Generating of Music Variations: Dynamical Systems Approach

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April 28, 2024

Abstract

This research introduces a novel approach to music composition aimed at mitigating composer burnout. Leveraging chaotic systems, specifically the Lorenz Equation known for its sensitivity to initial conditions, the proposed method maps musical data onto this system. By introducing variations in initial values, the method generates unpredictable compositions, fostering creativity and fresh perspectives.

1 Introduction

This research aims to address the prevalent issue of composer's burnout by introducing a novel approach to music composition utilizing chaotic systems. The proposed method leverages the Lorenz Equation, a chaotic system renowned for its sensitivity to initial conditions. By mapping musical data onto the Lorenz Equation and introducing variations in initial values, the method generates novel and unpredictable musical compositions, stimulating creativity and fostering fresh perspectives in music composition.

In reality, there are now many AI technologies that can create music. However, these technologies often require high computational resources, making them unable to run on devices with low processing power. Additionally, some AI music composition tools may produce music in limited styles. Well-known AI music generation platforms [1] today include:

- Mubert: An online AI music composition platform that allows users to create music easily without musical knowledge.
- Soundraw: An AI music creation tool that helps users quickly create background music for videos or games.
- Boomy: An AI music composition app that helps users create and distribute music on streaming platforms.
- AIVA: An AI assistant for music composition that helps users create music in various styles.
- Musicity: An AI music creation tool that utilizes deep learning models to generate music. This research project proposes a new approach that reduces the use of computational resources and creates music in a more diverse range of styles.

2 Literature Review

2.1 Chaotic System

Chaotic systems are like puzzles with clear rules, but tiny mistakes at the start (think a single misplaced piece) lead to wildly different solutions later on. Even though the rules are defined, the outcomes are unpredictable due to their sensitivity to initial conditions. This butterfly effect is seen in weather, fluid flow, and even some financial markets.

An example used in a Chaotic Systemg such as Modeling Natural Phenomena: Chaotic systems help model and understand various natural phenomena such as weather patterns, turbulent fluid flows, ecological systems, and population dynamics.[2] Finance and Economics: Chaotic dynamics are studied in finance and economics to model stock market fluctuations, economic cycles, and price dynamics, providing insights into the inherent unpredictability and nonlinear behavior of financial systems [3] Biomedical Systems: Chaotic systems are used to model and understand complex biological systems, including neural networks, cardiac rhythms, and gene regulatory networks, aiding in diagnosis, treatment, and understanding of diseases [4].

2.2 Lorenz Equation

The Lorenz equation is commonly defined as three coupled ordinary differential equation like:

$$\begin{aligned}\dot{x} &= \sigma(y - x),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= xy - bz,\end{aligned}$$

where x, y and z are the variables, and $\sigma > 0$, $r > 0$ and $b > 0$ are parameters.
When the Lorenz equation's parameters are set to $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$, it shows chaotic behavior, as illustrated in Figure 1. The Lorenz equations [5] have been applied across various domains such as Modeling Fluid Flows: A study by J.C. Sprott in Chaos and Stability in Nonlinear Analog Circuits (2003) [6] explores how the Lorenz system can be applied to model specific fluid flows. This demonstrates its use in understanding fluid dynamics beyond just atmospheric convection.[7]. Cryptography: The chaotic nature of the Lorenz system makes it a potential candidate for secure communication. A research paper by S. Baptista et al. titled "Lorenz System Parameter Determination and Application to Break the Security of Two-channel Chaotic Cryptosystems" (2006) explores this application. While this paper discusses limitations of specific implementations, it highlights the potential of the Lorenz system in cryptography [8] Engineering Applications: The Lorenz equations can be used to model certain electrical and mechanical systems that exhibit chaotic behavior. For instance, a paper by A.A. Fathy in "Chaos, Solitons Fractals" (2009) investigates the application of the Lorenz system to brushless DC motors [9]. This showcases the potential for the Lorenz system in analyzing and potentially controlling chaotic behavior in engineering systems.

2.3 Fourth-Order Runge–Kutta Method

While analytical solutions exist for some differential equations, many require numerical approaches to approximate their behavior. The field of mathematics offers a numerical methods for solving both single and systems of linear and nonlinear differential equations. Popular examples include the Euler method and Taylor series methods. However, when it comes to achieving a balance between accuracy and efficiency, the Runge-Kutta method reigns supreme for approximating solutions.

