Classical Mechanics [Taylor, J.R.] Solution Manual

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Lagrange's Equations

7.1 Write down the Lagrangian for a projectile (subject to no air resistance) in terms of its Cartesian coordinates (x, y, z), with z measured vertically upward. Find the three Lagrange equations and show that they are exactly what you would expect for the equations of motion.

Solution. The Lagrangian can be easily identified and written as such:

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \end{aligned} \tag{7.1}$$

A direct application of the Euler-Lagrange equations yields:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \quad \Rightarrow \quad m\ddot{x} = 0 \tag{7.2}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \quad \Rightarrow \quad m\ddot{y} = 0 \tag{7.3}$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \quad \Rightarrow \quad m\ddot{z} = -mg \tag{7.4}$$

All three of which are expected for a projectile in free fall. Note the negative sign on the z direction is due to the chosen sign convention. □

7.8 (a) Write down the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x_1}, \dot{x_2})$ for two particles of equal masses, $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = \frac{1}{2}kx^2$. [Here x is the extension of the spring, $x = (x_1 - x_2 - l)$, where l is the spring's unstretched length, and I assume that mass 1 remains to the right of mass 2 at all times.] (b) Rewrite $\mathcal{L}$ in terms of the new variables $X = \frac{1}{2}(x_1+x_2)$ (the CM position) and x (the extension), and write down the two Lagrange equations for X and x. (c) Solve for $X(t)$ and $x(t)$ and describe the motion.

Solution. (a) The Lagrangian is simply:

$$\mathcal{L} = \frac{1}{2}m(\dot{x_1}^2 + \dot{x_2}^2) - \frac{1}{2}k(x_1 - x_2 - l)^2 \tag{7.5}$$

(b) We note that we can manipulate the new variables $X = \frac{1}{2}(x_1 + x_2)$ and $x = (x_1 - x_2 - l)$, as such:

$$\dot{X} = \frac{1}{2}(\dot{x_1} + \dot{x_2}) \qquad \text{and} \qquad \dot{x} = \dot{x_1} - \dot{x_2} \tag{7.6}$$

$$\begin{aligned} \dot{x_1} &= \dot{X} + \frac{1}{2}\dot{x} \\ \dot{x_2} &= \dot{X} - \frac{1}{2}\dot{x} \end{aligned} \tag{7.7}$$

We can then rewrite the Lagrangian from Equation (7.5) as:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m\left[(\dot{X} + \frac{1}{2}\dot{x})^2 + (\dot{X} - \frac{1}{2}\dot{x})^2\right] - \frac{1}{2}kx^2 \\ &= \frac{1}{2}m\left(2\dot{X}^2 + \frac{1}{2}\dot{x}^2\right) - \frac{1}{2}kx^2 \end{aligned} \tag{7.8}$$

The Lagrange equations can be written fairly easily, one for each coordinate.

$$(1): \qquad \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{7.9}$$

$$-kx = \frac{1}{2}m\ddot{x} \quad \Rightarrow \quad \ddot{x} = -\frac{2k}{m}x \tag{7.10}$$

$$(2): \qquad \frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \tag{7.11}$$

$$0 = 2m\ddot{X} \quad \Rightarrow \quad \ddot{X} = 0 \tag{7.12}$$

(c) Now we are required to solve for $X(t)$ and $x(t)$. We tackle the easier of the two first:

$$\ddot{X} = 0 \quad \Rightarrow \quad X(t) = v_0 t + X_0 \tag{7.13}$$

where we have taken the liberty to introduce two integration constants that depend upon the initial conditions. As for the other coordinate x, we invoke the familiar SHM solution:

$$\ddot{x} = -\frac{2k}{m}x \quad \Rightarrow \quad x(t) = A\cos\left(\sqrt{\frac{2k}{m}} + \delta\right) \tag{7.14}$$

where we have introduced two other integration constants A and δ. The center of mass moves with constant velocity since no external forces are acting on it. The extension of the spring undergoes simple harmonic motion, which just means that the two masses are oscillating with respect to each other. $\square$

7.33 A bar of soap (mass m) is at rest on a frictionless rectangular plate that rests on a horizontal table. At time $t = 0$, I start raising one edge of the plate so that the plate pivots about the opposite edge with constant angular velocity ω, and the soap starts to slide toward the downhill edge. Show that the equation of motion for the soap has the form $\ddot{x} - \omega^2 x = -g \sin \omega t$, where x is the soap's distance from the downhill edge. Solve this for $x(t)$, given that $x(0) = x_0$. [You can easily solve the homogeneous equation; for a particular solution try $x = B \sin \omega t$ and solve for B.]

Solution. To obtain the equation of motion, we shall utilise the Lagrange equations. First, we write the Lagrangian, noting that the motion of the soap consists of both translational and rotational motion:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 - mgx\sin\omega t \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\omega^2 - mgx\sin\omega t\end{aligned} \tag{7.15}$$

Applying the Lagrange equations, we get

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x} \quad \Rightarrow \quad m\ddot{x} &= m\omega^2 x - mg\sin\omega t \\ \ddot{x} - \omega^2 x &= -g\sin\omega t \quad \text{(shown)}\end{aligned} \tag{7.16}$$

Note that Equation (7.16) is a second order, inhomogeneous differential equation. We first consider the homogeneous equation, which has a simple harmonic motion solution.

$$x_H(t) = A_1 e^{\omega t} + A_2 e^{-\omega t} \tag{7.17}$$

Now we consider the particular solution. We make a trigonometric ansatz $x_P = B\sin\omega t$. Substituting into Equation (7.16),

$$-\omega^2 B\sin\omega t - \omega^2 B\sin\omega t = -g\sin\omega t \tag{7.18}$$

$$\Rightarrow \quad B = \frac{g}{2\omega^2} \tag{7.19}$$

We can then compose our general solution for $x(t)$:

$$\begin{aligned}x(t) &= x_H + x_P \\ &= A_1 e^{\omega t} + A_2 e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t\end{aligned} \tag{7.20}$$

We have the initial conditions: $x(0) = x_0$ and $\dot{x}(0) = 0$, which yield us:

$$A_1 + A_2 = x_0 \qquad \text{and} \qquad \omega(A_1 - A_2) + \frac{g}{2\omega} = 0 \tag{7.21}$$

Using these conditions, we can solve for the integration constants A_1 and A_2:

$$A_1 = \frac{x_0}{2} - \frac{g}{4\omega^2} \tag{7.22}$$

$$A_2 = \frac{x_0}{2} + \frac{g}{4\omega^2} \tag{7.23}$$

Finally, we are able to construct our general equation as:

$$x(t) = \left(\frac{x_0}{2} - \frac{g}{4\omega^2}\right) e^{\omega t} + \left(\frac{x_0}{2} + \frac{g}{4\omega^2}\right) e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t \tag{7.24}$$

$$= x_0 \cosh \omega t - \frac{g}{2\omega^2} \sinh \omega t + \frac{g}{2\omega^2} \sin \omega t \tag{7.25}$$

□

7.40 The "spherical pendulum" is just a simple pendulum that is free to move in any sideways direction. (By contrast a "simple pendulum" is confined to a single vertical plane.) The bob of a spherical pendulum moves on a sphere, centered on the point of support with radius $r = R$, the length of the pendulum. A convenient choice of coordinates is spherical polars, r, θ, ϕ, with the origin at the point of support and the polar axis ($z-$axis) pointing straight down. The two variables θ and ϕ make a good choice of generalized coordinates. (a) Find the Lagrangian and the two Lagrange equations. (b) Explain what the ϕ equation tells us about the z component of angular momentum l_z. (c) For the special case that ϕ =const, describe what the θ equation tells us. (d) Use the ϕ equation to replace $\dot{\phi}$ by l_z in the θ equation and discuss the existence of an angle θ_0 at which θ can remain constant. Why is this motion called a conical pendulum? (e) Show that if $\theta = \theta_0 + \epsilon$, with ϵ small, then θ oscillates about θ_0 in harmonic motion. Describe the motion of the pendulum's bob.

Solution. (a) With reference to Figure 7.1, we can write the Lagrangian in spherical coordinates as:

$$\mathcal{L} = \frac{1}{2} m(R^2 \dot{\theta}^2 + R^2 \dot{\phi} \sin^2 \theta) + mgR \cos \theta \tag{7.26}$$

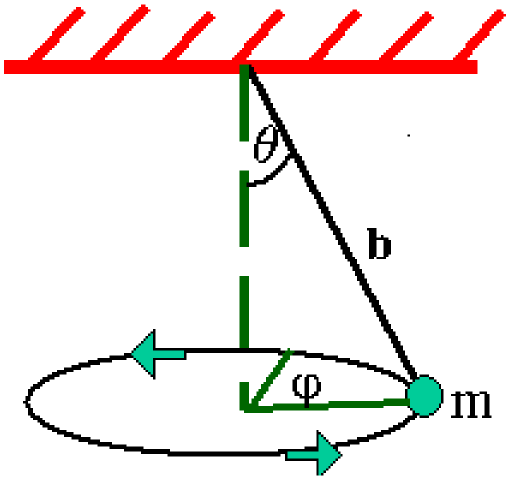


Figure 7.1: Sketch of the problem

Then, applying Lagrange equation to the two generalized coordinates:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\Rightarrow \qquad mR^2\ddot{\theta} = mR^2\dot{\phi}^2 \sin\theta\cos\theta - mgR\sin\theta \tag{7.27}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\Rightarrow \qquad mR^2\dot{\phi}\sin^2\theta = l_z = \text{const} \tag{7.28}$$

We leave it to the reader to simplify Equation (7.27). As for the latter equation, we have noted that the Lagrangian is independent of ϕ, such that $\frac{\partial \mathcal{L}}{\partial \phi} = 0$.

(b) Equation (7.28) tells us that the z component of the angular momentum is constant.

(c) If $\phi = \text{const}$, then $\dot{\phi} = 0$, and Equation (7.27) reduces to:

$$\ddot{\theta} = -\frac{g}{R}\sin\theta \tag{7.29}$$

That is, we can describe the motion of the pendulum like a simple pendulum, at a fixed plane characterized by $\phi = \phi_0$, for some fixed ϕ_0.

(d) We want to first express $\dot{\phi}$ in terms of l_z (using Equation (7.28), then substitute it into Equation (7.27).

$$\dot{\phi} = \frac{l_z}{mR^2 \sin^2\theta} \tag{7.30}$$

$$\begin{aligned} \ddot{\theta} &= \left(\frac{l_z}{mR^2 \sin^2\theta}\right)^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta \\ &= \frac{l_z^2 \cos\theta}{m^2R^4 \sin^3\theta} - \frac{g}{R}\sin\theta \end{aligned} \tag{7.31}$$

To explore a possible stationary solution of θ, we consider $\ddot{\theta} = 0$. Following from Equation (7.31), we obtain:

$$\frac{l_z^2 \cos\theta}{m^2R^4 \sin^3\theta} = \frac{g}{R}\sin\theta \tag{7.32}$$

$$\cos\theta = k\sin^4\theta = k(1-\cos^2\theta)^2 \tag{7.33}$$

where we have introduced the constant $k = gm^2R^3/l_z^2$. To analyze the solution, we substitute $x = \cos\theta$, into Equation (7.33) where $x \in [0,1]$. WolframAlpha gives us the following plot: We note

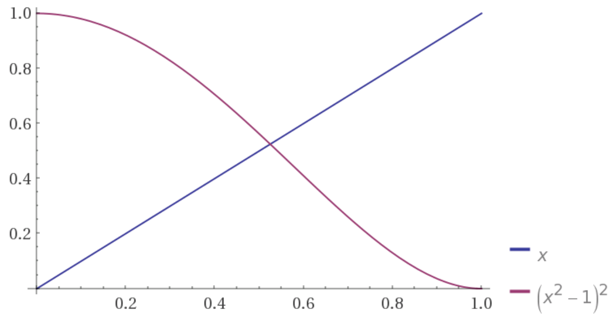


Figure 7.2: Plot of Equation (7.33)

that there is only one solution for which Equation (7.33) is satisfied. This would correspond to the solution θ_0, as specified in the question.

