1} < \frac{1}{2}$ True

IH: for positive int $n \geq 1$; $\sum_1^n \frac{1}{3^n} = x < \frac{1}{2}$

IS: Consider $\underline{n+1}$

By Induction, the theorem is true

$\hookrightarrow \underbrace{\sum_1^{n+1} \frac{1}{3^n}}$ $< \frac{1}{2}$ $\rightarrow$ $\frac{1}{3^2} < \frac{1}{6}$ ✓

$\uparrow n \geq 1$

$\hookrightarrow \underbrace{\sum_1^n \frac{1}{3^n}}_{IH} + \frac{1}{3^{n+1}} < \frac{1}{2}$ $\quad \frac{1}{3^{n+1}} < \frac{1}{6}$

$\uparrow$

$\hookrightarrow x + \frac{1}{3^{n+1}} < \frac{1}{2} \rightarrow \frac{1}{3^{n+1}} < \frac{1}{2} - \frac{1}{3}$

* note: largest number x can be is $\frac{1}{3}$. see Base case.

7) a) Show $\frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2}$

↳ $a_1+a_2 \geq 2\sqrt{a_1 a_2}$

↳ $a_1 - 2\sqrt{a_1 a_2} + a_2 \geq 0$

↳ $(\sqrt{a_1}-\sqrt{a_2})^2 \geq 0$

→ This enough information to prove our problem.
Any number squared is positive. So any number but
0 will fulfill this condition. And since $a_1 \neq a_2$,
this condition is guaranteed to be fulfilled

b) Assume $n = 2^k$ prove $n = 2^{k+1}$

taking from part a

→ $\frac{a_1+a_2}{2^1 \cdot 2^k} \geq \sqrt{a_1 a_2}$

↳ $\frac{1}{2^k}(a_1+a_2) - 2\sqrt{a_1 a_2} \geq 0$

↳ $a_1 - 2\sqrt{a_1 a_2} + a_2 \geq \frac{1}{2^k}(a_1+a_2)$

↳ $(\sqrt{a_1}-\sqrt{a_2})^2 \geq \frac{1}{2^k}(a_1+a_2)$

→ $2^k(\sqrt{a_1}-\sqrt{a_2})^2 \geq a_1+a_2$

↳ since k is a positive integer and with part a
knowledge, we can
say that by
induction,
this theorem
is true

c) for $k \geq 2$, AM-GM true for $n \geq k$; theorem

AM for $n = k$

$\hookrightarrow$ $a_k = \frac{a_1 + \dots a_{k-1}}{k-1} \geq \sqrt[k-1]{a_1 \dots a_{k-1}} = g_k$

$\hookrightarrow$ $\left(\frac{a_1 + \dots a_{k-1}}{k-1} \right)^{k-1} \geq a_1 \cdot \dots a_{k-1}$

$\hookrightarrow$ $\frac{(a_1 + \dots a_{k-1})^{k-1}}{(k-1)^{k-1}} \geq a_1 \dots a_{k-1}$

$\hookrightarrow$ $(k-1)^{k-1} \geq \frac{(a_1 + \dots a_{k-1})^{k-1}}{a_1 \cdot a_2 \cdot \dots a_{k-1}}$

$\rightarrow$ this is enough to prove by induction that this theorem is true

d) parts a-c imply AM-GM inequality for $n \geq 2$ is since part a showed that $n = 2$ is true, part b showed that $n = 2^k$ and $n = 2^{k-1}$, the domino effect of induction among part b and part c shows it works for the opposite $n = k-1$. collectively, they all prove that the AM-GM inequality works for $n \geq 2$ and n is all positive integers

8) Theorem: For any number of coins greater than 0 or $(n \geq 1)$, any move returns score $\frac{n(n-1)}{2}$

Proof by induction

BC $(n=1) \rightarrow \frac{1(1-1)}{2} = 0$ ✓ theorem works for base case

Note: for stacks with n coins, one stack will be size. If we fudge k nodes of, this will leave 2 stacks, k size k the other size $n-k$. the score is $k(n-k)$

IH: Score of the two stacks will be $\frac{n(n-1)}{2}$ and $\frac{(n-k)(n-1+k)}{2}$

IS: from $n+1$ coins

$\to k(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2}$

$\to \frac{2k(n+1-k)}{2} + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2}$

$\to 2kn + 2k - 2k^2 + k^2 - k + n^2 + n - kn - kn - k + k^2$

$\to \frac{n^2+n}{2} \to \frac{n(n+1)}{2}$ score

$\to$ Thus by induction, our theorem was true.