CS 70 HW #11

Sundays

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1) Maybe Lossy Maybe Not

a) Alice sends 7 packets

packet probability @ p

↳ if message length 6 → 6 more packets → decode 6 packets

↳ ok 7 packets → 6 packets or more

↳ $P[\text{decode}] = (1-p)^7 + \binom{7}{1} p (1-p)^6$

b) Correct 1 packet after deleting packets

Alice sends 10 packets

↳ $P[\text{decode}] = P[8] + P[9] + P[10]$

↳ $P[\text{decode}] = \binom{10}{2} p^2 (1-p)^8 + \binom{10}{1} p (1-p)^9$

$+ (1-p)^{10}$

c) Promoted w/ prob p, conversion prob q

Alice > 7 packets

↳ at 1 packet converted : Alice no enough packets (2 < 8)

↳ all packets come thru must be converted

↳ G must pay them

↳ $P[\text{delete}] = \binom{7}{1} p(1-p)^6 q ((1-q)^6$

$+ \binom{7}{1} p(1-p)^6 (1-q)^7$

$+ (1-p)^7 (1-q)^7$

2) Class Enrollment

a) # days for Logic follows
Geometric distribution w/ parameter p_g

↳ $P[G] = p_g(1-p)^{G-1}$; G positive

so for day G_1 probability ↑

b) conditional distribution $G > g$ is an
same as original distribution w/ parameter
p_g.

$$(1-p_g)^4 \frac{p_g}{1-(1-p_g)^5}$$

accounts for that a cal new information

c) Expected # days for logic to enroll given
groupings is p_h.

↳ Law total probability

$$E[H] = \sum (H \geq 1)\, p_g \frac{(1-p_g)^{G-1}}{p_h}$$

↳ $1/p_h$

d) – Distribution of G and H are still geometric distributions w/ parameters p_g and p_h

– However not independent because enrollment in History is now dependent on the enrollment status of Geoscience

e) Distribution A: min (p_g, p_h)

↳ $P[X=r] = 1-(1-p_g)^r(1-p_h)^r$

r = positive integer

f) Expected number of days for Lydia is

$E(B) = E(A) + E(H) \; b = A)$

↳ $E(A) = \dfrac{1}{p_g + p_h - p_g p_h}$

↳ $E(B) = \dfrac{1}{p_g + p_h - p_g p_h} + \dfrac{1}{p_h}$

Q1 Lydia can enroll in either 0, 1 or 2 classes at end of the 14 days

↳ p_0 = enrolled 0 class after 30 days

p_1 = enrolled in 1 class after 20 days

p_2 = enrolled 2 class after 20 days

↳ $p_0 = (1-p_g)^{30}(1-p_h)^{20}$

↳ $p_2 = 1-(1-p_y)^{20}(1-p_h)^{20}$

→ To find p_1, use total probability law

↳ $1-(1-p_y)^{20}(1-p_h)^{20} - \sum(k=1 \to 19)\, p_g$

$(1-p_y)^{k-1}(1-p_h)^{k}$

↳ expected number classes Lydia enrolled in after 14 days = $p_1 + 2p_2$

3) Swaps and Cycles

a) Define X_i: $X_i = 1$ if i partner in a switch

$X_i = 0$ if otherwise

define S: number of switches

$$S = \frac{1}{2}(X_1 + X_2 + \dots + X_n)$$

$\downarrow$ Linearity

$$E(S) = \frac{1}{2} n E(X_1)$$

* $(n-2)!$ permutations in which 1^{st} value is switched with 2^{nd}. Probability outcome: $\frac{1}{n(n-1)}$

$\downarrow$ accounting for switch of $1^{st}/2^{nd}$

$$E(S) = \frac{n}{2}(n-1)\left(\frac{1}{n(n-1)}\right)$$

$\hookrightarrow$ $1/2$

c) N_k: random variable for # k-cycles in random permutations

I_r : indicator random variable for presence of r^{th} cycle

↳ $N = \sum I_r$; I_r same distribution

from part a

$$E(N) = E(\sum I_r) = \binom{n}{k}(k-1)!$$

↳ simplification $\binom{n}{k}(k-1)! \times \frac{(n-k)!}{n!}$

$$\downarrow$$

$$1/k$$

4.1 Throwing Frisbees

a) Each trial has $n-1$ possible outcomes and each outcome occurs w/ probability of $\frac{1}{n-1}$.

$\downarrow$ Linearity of Expectation

expected # is $n-1$

b) Shazam now has $n-1$ trials but seeks a unique

↳ each time, $n-1$ possible outcomes each w/ probability of $\frac{1}{n-1}$.

↳ probability none stating frisbee is

$$1-\left(1-\frac{1}{(n-1)}\right)^{n-1}$$

↳ Linearity of expectation

↳ $1-\left(1-\frac{1}{(n-1)}\right)^{n-1}$

5) Balls and Bins

n balls m bins

X: number bins w/ 1 ball

Y: if X_i is 1 if has 1 ball and 0 otherwise

$\hookrightarrow$ $X = Y_1 + Y_2 + \dots + Y_n$

$\Big\downarrow$ linearity of expectation

$E(X) = E(Y_1) + E(Y_2) + \dots + E(Y_n)$

$\hookrightarrow$ $E(Y_i) = 1 P[Y_i = 1] + 0 P[Y_i = 0]$

$= P[Y_i = 1]$

$= P[\text{Bin } i \text{ is empty}]$

$= \left(1 - \frac{1}{m}\right)^n$

Since balls 1 … n needs to go in a bin not i

each w/ prob

$\downarrow$

* account for binomial random variables

↳ $\underbrace{n\left(1-\left(1-\frac{1}{m}\right)^{n}\right)}_{m^2}$

6) Will I get my Package

a) Define $X_i = \begin{cases} 1 : \text{if } i\text{-th customer gets package unopened} \\ 0 : \text{otherwise} \end{cases}$

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

↳ $E(X_i) = \Pr[X_i = 1] = \frac{1}{2n}$

* since each customer will get package w/ $\frac{1}{n}$ probability at its unopened probability of $\frac{1}{2}$ since packages are opened independently

↳ $E(X) = \frac{1}{2}$; $n \times \frac{1}{2n} = \frac{1}{2}$

b) $Var(X) = E(X^2) - E(X)^2$

↳ $E(X^2) = E((X_1 + \dots X_n)^2)$.

↳ $E(\sum_{i,j} X_j X_i)$

↓ Linearity of Expectation

$\sum_{i,j} E(X_i X_j)$

↳ If $i=j$, then:

$E(X_i X_j) = E(X_i^2) = \frac{1}{2n}$

↳ If $i \neq j$, then:

$(E X_i X_j) = Pr[X_i X_j = 1] \cdot 1 + Pr[X_i X_j = 0] \cdot 0$

$= \frac{1}{4n(n-1)}$

↓

Thus: $E(X^2) = n \cdot \frac{1}{2n} + n(n-1)\left(\frac{1}{4n(n-1)}\right)$

$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\hookrightarrow$ $Var(x) = E(x^2) - E(x)^2 =$

$$\frac{3}{4} - \frac{1}{4}$$

$$\downarrow$$

$$1/2$$