The Runge-Kutta method [10] is a numerical methods for approximating solutions to ordinary differential equations. These equations describe how a quantity changes with respect to

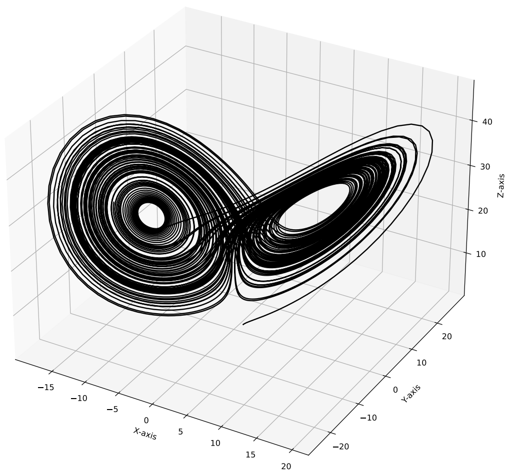


Figure 1: Chaotic behavior of the lorenz equation.

another variable, but often cannot be solved exactly. The Runge-Kutta method tackles these problems by breaking down the interval of interest into smaller subintervals and iteratively calculating the solution at each subinterval.

Let $\dot{y} = f(t, y)$. The approximation of y_{i+1} by fourth-order Runge–Kutta method is given by:

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ k_1 &= f(t_i, y_i), \\ k_2 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right), \\ k_3 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right), \\ k_4 &= f(t_i + h, y_i + hk_3), \end{aligned}$$

where $i = 0, 1, 2, ...$ and h is the step size, y is the variable and t is time.

2.4 Musical variations from a chaotic mapping: Diana S. Dabby

Diana S. Dabby's research [11] explores using chaotic systems to create musical variations. By leveraging chaotic trajectories' sensitivity to initial conditions, the study devises a method to introduce variability into a musical piece's pitch sequence. This allows for diverse musical variations while preserving the original composition's coherence. Dabby emphasizes the dynamic nature of this approach, offering composers a flexible tool for crafting a wide range of musical variations. Additionally, the study suggests extending this method beyond music to sequences of context-dependent symbols in various fields. Overall, the research showcases an innovative application of chaotic dynamics in music composition, yielding dynamic and versatile musical outcomes.

2.5 Melodic Variation

Melodic variation is essential in songwriting, adding depth and diversity to compositions. This article emphasizes its role in enhancing musicality and emotional depth. It cautions against monotony in melodies, stressing the importance of dynamic variation to engage listeners. Additionally, melodic variation contributes to a song's structure by creating contrast between sections. Two main methods are discussed: variations on a theme and countermelodies. Lastly, the article highlights the emotional impact of melodic variation and its ability to evoke a range of feelings in listeners. In conclusion, skillful use of melodic variation enables songwriters to craft compelling compositions that resonate deeply with audiences.

3 Main Result

This section explores the musical variations from a chaotic mapping method and combining musical variations from a chaotic mapping and melodic variation with expanded rhythm methd. We will demonstrate this method through two examples. In the first example, we will illustrate how a chaotic map can be used to generate variations in musical pitch. In the second example, we will combine this method with another method for expanding rhythm to create more interesting musical variations.

3.1 Musical Variations from a Chaotic Mapping

For a musical sheet, let m be a positive integer representing a number of notes, $P = \{p_0, p_1, \ldots, p_{m-1}\}$ be a sequence of music pitches and

$$\dot{x}(t) = f(t, x) \tag{1}$$

be a chaotic dynamical system with an initial condition $x(0) \in \mathbb{R}^n$, where $x(t) = (x_1(t), \ldots, x_n(t))$ is differntiable for all $t \geq 0$. Let $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Given a sequece $V = \{\phi_i(kh)\}_{k=0}^{m-1}$ for some $i \in \mathbb{N}_n$, where ϕ_i is a numerical solution in i-th component to (1) with a step size of h. Let $f : V \to P$ be a mapping defined by $f(v_k) = f(\phi_i(kh)) := p_k$ for all $k \in \{0\} \cup \mathbb{N}_{m-1}$.