(e) Consider a perturbation, $\theta = \theta_0 + \epsilon$. We can Taylor expand the following quantities about the equilibrium position x_0, and also write down the derivatives of θ:

$$\cos\theta \approx \cos\theta_0 - \epsilon\sin\theta_0 \tag{7.34}$$

$$\sin\theta \approx \sin\theta_0 + \epsilon\cos\theta_0 \tag{7.35}$$

$$\ddot{\theta} = \ddot{\epsilon} \tag{7.36}$$

Before we substitute these equations into Equation (7.31), we recall that at equilibrium, Equation (7.32) is satisfied. That is, we have:

$$\frac{l_z^2}{m^2R^4} = \frac{g}{R}\frac{\sin^4\theta_0}{\cos\theta_0} \tag{7.37}$$

which we can use to simplify our expression slightly later on. With the last 4 equations from above, Equation (7.31) gives us:

$$\ddot{\epsilon} \approx \frac{g}{R}\frac{\sin^4\theta_0}{\cos\theta_0}(\cos\theta_0 - \epsilon\sin\theta_0)(\sin\theta_0 + \epsilon\cos\theta_0)^{-3} - \frac{g}{R}(\sin\theta_0 + \epsilon\cos\theta_0) \tag{7.38}$$

$$= \frac{g}{R}\frac{\sin^4\theta_0}{\cos\theta_0}(\cos\theta_0 - \epsilon\sin\theta_0)(\sin^{-3}\theta_0 - 3\epsilon\sin^{-4}\theta_0\cos\theta_0) - \frac{g}{R}(\sin\theta_0 + \epsilon\cos\theta_0) \tag{7.39}$$

After some simplification and ignoring terms in ϵ^2, we obtain:

$$\ddot{\epsilon} = -\epsilon\frac{g}{R}\left(\frac{1 + 3\cos^2\theta_0}{\cos\theta_0}\right) \tag{7.40}$$

This shows that the pendulum will oscillate in a simple harmonic motion, with angular frequency approximately $\omega^2 = \frac{g}{R}\left(\frac{1 + 3\cos^2\theta_0}{\cos\theta_0}\right)$, about the equilibrium position θ_0. □

Chapter 8

Two-Body Central-Force Problems

8.13 Two particles whose reduced mass is μ interact via a potential energy $U = \frac{1}{2}kr^2$, where r is the distance between them. (a) Make a sketch showing $U(r)$, the centrifugal potential energy $U_{\text{cf}}(r)$, and the effective potential energy $U_{\text{eff}}(r)$. (Treat the angular momentum l as a known, fixed constant.) (b) Find the "equilibrium" separation r_0, the distance at which the two particles can circle each other with constant r. [Hint: This requires that dU_{eff}/dr be zero.] (c) By making a Taylor expansion of $U_{\text{eff}}(r)$ about the equilibrium point r_0 and neglecting all terms in $(r - r_0)^3$ and higher, find the frequency of small oscillations about the circular orbit if the particles are disturbed a little from the separation r_0.

Solution. (a) A sketch of all three graphs are shown in the figure below.

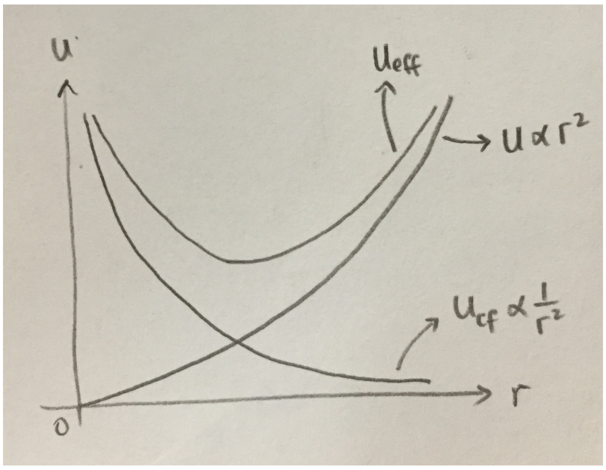


Figure 8.1: Graphs of the potentials $U(r)$, $U_{\text{cf}}(r)$ and $U_{\text{eff}}(r)$

We can write the potentials as such:

$$U(r) = \frac{1}{2}kr^2 \tag{8.1}$$

$$U_{\text{cf}}(r) = \frac{l^2}{2\mu r^2} \tag{8.2}$$

$$U_{\text{eff}}(r) = U(r) + U_{\text{cf}}(r) = \frac{1}{2}kr^2 + \frac{l^2}{2\mu r^2} \tag{8.3}$$

(b) The equilibrium separation by setting $\dfrac{dU_{\text{eff}}}{dr} = 0$. We obtain:

$$\frac{dU_{\text{eff}}}{dr} = kr_0 - \frac{l^2}{\mu r_0^3} = 0$$

$$\Rightarrow \qquad r_0 = \left(\frac{l^2}{\mu k}\right)^{1/4} \tag{8.4}$$

(c) Let us define $r = r_0 + \epsilon$. The Taylor expansion can be performed about the equilibrium r_0, noting that the first derivative was calculated before, and is equal to 0 at r_0:

$$\begin{aligned}
U_{\text{eff}}(r) &= U_{\text{eff}}(r_0) + \epsilon^2 \left.\frac{d^2U_{\text{eff}}(r)}{dr^2}\right|_{r=r_0+\epsilon} + \dots \\
&\approx \left(\frac{1}{2}kr_0^2 + \frac{l^2}{2\mu r_0^2}\right) + \epsilon^2 \left(k + \frac{3l^2}{\mu(r_0+\epsilon)^4}\right) \\
&\approx \left(\frac{1}{2}kr_0^2 + \frac{l^2}{2\mu r_0^2}\right) + \epsilon^2 \left(k + \frac{3l^2}{\mu}(r_0^{-4} + \epsilon\, r_0^{-5})\right) \\
&\approx \left(\frac{1}{2}kr_0^2 + \frac{l^2}{2\mu r_0^2}\right) + \epsilon^2 \left(k + \frac{3l^2}{\mu r_0^4}\right) + \epsilon^3 \frac{3l^2}{\mu r_0^5}
\end{aligned} \tag{8.5}$$

For small oscillations ϵ, we can ignore terms of ϵ^3 and above, and note that the effective potential takes on the form of a harmonic oscillator potential (e.g. one of an elastic spring). That is, it takes the form of $U(x) = U(0) + \frac{1}{2}k'x^2$, and the frequency of oscillation for such a system is $\omega = \sqrt{k'/m}$. Making such an observation, we can deduce the 'spring constant' k' and oscillation frequency ω:

$$k' = 2\left(k + \frac{3l^2}{\mu r_0^4}\right) = 2k + 6k = 8k$$

$$\omega = \sqrt{\frac{k'}{\mu}} = \sqrt{\frac{8k}{\mu}} \tag{8.6}$$

where we have used the result from Equation (8.4) to rewrite r_0 in terms of k, the central force constant. Do not confuse this with the spring constant k' used in the analogy earlier. □

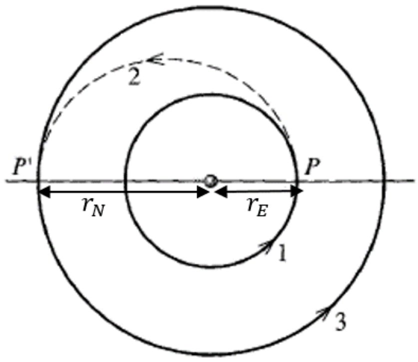


Figure 8.2: Transfer orbit for Problem 8.34

8.34 Suppose that we decide to send a spacecraft to Neptune, using the simple transfer described in Example 8.6 (c.f. textbook pg 318). The craft starts in a circular orbit close to the Earth (radius 1 AU) and is to end up in a circular orbit near Neptune (radius about 30 AU). Use Kepler's third law to show that the transfer will take about 31 years.

Solution. The transfer orbit in question is depicted in Figure 8.2, with the radius of the Earth orbit as r_E and the radius of the Neptune orbit as r_N.
Kepler's third law is stated as: $\dfrac{a^3}{\tau^3} = \dfrac{GM}{4\pi^2}$.
We know (from the figure) that the sum of the radii equals twice the semi-major axis a of the elliptical orbit. Explicitly, we have

$$r_E + r_N = 2a \qquad \Rightarrow \qquad a = 15.5 \text{ AU} \tag{8.7}$$

And so to get from P to P':

$$\begin{aligned} \text{Time of transfer} &= \frac{1}{2}\tau = \frac{1}{2}\sqrt{\frac{4\pi^2 a^3}{GM}} \\ &= 1.926 \times 10^9 \text{ s} \simeq 31 \text{ years} \qquad \text{(shown)} \end{aligned} \tag{8.8}$$

Note that I used the following constants: 1 AU $= 1.496 \times 10^{11}$ m, $G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}, $M_{\text{Sun}} = 1.989 \times 10^{30}$ kg. □

Chapter 9

Mechanics in Non-inertial Frames

9.1 No plans to do this chapter yet.

Chapter 10

Rotational Motion of Rigid Bodies

10.1 The result that $\sum m_\alpha \boldsymbol{r}'_\alpha = 0$, can be paraphrased to say that the position vector of the CM relative to the CM is zero, and, in this form, is nearly obvious. Nevertheless, to be sure that you understand the result, prove it by solving $\boldsymbol{r}_\alpha = \boldsymbol{R} + \boldsymbol{r}'_\alpha$ for $\boldsymbol{r}'_\alpha$ and substituting into the sum concerned.

Solution. We start with the equation given in the question, and substituting into the sum as suggested:

$$\boldsymbol{r}_\alpha = \boldsymbol{R} + \boldsymbol{r}'_\alpha \qquad \Rightarrow \qquad \boldsymbol{r}'_\alpha = \boldsymbol{r}_\alpha - \boldsymbol{R} \tag{10.1}$$

$$\sum m_\alpha \boldsymbol{r}'_\alpha = \sum (m_\alpha \boldsymbol{r}_\alpha) - \boldsymbol{R} \sum m_\alpha \tag{10.2}$$

where the sum is taken implicitly over α. We also have the following expression for the position of the center of mass:

$$\boldsymbol{R} = \frac{1}{M} \sum (m_\alpha \boldsymbol{r}_\alpha) \tag{10.3}$$

Using this relation, and noting that the term $\sum m_\alpha$ is simply the total mass M, Equation (10.2) becomes:

$$\sum m_\alpha \boldsymbol{r}'_\alpha = \boldsymbol{R}M - \boldsymbol{R}M = 0 \tag{10.4}$$

We have thus shown that the position of the CM relative to itself is zero. □

10.3 Five equal point masses are placed at the five corners of a square pyramid whose square base is centered on the origin in the xy plane, with side L, and whose apex is on the z axis at a height H above the origin. Find the CM of the five mass system.

