Homework #6

0) Sundry

Nobody

1) Union and Intersection

a) Always countable

↳ gets values in X which is always countable or, in other words, $X \cap Y$ subset of X which is always countable

b) Always uncountable

↳ will have values in Y which is always uncountable or, in other words, $X \cup Y$ superset of Y which is always uncountable

c) Always uncountable

↳ talking about at the i^{th} value of the set A. The def of S_i is that any value from A will be uncountable. This means that this question is uncountable

d) Sometimes countable, sometimes uncountable

countable: if both sets (A, A') in Question have no values in joint, this would lead to an empty set and essentially a countable set

uncountable: if both sets have values in common which leads to the join set being uncountable due to S_i

c) Sometimes countable, sometimes uncountable

countable: if both sets in Question (Y and B) are equal and have values in a joint set, this would lead to a countable set because we are given that B_i is countable

uncountable: if joint set has no values. This would lead to Y or being uncountable

f) Always countable

↳ We are given that B_i is a countable set which would be the set expressing in a original set

2) Finite and Infinite Graphs

a) finite graph $G = (V, E)$; $V \subseteq N$

Countable

↳ Since we bounded both V and E to finite sets and we defined V as natural numbers of all graphs with vertex set V, we can establish that it is a countable union of finite sets which makes it countable

b) set of all graphs, fixed countable infinite set of vertices where degree is 2. $\longrightarrow$ Z

Uncountable

taking a look at a graph with Natural numbers drawn infinite graph or G.

↳ attempt identify $\mathbb{N}$ with $\mathbb{N} \times \{0,1\}$ with bijection

- Z_a is subset graph

↳ Z_a edgeset defined as $\underbrace{\{(a,0),(a,1)\}}$ because the degree is 2.

↳ This edge set is uncountable which would fail the bijection and ultimately lead to Z being uncountable

c) $G = (V, E)$ isomorphic if bijection exists
$G' = (V', E')$ $f: V \to V'$
such that
$(u, v) \in E$ iff $(f(u), f(v)) \in E'$

Uncountable

Breaks down: Essentially we are looking at graphs G and G' and we trying to prove that they are uncountable

↳ issue with countability: unable to preserve bijection and degree vertices of G and G'

Contradiction

$G = C_6$; cycle with 6 vertices

$G' = 2 \cdot C_3$; 2 cycles with 3 vertices

↳ if a bijection between V of G and G' happens, we see that it cannot happen since G is a connected graph while G' isn't, thus failing our definition stated above.

3) Countability proof practice

a) disks : $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$

x_0, y_0 are $\in \mathbb{R}$, $r > 0$

↳ have set of disks in $\mathbb{R}^2$ where none overlap

relate to countable set $(\mathbb{Q} \times \mathbb{Q})$.

Set $(\mathbb{Q} \times \mathbb{Q})$ countable

Bijection $(\mathbb{Q} \times \mathbb{Q})$ and $\mathbb{R}^2$ of disks, exists

↳ since we know that none of the disks overlap and disks are filled in with vertical or horizontal to a set of countable numbers, we can see that the Bijection exists through cardinality

↓

Countable

h) circle $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2+(y-y_0)^2 \leq r^2\}$

$x_0, y_0, r \in \mathbb{R}, r>0$

have set of circle in $\mathbb{R}^2$ where none overlap

Potentially uncountable

in part a, we proved that disks in $\mathbb{R}^2$ would be countable. However, circles are not the same as disks because they aren't filled in.

↓

This leads to a potential scenario of all circles being centered by (0,0). Since they aren't technically overlapping but have the same center, we wouldn't be able to count all the sets, making it uncountable

c) $f: \mathbb{N} \to \mathbb{N}$; ($x \geq y$ then $f(x) \geq f(y)$)

Uncountable

Natural numbers defines its set as ons positive integer $\{1, 2, 3 \ldots \infty\}$. In the scenario we are given, we have a set containing multiple increasing functions. Since we are increasing to an uncountable set, this would mean through diagonalization that this set of function is uncountable

d) $f: \mathbb{N} \to \mathbb{N}$; ($x \geq y$ then $f(x) \leq f(y)$)

Countable

Natural numbers defines its set as ons positive integer $\{1, 2, 3 \ldots \infty\}$. In the scenario we are given, we have a set containing multiple decreasing functions. Since we are decreasing to a countable set or number, this would mean through cardinality that this set of function is countable

4) Fixed points

Determine program P has any fixed points

↳ Given any program P, a fixed point is an input x such that P(x) outputs x

a) uncomputable

We can reduce TestHalt to demonstrate this

↳ ReturnOutput

```
TestHalt(P, x):
    P'(x): run P(x) and return x
    if ReturnOutput(P', x):
        return True
    else:
        return False
```

↳ If ReturnOutput exists, then TestHalt must also exist by reduction. However, since it's established that TestHalt doesn't exist, ReturnOutput follows and doesn't exist.

b) Uncomputable

Reducing Returnoutput from part a to new program Return x or Null (P, x)

```
Returnoutput (P, x)
    for i in range (len(P)):
        if Return x or Null (P, x):
            return x
        else
            return Null
```

h) similar logic to part a. Reducing Returnoutput in turn makes this new program uncomputable

c) Computable

↳ needing the fixed point to be a natural number fixes part B. The "new" code is shown below

```
ReturnOutput(P, x)
    for i in range(len(P)):
        if ReturnOrNull(P, x) and is N:
            return x
        else
            return Null
```