Consequently, we introduce another sequence $\widetilde{V}_i = \left\{\tilde{\phi}_i(kh)\right\}_{k=0}^{m-1}$, where $\tilde{\phi}_i$ is a numerical solution with a new initial condition $\tilde{x}(0) \in \mathbb{R}^n$ i-th component to (1), when $\tilde{x}(0)$ start not far from $x(0)$, i.e., $\|x(0) - \tilde{x}(0)\| \leq d$ for some small positve number $d \in \mathbb{R}$. Then, we define another mapping $g : \widetilde{V} \to P$ by:

$$g(\tilde{v}_k) = g\left(\tilde{\phi}_i(kh)\right) := \begin{cases} f(\phi_i(b)) & \text{if } \exists a, b \in \operatorname{dom}\phi_i \text{ s.t. } \phi_i(a) < \tilde{\phi}(kh) \leq \phi_i(b) \\ f(\phi_i(a)) & \text{if } \tilde{\phi}(kh) < \phi_i(a) \text{ for all } a \in \operatorname{dom}\phi_i \\ f(\phi_i(b)) & \text{otherwise.} \end{cases}$$

This procedure yields, a new sequence of music pitches $\widetilde{P} = \{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n\}$.

Example 3.1. Let a sequence of pitches of 12 variations on Ah vous dirai-je Maman [13] in the first 3 bars in Figure 2 denoted by $P = \{C4, C4, G4, G4, A4, A4, G4, F4, F4, E4, E4\}$ in Figure 4a and Lorenz system be a dynamical system with chaotic behavior by giving Lorenz parameters $r = 28, \sigma = 10$ and $b = 2.6667$. Then the numerical solution of Lorenz system using fourth-order Runge–Kutta method with initial condition of $(1, 1, 1)$ is denoted by $X = \{1.00, 1.29, 2.13, 3.74, 6.54, 11.04, 16.69, 19.56, 15.37, 7.55, 1.20\}$ in Figure 4b, when X is a sequence of x-value from numerical solution of Lorenz system. Then, a mapping from music pitch to real value denoted by f result in Figure 4c as follows:

$f(x_i)$	$f(1.00)$	$f(1.29)$	$f(2.13)$	$f(3.74)$	$f(6.54)$	$f(11.04)$
p_i	C4	C4	G4	G4	A4	A4
$f(x_i)$	$f(16.69)$	$f(19.56)$	$f(15.37)$	$f(7.55)$	$f(1.20)$	
p_i	G4	F4	F4	E4	E4	

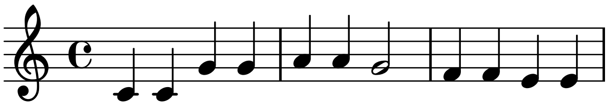


Figure 2: The original of 12 variations on Ah vous dirai-je Maman in the first 3 bars.

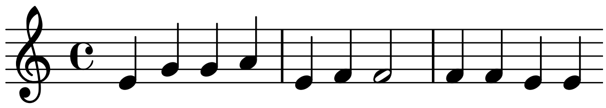


Figure 3: The new variation of 12 variations on Ah vous dirai-je Maman in the first 3 bars, generated by the Initial Condition $(1.01, 1, 1)$.

Next, We generating a new trajectory with an initial condition of $(1.01, 1, 1)$ and $X' = \{1.01, 1.30, 2.15, 3.76, 6.58, 11.10, 16.73, 19.55, 15.30, 7.48, 1.15\}$ is a sequence of x-value from new trajectory in Figure 4d. Then, a mapping from real value to music pitch denoted by g result in Figure 4e as follows:

$g(x'_i)$	$g(1.01)$	$g(1.30)$	$g(2.15)$	$g(3.76)$	$g(6.58)$	$g(11.10)$
p_i	E4	G4	G4	A4	E4	F4
$g(x'_i)$	$g(16.73)$	$g(19.55)$	$g(15.30)$	$g(7.48)$	$g(1.15)$	
p_i	F4	F4	F4	E4	E4	

This procedure yields, as shown in Figure 4, a new sequence of music pitches $\hat{P} = \{E4, G4, G4, A4, E4, F4, F4, F4, F4, E4, E4\}$ in Figure 4f. Which can be converted to sheet music in Figure 3. Since this method uses the same note duration and musical notes to create a new variation, so the resulting changes in the sequence compared to the original sequence might seem relatively small.

