basic solution: For a system of linear equations $Ax = b$ with n variables and $m \leq n$ constraints, set $n - m$ *non-basic* variables equal to zero and solve the remaining m *basic* variables.

basic feasible solutions (BFS): a basic solution that is feasible. That is $Ax = b$, $x \geq 0$ and x is a basic solution.

The feasible corner-point solutions to an LP are basic feasible solutions. The Simplex Method uses the *pivot* procedure to move from one BFS to an "adjacent" BFS with an equal or better objective function value.

The Pivot Procedure

1. Choose a *pivot element* a_{ij}
2. Divide row i of the augmented matrix $[A|b]$ by a_{ij}

$$\begin{bmatrix} \dots & a_{ij} & \dots & a_{i\ell} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & a_{kj} & \dots & a_{k\ell} & \dots \end{bmatrix} \rightarrow \begin{bmatrix} \dots & 1 & \dots & \frac{a_{i\ell}}{a_{ij}} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & a_{kj} & \dots & a_{k\ell} & \dots \end{bmatrix}$$

3. For each row k (other than row i), add $-a_{kj}\times$ row i to row k.
   The element in row k, column ℓ becomes $-a_{kj} \times a_{i\ell} + a_{k\ell}$.

$$\begin{bmatrix} \dots & 1 & \dots & \frac{a_{i\ell}}{a_{ij}} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & a_{kj} & \dots & a_{k\ell} & \dots \end{bmatrix} \rightarrow \begin{bmatrix} \dots & 1 & \dots & \frac{a_{i\ell}}{a_{ij}} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & a_{k\ell} - \frac{a_{kj}a_{i\ell}}{a_{ij}} & \dots \end{bmatrix}$$

Pivoting Example 1

Suppose we want to solve the following the LP with the Simplex method:

$$
\begin{array}{rrcrl}
\text{maximize} & x & + & 2y & \\
\text{s.t.} & x & + & y & \leq 4 \\
 & x & - & 2y & \leq 2 \\
 & -2x & + & y & \leq 2 \\
 & x, & & y & \geq 0
\end{array}
$$

First, we put the problem in standard form:

$$
\begin{array}{rrcrcrcrcrl}
\text{maximize} & x & + & 2y & & & & & & & \\
\text{s.t.} & x & + & y & + & s_1 & & & & & = 4 \\
 & x & - & 2y & & & + & s_2 & & & = 2 \\
 & -2x & + & y & & & & & + & s_3 & = 2 \\
 & x, & & y, & & s_1, & & s_2, & & s_3 & \geq 0
\end{array}
$$

Pivoting Example 1

Augmented matrix form of the constraints:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$

BFS 1:
Basic variables: $BV = \{s_1, s_2, s_3\}$
Non-basic variables: $NB = \{x, y\}$
Solution: $s_1 = 4$, $s_2 = 2$, $s_3 = 2$, $x = y = 0$
Objective function value: $x + 2y = 0 + 0 = 0$

Pivoting Example 1

Pivot on row 3, column 1.

First, divide row 3 by -2:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 \end{bmatrix}$$

Next, add -1 times row 3 to rows 1 and 2:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \frac{3}{2} & 1 & 0 & \frac{1}{2} & 5 \\ 0 & -\frac{3}{2} & 0 & 1 & \frac{1}{2} & 3 \\ 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 \end{bmatrix}$$

Pivoting Example 1

$$\left[\begin{array}{cccccc} 0 & \frac{3}{2} & 1 & 0 & \frac{1}{2} & 5 \\ 0 & -\frac{3}{2} & 0 & 1 & \frac{1}{2} & 3 \\ 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -1 \end{array}\right] \equiv \begin{array}{rrrrrrrrrr} & & \frac{3y}{2} & + & s_1 & & & + & \frac{s_3}{2} & = & 5 \\ & - & \frac{3y}{2} & & & + & s_2 & + & \frac{s_3}{2} & = & 3 \\ x & - & \frac{y}{2} & & & & & - & \frac{s_3}{2} & = & -1 \end{array}$$

BFS 2:

Basic variables: $BV = \{x, s_1, s_2\}$
Non-basic variables: $NB = \{y, s_3\}$
Solution: $x = -1$, $s_1 = 5$, $s_2 = 3$, $y = s_3 = 0$
Objective function value: $x + 2y = 0 + 0 = 0$
Solution is infeasible since $x < 0$.

This BFS is said to be *adjacent* to the first one since it shares two of the three basic variables.

Pivoting Example 1

Pivot on row 2, column 1.

