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Chapter 1

Introduction and Problem Statement

1.1 Overview

This report aims to solve a given difference equation using both theoretical analysis and practical implementation with LabVIEW. The primary focus is on understanding the system's behavior through impulse response, zero-state response, zero-input response, and total response. By comparing theoretical calculations with simulation results, we gain comprehensive insights into the system dynamics.

1.2 Objectives

The primary objectives of this report are:

- To solve the given difference equation theoretically by separating it into zero-input and zero-state components.
- To implement the solution in LabVIEW and generate plots for various responses.
- To compare theoretical results with LabVIEW simulations to validate the solution approaches.
- To analyze the system's behavior based on the combined findings.

1.3 Problem Statement

The given difference equation is:

$$y[n] - 0.5y[n-1] = x[n] + 2x[n-1] \tag{1.1}$$

with the initial condition:

$$y[-1] = 2 \tag{1.2}$$

and the input signal:

$$x[n] = u(n) \tag{1.3}$$

where $u(n)$ is the unit step function.

1.4 System Description

This system represents a discrete-time linear time-invariant (LTI) system of first order. The solution methodology involves splitting the response into homogeneous (zero-input) and particular (zero-state) parts. The homogeneous solution carries the effect of the initial condition, while the particular solution accounts for the response due to the external input. This approach allows for a comprehensive understanding of the system's behavior, including its transient and steady-state responses.

Chapter 2

Theoretical Solution

2.1 Methodology

The theoretical solution involves splitting the response into two parts:

- **Zero-Input Response ($y_{zi}[n]$):** The response due to the initial condition only, with $x[n] = 0$.
- **Zero-State Response ($y_{zs}[n]$):** The response due to the input when the initial condition is zero.

The total response is then the sum of these components:

$$y[n] = y_{zi}[n] + y_{zs}[n] \tag{2.1}$$

We also compute the impulse response $h[n]$, which is essential for characterizing the system's behavior.

> **Note:** The impulse response is defined as the output when the system is ”at rest” (i.e. with zero initial conditions) and the input is the Kronecker delta $\delta[n]$.

This approach uses the same ideas as those for differential equations (namely, finding the homogeneous solution and a particular solution) applied to our difference equation. In all cases we work step by step. Note that in discrete-time problems, ”differential equation” methods are replaced by ”difference equation” methods—but the procedure is analogous to the continuous case.

Our problem is defined by the following components:

- Difference equation: $y[n]-0.5\,y[n-1]=x[n]+2x[n-1]$
- Initial condition: $y[-1]=2$
- Input signal: $x[n]=u(n)$ (the unit step function)

We will find:

1. The impulse response $h[n]$
2. The zero-state response (response due solely to the forcing input with zero initial condition)
3. The zero-input response (response due solely to the initial condition when $x[n]=0$)
4. The total response (the sum of the above)

2.2 Solution Steps

2.2.1 Impulse Response $h[n]$

For the impulse response, we set $x[n]=\delta[n]$ and assume zero initial conditions. The difference equation becomes:

$$h[n]-0.5\,h[n-1]=\delta[n]+2\,\delta[n-1] \tag{2.2}$$

We can solve this difference equation directly in the time-domain using the standard approach of finding the homogeneous solution and then "injecting" the impulse at the appropriate time.

2.2.1.1 Step 1. Solve the Homogeneous Equation

The homogeneous part is

$$h[n]-0.5\,h[n-1]=0 \tag{2.3}$$

We look for solutions of the form $h[n]=r^n$. Plugging in, we obtain:

$$r^n-0.5\,r^{n-1}=0 \quad \Longrightarrow \quad r^{n-1}(r-0.5)=0 \tag{2.4}$$

For nonzero r^{n-1}, we must have

$$r - 0.5 = 0 \quad \Longrightarrow \quad r = 0.5 \tag{2.5}$$

Thus, the homogeneous solution is

$$h_h[n] = A\,(0.5)^n \tag{2.6}$$

where A is an arbitrary constant.

2.2.1.2 Step 2. Solve the Particular Equation

Since $x[n] = \delta[n]$, note that

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise,
- $\delta[n-1] = 1$ for $n = 1$ and 0 otherwise.