3.2 Melodic Variation with Expanded Rhythm Method

Given a note duration denoted by ϕ and divided into D equal parts, the duration, R, of each individual division is defined by the following equation:

$$R = \frac{\phi}{D}.$$

Note: In musical theory, equal parts refers to divisions that all have the same duration.

Example 3.2. Consider the music piece 12 variations on Ah vous dirai-je Maman, illustrated in Figure 5. The Figure shows that we already have 6 quarter notes and 1 half note.

If we want to divide each musical note into 4 parts, we can find the duration of each individual division as follows:

- Half note to 4 parts:

$$R = \frac{\phi}{D} = \frac{2}{4} = 0.5$$

i	1	2	3	4	5	6	7	8	9	10	11
p_i	$C4$	$C4$	$G4$	$G4$	$A4$	$A4$	$G4$	$F4$	$F4$	$E4$	$E4$

(a) The first 11 pitches of the 12 variations on Ah vous dirai-je Maman are marked below the pitch axis.

i	1	11	2	3	4	5	10	6	9	7	8
x_i	1.00	1.20	1.29	2.13	3.74	6.54	7.55	11.04	15.37	16.69	19.56

(b) The first 11 x-components of the numerical solution of Lorenz system with initial condition of $(1, 1, 1)$ are marked below the x axis.

i	1	11	2	3	4	5	10	6	9	6	7
$f(x_i)$	$C4$	$E4$	$C4$	$G4$	$G4$	$A4$	$E4$	$A4$	$F4$	$G4$	$F4$
x_i	1.00	1.20	1.29	2.13	3.74	6.54	7.55	11.04	15.37	16.69	19.56

(c) For each x-component x_i, apply the $f(x_i)$ mapping.

i	1	11	2	3	4	5	10	6	9	7	8
x'_i	1.01	1.15	1.30	2.15	3.76	6.58	7.48	11.10	15.30	16.73	19.55

(d) The first 11 x-components of the numerical solution of Lorenz system with new initial condition of $(1.01, 1, 1)$ are marked below the x axis.

i	1	11	2	3	4	5	10	6	9	6	7
$g(x'_i)$	$E4$	$E4$	$G4$	$G4$	$A4$	$E4$	$E4$	$F4$	$F4$	$F4$	$F4$
x'_i	1.01	1.15	1.30	2.15	3.76	6.58	7.48	11.10	15.30	16.73	19.55

(e) For each x-component x'_i, apply the $g(x'_i)$ mapping.

i	1	2	3	4	5	6	7	8	9	10	11
$\hat{p}_i$	$E4$	$G4$	$G4$	$A4$	$E4$	$F4$	$F4$	$F4$	$F4$	$E4$	$E4$

(f) The new variation of the first 11 pitches are marked below the pitch axis.

Figure 4: The visualizes how a chaotic mapping method can be used to generate musical variations

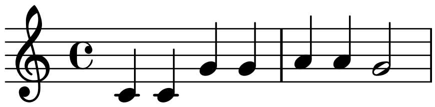


Figure 5: The original of 12 variations on Ah vous dirai-je Maman in the first 2 bars.

Figure 6: The melodic variation of Ah vous dirai-je, maman in the first 2 bars.

- Quarter note to 4 parts:

$$R = \frac{\phi}{D} = \frac{1}{4} = 0.25$$

Following this calculation, a half note can be divided into 4 eighth notes, and a quarter note can be divided into 4 sixteenth notes. This division is represented by the sequence $P = \{C4, C4, C4, C4, C4, C4, C4, C4, G4, G4, G4, G4, G4, G4, G4, G4, A4, A4, A4, A4, A4, A4, A4, A4, G4, G4, G4, G4\}$ which can be converted to sheet music, as shown in Figure 6.

