Z1 $V_0 = 2$ $V_1 = a$

$V_{n+1} = aV_n - bV_{n-1}$

Baza:
dla $n=1$ ok
$n=2$ ok.

Zał. że dla $2 \le n$:

$V_{2n} = (V_n)^2 - 2b^n$

$V_{2n+1} = V_n V_{n+1} - ab^n$

Pokażę że zachodzi też dla $n+1$ jest to prawda

$$V_{2n+2} = aV_{2n+1} - bV_{2n}$$

$$V_{2n+2} = a\left(aV_{2n} - bV_{2n-1}\right) - bV_{2n}$$

$$V_{2n+2} = a^2 V_{2n} - abV_{2n-1} - bV_{2n}$$

$$V_{2n+2} = a^2 V_{2n} - abV_{2n-1} - b\left(aV_{2n-1} - bV_{2n-2}\right) =$$

$$= a^2 V_{2n} - 2abV_{2n-1} + b^2 V_{2n-2} =$$

$$\cancel{a^2 V_{2n}}$$

$$a^2\left(V_n^2 - 2b^n\right) - 2ab\left(V_{n-1}V_n - ab^{n-1}\right) + b^2 V_{2n-2}$$

$$a^2 V_n^2 - 2a^2 b^n - 2abV_{n-1}V_n + 2a^2b^n + b^2 V_{2n-2}$$

$$a^2 V_n^2 - 2abV_{n-1}V_n + b^2\left(V_{n-1}^2 - 2b^{n-1}\right) =$$

$$a^2 V_n^2 - 2abV_{n-1}V_n + b^2 V_{n-1}^2 - 2b^{n+1} = V_{n+1}^2 - 2b^{n+1}$$

$$V_{2n+3} = aV_{2n+2} - bV_{2n+1} =$$

$$= a\left((V_{n+1})^2 - 2b^{n+1}\right) - b\left(V_{n+1}V_n - \right.$$

$$= a(V_{n+1})^2 - 2ab^{n+1} - bV_{n+1}V_n + ab^{n}$$

$$= V_{n+1}\left(aV_{n+1} - bV_n\right) + $$

$$= V_{n+1}V_{n+2} - ab^{n+1}$$

$$\left(1+\frac{1}{n}\right)^n = \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \dots + \binom{n}{n}\frac{1}{n^n} =$$

$$= \frac{1}{0!} + \frac{n}{1!\,n} + \frac{n(n-1)}{2!\,n^2} + \dots + \frac{n(n-1)(n-2)\cdot\ldots\cdot 1}{n!\cdot n^n} <$$

$$< 2 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \overset{(*)}{\leqslant} 2 + \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

$$< 3$$

Dowód (*)

$$n! \geqslant 2^{n-1}$$

Baza: dla $n = 2$

$2! \geqslant 2^1$ ok

Zał. że $n! \geqslant 2^{n-1}$ dla $n \geqslant 1$

Pokażę, że $n!(n+1) \geqslant 2^{n-1}\cdot 2$

$$n!(n+1) \geqslant 2^{n-1}(n+1) \geqslant 2^{n-1}\cdot 2 = 2^n$$

$\square$