Thus, the forcing term $\delta[n] + 2\,\delta[n-1]$ is nonzero only at $n = 0$ and $n = 1$:

- For $n = 0$: $\delta[0] + 2\,\delta[-1] = 1 + 2 \cdot 0 = 1$
- For $n = 1$: $\delta[1] + 2\,\delta[0] = 0 + 2 \cdot 1 = 2$
- For $n \geq 2$: the right-hand side is 0

We now determine $h[n]$ by considering the following:

For $n = 0$: Since the system is causal, we have $h[n] = 0$ for $n < 0$. Then, for $n = 0$:

$$h[0] - 0.5\,h[-1] = 1 \tag{2.7}$$

But $h[-1] = 0$, so

$$h[0] = 1 \tag{2.8}$$

For $n = 1$:

$$h[1] - 0.5\,h[0] = 2 \tag{2.9}$$

Substitute $h[0] = 1$:

$$h[1] - 0.5 = 2 \quad \Longrightarrow \quad h[1] = 2 + 0.5 = 2.5 \tag{2.10}$$

For $n \geq 2$: The input is zero for $n \geq 2$ so the equation becomes homogeneous:

$$h[n] - 0.5\,h[n-1] = 0 \quad \Longrightarrow \quad h[n] = 0.5\,h[n-1] \tag{2.11}$$

Starting with $h[1] = 2.5$, the solution for $n \geq 1$ is

$$h[n] = 0.5\,h[n-1] \tag{2.12}$$

Thus, for $n \geq 1$ we can write

$$h[n] = 2.5\,(0.5)^{n-1} \tag{2.13}$$

2.2.1.3 Final Impulse Response

Collecting the results:

$$\boxed{h[n] = \begin{cases} 1, & n = 0 \\ 2.5(0.5)^{n-1}, & n \geq 1 \end{cases}} \tag{2.14}$$

Or equivalently, writing 2.5 as $\frac{5}{2}$:

$$h[0] = 1, \quad h[n] = \frac{5}{2}\,(0.5)^{n-1}, \quad n \geq 1 \tag{2.15}$$

2.2.2 Zero-State Response $y_{zs}[n]$

For the zero-state response, we solve the equation with input $x[n] = u[n]$ and zero initial condition. The equation is:

$$y_{zs}[n] - 0.5\,y_{zs}[n-1] = u[n] + 2\,u[n-1] \tag{2.16}$$

2.2.2.1 Step 1. Write the Forcing Function

The zero-state response is the response when the system is initially at rest (all initial conditions are zero) and the input is given. Here, the input is $x[n] = u(n)$.

When solving for the zero-state response we assume that the system is ”at rest” before the input is applied. Thus, we take initial conditions (for the forced response) as:

$$y_{zs}[n] = 0 \quad \text{for } n < 0 \tag{2.17}$$

Notice that the forcing function $u[n] + 2u[n-1]$ has different values depending on n:

- For $n = 0$: $u(0) = 1$ and $u(-1) = 0$ so the forcing is $1 + 2 \cdot 0 = 1$
- For $n \geq 1$: $u(n) = 1$ and $u(n-1) = 1$ so the forcing is $1 + 2 \cdot 1 = 3$

2.2.2.2 Step 2. Find the Homogeneous Solution for $y_{zs}[n]$

The homogeneous equation is:

$$y_h[n] - 0.5\, y_h[n-1] = 0 \tag{2.18}$$

This is a first-order homogeneous difference equation with general solution:

$$y_h[n] = B\,(0.5)^n \tag{2.19}$$

where B is a constant determined by initial conditions.

2.2.2.3 Step 3. Find a Particular Solution $y_p[n]$

Since the right-hand side for $n \geq 1$ becomes constant (equal to 3), we can try a constant particular solution for $n \geq 1$:

$$y_p[n] = K \quad \text{for } n \geq 1 \tag{2.20}$$

Substituting into the original equation:

$$K - 0.5K = 3 \quad \Rightarrow \quad 0.5K = 3 \quad \Rightarrow \quad K = 6 \tag{2.21}$$

Thus, for $n \geq 1$, a particular solution is:

$$y_p[n] = 6 \tag{2.22}$$

2.2.2.4 Step 4. Form the General Zero-State Solution

The general solution is the sum of the homogeneous and particular solutions:

$$y_{zs}[n] = y_h[n] + y_p[n] = B\,(0.5)^n + 6 \quad \text{for } n \geq 1 \tag{2.23}$$