3.3 Combining Musical Variations from a Chaotic Mapping and Melodic Variation with Expanded Rhythm

For a musical sheet, let m be a positive integer representing a number of notes, $P = \{p_0, p_1, \ldots, p_{m-1}\}$ be a sequence of music pitches and

$$\dot{x}(t) = f(t, x) \tag{2}$$

be a chaotic dynamical system with an initial condition $x(0) \in \mathbb{R}^n$, where $x(t) = (x_1(t), \ldots, x_n(t))$ is differntiable for all $t \geq 0$. Next, let q be a positive integer representing a number of expanded notes, $P' = \{p'_0, p'_1, \ldots, p'_{q-1}\}$ be a sequence of expanded music pitches and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Given a sequece $V = \{\phi_i(kh)\}_{k=0}^{m-1}$ for some $i \in \mathbb{N}_n$, where ϕ_i is a numerical solution in i-th component to (2) with a step size of h. Let $f : V \to P$ be a mapping defined by $f(v_k) = f(\phi_i(kh)) := p_k$ for all $k \in \{0\} \cup \mathbb{N}_{m-1}$.

Consequently, we introduce another sequence $\widetilde{V}_i = \left\{\tilde{\phi}_i(kh)\right\}_{k=0}^{q-1}$, where $\tilde{\phi}_i$ is a numerical solution with a new initial condition $\tilde{x}(0) \in \mathbb{R}^n$ i-th component to (2), when $\tilde{x}(0)$ start not far from $x(0)$, i.e., $\|x(0) - \tilde{x}(0)\| \leq d$ for some small positve number $d \in \mathbb{R}$. Then, we define another mapping $g : \widetilde{V} \to P$ by:

$$g(\tilde{v}_k) = g\left(\tilde{\phi}_i(kh)\right) := \begin{cases} f(\phi_i(b)) & \text{if } \exists a, b \in \operatorname{dom}\phi_i \text{ s.t. } \phi_i(a) < \tilde{\phi}(kh) \leq \phi_i(b) \\ f(\phi_i(a)) & \text{if } \tilde{\phi}(kh) < \phi_i(a) \text{ for all } a \in \operatorname{dom}\phi_i \\ f(\phi_i(b)) & \text{otherwise.} \end{cases}$$

This procedure yields, a new sequence of music pitches $\widetilde{P} = \{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_q\}$.

Example 3.3. Let a sequence of pitches of 12 variations on Ah vous dirai-je Maman in the first 4 bars in Figure 7 denoted by $P = \{C4, C4, G4, G4, A4, A4, G4, F4, F4, E4, E4, D4, D4, C4\}$ in

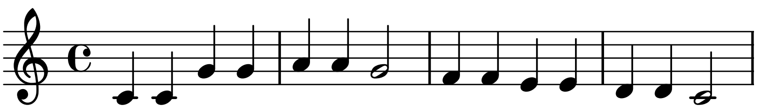


Figure 7: The original of 12 variations on Ah vous dirai-je Maman in the first 4 bars.

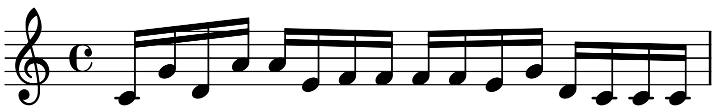


Figure 8: The new variation with melodic variation of Ah vous dirai-je, maman in the first bars, generated by the Initial Condition $(0.6, 0.5, 0.5)$.

Figure 9a and Lorenz system be a dynamical system with chaotic behavior by giving Lorenz parameters $r = 28, \sigma = 10$ and $b = 2.6667$. If we consider the first bar of 12 variations on Ah vous dirai-je Maman by melodic variation with expanded rhythm, $P' = \{C4, C4, C4, C4, C4, C4, C4, C4, G4, G4, G4, G4, G4, G4, G4, G4\}$ will be a sequence of expanded music pitches. Then the numerical solution of Lorenz system using fourth-order Runge–Kutta method with initial condition of $(0.5, 0.5, 0.5)$ is denoted by $X = \{0.50, 0.65, 1.08, 1.91, 3.41, 6.03, 10.33, 16.04, 19.69, 16.35, 8.51, 1.74, -2.45, -4.80\}$ in Figure 9b, when X is a sequence of x-value from numerical solution of Lorenz system. Then, a mapping from music pitch to real value denoted by f result in Figure 9c as follows:

