CS 70 HW #12

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1) Double-check your intuition again

a) i) $\text{cov}(X+Y, X-Y) = \text{cov}(X,X) + \text{cov}(X,Y) - \text{cov}(Y,X) - \text{cov}(Y,Y)$

$= \text{cov}(X,X) - \text{cov}(Y,Y)$

$= 0$

ii) $P(X+Y=7, X-Y=0) = 0$ since $-Y + X = 0$, then the sum of two dice rolls must be even. Since with $P(X+Y=7)$ ...

b) Yes: we says $\mu = E[X]$, then $0 = \text{var}(X)$

↳ then $E[(X-\mu)^2]$ so $(X-\mu)^2$ must be identically 0.

↳ $X = \mu$

c) No! $\text{var}(cX) = E[(cX - E[cX])^2]$

$= c^2 \text{var}(X)$

↳ if $\text{var}(X) \neq 0$ or $c \neq 0$ $c \neq 1$, then it is not true.

d) No.

$A = X + Y$

$B = X - Y$

→ since they $C(A, B)$ not constantly part B since they must be non zero variances which means they also have non zero derivations part a reinforces than their covariance is 0 which means they are uncorrelated → not independent

e) Yes: $\mathrm{cov}(X, Y) = 0 \to \mathrm{cov}(X, Y) = 0$

↳ we have $\mathrm{var}(X + Y) = \mathrm{cov}(X + Y, X + Y) =$

$\mathrm{var}(X) + \mathrm{var}(Y) + 2\,\mathrm{cov}(X, Y) =$

↳ $\mathrm{var}(X) + \mathrm{var}(Y)$

f) YES!

For all values (x) (y) → $\max(x,y)\min(x,y) = xy$

↳ This makes it true

g) NO.

ex

x : either 0 or 1 w/ probability 1/2 each

Y : independent from distribution

↳ $\text{cov}(X,Y) = 0$ since independence

↳ $E[\max(X,Y)] = 1\left(\frac{3}{4}\right) + 0\left(\frac{1}{4}\right) = \frac{3}{4}$

$E[\min(X,Y)] = 1\left(\frac{1}{4}\right) + 0\left(\frac{3}{4}\right) = \frac{1}{4}$

↳ This makes

$\text{cov}(\max(X,Y), \min(X,Y))$

↳ $1/16 \neq 0$

2) Unreliable Surveys

a) I would use poisson distribution

parameters: $\lambda = 4$ $\rightarrow$ Poisson(4)

b) use Poisson properties

↳ $E[X] = Var(X) = 4$

c) $P[S=0] = \frac{1}{e^4}$, $P[S=1] = \frac{4}{e^4}$, $P[S=2] = \frac{4^2}{2e^4}$

↳ $P[S<3] = P[S=0] + P[S=1] + P[S=2]$

↳ 0.2381

d) $P[S \geq 3] = 1 - 0.2381 = 0.7614$

3) Geometric and Poisson

$X \sim \text{Geometric}(p)$

$Y \sim \text{Poisson}(\lambda)$

Find $P[X > Y]$

$\hookrightarrow P(Y=n) = e^{-\lambda}\left(\frac{\lambda^n}{n!}\right)$

$P(X = n+1) = (1-p)^{(n+1)-1}$

$\downarrow$

$$P(X > Y) = \sum_n P(X > Y, Y = n) = \sum_n P(X > n, Y = n)$$

$$= \sum_n P(X \geq n+1)\, P(Y = n)$$

$$= \sum_n (1-p)^{n+1-1} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \sum_n e^{-\lambda} \times \frac{(\lambda(1-p))^n}{n!}$$

$$= e^{-\lambda} \times e^{\lambda(1-p)}$$

$$= e^{-\lambda p}$$

4) Coupon Collector Variance

X_i: number visits we'll have to make before we collect the ith unique Monopoly card (actually obtained)

& given: already collected $i-1$ unique cards

↳ since initial visits are independent, every time we get new card, we start clean

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

$$= \sum_{i=1}^{n} \frac{1-(n-i+1)/n}{[(n-i+1)/n]^2}$$

$$= \sum_{j=1}^{n} \frac{1-j/n}{(j/n)^2}$$

$$= \sum_{j=1}^{n} \frac{n^2}{j^2} - \sum_{j=1}^{n} \frac{n}{j}$$

$$= n^2 \left(\sum_{j=1}^{n} \frac{1}{j^2} \right) - E(X)$$

5) Probabilistically Buying Probability books

a) For any k, probabilities conditioned on $K = k$ must sum to 1

$$\sum_n P[N=n \mid K=k] = 1$$

↳ $1 = \sum_{n=1}^{k} P[N=n \mid K=k] = k \times \frac{c}{k} = c$

↳ $c = 1$

b) Joint distribution

↳ $P[N=n \mid K=k]$ and $P[K=k]$ for $k \in \{1, 2, 3\}$

Calculate $P[N=n \cap K=k]$ for each n, k

n/k	1	2	3
1	1/3	1/6	1/9
2	0	1/6	1/9
3	0	0	1/9

This gives us

$$P[N=n, K=k] = \begin{cases} \frac{1}{3k} & k \in \{1, 2, 3\}, n \in \{1, \dots k\} \\ 0 & \text{otherwise} \end{cases}$$

