1 Literature Review

1.1 Fourth-Order Runge–Kutta Method

While analytical solutions exist for some differential equations, many require numerical approaches to approximate their behavior. The field of mathematics offers a numerical methods for solving both single and systems of linear and nonlinear differential equations. Popular examples include the Euler method and Taylor series methods. However, when it comes to achieving a balance between accuracy and efficiency, the Runge-Kutta method reigns supreme for approximating solutions.

The Runge-Kutta method [**bose˙numerical˙2019**] is a numerical methods for approximating solutions to ordinary differential equations. These equations describe how a quantity changes with respect to another variable, but often cannot be solved exactly. The Runge-Kutta method tackles these problems by breaking down the interval of interest into smaller subintervals and iteratively calculating the solution at each subinterval. Let $\dot{y} = f(t, y)$. The approximation of y_{i+1} by fourth-order Runge–Kutta method is given by:

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ k_1 &= f(t_i, y_i), \\ k_2 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right), \\ k_3 &= f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right), \\ k_4 &= f(t_i + h, y_i + hk_3), \end{aligned}$$

where $i = 0, 1, 2, ...$ and h is the step size, y is the variable and t is time.

2 Main Result

This section explores the application of three techniques for generating musical variations, using the Ah vous dirai-je, Maman melody as a starting point and Lorenz Equation for chaotic trajectory.

2.1 Musical Variations from a Chaotic Mapping

Let a sequence of pitches be represented by $P = \{p_1, p_2, \dots, p_n\}$. Let $\dot{x} = f(x)$ be a dynamical system with chaotic behavior. Then the approximate solution of $\dot{x}$ is denoted by $V = \{v_0, v_1, \dots, v_n\}$, when $v_0 \in \mathbb{R}_n$ is an initial condition of this trajectory. Let K be a mapping from pitch to real value defined by

$$K(p_i) = v_i.$$

Given $V' = \{v'_1, v'_2, \dots, v'_n\}$ be a sequence of new trajectory by an initial condition of $v'_0 \in \mathbb{R}_n$.

For each element v' in V', Let L be a mapping from real value to pitch defines by

$$L(v'_i) = p_{j^*}.$$

Where j^* is the chosen index for the new pitch according to the following criteria:

1. If there exist the smallest v_i such that $v_i > v'_i$, then the new pitch must agree to use differ pitch, which is $j^* = j$ when j is a index of a nearest v_i value element such that $v_i > v_j$.

2. If there exist a highest value v'_i such that $v_i < v'_i$ for all v_i in V, then the new pitch must agree to use highest v_i value index.

3. If $v_i = v'_j$, then the new pitch must agree to use the same pitch with the original pitch.

4. Otherwise, the new pitch must agree to use lowest v_i value index.

This creates a new sequence of pitches which can be represented by $\hat{P} = \{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n\}$.

Example 1. *Let a sequence of pitches of 12 variations on Ah vous dirai-je Maman in the first 3 bars in Figure 1 be represented by* $P = \{C4, C4, G4, G4, A4, A4, G4, F4, F4, E4, E4\}$. *Let Lorenz system be a dynamical system with chaotic behavior by giving Lorenz parameters* $r = 28, \sigma = 10$ *and* $b = \frac{8}{3}$. *Then the approximate solution of Lorenz system using fourth-order Runge–Kutta method with initial condition of* $(1, 1, 1)$ *is denoted by* $X = \{1.00, 1.29, 2.13, 3.74, 6.54, 11.04, 16.69, 19.56, 15.37, 7.55, 1.20\}$ *when* X *is sequence of x-value of approximate solution of Lorenz system. Let* K *be a mapping from pitch to real value result as follows:*

$$\begin{aligned}
K(C4) &= 1.00,\\
K(C4) &= 1.29,\\
K(G4) &= 2.13,\\
K(G4) &= 3.74,\\
K(A4) &= 6.54,\\
K(A4) &= 11.04,\\
K(G4) &= 16.69,\\
K(F4) &= 19.56,\\
K(F4) &= 15.37,\\
K(E4) &= 7.55,\\
K(E4) &= 1.20.
\end{aligned}$$