Solution. The (x, y, z) coordinates of the 4 corners are $(L/2, L/2, 0)$, $(L/2, -L/2, 0)$, $(-L/2, L/2, 0)$, $(-L/2, -L/2, 0)$, while the coordinates of the apex is given by $(0, 0, H)$. Let m be the mass of one point mass. To calculate the CM of the system, we perform the following:

$$x_{\text{CM}} = \frac{m \sum_i x_i}{5m} = \frac{1}{5}\left(\frac{L}{2} + \frac{L}{2} - \frac{L}{2} - \frac{L}{2}\right) = 0 \tag{10.5}$$

$$y_{\text{CM}} = \frac{m \sum_i y_i}{5m} = \frac{1}{5}\left(\frac{L}{2} + \frac{L}{2} - \frac{L}{2} - \frac{L}{2}\right) = 0 \tag{10.6}$$

$$z_{\text{CM}} = \frac{m \sum_i z_i}{5m} = \frac{H}{5} \tag{10.7}$$

The coordinates of the CM is given by $(0, 0, H/5)$. □

10.4 The calculation of centers of mass or moments of inertia usually involves doing an integral, most often a volume integral, and such integrals are often best done in spherical polar coordinates. Prove that:

$$\int dV\, f(\boldsymbol{r}) = \int dr\, r^2 \int d\theta\, \sin\theta \int d\phi\, f(r, \theta, \phi).$$

[Think about the small volume dV enclosed between r and $r + dr$, θ and $\theta + d\theta$, and ϕ and $\phi + d\phi$.] If the volume integral on the left runs over all space, what are the limits of the three integrals on the right?

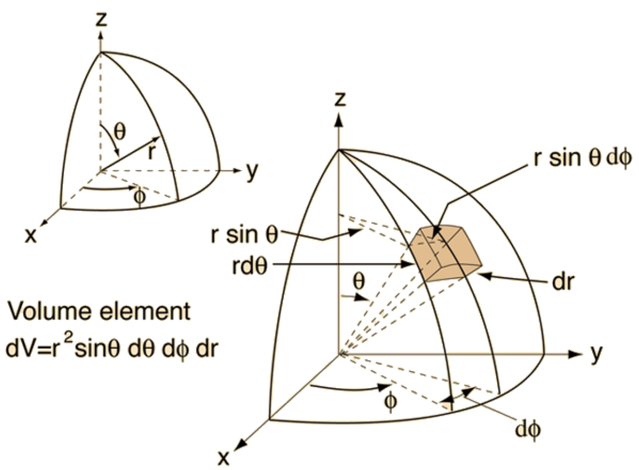


Figure 10.1: Taken from `http://astro-learned.blogspot.sg/`

Solution. Referring to the figure, a small volume element dV can be approximated to a cuboid with volume:

$$dV = (dr)(rd\theta)(r\sin\theta\, d\phi) = r^2 \sin\theta\, drd\theta d\phi \tag{10.8}$$

Substituting this into the integral, taking note of the independence (or dependence) of the integrand with respect to the 3 variables, we obtain, trivially, the result:

$$\int dV\, f(\boldsymbol{r}) = \int dr\, r^2 \int d\theta\, \sin\theta \int d\phi\, f(r,\theta,\phi). \tag{10.9}$$

Another approach to this question is by considering the Jacobian. We express the Cartesian coordinates in terms of the spherical polar coordinates:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta. \tag{10.10}$$

We then have the volume element as $dV = dxdydz = J(r,\theta,\phi)\, drd\theta d\phi$, with $J(r,\theta,\phi)$ being:

$$\begin{aligned} J(r,\theta,\phi) &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} \\ &= r^2\sin\theta \end{aligned} \tag{10.11}$$

This yields us the same result as above. The integration limits for a volume integral over all space are: $0 \leq r \leq \infty$, $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$. □

10.5 A uniform solid hemisphere of radius R has its flat base in the xy plane, with its center at the origin. Use the result of Problem 10.4 to find the center of mass.

Solution. Due to the inherent spherical symmetry of the object, one might expect the x and y coordinates of the center of mass to be zero. Nevertheless, they can be computed by:

$$x_{\text{CM}} = \frac{\int x\ \mathrm{d}m}{\int\ \mathrm{d}m} = 0 \qquad\qquad y_{\text{CM}} = \frac{\int y\ \mathrm{d}m}{\int\ \mathrm{d}m} = 0 \tag{10.12}$$

Let ρ be the density of the hemisphere. The z coordinate can be computed in a similar fashion:

$$
\begin{aligned}
z_{\text{CM}} &= \frac{\int z\ \mathrm{d}m}{\int\ \mathrm{d}m} = \frac{\rho\int r\cos\theta\,\mathrm{d}V}{\rho\int\ \mathrm{d}V} \\
&= \frac{\int_0^R r\cdot r^2\ \mathrm{d}r \int_0^{\pi/2}\sin\theta\cos\theta\ \mathrm{d}\theta\int_0^{2\pi}\ \mathrm{d}\phi}{\int_0^R r^2\ \mathrm{d}r\int_0^{\pi/2}\sin\theta\ \mathrm{d}\theta\int_0^{2\pi}\ \mathrm{d}\phi} \\
&= \frac{(R^4/4)\ (1/2)\ (2\pi)}{(R^3/3)\ (1)\ (2\pi)} = \frac{3R}{8}
\end{aligned}
\tag{10.13}
$$

□

10.6 (a) Find the CM of a uniform hemispherical shell of inner and outer radii a and b and mass M positioned as in Problem 10.5. (b) What becomes of your answer when $a = 0$? (c) What if $b \to a$?

Solution. (a) Once again, by symmetry we have $x_{\text{CM}} = y_{\text{CM}} = 0$. We set up a similar integral to compute z_{CM}, but with modified limits:

$$
\begin{aligned}
z_{\text{CM}} &= \frac{\int z\,dm}{\int dm} = \frac{\rho\int r\cos\theta dV}{\rho\int dV} \\
&= \frac{\int_a^b r\cdot r^2\,dr\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{2\pi}d\phi}{\int_a^b r^2\,dr\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{2\pi}d\phi} \\
&= \frac{\frac{1}{4}(b^4-a^4)\ (1/2)\ (2\pi)}{\frac{1}{3}(b^3-a^3)\ (1)\ (2\pi)} \\
&= \frac{3}{8}\left(\frac{b^4-a^4}{b^3-a^3}\right)
\end{aligned}
\tag{10.14}
$$

(b) Note that when $a = 0$, we are simply computing the CM of a uniform hemisphere (as in Problem 10.4), with $b = R$. It is then no surprise that our answer here is $\frac{3}{8}b$, which corresponds with our answer from the previous problem.

(c) When $b \to a$, the hemispherical shell becomes infinitely thin, with radius a. We would expect that z_{CM} approaches $a/2$. Now let us compute it. Note that here, a is simply a number, and so we are only treating b as a variable as it tends towards a. Since we have the limit of [0]/[0], we can use L'Hopital's rule:

$$
\begin{aligned}
z_{\text{CM}} &= \lim_{b\to a}\frac{3}{8}\left(\frac{b^4-a^4}{b^3-a^3}\right) \\
&= \frac{3}{8}\lim_{b\to a}\left(\frac{4b^3}{3b^2}\right) \\
&= \frac{3}{8}\left(\frac{4a^3}{3a^2}\right) = \boxed{\frac{a}{2}}
\end{aligned}
\tag{10.15}
$$

□

10.9 The moment of inertia of a continuous mass distribution with density ϱ is obtained by converting the sum $I_z = \sum m_\alpha \rho_\alpha^2$ into the volume integral $\int \rho^2 \, \mathrm{d}m = \int \rho^2 \varrho \, \mathrm{d}V$. Find the moment of inertia of a uniform circular cylinder of radius R and mass M for rotation about its axis. Explain why the products of inertia are zero.

Solution. For continuous masses, we have:

$$\begin{aligned} I_z &= \varrho \int \rho^2 \, \mathrm{d}V = \varrho \int_0^h \int_0^{2\pi} \int_0^R \rho^2 \cdot \rho \;\; \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}z \\ &= \varrho \, (h)(2\pi) \left(\frac{R^4}{4} \right) \end{aligned} \tag{10.16}$$

Since the density $\varrho = \dfrac{M}{\pi R^2 h}$, we finally obtain the moment of inertia I_z:

$$I_z = \boxed{\frac{MR^2}{2}} \tag{10.17}$$

The products of inertia are zero because of cylindrical symmetry. More explicitly, we have, for example:

$$\begin{aligned} I_{xy} &= \varrho \int -xy \;\; \mathrm{d}V \\ &= -\varrho \int_0^h \mathrm{d}z \int_0^R \rho^2 \cdot \rho \, \mathrm{d}\rho \underbrace{\int_0^{2\pi} \sin\theta \cos\theta \;\; \mathrm{d}\theta}_{=0} \\ &= 0 \end{aligned}$$

□

10.15 (a) Write down the integral (as in Problem 10.9) for the moment of inertia of a uniform cube of side a and mass M, rotating about an edge, and show that it is equal to $\frac{2}{3}Ma^2$. (b) If I balance the cube on an edge in unstable equilibrium on a rough table, it will eventually topple and rotate until it hits the table. By considering the energy of the cube, find its angular velocity just before it hits the table. (Assume the edge does not slide on the table.)

Solution. (a) The details of this integral can be found in the textbook's Example 10.2. I shall present the final answer here without proof:

$$\begin{aligned}
I &= \rho \int_{0,0,0}^{a,a,a} \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} dV \\
&= \frac{M}{a^3} \begin{pmatrix} \frac{2}{3}a^5 & -\frac{1}{4}a^5 & -\frac{1}{4}a^5 \\ -\frac{1}{4}a^5 & \frac{2}{3}a^5 & -\frac{1}{4}a^5 \\ -\frac{1}{4}a^5 & -\frac{1}{4}a^5 & \frac{2}{3}a^5 \end{pmatrix} \\
&= \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \qquad (10.18)
\end{aligned}$$

Hence we can directly read off the moment of inertia about an edge: $I = \frac{2}{3}Ma^2$.