↳ some issues before is that the set of numbers could have been uncountable, such as $\mathbb{R}$, which in turn make it uncountable. However, with this alternative, it fixes the issues in part B making it computable

5) Kolmogorov Complexity

↳ string K(x) is length of optimally compressed x

↳ shortest program length of x

a) "notion of smallest positive integer not defined under 280 characters" is paradoxical

↳ This statement is paradoxical because if this statement is True, that would imply that the smallest possible integer is not under 280 characters, or that 1 character doesn't exist which implies False. In this sense, the above statement is paradoxical.

b) Theorem: Any length n, there is at least one string of bits that cannot be compressed to less than n bits

↳ Assume that number of strings of length n is 2^n.

By definition of the establishment proven earlier, number of descriptions of length $2n$ is

$2^0 + 2^1 + \ldots + 2^{n-1} \rightarrow 2^n - 1$. This tells me that of the 2^n number of strings of length n which is 2^n, there is at least 1 string of bits that cannot be compressed to less than n bits c)

```
TestHalt (P, n):
  P'(n): run P(n) and return a binary string
  if ReturnOutputHistory (P', n):
      return P'(n)
  else:
      return null
```

This code would spit out a binary string output with an infinite n input

d) Since we are using TestHalt and trying to make a bijection, this would imply that the code will not work as they're uncomputable

6) Counting, Counting, and more Counting

a) $\binom{n+k}{k}$

b) 19 digits with no same number adjacent

↳ $\binom{3}{1}$ $\binom{2}{1}$

1st 2nd → 19th

↳ $3 \times 2^{18} = 786,432$

c) i) $\binom{52}{13} \to \frac{52!}{(52-13)!\,13!} = 635,013,559,600$

ii) $\binom{48}{13} \to \frac{48!}{(48-13)!\,13!}$

iii) $\binom{48}{9} \times \binom{4}{4} \to \frac{48!}{(48-9)!\,9!} \times \frac{4!}{(4-4)!\,4!}$

iv) 13 spades in total

$\binom{39}{9} \times \binom{13}{4}$

d) 104! different combinations

52 duplicate card possibilities

↳ $\frac{104!}{(2!)^{52}}$

more   more

c) number of 0s = number of 1s

$$\downarrow \qquad\qquad \downarrow$$

$$x = y$$

z → number with same occurrence of 0s and 1s

↳ total number of number combinations with 99 bits (since bit same only 1 and 0) 2^{99}

→ $2^{99} = x + y + z$; $x \Rightarrow$ $z = \binom{99}{49}$

↳ $2^{99} - \binom{99}{49} = 2x$

↳ $2^{98} - \left(\frac{1}{2}\right)\binom{99}{49} = x$

f) i) ALABAMA

↳ 4 A's 1 L 1 B 1 M

↳ total 7 letters

→ 7! different unordered ways to order Alabama

→ need to account for duplicate A's 4!

↳ $\frac{7!}{4!}$ → $7 \times 5 \times 6$ → 210

ii) MONTANA

↳ 2 A's 2 N's 1 M 1 T 1 O

↳ 2 letters

→ 7! different unordered ways to order

→ need to account for duplicate A's N's

↳ $\frac{7!}{(2!)(2!)}$

g) i) need c left of c

↳ 6 letters ; 6! total w/o restriction

↳ order doesn't matter

→ need to account for restriction

↳ $\frac{6!}{2!}$

ii) $\frac{6!}{2!} \cdot 4!$

h) 25^8

i) star/bar : 8 stars, 24 bins $\rightarrow \binom{24+8}{8} \rightarrow \binom{32}{8}$

j) 8 − 6 = 2 balls remaining, 6 bins

$k = 2$
$n = 6$
$\rightarrow \binom{6-1+2}{2} \rightarrow \binom{7}{2}$

(5) 20 students enrolled

↳ different ways to pair students

↳ need to account for duplicates

$20!$ pairings

→ Account for pair ordering

↳ 2^{10}

→ Account for pair placement orders

↳ count each pairing $10!$ times

→ $\frac{20!}{2^{10} \cdot 10!}$

l) $x_0 + x_1 + \dots + x_k = n$ solutions if x nonnegative integer

k stars

n groups ⇝ $\binom{n-1+k}{k}$

m) $x_0 + x_1 = n$ solutions if each x strictly positive

2 stars

n groups ⇝ $\binom{n-1+2}{2} \to \binom{n+1}{2}$

n) $x_0 + x_1 + \dots + x_k = n$ solution if x strictly nonzero integer

k stars

n groups ⇝ $\binom{n-1+k}{k}$

7) Fermat's [illegible]

a) n^p

b) Since a bowl without at least 2 colors would be single colored, I would just subtract that instance from all instances

↳ $n^p - n^1$

c) Since rotating the bowl p times will produce equivalent bowls, dividing the total by p

↳ $\dfrac{n^p - n^1}{p}$

d) FLT

↳ if p is prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \bmod p$

$$\rightarrow \frac{n^p - n}{p} \rightarrow \frac{n(n^{p-1} - 1)}{p}$$

↳ we are given that p is prime. Now looking at if it is divisible, we know it is because the question tells us that as a given. Our concern is if the two are coprime but if they are, then we know FLT holds