CS 70 HW #8

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1) Probability Warm Up

a) $$P(A) = \frac{\binom{15}{15} \cdot 30^{15} \cdot 70^{20-15}}{100^{20}} = \binom{20}{15}\left(\frac{3}{10}\right)^{15}\left(\frac{7}{10}\right)^{20-15}$$

b) from part a

↳ $$P(A) = \frac{\binom{30}{15}\binom{70}{15-15}}{\binom{100}{15}}$$

c) get one value twice

↳ complement: no value repeated

→ $$\frac{6!}{(6-5)!} = 720 \qquad \Omega = 6^5 = 7776$$

↳ $$\frac{720}{7776} = 9.26\%$$

↳ $100\% - 9.26\% = 90.74\%$ chance

repeat at least 1 happens

2) Five Up

a) $\Omega = (2)^5 = 32$

b)

$$\left\{\begin{matrix} (H\ H\ H\ T\ T) & (H\ T\ T\ H\ H) \\ (H\ H\ T\ H\ T) & (T\ H\ H\ H\ T) \\ (H\ H\ T\ T\ H) & (T\ H\ H\ T\ H) \\ (H\ T\ H\ H\ T) & (T\ H\ T\ H\ H) \\ (H\ T\ H\ T\ H) & (T\ T\ H\ H\ H) \end{matrix}\right\}$$

$\frac{10}{32}$

c)

$$\left\{\begin{matrix} (H\ H\ H\ H\ T) \\ (H\ H\ H\ T\ H) \\ (H\ H\ T\ H\ H) \\ (H\ T\ H\ H\ H) \\ (T\ H\ H\ H\ H) \\ (H\ H\ H\ H\ H) \end{matrix}\right\}$$

$\frac{10+6}{32} = \frac{16}{32} = \frac{1}{2}$

d) Only $1/32$ (H H H T T)

Only $1/32$ (H H H H T)

e) complement: (all T) → (T T T T T)

↳ 31/32 for at least 1 head

f) 1/2, shown in part c

g) (H H H T T)

$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = 3.29\%$

(H H H H T)

$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = 6.58\%$

h) complement: (all T) → (T T T T T)

$\left(\frac{1}{3}\right)^5 = 4.12\%$

↳ $100\% - 4.12\% = 99.59\%$

i) 1/2 from part f

↳ since odds are now 2/3 for heads,

that's new probability

↳ 2/3

3) Past Probabilities

a) i) $\Omega = (q)^2$

ii) Take one dice $x = (0 \ldots q-1)$

another dice $y = (0 \ldots q-1)$

↳ for each mult (xy) mod q, going to be q possible values for the product

↳ $p[w] : \frac{1}{q^2}$

iii) E_1 = resulting product is 0

Need to find product that is not equal to 0 mod q

→ x has $q-1$ values

→ y has $q-1$ values

$\} \; q + q - 2 + 1$

↳ $2q - 1$

↳ $\frac{2q-1}{q^2}$

iv) $E_2 =$ product is $\frac{(a-1)}{2}$

Need to fulfill $xy \equiv \frac{(a-1)}{2} \mod a$

→ since a is odd, we get an integer

↳ if xy is even, then $xy \equiv 0 \mod a$

↳ if xy odd, then $xy \equiv 1 \mod a$

↳ For $xy \equiv \frac{(a-1)}{2} \mod a$ to work,

↳ we need xy to be odd

↳ $xy = a(a-1)$ $\Omega = a^2$

↳ $\frac{a(a-1)}{a^2} \Rightarrow \frac{(a-1)}{a}$

b) i) Ω = set of all graphs on n vertices

ii) probability of each individual occurrence is $1/N$ since each coin flip is just as likely

iii) E_1 = Graph is complete

↳ There can only be 1 complete graph

↳ from part b) ii) → $1/N$

iv) E_2 = vertex i has degree d

exactly d of $n-1$ possible adjacent edges must be present.