$f(x_i)$	$f(0.50)$	$f(0.65)$	$f(1.08)$	$f(1.91)$	$f(3.41)$	$f(6.03)$	$f(10.33)$
p_i	C4	C4	G4	G4	A4	A4	G4
$f(x_i)$	$f(16.04)$	$f(19.69)$	$f(16.35)$	$f(8.51)$	$f(1.74)$	$f(-2.45)$	$f(-4.80)$
p_i	F4	F4	E4	E4	D4	D4	C4

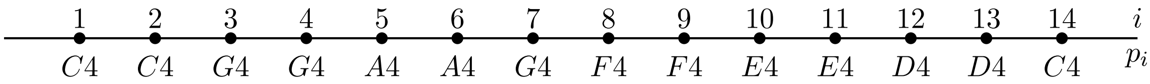
Next, We generating a new trajectory with an initial condition of $(0.6, 0.5, 0.5)$ and $X' = \{0.60, 0.73, 1.21, 2.13, 3.79, 6.67, 11.30, 17.02, 19.67, 15.07, 7.11, 0.81, -2.98, -5.09, -6.33, -7.21\}$ is a sequence of x-value from new trajectory in Figure 9d. Then, a mapping from real value to music pitch denoted by g result in Figure 9e as follows:

$g(x'_i)$	$g(0.60)$	$g(0.73)$	$g(1.21)$	$g(2.13)$	$g(3.79)$	$g(6.67)$	$g(11.30)$	$g(17.02)$
p_i	C4	G4	D4	A4	A4	E4	F4	F4
$g(x'_i)$	$g(19.67)$	$g(15.07)$	$g(7.11)$	$g(0.81)$	$g(-2.98)$	$g(-5.09)$	$g(-6.33)$	$g(-7.21)$
p_i	F4	F4	E4	G4	D4	C4	C4	C4

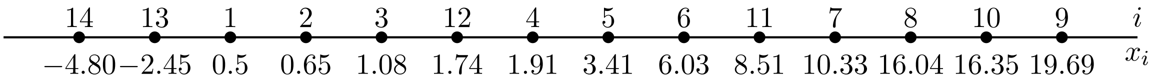
This procedure yields, as shown in Figure 9, a new sequence of music pitches $\hat{P} = \{C4, G4, D4, A4, A4, E4, F4, F4, F4, F4, E4, G4, D4, C4, C4, C4\}$ in Figure 9f. Which can be converted to sheet music in Figure 8.

4 Discussion

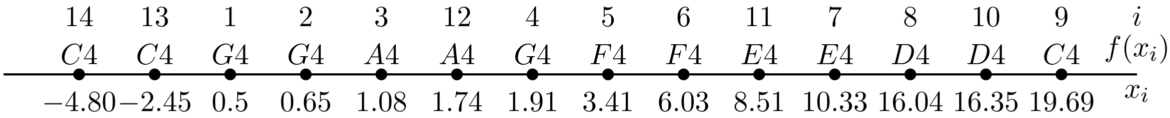
This section explores the potential of Combination of Musical Variations from a Chaotic Mapping and Melodic Variation with Expanded Rhythm technique for generating musical variations.



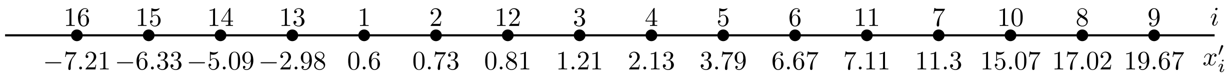
(a) The first 14 pitches of the 12 variations on Ah vous dirai-je Maman are marked below the pitch axis.



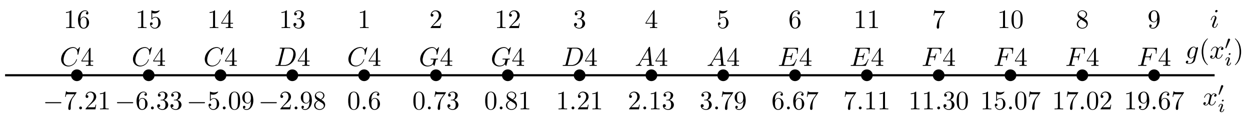
(b) The first 14 x-components of the numerical solution of Lorenz system with initial condition of $(0.5, 0.5, 0.5)$ are marked below the x axis.



