Integration

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When the integral around C is written, as usual, as the sum of three integrals, the integral along the ray $z=r\exp(2\pi i/n)$ may be written as

$$\int_R^0 \frac{r^m \exp[(m+1)2\pi i/n]dr}{r^n+a} = -\int_0^R \frac{x^m \exp[(m+1)2\pi i/n]dx}{x^n+a}.$$

Combining this last integral with the integral along the real axis from $x=0$ to $x=R$ and letting $R\to\infty$, we have, for $0<m+1<n$,

$$\int_0^\infty \frac{x^m dx}{x^n+a} = \frac{2\pi i}{\{1-\exp[(m+1)2\pi i/n]\}na^{(n-m-1)/n}\exp[(n-m-1)\pi i/n]}$$

$$= \frac{\pi}{na^{(n-m-1)/n}\sin[(m+1)\pi/n]}.$$

INTEGRATION

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Although we frequently think of indefinite integration as being more complex than the determination of roots of algebraic equations, we shall see that in some respects this is not the case. It is well known that solutions of some algebraic equations cannot be written in terms of radicals, and also that certain functions do not possess elementary integrals. However, due to the length and depth of the proofs of these facts, few texts go beyond merely exhibiting some examples. Confusing a sufficient condition with one which is both necessary and sufficient, many a student finds security in knowing that all equations of degree $\leqq 4$ have been solved. With respect to integration, however, he is well aware of his rather unhappy position, that of knowing that some functions cannot be integrated (in finite terms) but of possessing no way of determining whether a particular function is in this class or not. This latter situation need not be accepted, for Liouville obtained a test [1] which is both necessary and sufficient for the integrability of any one of a rather large class of functions. We shall recall this test and show both how it can be applied by the college freshman and its utility as an integration technique.

The foundation for the test is the following theorem which is sufficiently "natural" to be readily accepted and easily remembered.

Liouville's Theorem. *If $\int fe^g dx$ is an elementary function, f and g are rational functions of x, and the degree of $g>0$, then*

$$\int fe^g dx = Re^g \tag{1}$$

where R is a rational function of x.

This is actually a special case of Liouville's original theorem (see [1] page 114 or [2] page 47), but it is sufficiently general for our purposes.

In this note, by rational function we mean the quotient of two polynomials with coefficients in any field of characteristic zero, for example, the complex numbers. The term elementary function is more difficult to define (indeed, Ritt takes 12 pages to do so). However, the student is willing to accept as meaningful such statements as "the general algebraic equation of degree 5 cannot be solved in terms of radicals," although we know that "in terms of radicals" requires some preliminary discussion. Therefore, he finds no difficulty with the following definition: An elementary function is one which can be constructed by means of any finite combination of the operations addition, subtraction, multiplication, division, raising to powers, taking roots, forming trigonometric functions and their inverses, taking exponentials and logarithms. In short, no matter how complicated the function, if we can write down all of its terms, the function is elementary. (Actually, the construction of elementary functions includes the forming of algebraic functions, but it seems advisable to omit this generality for the beginning calculus student.)

We return to the test. If we wish to determine whether fe^g can be integrated (*i.e.* has an integral which is an elementary function, or, as we shall also say, $\int fe^g dx$ is elementary), we know, by Liouville's Theorem the form of the integral. Differentiating equation (1) and cancelling the nonzero e^g we find $f=R'+Rg'$ or, letting $R=P/Q$, where P and Q are relatively prime polynomials in x,

$$fQ^2 = P'Q - PQ' + PQg'. \tag{2}$$

Thus $\int fe^g dx$ is elementary if and only if there exist polynomials P and Q satisfying the differential equation (2).

Besides Liouville's theorem, the test requires only one further fact; namely, the following

LEMMA. *If the polynomial $f(x)$ has an r-fold zero at $x=\alpha$ and $r>0$, then $f'(x)$ has an $(r-1)$-fold zero at $x=\alpha$; in other words, if $f(x)=(x-\alpha)^r h(x)$ where $r>0$, $h(x)$ is a polynomial and $h(\alpha)\neq 0$, then $f'(x)=(x-\alpha)^{r-1}k(x)$ where $k(\alpha)\neq 0$.*

The proof of the lemma is a simple differentiation exercise which the student can supply.

It will make things easier if we define the term multiplicity. The number α is called a zero of the polynomial $f(x)$ of multiplicity r (or a root of $f(x)=0$ of multiplicity r) if $f(x)=(x-\alpha)^r h(x)$, where the polynomial $h(\alpha)\neq 0$. In terms of multiplicity the lemma reads: If α is a zero of the polynomial $f(x)$ of multiplicity $r>0$, then α is a zero of $f'(x)$ of multiplicity $r-1$.