(b) When the cube is resting stably on the table, the height of the CM above the table is $a/2$. When it is held at an unstable equilibrium (assuming balanced on an edge at a 45° angle), the height of the CM above the table is $\dfrac{a}{2\tan 45^\circ} = \dfrac{a}{\sqrt{2}}$. Hence we can calculate the change in the potential energy of the cube as it falls:

$$\Delta P.E. = Mg\left(\frac{a}{\sqrt{2}} - \frac{a}{2}\right) = Mga\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) \qquad (10.19)$$

The corresponding gain in kinetic energy is $\Delta K.E. = \dfrac{1}{2}I\omega^2$, where ω is the angular velocity just before it hits the table and I is the moment of inertia about an edge (calculated in (a)). Equating the two energies together, and solving for ω:

$$\omega^2 = \frac{3g}{2a}(\sqrt{2}-1) \qquad (10.20)$$

□

10.23 Consider a rigid plane body or "lamina", such as a flat piece of sheet metal, rotating about a point O in the body. If we choose axes so that the lamina lies in the xy plane, which elements of the inertia tensor I are automatically zero? Prove that $I_{zz} = I_{xx} + I_{yy}$.

Solution. If the lamina is on the xy plane, then $z = 0$ automatically. Hence, products of inertia containing z is thus automatically zero. Explicitly, $I_{xz} = I_{zx} = I_{zy} = I_{yz} = 0$. We will not be able to deduce anything about I_{yx} and I_{xy}.

Consider:

$$
\begin{aligned}
I_{zz} &= \rho \int x^2 + y^2 dV \\
&= \rho \int x^2 + z^2 dV + \rho \int y^2 + z^2 dV \\
&= I_{yy} + I_{xx}
\end{aligned} \tag{10.21}
$$

We can perform the second step because $z^2 = 0$. □

10.25 (a) Find all nine elements of the moment of inertia tensor with respect to the CM of a uniform cuboid (a rectangular brick shape) whose sides are $2a, 2b$ and $2c$ in the x, y, z directions and whose mass is M. Explain clearly why you could write down the off-diagonal elements without doing any integration. (b) Combine the results of part (a) and Problem 10.24 to find the moment of inertia tensor of the same cuboid with respect to the corner A at (a, b, c). (c) What is the angular momentum about A if the cuboid is spinning with angular velocity ω around the edge through A and parallel to the x axis?

Solution. (a)

$$
I = \rho \iiint_{-c,-b,-a}^{c,\ b,\ a} \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} dV \tag{10.22}
$$

Identify the density to be $\rho = \frac{M}{8abc}$. We compute the diagonal elements first.

$$
\begin{aligned}
I_{xx} &= \rho \int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} y^2 + z^2 \ dxdydz \\
&= \frac{M}{8abc} \int_{-c}^{c} \int_{-b}^{b} 2a(y^2 + z^2) \ dydz \\
&= \frac{M}{4bc} \int_{-c}^{c} \frac{2b^3}{3} + 2bz^2 \ dz \\
&= \frac{M}{2c} \left[\frac{2b^2 c}{3} + \frac{2c^3}{3} \right] = \frac{M}{3}(b^2 + c^2) \\
I_{yy} &= \cdots = \frac{M}{3}(a^2 + c^2) \\
I_{zz} &= \cdots = \frac{M}{3}(a^2 + b^2)
\end{aligned}
$$

where the other two moments of inertia can be directly calculated, or inferred from I_{xx} (due to the 'symmetry' present).

The off-diagonal elements are zero because: $-xy, -yz, -xz$ etc. are all odd functions in x, y, z. Integrating from $-a$ to a of x, for example, will yield 0.

(b) We shift the 'origin' by $\boldsymbol{d} = (a, b, c)$. Using the results from Problem 10.24, the modified moment of inertia is given by:

$$I' = \frac{M}{3}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} + M\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ba & a^2+c^2 & -bc \\ -ca & -cb & a^2+b^2 \end{pmatrix}$$
$$= \frac{M}{3}\begin{pmatrix} 4b^2+4c^2 & -3ab & -3ac \\ -3ba & 4a^2+4c^2 & -3bc \\ -3ca & -3cb & 4a^2+4b^2 \end{pmatrix} \tag{10.23}$$

(c) If the angular velocity $\boldsymbol{\omega}$ is parallel to the x axis, we can write $\boldsymbol{\omega} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$. The angular momentum is then given as the matrix multiplication:

$$\boldsymbol{L} = I\boldsymbol{\omega} = \frac{M\omega}{3}\begin{pmatrix} 4(b^2+c^2) \\ -3ba \\ -3ca \end{pmatrix} \tag{10.24}$$

□

10.26 (a) Prove that in cylindrical polar coordinates, a volume integral takes the form

$$\int \mathrm{d}V f(\boldsymbol{r}) = \int \rho\,\mathrm{d}\rho \int \mathrm{d}\phi \int \mathrm{d}z f(\rho, \phi, z).$$

(b) Show that the moment of inertia of a uniform solid cone pivoted at its tip and rotating about its axis is given by the integral:

$$I_{zz} = \varrho \int_V \mathrm{d}V \rho^2 = \varrho \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi \int_0^r \rho\,\mathrm{d}\rho\,\rho^2\,,$$

explaining clearly the limits of integration. Show that the integral evaluates to $\frac{3}{10}MR^2$. (c) Prove also that $I_{xx} = \frac{3}{20}M(R^2+4h^2)$.

Solution. (a) Using cylindrical polar coordinates, we have the following coordinate transformations: $x = \rho\cos\phi$, $y = \rho\sin\phi$. Hence consider the Jacobian for a small volume element:

$$J(\rho, \phi, z) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \tag{10.25}$$

And so we have $\mathrm{d}V = \rho\ \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}z$, which then yields us the integral as required.

(b) If we were to use cylindrical coordinates, I_{zz} will be calculated as:

$$I_{zz} = \varrho \int_V x^2 + y^2\, \mathrm{dV} = \varrho \int_V \rho^2\, \mathrm{dV}$$

Utilising the result from part (a), we then have:

$$I_{zz} = \varrho \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi \int_0^r \rho^3\, \mathrm{d}\rho \tag{10.26}$$

where we have the density: $\varrho = \dfrac{M}{1/3\ \pi R^2 h}$. The volume could have been obtained by considering the triple integral $\iiint\ \mathrm{d}V$, but here we quote a direct result.

The limits of integration are as such:

- h is the height of the cone.
- 2π is taken due to the cylindrical symmetry.
- $r = R \cdot \dfrac{z}{h}$ is derived from considering a set of similar triangles in Figure 10.2, with R being the radius of the base of the cone.

Let us now evaluate this integral.

$$\begin{aligned}
I_{zz} &= \varrho \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi\ \frac{R^4 z^4}{4h^4} \\
&= \varrho \int_0^h \mathrm{d}z\ \frac{\pi R^4 z^4}{2h^4} \\
&= \frac{3M}{\pi R^2 h} \cdot \frac{\pi R^4 h}{10} \\
&= \frac{3}{10} M R^2 \qquad \text{(shown)}
\end{aligned} \tag{10.27}$$

(c) Using the same notations as before, I_{xx} is calculated as:

$$\begin{aligned}
I_{xx} &= \varrho \int_V y^2 + z^2\, \mathrm{d}V = \varrho \int_V \rho^2 \sin^2\phi + z^2\, \mathrm{d}V \\
&= \varrho \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\phi \int_0^r \rho^3 \sin^2\phi + \rho z^2\, \mathrm{d}\rho \\
&= \varrho \int_0^h \mathrm{d}z \int_0^{2\pi} \frac{R^4 z^4}{4h^4} \sin^2\phi + \frac{R^2 z^4}{2h^2}\, \mathrm{d}\phi \\
&= \varrho \int_0^h \frac{\pi R^4 z^4}{4h^4} + \frac{\pi R^2 z^4}{h^2}\, \mathrm{d}z \\
&= \frac{3}{20} M (R^2 + 4h^2) \qquad \text{(shown)}
\end{aligned}$$

□

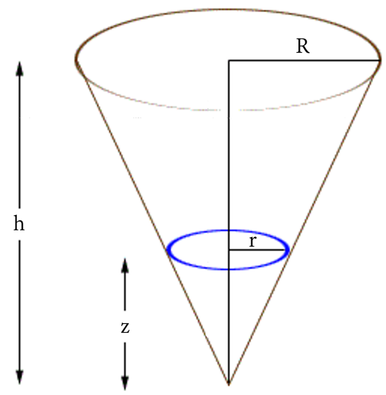


Figure 10.2: Similar triangles in a cone for Problems 10.26 and 10.27

10.27 Find the inertia tensor for a uniform, thin hollow cone, such as an ice-cream cone, of mass M, height h, and base radius R, spinning about its pointed end.

Solution. At any particular z, we have the corresponding radius $r = R \cdot \dfrac{z}{h}$. We obtain this ratio by considering a set of similar triangles, as seen in Figure 10.2. Using cylindrical coordinates (r, θ, z), we first calculate the density $\varrho = M/A$, where A is the curved surface area of the cone.

$$\begin{aligned} A = \int \mathrm{d}A &= \int_0^h \int_0^{2\pi} r \,\mathrm{d}\theta \,\mathrm{d}z \\ &= \pi R h \end{aligned} \tag{10.28}$$

$$\Rightarrow \qquad \varrho = \frac{M}{\pi R h} \tag{10.29}$$

The inertia tensor is then given by:

$$\begin{aligned} I &= \varrho \iint \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix} dA \\ &= \varrho \int_0^h \int_0^{2\pi} \begin{pmatrix} r^2\sin^2\theta + z^2 & -r^2\sin\theta\cos\theta & -rz\cos\theta \\ -r^2\sin\theta\cos\theta & r^2\cos^2\theta + z^2 & -rz\sin\theta \\ -rz\cos\theta & -rz\sin\theta & r^2 \end{pmatrix} r \,\mathrm{d}\theta \,\mathrm{d}z \end{aligned} \tag{10.30}$$

Note that off-diagonal elements equal to zero because

$$\int_0^{2\pi} \sin\theta \,\mathrm{d}\theta = \int_0^{2\pi} \cos\theta \,\mathrm{d}\theta = \int_0^{2\pi} \sin\theta\cos\theta \,\mathrm{d}\theta = 0$$

Using the fact that $\displaystyle\int_0^{2\pi} \sin^2\theta \,\mathrm{d}\theta = \int_0^{2\pi} \cos^2\theta \,\mathrm{d}\theta = \pi$, we have the diagonal components of the inertia tensor to be:

$$\begin{aligned} I_{xx} = I_{yy} &= \varrho \int_0^h \pi \frac{R^3 z^3}{h^3} + 2\pi \frac{R z^3}{h} \,\mathrm{d}z \\ &= \frac{M}{4}(R^2 + 2h^2) \end{aligned} \tag{10.31a}$$

$$\begin{aligned} I_{zz} &= \varrho \int_0^h 2\pi \frac{R^3 z^3}{h^3} \,\mathrm{d}z \\ &= \frac{MR^2}{2} \end{aligned} \tag{10.31b}$$

Finally we obtain the inertia tensor for a hollow cone to be:

$$I = \frac{M}{4} \begin{pmatrix} R^2 + 2h^2 & 0 & 0 \\ 0 & R^2 + 2h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix} \tag{10.32}$$

□

10.35 A rigid body consists of three masses fastened as follows: m at $(a, 0, 0)$, $2m$ at $(0, a, a)$ and $3m$ at $(0, a, -a)$. (a) Find the inertia tensor $\boldsymbol{I}$. (b) Find the principal moments and a set of orthogonal principle axes.