Given $X' = \{1.01, 1.30, 2.15, 3.76, 6.58, 11.10, 16.73, 19.55, 15.30, 7.48, 1.15\}$ *be a sequence of new trajectory with an initial condition of* $(1.01, 1, 1)$ *Let* L *be a mapping from real value to pitch result as follows:*

$$\begin{aligned}
L(1.01) &= C4,\\
L(1.30) &= E4,\\
L(2.15) &= G4,\\
L(3.76) &= G4,\\
L(6.58) &= A4,\\
L(11.10) &= A4,\\
L(16.73) &= F4,\\
L(19.55) &= G4,\\
L(15.30) &= A4,\\
L(7.48) &= A4,\\
L(1.15) &= C4.
\end{aligned}$$

This creates a new sequence of pitches which can be represented by $\hat{P} = \{C4, E4, G4, G4, A4, A4, F4, G4, A4, A4, C4\}$. *Which can be converted to sheet music, as shown in Figure 2. Since this method uses the same note duration and musical notes to create a new variation, so the resulting changes in the sequence compared to the original sequence might seem relatively small.*

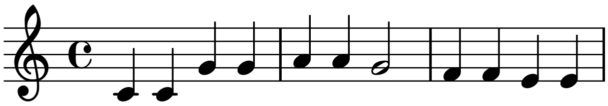

Figure 1: The original of 12 variations on Ah vous dirai-je Maman in the first 3 bars.

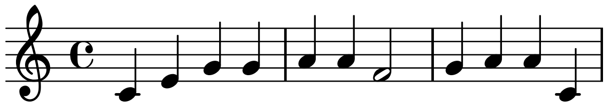

Figure 2: The new variation of Ah vous dirai-je, maman in the first 3 bars, generated by the Initial Condition $(1.01, 1, 1)$.

2.2 Melodic Variation with Expanded Rhythm Method

Given a note duration denoted by ϕ and divided into D equal parts, the duration, R, of each individual division is defined by the following equation:

$$R = \frac{\phi}{D}.$$

Note: In musical theory, equal parts refers to divisions that all have the same duration.

Example 2. *Consider the music piece 12 variations on Ah vous dirai-je Maman, illustrated in Figure 3. The Figure shows that we already have 6 quarter notes and 1 half note.*

If we want to divide each musical note into 4 parts, we can find the duration of each individual division as follows:

- *Half note to 4 parts:*
$$R = \frac{\phi}{D} = \frac{2}{4} = 0.5$$
- *Quarter note to 4 parts:*
$$R = \frac{\phi}{D} = \frac{1}{4} = 0.25$$

Following this calculation, a half note can be divided into 4 eighth notes, and a quarter note can be divided into 4 sixteenth notes. This division is represented by the sequence $P = \{C4, C4, C4, C4, C4, C4, C4, C4, G4, G4, G4, G4, G4, G4, G4, G4, A4, A4, A4, A4, A4, A4, A4, A4, G4, G4, G4, G4\}$ *which can be converted to sheet music, as shown in Figure 4.*

2.3 Combining Musical Variations from a Chaotic Mapping and Melodic Variation with Expanded Rhythm

Let a sequence of pitches be represented by $P = \{p_1, p_2, \ldots, p_n\}$ and a sequence of expanded rhythm pitches be represented by $P^* = \{p_1^*, p_2^*, \ldots, p_n^*\}$. Let $\dot{x} = f(x)$ be a dynamical system with chaotic behavior. Then the approximate solution of $\dot{x}$ is denoted by $V = \{v_0, v_1, \ldots, v_n\}$,

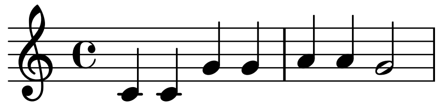

Figure 3: The original of 12 variations on Ah vous dirai-je Maman in the first 2 bars.

Figure 4: The melodic variation of Ah vous dirai-je, maman in the first 2 bars.

when $v_0 \in \mathbb{R}_n$ is an initial condition of this trajectory. Let K be a mapping from pitch to real value defined by

$$K(p_i) = v_i.$$

Given $V' = \{v'_1, v'_2, \ldots, v'_n\}$ be a sequence of new trajectory with the same number of members as P^* by an initial condition of $v'_0 \in \mathbb{R}_n$.