2.2.2.5 Step 5. Apply the Initial Condition for the Forced (Zero-State) Case

For the zero-state response, we assume ”at rest” initial conditions:

$$y_{zs}[n] = 0 \quad \text{for } n < 0 \tag{2.24}$$

At $n = 0$, using the original difference equation:

$$y_{zs}[0] - 0.5\,y_{zs}[-1] = u(0) + 2\,u(-1) \tag{2.25}$$

For the zero-state response, $y_{zs}[-1] = 0$ (initial rest) and $u(0) = 1$ while $u(-1) = 0$. Therefore:

$$y_{zs}[0] = 1 \tag{2.26}$$

But our general solution at $n = 0$ is:

$$y_{zs}[0] = B\,(0.5)^0 + 6 = B + 6 \tag{2.27}$$

Setting this equal to 1:

$$B + 6 = 1 \quad \Rightarrow \quad B = -5 \tag{2.28}$$

Thus, for $n \geq 0$ the zero-state response is:

$$\boxed{y_{zs}[n] = 6 - 5\,(0.5)^n \quad \text{for } n \geq 0} \tag{2.29}$$

2.2.2.6 Quick Check

Let's verify our solution at a few points:

- For $n = 0$: $y_{zs}[0] = 6 - 5(0.5)^0 = 6 - 5 = 1$
- For $n = 1$: $y_{zs}[1] = 6 - 5(0.5)^1 = 6 - 2.5 = 3.5$

This sequence will approach 6 as $n \to \infty$, which is consistent with the steady-state response we expect.

2.2.3 Zero-Input Response $y_{zi}[n]$

For the zero-input response, we solve the homogeneous equation with the given initial condition $y[-1] = 2$ and with input $x[n] = 0$ for all n:

$$y_{zi}[n] - 0.5\,y_{zi}[n-1] = 0 \tag{2.30}$$

2.2.3.1 Step 1. Solve the Homogeneous Equation

The zero-input response is the response due solely to the initial condition with the input set to zero. The difference equation reduces to:

$$y_{zi}[n] - 0.5\,y_{zi}[n-1] = 0 \tag{2.31}$$

As with our previous solutions, the homogeneous solution is of the form:

$$y_{zi}[n] = A\,(0.5)^n \tag{2.32}$$

where A is a constant to be determined from the initial condition.

2.2.3.2 Step 2. Determine the Constant A from the Initial Condition

Be careful with the indexing: the given initial condition is $y[-1] = 2$. We need to use this to find the proper value of A.

For $n = 0$ (with $x[0] = 0$):

$$y_{zi}[0] - 0.5\,y_{zi}[-1] = 0 \tag{2.33}$$

Substitute $y_{zi}[-1] = 2$:

$$y_{zi}[0] = 0.5\,(2) = 1 \tag{2.34}$$

Now, our homogeneous solution for $n \geq 0$ is:

$$y_{zi}[n] = A\,(0.5)^n \tag{2.35}$$

At $n = 0$:

$$y_{zi}[0] = A = 1 \tag{2.36}$$

Thus, the zero-input response is:

$$\boxed{y_{zi}[n] = (0.5)^n \quad \text{for } n \geq 0} \tag{2.37}$$

This expression shows that the zero-input response decays exponentially toward zero as n increases, which is expected for a stable system with $|0.5| < 1$.

2.2.4 Total Response $y[n]$

The total response is the sum of the zero-state and zero-input responses:

$$y[n] = y_{zs}[n] + y_{zi}[n] = (6 - 5(0.5)^n) + (0.5)^n = 6 - 4(0.5)^n \tag{2.38}$$

The final expression is:

$$\boxed{y[n] = 6 - 4\,(0.5)^n \quad \text{for } n \geq 0} \tag{2.39}$$

This shows that the total response also approaches a steady-state value of 6, but with a different transient component than the zero-state response.