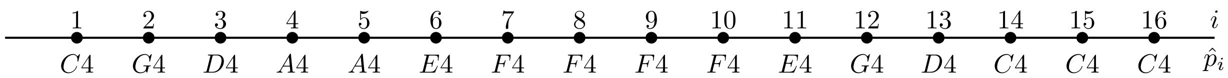
(c) For each x-component x_i, apply the $f(x_i)$ mapping.



(d) The first 16 x-components of the numerical solution of Lorenz system with new initial condition of $(0.6, 0.5, 0.5)$ are marked below the x axis.



(e) For each x-component x'_i, apply the $g(x'_i)$ mapping.



(f) The new variation of the first 11 pitches are marked below the pitch axis.

Figure 9: The visualizes how a chaotic mapping with melodic variation method can be used to generate musical variations

Figure 10: The original of Pachelbel's Canon in the first 10 bars.

Figure 11: The new variation with melodic variation of Pachelbel's Canon in the first 10 bars (including 2 additional bars at the end of the 10th bar).

It analyzes the impact of this approach on the resulting variations compared to traditional melodic variation techniques.

4.1 Strengths of the Combined Approach

The combined approach shows potential in generating interesting musical variations, as shown in the new variations with melodic variation of Pachelbel's Canon [16] (Figure 11) compared to the original melody (Figure 10).

This method appears to be particularly effective for pieces with a larger range of musical pitches, as exemplified by Pachelbel's Canon. The additional pitches provide more material for the chaotic mapping and melodic variation techniques to manipulate, leading to richer and more diverse variations.

4.2 Weaknesses of the Combined Approach

Let's consider the music sheet for Vanessa Carlton's A Thousand Miles (Figure 12) and the new variation with melodic variation (Figure 13). This result shows that using music sheets with too many musical note durations can lead to new variations with melodies that are difficult to play and listen to. This is a major weakness of this method.

4.3 Limitations and Considerations

Further evaluation with a wider range of musical pieces is necessary to determine the generalizability of these observations. The effectiveness of the combined approach might vary depending

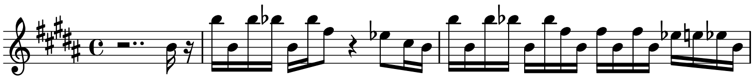


Figure 12: The original of Vanessa Carlton - A Thousand Miles in the first 3 bars.

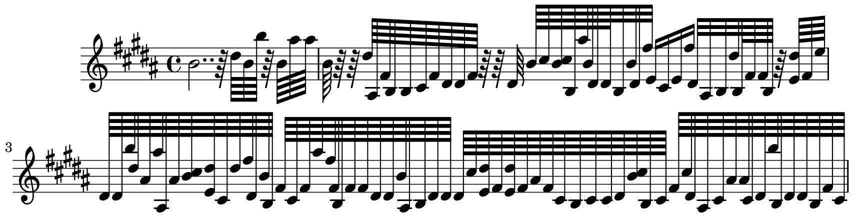


Figure 13: The new variation with melodic variation of Vanessa Carlton - A Thousand Miles in the first 3 bars.

on the musical style and characteristics of the original piece.

The computational complexity of the chaotic mapping technique should also be considered. While it offers a powerful approach for variation, it might require more processing power compared to simpler melodic variation methods.

4.4 Future Directions

Exploring different parameters and configurations within the chaotic mapping and melodic variation techniques could potentially lead to a wider range of variation styles and outcomes.

Investigating methods for user control over the variation process would be valuable. This could allow musicians to tailor the generated variations to their specific creative goals.

Integrating this combined approach into music composition tools could provide composers with a valuable tool for generating new musical ideas and exploring creative possibilities.

5 Conclusion

The proposed method, leveraging the power of chaotic systems, offers a promising approach to music composition, addressing the limitations of existing AI techniques and providing composers with a tool to generate novel, unpredictable, and diverse musical compositions while reducing computational resource demands. This method has the potential to stimulate creativity, alleviate composer's burnout, and expand the boundaries of musical expression, paving the way for a new era of music creation driven by chaos and innovation.

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