c) marginal distribution of N

As $P[N=n] = P[N=n \wedge K=1] + P[N=n \wedge K=2] +$

$P[N=n \wedge K=3]$

$$\Downarrow$$

$$P[N=n] = \begin{cases} \frac{1}{3} + \frac{1}{6} + \frac{1}{9} & \text{if } n=1 \\ \frac{1}{6} + \frac{1}{9} & \text{if } n=2 \\ \frac{1}{9} & \text{if } n=3 \end{cases} \quad \begin{cases} \frac{11}{18} & \text{if } n=1 \\ \frac{5}{18} & \text{if } n=2 \\ \frac{2}{18} & \text{if } n=3 \end{cases}$$

d) conditional distribution given $N=1$

$$\begin{cases} \frac{1}{3}\left(\frac{18}{11}\right) = \frac{6}{11} & \text{if } k=1 \\ \frac{1}{6}\left(\frac{18}{11}\right) = \frac{3}{11} & \text{if } k=2 \\ \frac{1}{9}\left(\frac{18}{11}\right) = \frac{2}{11} & \text{if } k=3 \\ 0 & \text{otherwise} \end{cases}$$

e) A : event that $1 \leq N \leq 2$

$\hookrightarrow \dfrac{\Pr(K=k, A)}{\Pr(A)}$

$\hookrightarrow \Pr(A) = P[N=1] + P[N=2] = \frac{11}{18} + \frac{5}{18} = \frac{8}{9}$

$$\hookrightarrow \Pr(K=k, A) = \begin{cases} \frac{1}{3(1)} & \text{if } k=1 \\ \frac{1}{3(3)} + \frac{1}{2(4)} & \text{if } k=2 \\ \frac{1}{3(4)} + \frac{1}{2(4)} & \text{if } k=3 \end{cases}$$

$\downarrow$

$$
\begin{cases} 3/16 & \text{if } K = 1 \\ 1/4 & \text{if } K = 1 \\ 3/16 & \text{if } K = 3 \end{cases}
$$

Symmetric around $K = 2$

↳ 2 conditional mean

$$\mathrm{var}(K \mid A) = E[X^2 \mid A] - E[X \mid A]^2$$

$$= \frac{3}{16}(1-2)^2 + \frac{1}{4} \times 0 + \frac{3}{16} \times (3-2)^2$$

$$= \frac{3}{8}$$

↳ $\frac{3}{8}$ conditional variance

f) cost each books mem ?

total cost ?

$E[C_i] = 3$

T : total cost ; $T = C_1 + \dots C_n$

using all knowledge

$$E[T] = E[E[T \mid N]]$$
$$= E\left[E\left[\sum_{i=1}^{N} C_i \mid N\right]\right]$$
$$= E[N \times 3] = 3E[N]$$
$$= 3 \times \left(1 \times \left(\frac{11}{18}\right) + 2\left(\frac{5}{18}\right) + 3\left(\frac{2}{18}\right)\right)$$
$$= 4.5$$

h) expectation total cost = \$4.5

6) Dice games

a) X : a random integer between 0 and 100 inclusive

fair odds

Y : random fair odd number between 0 and X inclusive

$\hookrightarrow E[X] = \frac{0+100}{2} = 50$

$E[Y \mid X] \leq \frac{0+X}{2} = \frac{X}{2}$

$\hookrightarrow \frac{50}{2} = 25$

$\hookrightarrow E[Y] = E[E[Y \mid X]]$ or $E[\frac{X}{2}]$

$\hookrightarrow \frac{1}{2} E[X]$

b) X: Total number of rolls she makes (inclusive)

Y: number of rolls where even number

$E[X] = 6$

Geo mean $= 1/p$

$E[Y|X] = nC$

$X \sim Geo(1/6)$

$\hookrightarrow X = \frac{1}{6} + \frac{5}{6} \times (X+1)$

$\hookrightarrow X + 2$

$\hookrightarrow E[Y] = E[E[Y|X]]$

$E\left[\frac{3(X-1)}{5}\right]$

$\hookrightarrow \frac{3}{5} E[X] - E\left[\frac{3}{5}\right] = \frac{3}{5}(6) - \frac{3}{5}$

$\hookrightarrow E[Y] = \frac{18}{5} - \frac{3}{5}$

$= \frac{15}{5} = 3$