Solution. (a) The inertia tensor of a rigid body with discrete masses is given by:

$$\begin{aligned} I &= \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & x_i^2 + z_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & x_i^2 + y_i^2 \end{pmatrix} \\ &= \begin{pmatrix} m(0) + 2m(2a^2) + 3m(2a^2) & 0 & 0 \\ 0 & m(a^2) + 2m(a^2) + 3m(a^2) & -2m(a^2) - 3m(-a^2) \\ 0 & -2m(a^2) - 3m(-a^2) & m(a^2) + 2m(a^2) + 3m(a^2) \end{pmatrix} \\ &= ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} \end{aligned}$$

(b) We have to find directions of $\boldsymbol{\omega}$ such that $I\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$, where λ is the moment of inertia about the principal axes. This eigenvalue problem has a non-trivial solution when $\det(I - \lambda\mathbf{1}) = 0$.

Firstly, by inspection, we know that $\boldsymbol{\omega}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector, or equivalently, one of the principal axes directions, with $\lambda_1 = 10ma^2$. To check that this is true, one only needs to substitute

$\boldsymbol{\omega_1}$ into $I\boldsymbol{\omega}$, and verify that one obtains $\lambda_1\boldsymbol{\omega}$.

For the other two solutions, we just need to solve:

$$\begin{vmatrix} 6ma^2-\lambda & ma^2 \\ ma^2 & 6ma^2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \qquad (6ma^2-\lambda^2)^2-(ma^2)^2=0$$

And so the eigenvalues are:

$$\lambda_2 = 5ma^2 \qquad \text{and} \qquad \lambda_3 = 7ma^2$$

When $\lambda_2 = 5ma^2$, we have:

$$ma^2\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\omega_y\\ \omega_z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\Rightarrow \qquad \omega_y = 1,\ \omega_z=-1$$

Normalizing, we obtain

$$\boldsymbol{\omega}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} \tag{10.33}$$

Then when $\lambda_3 = 7ma^2$, we have, instead:

$$ma^2\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\omega_y\\ \omega_z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\Rightarrow \qquad \omega_y = 1,\ \omega_z=1$$

Normalizing, we obtain

$$\boldsymbol{\omega}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \tag{10.34}$$

Summarizing, with the set of orthogonal principle axes $\{\boldsymbol{\omega}_1,\ \boldsymbol{\omega}_2,\ \boldsymbol{\omega}_3\}$, the inertia tensor consisting of the principal moments is given by:

$$I' = \begin{pmatrix}10ma^2 & 0 & 0\\ 0 & 5ma^2 & 0\\ 0 & 0 & 7ma^2\end{pmatrix} \tag{10.35}$$

□

10.36 A rigid body consists of three equal masses (m) fastened at the positions $(a,0,0)$, $(0,a,2a)$, and $(0,2a,a)$. (a) Find the inertia tensor I. (b) Find the principal moments and a set of orthogonal principal axes.

Solution. (a) We take reference from the previous problem (Problem 10.35). A similar, trivial calculation can be performed to obtain the inertia tensor as:

$$\begin{aligned} I &= \sum_i m_i \begin{pmatrix} y_i^2+z_i^2 & -x_iy_i & -x_iz_i \\ -y_ix_i & x_i^2+z_i^2 & -y_iz_i \\ -z_ix_i & -z_iy_i & x_i^2+y_i^2 \end{pmatrix} \\ &= ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix} \end{aligned}$$

(b) Similar to the previous problem, we have to find directions of $\boldsymbol{\omega}$ (principal axes), such that we have $I\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$, with λ being the moment of inertia about each principal axis. This eigenvalue equation has non-trivial solutions if and only if $\det(I - \lambda\mathbf{1}) = 0$.

From the form of the inertia tensor above, we observe by inspection that the first eigenvector is $\boldsymbol{\omega}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, with an eigenvalue of $\lambda_1 = 10ma^2$.

For the other 2 axes, we consider the determinant:

$$\begin{aligned} & \begin{vmatrix} 6ma^2-\lambda & -4ma^2 \\ -4ma^2 & 6ma^2-\lambda \end{vmatrix} = 0 \\ \Rightarrow \qquad & (6ma^2-\lambda^2)^2 - (4ma^2)^2 = 0 \end{aligned}$$

And so the eigenvalues are:

$$\lambda_2 = 10ma^2 \qquad \text{and} \qquad \lambda_3 = 2ma^2$$

When $\lambda_2 = 10ma^2$, we have:

$$\begin{aligned} ma^2 \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} \omega_y \\ \omega_z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \qquad \omega_y = 1,\ \omega_z &= -1 \end{aligned}$$

Normalizing, we obtain

$$\boldsymbol{\omega}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{10.36}$$

Then when $\lambda_3 = 2ma^2$, we have, instead:

$$\begin{aligned} ma^2 \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} \omega_y \\ \omega_z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \qquad \omega_y = 1,\ \omega_z &= 1 \end{aligned}$$

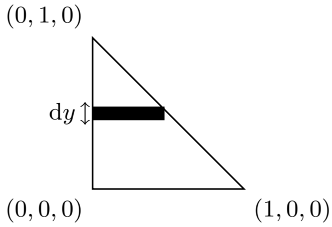


Figure 10.3: Metal Triangle for Problem 10.37

Normalizing, we obtain

$$\boldsymbol{\omega}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \tag{10.37}$$

Summarizing, with the set of orthogonal principle axes $\{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3\}$, the inertia tensor consisting of the principal moments is given by:

$$I' = \begin{pmatrix}10ma^2 & 0 & 0\\ 0 & 10ma^2 & 0\\ 0 & 0 & 2ma^2\end{pmatrix} \tag{10.38}$$

□

10.37 A thin, flat, uniform metal triangle lies in the xy plane with its corners at $(1,0,0)$, $(0,1,0)$ and the origin. Its surface density (mass/area) is $\sigma = 24$. (a) Find the triangle's inertia tensor I. (b) What are its principal moments and the corresponding axes?

Solution. (a) The inertia tensor is given by:

$$\begin{aligned}I &= \sigma\int_A \begin{pmatrix}y^2+z^2 & -xy & -xz\\ -yx & x^2+z^2 & -yz\\ -zx & -zy & x^2+y^2\end{pmatrix} dA\\ &\overset{(z=0)}{=} \sigma\int_A \begin{pmatrix}y^2 & -xy & 0\\ -yx & x^2 & 0\\ 0 & 0 & x^2+y^2\end{pmatrix} dA\end{aligned} \tag{10.39}$$

We shall compute the elements of the inertia tensor component wise. Take reference from Figure 10.3, noting that the hypotenuse is characterized as: $y = 1 - x$:

$$\begin{aligned}I_{xx} &= \sigma\int_0^1\int_0^{1-y} y^2\,\mathrm{d}x\,\mathrm{d}y\\ &= \sigma\int_0^1 y^2(1-y)\,\mathrm{d}y = 2\end{aligned} \tag{10.40}$$

$$\begin{aligned}I_{yy} &= \sigma\int_0^1\int_0^{1-x} x^2\,\mathrm{d}y\,\mathrm{d}x\\ &= \sigma\int_0^1 x^2(1-x)\,\mathrm{d}x = 2\end{aligned} \tag{10.41}$$

$$\begin{aligned} I_{zz} &= \sigma \int_0^1 \int_0^{1-y} x^2 + y^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= I_{xx} + I_{yy} = 4 \qquad\qquad \text{(c.f. Problem 10.23)} \end{aligned} \tag{10.42}$$

$$\begin{aligned} I_{xy} = I_{yx} &= \sigma \int_0^1 \int_0^{1-y} -xy \,\mathrm{d}x\,\mathrm{d}y \\ &= \sigma \int_0^1 -y\,\frac{(1-y)^2}{2}\,\mathrm{d}y = -1 \end{aligned} \tag{10.43}$$

Putting all the numbers together, we get the inertia tensor to be:

$$I = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{10.44}$$

(b) For the set of principal axes $\{\boldsymbol{\omega_1}, \boldsymbol{\omega_2}, \boldsymbol{\omega_3}\}$, we necessarily have $I\boldsymbol{\omega_i} = \lambda_i\boldsymbol{\omega_i}$ for $i = 1, 2, 3$. Note that λ_i is the moment of inertia about the principal axis in the direction of $\boldsymbol{\omega}_i$. The eigenvalue equation has non-trivial solutions if and only if $\det(I - \lambda\mathbf{1}) = 0$.

By inspection, we have that $\boldsymbol{\omega}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector, with corresponding eigenvalue/ moment of inertia $\lambda_3 = 4$. Verify this by ensuring the eigenvalue equation stated above is satisfied for this choice of $\boldsymbol{\omega_3}$.

As for the remaining two principal axes, we shall consider:

$$\begin{aligned} &\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \\ \Rightarrow \qquad &(2-\lambda)^2 - 1 = 0 \end{aligned}$$

And so the eigenvalues are:

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = 3$$

When $\lambda_1 = 1$, we have:

$$\begin{aligned} &\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \qquad &\omega_x = 1,\ \omega_y = 1 \end{aligned}$$

Normalizing, we obtain

$$\boldsymbol{\omega}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{10.45}$$

Then when $\lambda_2 = 3$, we have:

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \qquad \omega_x = 1,\ \omega_y = -1$$

Normalizing, we obtain

$$\boldsymbol{\omega}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{10.46}$$

Summarizing, with the set of orthogonal principle axes $\{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3\}$, the inertia tensor consisting of the principal moments is given by:

$$I' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{10.47}$$

□

Chapter 11

Coupled Oscillators and Normal Modes

11.5 (a) Find the normal frequencies, ω_1 and ω_2, for the two carts shown in Figure 11.1, assuming that $m_1 = m_2$ and $k_1 = k_2$. (b) Find and describe the motion for each of the normal modes in turn.

Solution. (a) Let $m_1 = m_2 = m$, $k_1 = k_2 = k$, and rightwards be positive. We can write down the forces acting on each cart:

$$F_1 = k_2 x_2 + (-k_1 - k_2)x_1 = kx_2 - 2kx_1 \tag{11.1}$$

$$F_2 = -k_2 x_2 + k_2 x_1 = -kx_2 + kx_1 \tag{11.2}$$

Using Newton's second law, we express the above two equations as a single matrix equation:

$$\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{11.3}$$

We look for the normal mode solutions that take the form $\boldsymbol{x} = \boldsymbol{A}e^{i\omega t}$, where $\boldsymbol{x}$ and $\boldsymbol{A}$ are column vectors. Substituting into Equation (11.3),

$$-m\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{11.4}$$

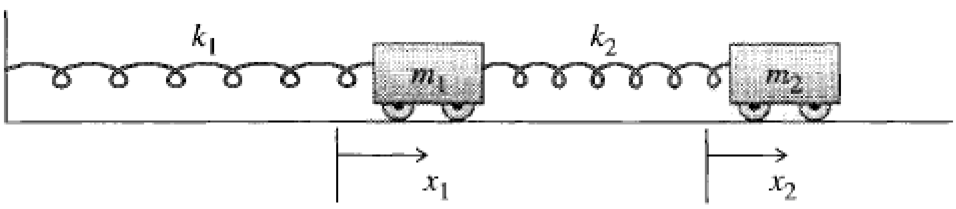


Figure 11.1: Two carts for Problems 11.5 and 11.6

For non-trivial solutions, we have:

$$\begin{vmatrix} -2k + m\omega^2 & k \\ k & -k + m\omega^2 \end{vmatrix} = 0$$
$$\Rightarrow \qquad k^2 - 3km\omega^2 + (m\omega^2)^2 = 0$$

We obtain the eigenfrequencies to be:

$$\omega^2 = \frac{(3 \pm \sqrt{5})}{2}\omega_0 \, , \tag{11.5}$$

where $\omega_0 = k/m$. This works out to be $\omega_1 = 0.62\omega_0$ and $\omega_2 = 1.62\omega_0$ approximately.