By examining some of the examples most frequently quoted in texts, we shall show how easily the analysis of equation (2) can be carried out, even by the student who has not previously encountered differential equations.

Example 1: e^{-x^2}. If $\int e^{-x^2}dx$ is elementary, then $\int e^{-x^2}dx = Re^{-x^2}$ or $1=R'-2xR$. Letting $R=P/Q$, where P and Q are relatively prime polynomials and $Q\neq 0$, we find:

$$Q^2 = QP' - PQ' - 2xPQ \tag{1.2}$$

which is equation (2). Rearranging, we obtain

$$Q(Q - P' + 2xP) = - PQ'. \tag{1.3}$$

Let us assume that the degree of Q is positive. Then $Q=0$ has a root; let α be such a root and call its multiplicity $r(r>0)$. Since P and Q are relatively prime, $P(\alpha)\neq 0$. Now, α is a zero of the left side of (1.3) of multiplicity $\geqq r$ but α is a zero of the right side of multiplicity $r-1$. This is a contradiction, hence our assumption that the degree of Q is positive must be false. Q is a constant ($\neq 0$) which we can assume is unity.

From (1.3) we obtain

$$P' - 2xP = 1. \tag{1.4}$$

Since P is a polynomial in x, it is clear that the degree of $-2xP>$ degree of P', and the degree of $-2xP>0$. The degree of the left side of (1.4) is always greater than the degree of the right side, which is a contradiction. We have proved that there is no polynomial P satisfying (1.4), hence no rational function satisfying (1.2). Consequently $\int e^{-x^2}dx$ is not elementary.

Example 2: e^{bx}/x, with b a nonzero constant. If $\int(e^{bx}/x)dx$ is elementary, then $\int(e^{bx}/x)dx=Re^{bx}$ or $(1/x)=R'+bR$. Letting $R=P/Q$, where P and Q are relatively prime polynomials, $Q\neq 0$, we find:

$$Q^2 = xQP' - xPQ' + xbPQ \tag{2.2}$$

$$Q(Q - xP' - bxP) = - xPQ'. \tag{2.3}$$

If we assume that Q has positive degree, then $Q=0$ has a root. Let α be such a root and call its multiplicity r. If $\alpha\neq 0$, we encounter the same contradiction met in the first example, that α is a zero of the left side of (2.3) of multiplicity $\geqq r$, while α is a zero of the right side of multiplicity $r-1$. Thus α must be zero, and $Q=cx^r$, for some $c\neq 0$. Putting this expression for Q in (2.2) we have $cx^{r+1}(cx^{r-1}-P'-bP)=-crx^rP$. Again there is a contradiction, for the number 0 is a zero of the left side of multiplicity $\geqq r+1$, while it is a zero of the right side of multiplicity r. Our assumption that Q has positive degree is no longer tenable; hence Q is a constant, which we can assume is unity.

From (2.3) we obtain

$$xP' + bxP = 1. \tag{2.4}$$

As before, since P is a polynomial in x, the degree of the left side = degree of $(bxP)>0=$ degree of the right side. We have proved that there is no polynomial P satisfying (2.4), hence no rational function satisfying (2.2). Consequently $\int(e^{bx}/x)dx$ with $b\neq 0$ is not elementary.

Example 3: $(\sin x)/x$. It is clear that if $f(x)=u(x)+iv(x)$, where $u(x)$ and $v(x)$ are real valued functions, then

$$\mathcal{R}\int f(x)dx = \int \mathcal{R}f(x)dx = \int u(x)dx,$$

$$\mathcal{I}\int f(x)dx = \int \mathcal{I}f(x)dx = \int v(x)dx,$$

and if $\int f(x)dx$ is elementary, both $\int u(x)dx$ and $\int v(x)dx$ are elementary. ($\mathcal{R}$ and $\mathcal{I}$ stand for "the real and imaginary parts of," respectively.)

Although $(\sin x)/x$ is not in the form of Liouville's Theorem, by Euler's relation ($e^{ix}=\cos x+i\sin x$), we have $(\sin x)/x=\mathcal{I}(e^{ix}/x)$. Since e^{ix}/x does not possess an elementary integral, by example 2, neither does $\mathcal{I}(e^{ix}/x)=(\sin x)/x$.