2.3 Theoretical Values

2.3.1 Side-by-Side Theoretical Tables

To provide a more compact representation of the theoretical results, Tables 2.1 and 2.2 show the theoretical values side by side.

n	$h[n]$	$y_{zs}[n]$
0	1	1
1	2.5	3.5
2	1.25	4.75
3	0.625	5.375
4	0.3125	5.6875
5	0.15625	5.84375
6	0.078125	5.921875
7	0.0390625	5.9609375
8	0.01953125	5.98046875
9	0.009765625	5.990234375

Table 2.1: Theoretical Values for $h[n]$ and $y_{zs}[n]$

n	$y_{zi}[n]$	$y[n]$
0	1	2
1	0.5	4
2	0.25	5
3	0.125	5.5
4	0.0625	5.75
5	0.03125	5.875
6	0.015625	5.9375
7	0.0078125	5.96875
8	0.00390625	5.984375
9	0.001953125	5.9921875

Table 2.2: Theoretical Values for $y_{zi}[n]$ and $y[n]$

Chapter 3

LabVIEW Implementation

3.1 LabVIEW Setup

The LabVIEW environment was configured to simulate the given difference equation. Each response type (impulse, zero-state, zero-input, and total) was implemented as a separate block diagram using LabVIEW's graphical programming capabilities. The implementation consists of:

- For Loops to simulate the discrete-time steps
- Shift Registers to implement recursive calculations and memory elements
- Mathematical operations (addition, multiplication) to compute the difference equation
- Array controls and indicators to display the results

For all simulations, we set the number of iterations to 10, allowing us to observe the system behavior over the first 10 time steps ($n = 0$ to $n = 9$).

3.2 Implementation of Responses

3.3 Block Diagram Explanation

Each LabVIEW block diagram implements a specific response type for the given difference equation. The diagrams utilize LabVIEW's visual programming environment to implement the recursion formulas derived theoretically.

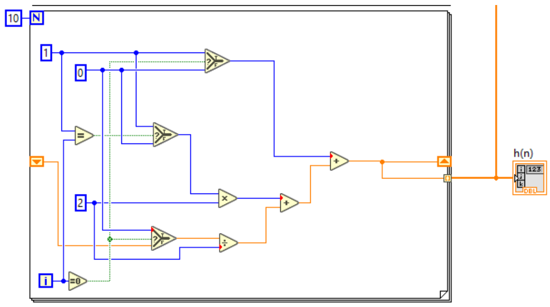

Figure 3.1: LabVIEW Block Diagram for $h(n)$

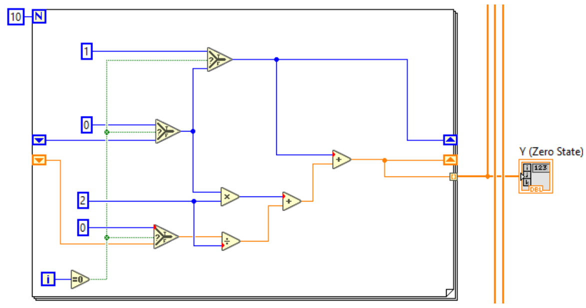

Figure 3.2: LabVIEW Block Diagram for $Y_{ZS}(n)$

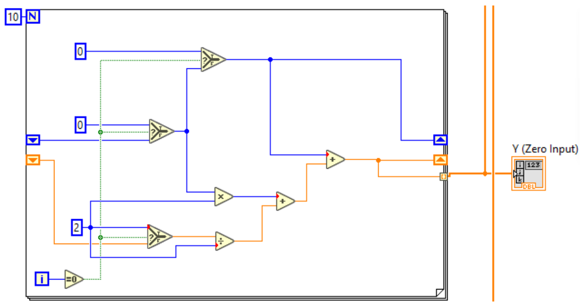

Figure 3.3: LabVIEW Block Diagram for $Y_{ZI}(n)$

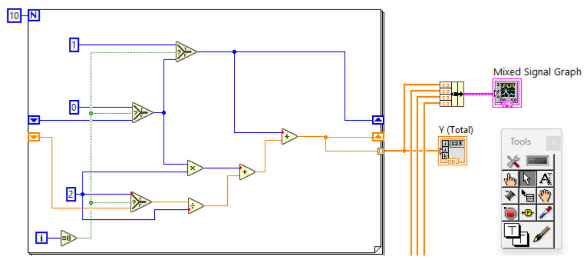

Figure 3.4: LabVIEW Block Diagram for $Y_T(n)$

3.3.1 Impulse Response Implementation

The impulse response diagram (Figure 3.1) implements $h[n] - 0.5\,h[n-1] = \delta[n] + 2\,\delta[n-1]$. Key components include:

