CS 70 HW #2

1) Universal Preference

a) n jobs preferences: $C_1 > C_2 > \dots > C_n$

n candidates preferences: $J_1 > J_2 > \dots > J_n$

Lemma (improvement)

Pairings: $(C_1, J_1), (C_2, J_2) \dots (C_{n-1}, J_{n-1}), (C_n, J_n)$

↳ Proof by Induction

BC ($n=1$): C_1 and J_1 only candidates

↳ (C_1, J_1) pair occurs

IH: Assume for $n = k$, candidate C_n will accept an offer from its preferred job J_n and J_n will offer to its most preferred candidate C_n. (C_n, J_n)

IS $(k+1)$: $(C_1, J_1) \dots \underbrace{(C_{k+1}, J_{k+1})}_{\text{IH}}$

$(C_1, J_1) \dots (C_n, J_n)$

↳ By Induction, it is true

b) We get essentially the same matching.

i) The candidates are going to apply for the most desirable job

ii) The job is going to select the best candidate

iii) the remaining candidates apply for next most desirable job

iv) that job selects most suited candidate

v) steps iii) and iv) repeat till all get assigned

result $\rightarrow (J_n, C_n), (J_{n-1}, C_{n-1}) \ldots (J_1, C_1)$

c) There can only be at least 1 stable matching.

$\hookrightarrow$ Since all candidates and jobs really only have 1 top preference, this means that only 1 matching will be of top preference or max. The others will stable but not of most preference. There shouldn't be any rogue couples.

2| Nothing can be better than something

a| Theorem: a stable pairing exists in the case where we allow unmatched entities

Proof by contradiction

Assume $\neg P$: a stable pairing doesn't exist in the case where we allow unmatched entities

$C_1: J_1 > J_2$ $\quad$ $J_1: C_1 > C_2$

$C_2: J_2 > J_1$ $\quad$ $J_2: C_2 > 1$

↳ stable matchings: (C_1, J_1) (C_2, J_2)

→ Thus, $\neg P$ is False

↳ Thus by contradiction, the theorem is true.

b) Theorem: if being unmatched is stable, then they must remain unmatched in any other matching

Proof by contradiction

Assume $\neg P$: if an entity remains unmatched in one stable matching, they must match in any other stable matching

A : 1 > 2 > 3
B : 2 > 1 > 3
C : 1 > 2 > 3

1 : A > B > C
2 : B > A > C
3 : B > A > C

(A, 1) (B, 1)

unmatched

C } 3

(1, A) (2, B)

unmatching

C } 3

↳ $\neg P$ is false

↳ by contradiction, this theorem is true

3) A better stable matching

a) $R \wedge R'$

Chart

	R	R'	Preference
A	4	3	3
B	3	4	4
C	1	2	2
D	2	1	1

$\hookrightarrow R \wedge R' = \{(A,3), (B,4), (C,2), (D,1)\}$

$\rightarrow$ $R \wedge R'$ is stable because there are no rogue couples in this solution

b) When comparing $R \wedge R'$, R and R' (out of possibilities in R and R'), we can observe that $R \wedge R'$ has the highest possible preference for each job to its candidate. This makes $R \wedge R'$ the more effective list in comparison to R and R' where not all of their matching have higher matchs than $R \wedge R'$

Chart

	R	R'	R ∧ R'
A	4	3	3
B	3	4	4
C	1	2	2
D	2	1	1

Preferences

A: 1 > 2 > 3 > 4

B: 2 > 1 > 4 > 3

C: 3 > 4 > 1 > 2

D: 4 > 3 > 2 > 1

↳ overall, R ∧ R' is more consistent with preference

c) Let:

a) J and C denote sets of jobs/candidates that prefer R to R'

b) J' and C' denote sets of jobs/candidates that prefer R' to R

→ Total number of jobs/candidates equal. This means:

$|J| + |J'| = |C| + |C'| =$ total jobs/candidates

↳ $|J| \leq |C'|$ and $|J'| \leq |C|$

since there needs to be at least as many jobs as candidates

↳ $|J'| \geq |C|$ and $|J| \geq |C'|$

from the previous part, we can deduce that they must equal

↳ Assume: J prefers R to R' and C prefers R' to R and
J' prefers R to R and C prefers R to R

↳ The theorem is true

D) i) Prove $R \wedge R'$ is a pairing

From part c

↳ J prefers R to R' and C prefers R' to R or

J' prefers R to R and C prefers R to R'

↳ $|J| + |J'| = |C| + |C'|$

↳ $|J'| \geq |C|$ and $|J| \geq |C'|$

→ From this, I can deduce that no job or candidate there are no unmatched pairs and no job/candidates is used multiple times in $R \wedge R'$. This is enough proof to demonstrate that $R \wedge R'$ is pairing

ii) any (J, C) pairing in $R \wedge R'$ implies that that pair was found either in $(J, C) \in R$ or $(J, C) \in R'$. Since in these two respective lists R any R' they were explicitly stated as stable pairings, we can assume that applies to this situation as well.