Example 4: $1/\log x$. Again Liouville's theorem is not immediately applicable. If $y=\log x$, then $\int(1/\log x)dx=\int(e^y/y)dy$. By Example 2, the latter integral is not elementary, hence the same is true of the former.

Example 5: $(x^2+ax+b)e^x/(x-1)^2$, with a and b constants. Not only do the usual integration techniques require a considerable amount of skill, but there is no a priori assurance that we could find the integral, if it exists. Let us apply the test.

Assume $\int[(x^2+ax+b)e^x/(x-1)^2]dx=Re^x=Pe^x/Q$. Then

$$(5.2) \qquad (x^2 + ax + b)Q^2 = (P'Q - Q'P + PQ)(x - 1)^2$$

$$(5.3) \qquad Q(Q(x^2 + ax + b) - (x - 1)^2P' - (x - 1)^2P) = - Q'P(x - 1)^2.$$

Assume Q has positive degree, and let α be a zero of Q of multiplicity r. If $\alpha\neq 1$, α is a zero of the left side of multiplicity $\geqq r$, but a zero of the right side of multiplicity $r-1$. This is a contradiction, hence $\alpha=1$ and $Q=(x-1)^r$. Substituting this into (5.3) we find

$$(x - 1)^r[(x - 1)^r(x^2 + ax + b) - (x - 1)^2P' - (x - 1)^2P] = - r(x - 1)^{r+1}P.$$

Using the fact that the multiplicities of 1 as a zero of the left and right sides must be the same, we see that $r=1$; *i.e.* $Q=(x-1)$. In the last equation we can cancel a common factor of $(x-1)^2$ from both sides, giving

$$(x^2 + ax + b) - (x - 1)P' - (x - 1)P = - P,$$

$$(x - 1)P' + (x - 2)P = x^2 + ax + b.$$

P is clearly linear, $P=cx+d$, which means

$$cx - c + cx^2 + dx - 2cx - 2d = x^2 + ax + b.$$

Since these two polynomials are identical, the coefficients of like powers are the same, $c=1$, $d-c=a$, $-c-2d=b$, whence $b=-2a-3$.

Consequently $\int[(x^2+ax+b)e^x/(x-1)^2]dx$ is elementary if and only if $b=-2a-3$, in which case the integral is $e^x(x+a+1)/(x-1)+C$.

By using one unproved theorem, we have seen how it is possible for even the beginning calculus student to test the integrability of certain transcendental functions, and the logical structure of the test, though not trivial, is sufficiently similar to others he has seen for it to be easily grasped (*e.g.*, the test for the rationality of say $\sqrt{2}$). Although it is obvious that a table of functions which cannot be integrated could be constructed by a careful analysis of examples, to do so for the student would be no better than is done at present. Rather we feel that the student, by applying the test to a few functions, will have a fruitful introduction to differential equations and will gain well-founded confidence in his ability to follow and reconstruct proofs requiring more than one or two steps. Also, the last example illustrates the usefulness of the test as an integration technique.

References

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THE INTEREST RATE IN INSTALLMENT CONTRACTS

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1. Introduction. In an installment contract* in which the time price differential is an add-on, the yield return to the lender increases as the contract gets longer up to a point. Thereafter the yield declines as the length of the contract increases.

Example. For a 6% add-on rate (*i.e.* $\frac{1}{2}$% of the original loan is added to the loan cost for each month of the duration of the loan) in an installment contract the effective rate is 8.98% per year for a 3-month contract and 10.21% for a 6-month contract. The yield builds up to a maximum of 11.13% around a 26-month contract. Thereafter it declines. The yield is 10.21% for a 120-month contract and it continues to decline to 6.2% in a 500-year contract.

The formula for the present value of a loan on a monthly basis is

$$B = Ra_{\overline{n}|} \text{ at } (i) \tag{1}$$

where B = cash price or present value of loan, R = monthly payment, n = number of months, and i = interest rate per month. Let c = per cent of add-on per month (*i.e.* at a 6% yearly add-on, $c=.005$); then $R=B(1+cn)/n$ and (1) becomes $B=B(1+cn)[1-(1+i)^{-n}]/ni$, or it can be put in the form

$$(1 + cn - in)(1 + i)^n - (1 + cn) = 0. \tag{2}$$

2. Maximum value of i for a constant value of c. The value of i as given by (2) is expressed approximately [1] [2] by

$$i_b = \frac{6cn}{3(n + 1) + cn(n - 1)}, \tag{3}$$

* This problem was presented to me by M. R. Neifeld, Beneficial Management Corporation, Morristown, New Jersey.