- A For Loop that iterates 10 times to compute values for $n = 0$ to $n = 9$
- Shift registers to store the previous value $h[n-1]$
- A case structure that sets $\delta[0] = 1$ for the first iteration and $\delta[n] = 0$ elsewhere
- A delayed unit impulse for creating $\delta[n-1]$
- Multiplication by 0.5 for the feedback term and by 2 for the delayed impulse

3.3.2 Zero-State Response Implementation

The zero-state response diagram (Figure 3.2) implements the response to a unit step input with zero initial conditions. It features:

- Constant input of 1 representing $u[n] = 1$ for all $n \geq 0$
- Shift registers initialized to 0 to implement the zero initial condition
- Recursive implementation of $y_{zs}[n] = 0.5\,y_{zs}[n-1] + u[n] + 2\,u[n-1]$
- Output array collecting all computed values of $y_{zs}[n]$

3.3.3 Zero-Input Response Implementation

The zero-input response diagram (Figure 3.3) computes the response due solely to the initial condition with zero input:

- Shift register initialized to the value 2, representing $y[-1] = 2$
- No external input ($x[n] = 0$ for all n)
- Simple recursive calculation of $y_{zi}[n] = 0.5\,y_{zi}[n-1]$
- Output array storing the resulting decay sequence

3.3.4 Total Response Implementation

The total response diagram (Figure 3.4) implements the complete difference equation:

- Combines both the effect of the initial condition $y[-1] = 2$ and step input $u[n]$
- Implements the full equation $y[n] = 0.5\,y[n-1] + u[n] + 2\,u[n-1]$
- Uses shift registers for both $y[n-1]$ and $u[n-1]$ terms
- Output array showing the combined response

These implementations demonstrate how theoretical difference equations can be directly translated into visual programming constructs in LabVIEW, providing both numerical results and graphical visualization.

3.4 Results and Observations

3.4.1 Combined Plot Analysis

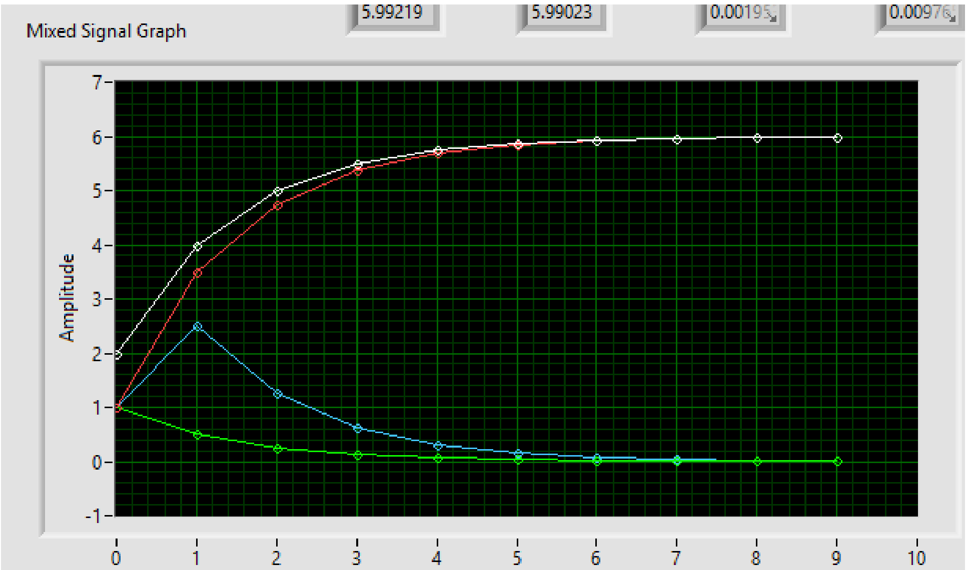

Figure 3.5: Combined Plot of $h(n)$, $Y_{ZS}(n)$, $Y_{ZI}(n)$, and $Y_T(n)$

The combined plot reveals several important observations:

- The impulse response $h[n]$ shows an initial value of 1, jumps to 2.5 at $n = 1$, and then decays exponentially.
- The zero-state response $y_{zs}[n]$ starts at 1 and increases asymptotically toward 6.
- The zero-input response $y_{zi}[n]$ begins at 1 (from $y[-1] = 2$) and decays exponentially toward 0.
- The total response $y[n]$ starts at 2 and increases toward the steady-state value of 6.

