Z1 $\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}$

To wyrażenie jest jednorodne, więc możemy analizować dla $a=1$ $b,c\in\mathbb{R}_+$

$$\frac{1}{1+b}+\frac{b}{b+c}+\frac{c}{c+1}\le 2$$

$$\Updownarrow$$

$$(b+c)(c+1)+b(1+b)(1+c)+c(b+c)(1+b)\le 2(1+b)(b+c)(c+1)$$

$\cancel{bc+b+c^2+c+}$ $\cancel{(1+2b)(bc+b+c^2+c)}$

$\cancel{(b+c)(c+1)+1+b+c+bc+b+b^2+c+bc\le (b+c)}$

$\cancel{1+(b+c)}$

$\cancel{1+b+c+bc+b+b^2+c+bc\le bc+b+c^2+c+2b^2+2b^2c+2bc^2}$

$\cancel{1+b+c\le c^2+b^2+2b^2c+2bc^2}$

$\cancel{0\le c^2+b^2+2b^2c+2bc^2-1-b-c}$

$\cancel{0\le 1}$

$$\Updownarrow$$

$b+b^2+c+bc$ $\quad +2b(b+c)(c+1)$

$$\cancel{(b+c)(c+1)}+b(1+b+c+bc)+c(b+c)(1+b)\le 2(b+c)(c+1)+$$

$$\Updownarrow$$

$$\cancel{b}+b^2+\cancel{bc}+b^2c+bc+b^2c+\cancel{c^2}+bc^2\le \cancel{bc}+\cancel{b}+c+\cancel{c^2}+2b(bc+b+c^2+c)$$

$$\Updownarrow$$

$$\cancel{b^2}+\cancel{2b^2c}+\cancel{bc}+\cancel{bc^2}\le c+\cancel{2}b^2c+\cancel{2}b^2+\cancel{2}bc+\cancel{2bc^2}$$

$$\Updownarrow$$

$$0\le \underset{\geq 0}{c}+\underset{\geq 0}{b^2c}+\underset{\geq 0}{b^2}+\underset{\geq 0}{bc}\quad \blacksquare$$

$$\frac{1}{1+b} + \frac{b}{b+c} + \frac{c}{c+1} \geq 1$$

$$\Updownarrow$$

$$(b+c)(c+1) + b(1+b)(c+1) + c(1+b)(b+c) \geq 1(1+b)(b+c)(c+1)$$

$$\Updownarrow$$

$$bc+b+c^2+c+\cancel{b(1+b)(c+1)}+c(b+c+b^2+bc) \geq c\underbrace{(1+b)(1+c)}^{1+b+c+bc}$$

$$\Updownarrow$$

$$bc+b+c^2+\cancel{c}+\cancel{cb}+\cancel{c^2}+b^2c+\cancel{bc^2} \geq \cancel{c}\,\cancel{1}+\cancel{bc}+\cancel{c^2}+\cancel{bc^2}$$

$$\Updownarrow$$

$$\underset{\geq 0}{bc}+\underset{\geq 0}{b}+\underset{\geq 0}{c^2}+\underset{\geq 0}{b^2c} \geq 0$$

dla $a=1,\ b \to \infty,\ c \to 0$

$$\cancel{1}\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right) \to 2$$

dla $a=1,\ b\to 0,\ \cancel{c\to b}\ c=b^2$

$$\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right) \to 1$$

$$\frac{a}{a+b} \to 1, \qquad \frac{b}{b+c} = \frac{1b}{1b+b^2} \to 0, \qquad \frac{c}{c+a} \to 0$$

więc $\sup = 2$ i $\inf = 1$

22. $0^* = \{ p : p \in \mathbb{Q},\ p < 0 \}$

- $(\alpha + 0^* \subseteq \alpha)$

  Weźmy $p \in \alpha + 0^*$

  czyli $p = q + r$, $q \in \alpha$ $r \in 0^*$

  $p = q - |r| < q \in \alpha \Rightarrow p \in \alpha$

- $(\alpha \subseteq \alpha + 0^*)$

  Weźmy $p \in \alpha$

  Wtedy istnieje $q \in \alpha$ t. że $p < q$,

  czyli $0 > p - q$, czyli $p - q \in 0^*$

  Czyli istnieje q, $p - q \in 0^*$ i $q \in \alpha$ t. że

  $(p-q) + q = p \in \alpha + 0^*$

czyli $\alpha = \alpha + 0^*$