4) Build up error

→ If (vertex degree ≥ 1) → graph connected

False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof? We use induction on the number of vertices $n \geq 1$.

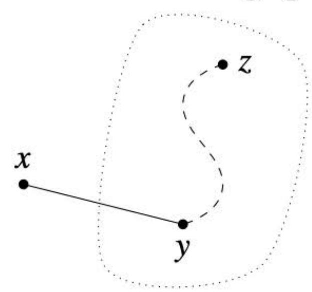
$n=1$

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$. weak IH

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.

n-vertex graph



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$. □

→ The premise of this proof derives that the graph is connected of (vertex degree ≥ 1). Besides the IH being weak, The IS has many flaws mainly false assumptions involved. When the n-vertex graph is created, there is nothing involved in the logic that forces every vertex to be connected. Furthermore, when considering the $n+1$ graph, there is nothing that is allowing an edge to be formed into connecting the $n+1$ vertex and any other vertex. Essentially, there are very many assumptions and methods used without any reasoning or support from BC or IH.

5) Proofs in Graphs

a) Givens: Old California has $(n \geq 2)$ cities

x has road to y and vice versa; all one ways
Directional graph

theorem: There exists a city which can be reached by every other city by traveling through at most 2 roads

Proof by Induction

BC $(n=2)$: city 1 and city 2 will have a road that goes to and from them, making the base case true
↳ $m = (n-1) = 1$; m = vertex degree

IH: Assume for $n = k$, there will be at least 1 city that can be reached from every city by traveling through at max 2 roads. → $m = n-1$

IS: In a directed graph with every pair of cities having a one way street, we know that all the graph is connected. This means that every vertex has a degree of $n-1$ or $m = n-1$; m = vertex degree,
$k+1 = n$ when we implement the $k+1$ we get
↳ $m = (k+1) - 1 = \underline{k}$. This shows us that for $k+1 = n$, the vertex degree (m) of every vertex is equal to k. Using our induction hypothesis, we are able to say there will be at least 1 city that can be reached from every city by traveling through at max 2 roads.
↳ By Induction our theory is True

b) Theorem: m walks that together cover edges of G

Graph G has 2m vertices representing it as:

$a_2, a_4, a_6, a_8 \ldots a_m$; $m > 0$

→ Given m edges to connect 2m vertices

↳ $\{a_2, a_4\}$ $\{a_6, a_8\}$ $\{a_{m-2}, a_m\}$

where $\{\ \}$ represents edges between 2 vertices

↳ This results in all vertices having an even degree.

↳ Eulerian trail

↳ This proves the theorem

c) G is bipartite with vertex sets V_1 and V_2

↳ every step must be from $V_1 \to V_2$ or $V_2 \to V_1$.

If you want to end exactly where you started, you will need an even length because the G is bipartite on one direction.

b) Bipartite Graphs

a) (Sum of vertices degrees = 2 · Edges)

→ Bipartite Graph

↳ every edge connects a vertex in disjointed set L to a vertex in disjointed set R.

↳ Sum of vertices degrees in L = Sum of vertices degrees in R

↳ $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$

b) Given $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$

↳ average degree of vertices in L/R or (s = t)

→ $|L|$ = # vertices in L

→ $|R|$ = # vertices in R

↳ Total # edges = $|L| \cdot |R|$

$\rightarrow S = \dfrac{2 \cdot (|L| \cdot |R|)}{|L|} \rightarrow 2 \cdot |R|$

$\hookrightarrow \dfrac{S}{t} = \dfrac{2 \cdot |R|}{2 \cdot |L|} \rightarrow \dfrac{|R|}{|L|}$

Def Bipartite graph

c) a graph is bipartite if it can be colored in two different colors such that no two adjacent vertices have the same color

$\hookrightarrow$ for this example, set L would be colored one color and set R would be colored a different color

$\rightarrow$ no edges connecting vertices with same color, meaning it's bipartite

$\rightarrow$ this means that the graph can be 2-colored.