The plot confirms that $y[n] = y_{zs}[n] + y_{zi}[n]$ for all $n \geq 0$, validating our implementation.

3.4.2 Numerical Values

The table of numerical values provides precise data points for the first 10 iterations. Key observations:

- The zero-input response $y_{zi}[n]$ decays by a factor of 0.5 in each step.

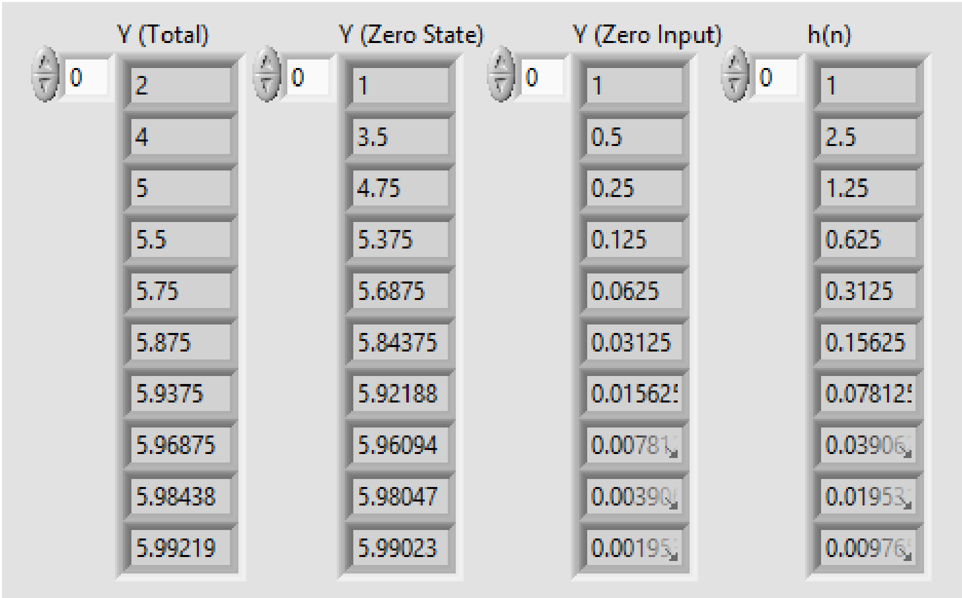

Figure 3.6: Table of Values for $h(n)$, $Y_{ZS}(n)$, $Y_{ZI}(n)$, and $Y_T(n)$ (First 10 Iterations)

- The zero-state response $y_{zs}[n]$ approaches its steady-state value of 6 as $(0.5)^n$ approaches 0.
- The total response $y[n]$ approaches 6 more rapidly than $y_{zs}[n]$ due to the initial condition contribution.
- By $n = 9$, both $y_{zs}[n]$ and $y[n]$ are very close to their steady-state values (5.99 and 5.99 respectively).

These numerical values will be compared with the theoretical results in the next chapter.

Chapter 4

Comparison and Conclusion

4.1 Comparison of Results

After implementing the system in LabVIEW and solving it theoretically, we can now compare the results to validate our approaches.

4.1.1 Numerical Comparison

Comparing the numerical values from the LabVIEW implementation (Figure 3.6) with the theoretical calculations (Tables 2.1 and 2.2), we observe excellent agreement between both approaches. The small differences that might exist can be attributed to numerical precision in the computations.

- **Impulse Response:** The LabVIEW simulation correctly produces $h[0] = 1$ and $h[1] = 2.5$, followed by exponential decay exactly as predicted by the theoretical formula $h[n] = 2.5\,(0.5)^{n-1}$ for $n \geq 1$.
- **Zero-State Response:** The LabVIEW values match the theoretical prediction $y_{zs}[n] = 6 - 5\,(0.5)^n$, starting at 1 and asymptotically approaching 6.
- **Zero-Input Response:** The LabVIEW simulation correctly shows the exponential decay $y_{zi}[n] = (0.5)^n$ starting from $y_{zi}[0] = 1$.
- **Total Response:** The LabVIEW total response values agree with the theoretical prediction $y[n] = 6 - 4\,(0.5)^n$.