(b) When $\omega = \omega_1$, we have:

$$\begin{pmatrix} \frac{-1-\sqrt{5}}{2}k & k \\ k & \frac{-1-\sqrt{5}}{2}k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow A_1 = 1 \qquad \text{and} \qquad A_2 = \frac{1+\sqrt{5}}{2} \tag{11.6}$$

And so we have the amplitudes (the reader can normalize the expression if he so wishes) to be:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \tag{11.7}$$

This means that both masses oscillate in phase, with the amplitude of mass 2 to be approximately 1.62 times that of mass 1.

When $\omega = \omega_2$, we then have:

$$\begin{pmatrix} \frac{-1+\sqrt{5}}{2}k & k \\ k & \frac{-1+\sqrt{5}}{2}k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow A_1 = 1 \qquad \text{and} \qquad A_2 = \frac{1-\sqrt{5}}{2} \tag{11.8}$$

The amplitudes, when written in vector form is:

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \tag{11.9}$$

In this situation, both masses oscillate out of phase, with the amplitude of mass 2 being approximately 0.62 times that of mass 1. $\square$

11.6 Answer the same questions as in Problem 11.5 but for the case that $m_1 = m_2$ and $k_1 = 3k_2/2$. (Write $k_1 = 3k$ and $k_2 = 2k$.) Explain the motion in the two normal modes.

Solution. We use the same sign conventions as Problem 11.5, with $m_1 = m_2 = m$ and $k_1 = 3k$, $k_2 = 2k$. The forces on each mass is then:

$$F_1 = k_2x_2 + (-k_1 - k_2)x_1 = 2kx_2 - 5kx_1 \tag{11.10}$$

$$F_2 = -k_2x_2 + k_2x_1 = -2kx_2 + 2kx_1 \tag{11.11}$$

We then use Newton's second law to express the two equations above in a single matrix equation:

$$\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -5k & 2k \\ 2k & -2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{11.12}$$

Normal mode solutions: $\boldsymbol{x} = \boldsymbol{A}e^{i\omega t}$. Substituting into the above equation, we obtain:

$$-m\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -5k & 2k \\ 2k & -2k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{11.13}$$

For non-trivial solutions, we have the determinant vanishing:

$$\begin{vmatrix} -5k + m\omega^2 & 2k \\ 2k & -2k + m\omega^2 \end{vmatrix} = 0$$
$$\Rightarrow \qquad 6k^2 - 7km\omega^2 + (m\omega^2)^2 = 0$$

We obtain the eigenfrequencies to be:

$$\omega^2 = 6\omega_0 \qquad \text{or} \qquad \omega_0\,, \tag{11.14}$$

where $\omega_0 = k/m$. And so $\omega_1 = \sqrt{6}\omega_0$ and $\omega_2 = \omega_0$.

When $\omega_1 = \sqrt{6}\omega_0$, we have:

$$\begin{pmatrix} k & 2k \\ 2k & 4k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \qquad A_1 = 2 \qquad \text{and} \qquad A_2 = -1 \tag{11.15}$$

In vector form,

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{11.16}$$

With this normal mode, mass 1 and mass 2 are oscillating out of phase, with the amplitude of mass 1 being twice that of mass 2.

We do the same for $\omega_2 = \omega_0$. We get:

$$\begin{pmatrix} -4k & 2k \\ 2k & -k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \qquad A_1 = 1 \qquad \text{and} \qquad A_2 = 2 \tag{11.17}$$

In vector form,

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{11.18}$$

With this normal mode, mass 1 and mass 2 are oscillating in phase, with the amplitude of mass 1 being half that of mass 2. $\square$

Chapter 12

Nonlinear Mechanics and Chaos

12.1 No plans to do this chapter yet.

Chapter 13

Hamiltonian Mechanics

13.3 Consider the Atwood Machine of Fig 13.1, but suppose that the pulley is a uniform disc of mass M and radius R. Using x as your generalized coordinate, write down the Lagrangian, the generalized momentum p, and the Hamiltonian $\mathcal{H} = p\dot{x} - \mathcal{L}$. Find Hamilton's equations and use them to find the acceleration $\ddot{x}$.

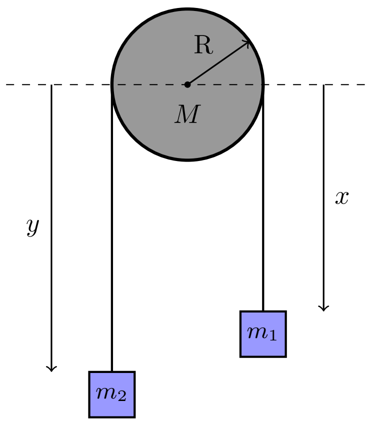


Figure 13.1: The Atwood Machine

Solution. The moment of inertia of the disc is $\frac{MR^2}{2}$, and angular velocity of the disc is given by $\omega = \frac{\dot{x}}{R}$. Since the length of the string is fixed (say l), we can write the following equation to incorporate the constraint:

$$y + x + \pi R = l \quad \Rightarrow \quad y = -x + \text{const} \tag{13.1}$$

Now we can express the kinetic and potential energies in terms of a single coordinate x.

$$\begin{aligned} T &= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}\frac{MR^2}{2}\left(\frac{\dot{x}}{R}\right)^2 \\ &= \frac{1}{2}\left(m_1+m_2+\frac{1}{2}M\right)\dot{x}^2 && (13.2) \\ U &= -m_2gy - m_1gx \\ &= (m_2-m_1)gx + \text{const} && (13.3) \end{aligned}$$

The Lagrangian is therefore:

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}\left(m_1+m_2+\frac{1}{2}M\right)\dot{x}^2 + (m_1-m_2)gx + \text{const} && (13.4) \end{aligned}$$

The generalized momentum p can be written as such:

$$p = \frac{\partial\mathcal{L}}{\partial\dot{x}} = \left(m_1+m_2+\frac{1}{2}M\right)\dot{x} \tag{13.5}$$

The Hamiltonian is then given by:

$$\begin{aligned} \mathcal{H} &= p\dot{x} - \mathcal{L} \\ &= \frac{p^2}{m_1+m_2+\frac{1}{2}M} - \frac{1}{2}\left(\frac{p^2}{m_1+m_2+\frac{1}{2}M}\right) - (m_1-m_2)gx - \text{const} \\ &= \frac{1}{2}\left(\frac{p^2}{m_1+m_2+\frac{1}{2}M}\right) - (m_1-m_2)gx - \text{const} && (13.6) \end{aligned}$$

We note that $\mathcal{H}$ is just the total energy of the system, that is, $\mathcal{H} = T + U$. The Hamilton's equations are given by:

$$\begin{aligned} \dot{x} &= \frac{\partial\mathcal{H}}{\partial p} \\ &= \frac{p}{m_1+m_2+\frac{1}{2}M} && (13.7) \\ \dot{p} &= \frac{\partial\mathcal{H}}{\partial x} \\ &= (m_2-m_1)g && (13.8) \end{aligned}$$

We can obtain the acceleration rather trivially from Hamilton's equations (Eqns 13.7 and 13.8).

$$\begin{aligned} \ddot{x} &= \frac{\dot{p}}{m_1+m_2+\frac{1}{2}M} \\ &= \frac{(m_2-m_1)g}{m_1+m_2+\frac{1}{2}M} && (13.9) \end{aligned}$$

□

13.5 A bead of mass m is threaded on a frictionless wire that is bent into a helix with cylindrical polar coordinates (ρ, ϕ, z) satisfying $z = c\phi$ and $\rho = R$, with c and R constants. The z axis points vertically up and gravity vertically down. Using ϕ as your generalized coordinate, write down the kinetic and potential energies, and hence the Hamiltonian $\mathcal{H}$ as a function of ϕ and its conjugate momentum p. Write down Hamilton's equations and solve for $\ddot{\phi}$ and hence $\ddot{z}$. Explain your result in terms of Newtonian mechanics and discuss the special case that $R = 0$.

Solution. In cylindrical coordinates, we have, in general, the kinetic energy T to be:

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \tag{13.10}$$

Substituting in the constraints $z = c\phi$ and $\rho = R$, we obtain

$$T\left(\dot{\phi}\right) = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \left(c\dot{\phi}\right)^2\right) \tag{13.11}$$

The potential energy is simply given by the gravitational potential energy:

$$U(\phi) = mgz = mgc\phi \tag{13.12}$$

We can write the Lagrangian $\mathcal{L}$ as:

$$\mathcal{L} = T - U = \frac{1}{2}m\left(R^2 + c^2\right)\dot{\phi}^2 - mgc\phi \tag{13.13}$$

The next step is to find the canonical momentum p and substitute it into the general expression for the Hamiltonian.

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\left(R^2 + c^2\right)\dot{\phi} \tag{13.14}$$

$$\mathcal{H} = p\dot{\phi} - \mathcal{L} \tag{13.15}$$

$$= \frac{p^2}{mR^2\left(R^2 + c^2\right)} - \left[\frac{p^2}{2m\left(R^2 + c^2\right)} - mgc\phi\right] \tag{13.16}$$

$$= \frac{p^2}{2m\left(R^2 + c^2\right)} + mgc\phi \tag{13.17}$$

Now, we turn to Hamilton's equations by taking appropriate derivatives of the Hamiltonian.

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m\left(R^2 + c^2\right)} \tag{13.18}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial \phi} = -mgc \tag{13.19}$$

We are able to solve for $\ddot{\phi}$ and $\ddot{z}$ as such:

$$\ddot{\phi} = \frac{\dot{p}}{m\left(R^2 + c^2\right)} = -\frac{gc}{R^2 + c^2} \tag{13.20}$$

$$\ddot{z} = c\ddot{\phi} = -\frac{gc^2}{R^2 + c^2} \tag{13.21}$$

Let us unwrap the helix into a 2-D plane. When go round the helix an angle of $\Delta\phi = 2\pi$, the vertical height we have gained is $z = 2\pi c$, while the 'horizontal' distance travelled is $2\pi R$. That is, we have the following:

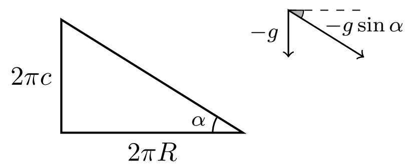


Figure 13.2: Unwrapped helix into a 2-D plane

where the angle $\tan\alpha$ is given by c/R. From this, we note that we can write Eqn 13.21 as:

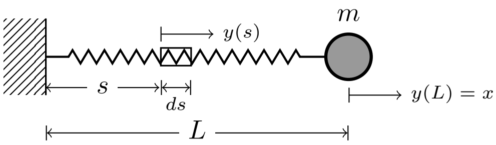
$$\ddot{z} = -g\sin^2\alpha \tag{13.22}$$

An object sliding down a slope of angle α has a component of acceleration down the slope given by $a_{tang} = g\sin\alpha$. So the bead has a tangential acceleration a_{tang} along the wire. The vertical component of this acceleration (along the z axis) is $\ddot{z} = a_{tang}\,\sin\alpha = g\sin^2\alpha$. This is precisely the acceleration calculated in Equation (13.22).