For each element v' in V', Let L be a mapping from real value to pitch defines by

$$L(v'_i) = p_{j^*}.$$

Where j^* is the chosen index for the new pitch according to the following criteria:

1. If there exist the smallest v_i such that $v_i > v'_i$, then the new pitch must agree to use differ pitch, which is $j^* = j$ when j is a index of a nearest v_i value element such that $v_i > v_j$.
2. If there exist a highest value v'_i such that $v_i < v'_i$ for all v_i in V, then the new pitch must agree to use highest v_i value index.
3. If $v_i = v'_j$, then the new pitch must agree to use the same pitch with the original pitch.
4. Otherwise, the new pitch must agree to use lowest v_i value index.

This creates a new sequence of pitches which can be represented by $\hat{P} = \{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n\}$.

Example 3. *Let a sequence of pitches of 12 variations on Ah vous dirai-je Maman in the first 2 bars in Figure 3 be represented by* $P = \{C4, C4, G4, G4, A4, A4, G4\}$ *and a sequence of expanded rhythm pitches on Figure 4 be represented by* $P^* = \{C4, C4, C4, C4, C4, C4, C4, C4, G4, G4, G4, G4, G4, G4, G4, G4, A4, A4, A4, A4, A4, A4, A4, A4, G4, G4, G4, G4\}$*. Let Lorenz system be a dynamical system with chaotic behavior by giving Lorenz parameters* $r = 28, \sigma = 10$ *and* $b = \frac{8}{3}$*. Then the approximate solution of Lorenz system using fourth-order Runge–Kutta method with initial condition of* $(1, 1, 1)$ *is denoted by* $X = \{1.00, 1.29, 2.13, 3.74, 6.54, 11.04, 16.69, 19.56, 15.37, 7.55, 1.20\}$ *when* X *is sequence of x-value of approximate solution of Lorenz system. Let* K *be a mapping from pitch to real value result as follows:*

Figure 5: The new variation with melodic variation of Ah vous dirai-je, maman in the first 2 bars, generated by the Initial Condition $(1.01, 1, 1)$.

$$\begin{aligned}
K(C4) &= 1.00,\\
K(C4) &= 1.29,\\
K(G4) &= 2.13,\\
K(G4) &= 3.74,\\
K(A4) &= 6.54,\\
K(A4) &= 11.04,\\
K(G4) &= 16.69.
\end{aligned}$$

Given $X' = \{1.01, 1.30, 2.15, 3.76, 6.58, 11.10, 16.73, 19.55, 15.30, 7.48, 1.15, -2.70, -4.85, -6.13, -7.06, -7.87, -8.64, -9.29, -9.70, -9.73, -9.37, -8.76, -8.07, -8.50, -7.17, -7.12, -7.37, -7.85\}$ *be a sequence of new trajectory with an initial condition of* $(1.01, 1, 1)$ *Let* L *be a mapping from real value to pitch result as follows:*

$$\begin{aligned}
L(1.01) &= C4, & L(-7.06) &= C4,\\
L(1.30) &= C4, & L(-7.87) &= C4,\\
L(2.15) &= G4, & L(-8.64) &= C4,\\
L(3.76) &= G4, & L(-9.29) &= C4,\\
L(6.58) &= A4, & L(-9.70) &= C4,\\
L(11.10) &= A4, & L(-9.73) &= C4,\\
L(16.73) &= A4, & L(-9.37) &= C4,\\
L(19.55) &= G4, & L(-8.76) &= C4,\\
L(15.30) &= A4, & L(-8.07) &= C4,\\
L(7.48) &= A4, & L(-8.50) &= C4,\\
L(1.15) &= C4, & L(-7.17) &= C4,\\
L(-2.70) &= C4, & L(-7.12) &= C4,\\
L(-4.85) &= C4, & L(-7.37) &= C4,\\
L(-6.13) &= C4, & L(-7.85) &= C4.
\end{aligned}$$

This creates a new sequence of pitches which can be represented by $\hat{P} = \{C4, C4, G4, G4, A4, A4, A4, G4, A4, A4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4, C4\}$. *Which can be converted to sheet music, as shown in Figure 5.*