↳ $\binom{n-1}{d}$ choices for such edges

→ $2^{\binom{n}{2}-(n-1)}$ graphs w/ choice

$$\rightarrow P(E_2) = \frac{\binom{n-1}{d} 2^{\binom{n}{2}-(n-1)}}{2^{\binom{n}{2}}}$$

4) Cliques in Random Graphs

a) $\Omega = 2^{\binom{n}{2}}$

↳ Between every vertex pair, there is either an edge or no edge.

b) For a fixed set of 15 vertices to be a clique, we know the $\binom{15}{2}$ pairs of those vertices have to be connected by an edge.

↳ $\frac{1}{2}^{\binom{n}{2}}$

c) E_{v_1} and E_{v_2} are independent

→ cond: V_1 and V_2 need to share at most 1 vertex

↳ if they do, then we know that

$$P(E_{v_1} \cap E_{v_2}) = P(\text{all edges } V_1\ V_2) = \left(\tfrac{1}{2}\right)^{\binom{v_1}{2}} \left(\tfrac{1}{2}\right)^{\binom{v_2}{2}}$$

$$= P(E_{v_1})P(E_{v_2})$$

d) Since $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!}$$

$$\leq n \cdot (n-1) \cdots (n-k+1)$$

$$\leq n^k$$

e) Since there are $\binom{n}{k}$ ways to choose a subset of size k, we can find that

$\binom{n}{k} \div 2^{\binom{k}{2}}$ from part a

$\hookrightarrow 2^{\binom{k}{2}} \to 2^{k(k-1)/2}$

$\hookrightarrow \frac{n^k}{(2^{(k-1)/2})^k} \to \frac{n^k}{(2^{(2\log n+1-1)/2})^k} \to \frac{n^k}{(2^{2\log n})^k}$

$\to \frac{n^k}{n^{2k}} = \frac{1}{n^k} \leq \frac{1}{n}$

5) PIE extended

Let B be the event that an odd number of

A1, A2 ... occur.

↳ Express B in terms of the events A_i as follows

$P(B) = \sum (-1)^{k-1} \sum (M) k$; $M(A)$ denotes number of k elements in subset {A1, A2 ... An}

x terms in 1st sum correspond to coefficients in expansion of $(1-1)^n = 0$

↓

$\sum P(A_i) - 2 \sum P(A_i \cap A_j) + 4 \sum P(A_i \cap A_j \cap A_k) \ldots$

then combine over set of B

$\sum P(A_i) - 2 \sum P(A_i \cap A_j) + \ldots = P(B) + P(A_1 \cap A_2) \ldots$

↳ Substituting this into P(B) and we get

$\sum P(A_i) - 2 \sum P(A_i \cap A_j) + \ldots P(A_1 \cap A_2 \cap \ldots A_n) \geq 0$

which is the answer

6) Independent Complements

Let Ω be sample space and $A, B \subseteq \Omega$ be 2 independent events

a) Prove: $\bar{A}$ and $\bar{B}$ must be independent

$$P(A \cap B) = P(A) P(B)$$

$$\hookrightarrow P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= (1 - P(A))(1 - P(B))$$

$$= P(\bar{A}) P(\bar{B})$$

b) Prove: A and $\bar{B}$ must be independent

$$\hookrightarrow P(A \cap \bar{B}) = P(A - (A \cap B))$$

$$= P[A] - P(A \cap B]$$

$$= P(A) - P(A) P(B)$$

$$= P(A)(1 - P(B))$$

$$= P(A) P(\bar{B})$$

C) **Disprove**: A and $\bar{A}$ must be independent

↳ If $0 < P(A) < 1$ then $P(A \cap \bar{A}) = 0$

but $P(A)P(\bar{A}) > 0$ so $P(A \cap \bar{A}) \neq P(A)P(\bar{A})$

D) **Prove**: Possible that $A = B$

↳ If $P(A) = P(B) = 0$, then $P(A \cap B) = 0 \times 0$