If $R = 0$, then by Equation (13.21), $\ddot{z} = -g$, as expected, since $\alpha = \pi/2$. $\square$

13.6 In discussing the oscillation of a cart on the end of a spring, we almost always ignore the mass of the spring. Set up the Hamiltonian $\mathcal{H}$ for a cart of mass m on a spring (force constant k) whose mass M is *not* negligible, using the extension x of the spring as the generalized coordinate. Solve Hamilton's equations and show that the mass oscillates with angular frequency $\omega = \sqrt{k/(m + M/3)}$. That is, the effect of the spring's mass is to add $M/3$ to m. (Assume that the spring's mass is distributed uniformly and that it stretches uniformly.)

Solution. With reference to the figure, let L be the length of the spring, while ρ be the mass per unit length of spring.



Consider an element A of spring of length ds and mass ρds, at a distance s from the fixed end. Let $y(s)$ be the displacement of the element A, and $y(L) = x$ be the displacement of mass m at $s = L$. Assume that the spring stretches uniformly, i.e. $y(s) = xs/L$, with $y(0) = 0$, $y(L) = x$. We can then express the velocity and kinetic energy of element A as:

$$\dot{y} = \frac{\dot{x}s}{L}$$
$$T_{\mathrm{A}} = \frac{1}{2}(\rho\, ds)\dot{y}^2 = \frac{\rho\, \dot{x}^2 s^2}{2L^2}\, ds \tag{13.23}$$

Thus, the kinetic energy of the spring is given by:

$$\begin{aligned} T_{\text{spring}} &= \frac{\rho\, \dot{x}^2}{2L^2}\int_0^L s^2\, ds \\ &= \frac{1}{6}\rho\, L\dot{x}^2 = \frac{1}{6}M\dot{x}^2 \end{aligned} \tag{13.24}$$

where in the last line we have identified the total mass of the spring as $M = \rho L$.

With the kinetic energy of the cart as $T_{\text{cart}} = \frac{1}{2}m\dot{x}^2$, we have the total KE:

$$\begin{aligned} T = T_{\text{spring}} + T_{\text{cart}} &= \frac{1}{6}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 \\ &= \frac{1}{2}m_{\text{eff}}\dot{x}^2 \end{aligned} \tag{13.25}$$

where $m_{\text{eff}} = m + \frac{1}{3}M$ is the effective mass.

The potential energy can be identified to be $U = \frac{1}{2}kx^2$, and thus the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m_{\text{eff}}\dot{x}^2 - \frac{1}{2}kx^2 \tag{13.26}$$

We use the Lagrangian to derive the momentum in terms of the velocities, and then use the inverse relation to obtain the Hamiltonian $\mathcal{H}(p_x, x)$, as such:

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m_{\text{eff}}\dot{x} \tag{13.27}$$
$$\mathcal{H} = p_x\dot{x} - \mathcal{L} = \frac{p_x^2}{2m_{\text{eff}}} + \frac{1}{2}kx^2 \tag{13.28}$$

The Hamilton's equations of motions give us:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m_{\text{eff}}} \qquad\qquad \dot{p_x} = -\frac{\partial \mathcal{H}}{\partial x} = -kx \tag{13.29}$$

Differentiating the first of the two equations with respect to time, and using the second equation, we obtain:

$$\ddot{x} = \frac{\dot{p_x}}{m_{\text{eff}}} = -\frac{k}{m_{\text{eff}}}x = -\omega^2 x \tag{13.30}$$

where $\omega^2 = k/m_{\text{eff}}$. More explicitly, this is the equation of simple harmonic motion with frequency

$$\omega = \sqrt{\frac{k}{m_{\text{eff}}}} = \sqrt{\frac{k}{m + M/3}} \tag{13.31}$$

□

13.17 Consider the mass confined to the surface of a cone described in Example 13.4 (page 533). We saw that there are solutions for which the mass remains at the fixed height $z = z_0$, with fixed angular velocity $\dot{\phi}_0$, say. (a) For any chosen value of p_ϕ, use (13.34) to get an equation that gives the corresponding value of the height z_0. (b) Use the equations of motion to show that this motion is stable. That is, show that if the orbit has $z = z_0 + \epsilon$, with ϵ small, then ϵ will oscillate about zero. (c) Show that the angular frequency of these oscillations is $\omega = \sqrt{3}\dot{\phi}_0 \sin\alpha$, where α is the half angle of the cone ($\tan\alpha = c$ where c is the constant in $\rho = cz$). (d) Find the angle α for which the frequency of oscillation ω is equal to the orbital angular velocity $\dot{\phi}_0$, and describe the motion in this case.

Solution. The two equations in (13.34) from the textbook (pg 534) are:

$$\dot{z} = \frac{p_z}{m(c^2+1)} \tag{13.32}$$

$$\dot{p_z} = \frac{p_\phi^2}{mc^2 z^3} - mg \tag{13.33}$$

The above equations were obtained from constructing the Lagrangian, and writing the Lagrange equations. For completeness, the Hamiltonian of the system can be written as:

$$\mathcal{H} = \frac{1}{2m}\left[\frac{p_z^2}{c^2+1} + \frac{p_\phi^2}{c^2 z^2}\right] + mgz \tag{13.34}$$

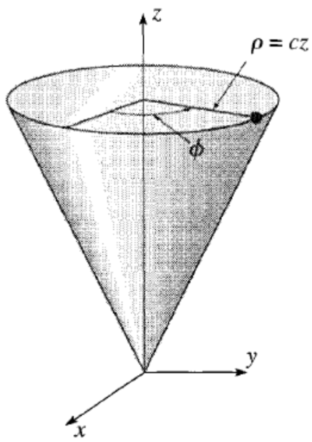


Figure 13.3: A mass m constrained to move on the surface of a cone. Taken from textbook pg 533.

(a) To maintain at a fixed height, we require $\dot{z} = 0$, which from Equation (13.32), we infer that it is necessary that $p_z = 0$. And so $\dot{p_z} = 0$. From Equation (13.33), we see that the height z_0 which corresponds to a particular value of p_ϕ is:

$$z_0 = \left(\frac{p_\phi^2}{m^2 c^2 g} \right)^{1/3} \tag{13.35}$$

(b) From Equation (13.32), we differentiate once with respect to time to get an equation of motion for z as:

$$\begin{aligned} \ddot{z} &= \frac{\dot{p_z}}{m(c^2+1)} \\ &= \frac{1}{m(c^2+1)} \left(\frac{p_\phi^2}{mc^2 z^3} - mg \right) \end{aligned} \tag{13.36}$$

We consider a perturbation of z about the equilibrium z_0, that is, $z = z_0 + \epsilon$, for small ϵ. We have the following approximations:

$$\ddot{z} = \ddot{\epsilon} \tag{13.37}$$

$$z^{-3} = z_0^{-3} \left(1 - 3\frac{\epsilon}{z_0} \right) \tag{13.38}$$

Substituting these relations into Equation (13.36), we obtain:

$$\ddot{\epsilon} = -\frac{3}{m(c^2+1)} \frac{p_\phi^2}{mc^2 z_0^4} \, \epsilon \tag{13.39}$$

where we have used the fact that $\dfrac{p_\phi^2}{mc^2 z_0^3} = mg$ (from Equation (13.35) in an intermediate step. To simplify Equation (13.39) further, we recall a relation between $\dot{\phi}$ and p_ϕ derived from the Hamilton's equations.

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mc^2 z^2} \tag{13.40}$$

We use this to replace p_ϕ in Equation (13.39):

$$\ddot{\epsilon} = -\frac{3\dot{\phi}^2 c^2}{c^2+1}\,\epsilon \tag{13.41}$$

This takes on the form of a SHM equation. (shown)

(c) The angular frequency of the oscillation can be simply obtained from the SHM equation:

$$\omega = \sqrt{\frac{3\dot{\phi}^2 c^2}{c^2+1}} = \sqrt{3}\,\dot{\phi}\,\frac{c}{\sqrt{c^2+1}} \tag{13.42}$$

Finally, we can note that the fraction above is none other than $\sin\alpha$, where α is defined in the following way (c.f. Figure 13.3): $\tan\alpha = cz/z$. Hence, we have shown that $\omega = \sqrt{3}\dot{\phi}_0 \sin\alpha$. $\square$

(d) In order for $\omega = \dot{\phi}_0$, we require $\sin\alpha = 1/\sqrt{3}$, or:

$$\alpha = 35.26^\circ \tag{13.43}$$

In this case, the mass's rotational frequency equals its oscillation frequency. As a result, its orbit is closed.

13.23 Consider the modified Atwood machine shown in Figure 13.4. The two weights on the left have equal masses m and are connected by a massless spring of force constant k. The weight on the right has mass $M = 2m$, and the pulley is massless and frictionless. The coordinate x is the extension of the spring from its equilibrium length; that is, the length of the spring is $l_e + x$ where l_e is the equilibrium length (with all the weights in position and M held stationary). (a) Show that the total potential energy (spring + gravitational) is just $U = \frac{1}{2}kx^2$ (plus a constant that we can take to be zero). (b) Find the two momenta conjugate to x and y. Solve for $\dot{x}$ and $\dot{y}$, and write down the Hamiltonian. Show that the coordinate y is ignorable. (c) Write down the four Hamilton equations and solve them for the following initial conditions: You hold the mass M fixed with the whole system in equilibrium and $y = y_0$. Still holding M fixed, you pull the lower mass m down a distance x_0 and at time $t = 0$ you let go of both masses. Describe the motion. In particular, find the frequency with which x oscillates.

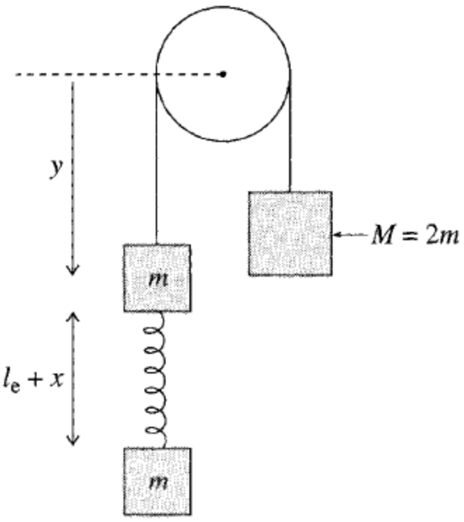


Figure 13.4: Modified Atwood machine for Problem 13.23

Solution. (a) Let string length be L. We take downwards to be positive, and the origin to be the centre of the pulley wheel. We also assume the radius of the pulley to be 0, without any loss in generality. In addition, let us define the original, unstretched length of the spring (without load) to be l_0. That is, $l_e = l_0 + mg/k$, where k is the spring constant.