This agreement confirms the validity of both our theoretical derivation and LabVIEW implementation.

4.1.2 Graphical Comparison

The combined plot (Figure 3.5) from LabVIEW visually confirms the behavior we predicted theoretically:

- The exponential decay of the impulse response
- The asymptotic approach of the zero-state response to the steady-state value of 6
- The exponential decay of the zero-input response to zero
- The total response approaching the same steady-state value as the zero-state response, but with a different transient behavior

The plot clearly illustrates the principle of superposition, showing that $y[n] = y_{zs}[n] + y_{zi}[n]$ at every time step.

4.2 System Behavior Analysis

4.2.1 Stability Analysis

The system's homogeneous solution involves the term $(0.5)^n$, which decays to zero as n increases since $|0.5| < 1$. This confirms that the system is stable, as any response due to initial conditions will eventually diminish to zero.

4.2.2 Steady-State Analysis

The total response approaches a steady-state value of 6 as n increases. This can be interpreted as the DC gain of the system multiplied by the step input amplitude:

$$y_{ss} = \lim_{n\to\infty} y[n] = \lim_{n\to\infty} (6 - 4(0.5)^n) = 6 \tag{4.1}$$

This steady-state value can also be derived directly from the difference equation by assuming

$y[n] = y[n-1] = y_{ss}$ and $x[n] = x[n-1] = 1$ for large n:

$$y_{ss} - 0.5y_{ss} = 1 + 2 \Rightarrow 0.5y_{ss} = 3 \Rightarrow y_{ss} = 6 \tag{4.2}$$

4.2.3 Impact of Initial Condition

The initial condition $y[-1] = 2$ affects the transient behavior of the system but not its steady-state response. Specifically:

- It introduces the zero-input component $(0.5)^n$
- It causes the total response to start at $y[0] = 2$ instead of $y[0] = 1$ (as in the zero-state case)
- The effect diminishes over time as $(0.5)^n$ approaches zero

This demonstrates an important property of stable LTI systems: the influence of initial conditions eventually fades away, leaving only the forced response.

4.3 Final Summary

This assignment has demonstrated the application of difference equation solution techniques using both theoretical and simulation approaches. Key accomplishments include:

- Successfully derived closed-form expressions for the impulse response, zero-state response, zero-input response, and total response
- Implemented the difference equation in LabVIEW using proper recursive structures
- Verified the principle of superposition by showing that $y[n] = y_{zs}[n] + y_{zi}[n]$
- Analyzed the system's stability and steady-state behavior

The agreement between theoretical and simulation results validates both approaches and reinforces our understanding of discrete-time system analysis.

List of Abbreviations

BIBO	Bounded-Input Bounded-Output (stability criterion)
CT	Continuous-Time
DC	Direct Current (refers to steady-state value)
DFT	Discrete Fourier Transform
DSP	Digital Signal Processing
DT	Discrete-Time
DTFT	Discrete-Time Fourier Transform
FFT	Fast Fourier Transform
FIR	Finite Impulse Response
GUI	Graphical User Interface
IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
NI	National Instruments (manufacturer of LabVIEW)
ODE	Ordinary Differential Equation
ROC	Region of Convergence
VI	Virtual Instrument (LabVIEW program)
ZIR	Zero-Input Response
ZSR	Zero-State Response

Bibliography

[1] National Instruments, *LabVIEW User Manual*, January 1998 Edition, Part Number 320999B-01. Available at: `http://www.natinst.com`.

[2] Alan V. Oppenheim, Alan S. Willsky, and S. Hamid Nawab, *Signals and Systems*, 2nd Edition, Prentice Hall, 1996.

[3] John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th Edition, Prentice Hall, 2007.

[4] National Instruments, *LabVIEW Digital Signal Processing and Digital Communications*. Available at: `https://www.ni.com/en-us/shop/labview/digital-signal-processing-and-communications.html`.