The two components of the potential energy can be written as:

$$\begin{aligned} U_{\text{grav}} &= -mgy - mg(y + l_e + x) - 2m(L - y)g \\ &= -mgl_e - 2mLg - mgx \qquad (13.44) \\ U_{\text{spring}} &= \frac{1}{2}k(x + mg/k)^2 \qquad (13.45) \end{aligned}$$

Summing the two potential energies give us:

$$\begin{aligned} U &= (-mgl_e - 2mLg - mgx) + \left(\frac{1}{2}kx^2 + mgx + \frac{(mg)^2}{2k}\right) \\ &= \frac{1}{2}kx^2 + \text{const} \qquad \text{(shown)} \qquad (13.46) \end{aligned}$$

Note that we can ignore the constant, and take it to be equal to zero. Henceforth, the potential energy shall be computed only with $U = \frac{1}{2}kx^2$.

(b) We first complete the expression for the Lagrangian ($\mathcal{L} = T - U$) by considering the kinetic energy T of the system:

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}(2m)\dot{y}^2 + \frac{1}{2}m(\dot{y} + \dot{x})^2 \qquad (13.47)$$

Then, we can write the momentum conjugates as:

$$\begin{aligned} p_x &= \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m(\dot{y}+\dot{x}) \\ p_y &= \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial T}{\partial \dot{y}} = 3m\dot{y} + m(\dot{y}+\dot{x}) = m(4\dot{y}+\dot{x}) \end{aligned} \tag{13.48}$$

Solving the above equation for $\dot{x}$ and $\dot{y}$, we find their expressions:

$$\begin{aligned} \dot{x} &= \frac{4p_x - p_y}{3m} \\ \dot{y} &= \frac{p_y - p_x}{3m} \end{aligned} \tag{13.49}$$

The Hamiltonian can thus be written as:

$$\begin{aligned} \mathcal{H} &= \dot{x}p_x + \dot{y}p_y - \mathcal{L} \\ &= \frac{1}{2m}\left[\frac{1}{3}(p_y - p_x)^2 + p_x^2\right] + \frac{1}{2}kx^2 \end{aligned} \tag{13.50}$$

after some simplification. Note that this is simply the sum of the kinetic energies and potential energy. Thus $\mathcal{H}$ represents the total energy of the system.

Since $\dot{p_y} = -\dfrac{\partial \mathcal{H}}{\partial y} = 0$, p_y is constant and thus y must be an ignorable coordinate. (shown)

(c) The four Hamilton equations are:

$$\dot{p_x} = -\frac{\partial \mathcal{H}}{\partial x} = -kx \qquad \dot{p_y} = -\frac{\partial \mathcal{H}}{\partial y} = 0 \tag{13.51}$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{4p_x - p_y}{3m} \qquad \dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y - p_x}{3m} \tag{13.52}$$

noting that the expressions in Equation (13.52) matches with those from Equation (13.49).

To solve these equations, we shall use the following initial conditions: $x(0) = x_0$, $y(0) = y_0$, $\dot{x}(0) = \dot{y}(0) = 0$. Furthermore, with Equation (13.48), we also have $p_x(0) = p_y(0) = 0$.

For the coordinate in x, we differentiate the expression for $\dot{x}$ to get:

$$\ddot{x} = \frac{4\dot{p_x} - \dot{p_y}}{3m} = -\frac{4k}{3m}x, \tag{13.53}$$

where we have substituted the expressions for $\dot{p_x}$ and $\dot{p_y}$ from Equation (13.51). One can easily recognise this taking the form of a simple harmonic motion, where the position and velocity is given by:

$$x(t) = A\cos\left(\sqrt{\frac{4k}{3m}}t + \delta\right) \tag{13.54}$$

$$\dot{x}(t) = -\sqrt{\frac{4k}{3m}}A\sin\left(\sqrt{\frac{4k}{3m}}t + \delta\right) \tag{13.55}$$

where A and δ can be found by considering the initial conditions. It turns out that $\delta = 0$ and $A = x_0$, after a quick substitution of the initial conditions. As such, the mass attached to the spring is undergoing simple harmonic motion described by:

$$\boxed{x(t) = x_0 \cos\left(\sqrt{\frac{4k}{3m}}t\right)} \tag{13.56}$$

with angular frequency $\boxed{\omega = \sqrt{\frac{4k}{3m}}t}$. As for the motion characterized by the coordinate y, since we know that (i) p_y is constant in time, and (ii) $p_y(0) = 0$, we have, necessarily, that $p_y = 0$ for all time t. From the second equation in Equation (13.48), we have:

$$\dot{y} = -\frac{\dot{x}}{4} \qquad \Rightarrow \qquad \dot{y} = \frac{1}{4}\sqrt{\frac{4k}{3m}}x_0 \sin\left(\sqrt{\frac{4k}{3m}}t\right) \tag{13.57}$$

A quick integration (and applying initial conditions) yields us:

$$\boxed{y(t) = (y_0 + \frac{1}{4}x_0) - \frac{1}{4}x_0 \cos\left(\sqrt{\frac{4k}{3m}}t\right)} \tag{13.58}$$

It is not surprising to see that y is executing simple harmonic motion as well. □

13.25 Here is another example of a canonical transformation, which is still too simple to be of any real use, but does nevertheless illustrate the power of these changes of coordinates. (a) Consider a system with one degree of freedom and Hamiltonian $\mathcal{H} = \mathcal{H}(q, p)$ and a new pair of coordinates Q and P defined so that $q = \sqrt{2P}\sin Q$ and $p = \sqrt{2P}\cos Q$. Prove that if $\frac{\partial \mathcal{H}}{\partial q} = -\dot{p}$ and $\frac{\partial \mathcal{H}}{\partial p} = \dot{q}$, it automatically follows that $\frac{\partial \mathcal{H}}{\partial Q} = -\dot{P}$ and $\frac{\partial \mathcal{H}}{\partial P} = \dot{Q}$. In other words, the Hamiltonian formalism applies just as well to the new coordinates as to the old. (b) Show that the Hamiltonian of a one-dimensional harmonic oscillator with mass $m = 1$ and force constant $k = 1$ is $\mathcal{H} = \frac{1}{2}(q^2 + p^2)$. (c) Show that if you rewrite this Hamiltonian in terms of the coordinates Q and P defined above, then Q is ignorable. What is P? (d) Solve the Hamiltonian equation for $Q(t)$ and verify that, when rewritten for q, your solution gives the expected behaviour.

Solution. (a) We have the following relation given in the question:

$$\begin{aligned} q &= \sqrt{2P}\sin Q \\ p &= \sqrt{2P}\cos Q \end{aligned} \tag{13.59}$$

We compute the relevant derivatives below:

$$\frac{\partial q}{\partial P} = \frac{\sin Q}{\sqrt{2P}} \qquad \frac{\partial q}{\partial Q} = \sqrt{2P}\cos Q \tag{13.60}$$

$$\frac{\partial p}{\partial P} = \frac{\cos Q}{\sqrt{2P}} \qquad \frac{\partial p}{\partial Q} = -\sqrt{2P}\sin Q \tag{13.61}$$

Consider the Hamiltonian equation for Q, recalling that $\dfrac{\partial \mathcal{H}}{\partial p} = \dot{q}$ and $\dfrac{\partial \mathcal{H}}{\partial q} = -\dot{p}$:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial Q} &= \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial Q} + \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial Q} \\
&= \dot{q}\frac{\partial p}{\partial Q} - \dot{p}\frac{\partial q}{\partial Q} \\
&= \left(\frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial P}\dot{P}\right)\frac{\partial p}{\partial Q} - \left(\frac{\partial p}{\partial Q}\dot{Q} + \frac{\partial p}{\partial P}\dot{P}\right)\frac{\partial q}{\partial Q} \\
&= \dot{Q}\left(\frac{\partial q}{\partial Q}\frac{\partial p}{\partial Q} - \frac{\partial p}{\partial Q}\frac{\partial q}{\partial Q}\right) + \dot{P}\left(\frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} - \frac{\partial p}{\partial P}\frac{\partial q}{\partial Q}\right) \\
&= 0 + \dot{P}\left(\frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} - \frac{\partial p}{\partial P}\frac{\partial q}{\partial Q}\right) \\
&= \dot{P}\left(-\sin^2 Q - \cos^2 Q\right) = -\dot{P} \qquad \text{(shown)}
\end{aligned}$$

Similarly, we can consider the Hamiltonian equation for P:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial P} &= \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial P} \\
&= \dot{q}\frac{\partial p}{\partial P} - \dot{p}\frac{\partial q}{\partial P} \\
&= \left(\frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial P}\dot{P}\right)\frac{\partial p}{\partial P} - \left(\frac{\partial p}{\partial Q}\dot{Q} + \frac{\partial p}{\partial P}\dot{P}\right)\frac{\partial q}{\partial P} \\
&= \dot{Q}\left(\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial p}{\partial Q}\frac{\partial q}{\partial P}\right) + \dot{P}\left(\frac{\partial q}{\partial P}\frac{\partial p}{\partial P} - \frac{\partial p}{\partial P}\frac{\partial q}{\partial P}\right) \\
&= \dot{Q}\left(\frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial p}{\partial Q}\frac{\partial q}{\partial P}\right) + 0 \\
&= \dot{Q}\left(\cos^2 Q + \sin^2 Q\right) = \dot{Q} \qquad \text{(shown)}
\end{aligned}$$

(b) The Lagrangian for the 1-D harmonic oscillator is simply given by the quantity $T - U$:

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{13.62}$$

We can express the velocity as a function of the momentum:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \quad \Rightarrow \; \dot{q} = \frac{p}{m} \tag{13.63}$$

Thus the Hamiltonian can be written as such:

$$\begin{aligned}
\mathcal{H} = T + U &= \frac{1}{2}m\left(\frac{p}{m}\right)^2 + \frac{1}{2}kq^2 \\
&= \frac{1}{2}(q^2 + p^2) \qquad \text{(shown)}
\end{aligned} \tag{13.64}$$

where the last line was obtained by setting $k = m = 1$, as given in the question.

(c) Using the coordinate transformations defined in Equation (13.59), we get:

$$\mathcal{H} = \frac{1}{2}\left(2P\sin^2 Q + 2P\cos^2 Q\right) = P \tag{13.65}$$

And so P is the Hamiltonian of the system. It can also be observed that Q is no longer present in the Hamiltonian, and so, is an ignorable coordinate.

(d) We have from Hamilton's equations:

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 1 \tag{13.66}$$

As such, we can easily find $Q(t)$ by integrating, then solving for q:

$$Q = t + \delta \tag{13.67}$$

$$q = \sqrt{2}\sin t + \delta \tag{13.68}$$

where δ is a constant of integration. This gives the expected behaviour (a simple harmonic oscillation). $\square$

Chapter 14

Collision Theory

14.1 No plans to do this chapter yet.

Chapter 15

Special Relativity

15.1 No plans to do this chapter yet.

Chapter 16

Continuum Mechanics

16.1 No plans to do this chapter yet.