Add -1 times row 2 to rows 1 and two times row 1 to row 3:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 & 1 & -1 & 0 & 2 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ 0 & -3 & 0 & 2 & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 1 & -1 & 0 & 2 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ 0 & -3 & 0 & 2 & 1 & 6 \end{bmatrix} \equiv \begin{array}{rcrcrcrcrcl} & & 3y & + & s_1 & - & s_2 & & & = & 2 \\ x & - & 2y & & & + & s_2 & & & = & 2 \\ & - & 3y & & & + & 2s_2 & + & s_3 & = & 6 \end{array}$$

BFS 3:

Basic variables: $BV = \{x, s_1, s_3\}$

Non-basic variables: $NB = \{y, s_2\}$

Solution: $x = 2$, $s_1 = 2$, $s_3 = 6$, $y = s_2 = 0$

Objective function value: $x + 2y = 2 + 0 = 2$

Pivoting Example 1

Pivot on row 3, column 2.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & -2 & 0 & 1 & 0 & 2 \\ -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & 1 & 2 & 6 \\ -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & 1 & 2 & 6 \\ -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \equiv \begin{array}{rcrcrcrcrcl} 3x & & & + & s_1 & & & - & s_3 & = & 2 \\ -3x & & & & & + & s_2 & + & 2s_3 & = & 6 \\ -2x & + & y & & & & & + & s_3 & = & 2 \end{array}$$

BFS 4:

Basic variables: $BV = \{y, s_1, s_2\}$

Non-basic variables: $NB = \{x, s_3\}$

Solution: $y = 2$, $s_1 = 2$, $s_2 = 6$, $x = s_3 = 0$

Objective function value: $x + 2y = 0 + 4 = 4$

Row-Zero Form of an LP

Standard form:

$$\begin{array}{rrcrcrcrcrl}
\text{maximize} & x & + & 2y & & & & & & & \\
\text{s.t.} & x & + & y & + & s_1 & & & & & = 4 \\
& x & - & 2y & & & + & s_2 & & & = 2 \\
& -2x & + & y & & & & & + & s_3 & = 2 \\
& x, & & y, & & s_1, & & s_2, & & s_3 & \geq 0
\end{array}$$

Row-Zero form:

$$\begin{array}{rrcrcrcrcrcrl}
\text{maximize} & z & & & & & & & & & & & \\
\text{s.t.} & z & - & x & - & 2y & & & & & & & = 0 \\
& & & x & + & y & + & s_1 & & & & & = 4 \\
& & & x & - & 2y & & & + & s_2 & & & = 2 \\
& & & -2x & + & y & & & & & + & s_3 & = 2 \\
& & & x, & & y, & & s_1, & & s_2, & & s_3 & \geq 0
\end{array}$$

- A system of linear equations is in *canonical form* if each equation has a variable x_j with a coefficient of 1 in that equation such that the coefficient x_j is 0 in all other equations.
- If an LP is in canonical form, then we can find a basic solution by inspection.
- If an LP is in canonical form and all the constraints have non-negative right-hand sides, then we can find a basic feasible solution by inspection.
- If an LP is in Row-Zero form and the row 1, row 2, . . ., row m constraints have non-negative right-hand sides,then we can find BFS and its objective function variable by inspection.

Row-Zero Form BFS

Row-Zero form:

$$
\begin{array}{rrrrrrrrrrrrl}
\text{maximize} & z & & & & & & & & & & & \\
\text{s.t.} & z & - & x & - & 2y & & & & & & & = 0 \\
 & & & x & + & y & + & s_1 & & & & & = 4 \\
 & & & x & - & 2y & & & + & s_2 & & & = 2 \\
 & & - & 2x & + & y & & & & & + & s_3 & = 2 \\
 & & & x, & & y, & & s_1, & & s_2, & & s_3 & \geq 0
\end{array}
$$

Basic variables: $BV = \{z, s_1, s_3, s_2\}$
Non-basic variables: $NB = \{x, y\}$
Solution: $z = 0$, $s_1 = 4$, $s_2 = 2$, $s_3 = 2$, $x = y = 0$
Objective function value: $z = x + 2y = 0 + 0 = 0$

Row-Zero Form and Augmented Matrix

Row-Zero form:

$$\begin{array}{lrcrcrcrcrcrl}
\text{maximize} & z & & & & & & & & & & & \\
\text{s.t.} & z & - & x & - & 2y & & & & & & & = 0 \\
 & & & x & + & y & + & s_1 & & & & & = 4 \\
 & & & x & - & 2y & & & + & s_2 & & & = 2 \\
 & & - & 2x & + & y & & & & & + & s_3 & = 2 \\
 & & & x, & & y, & & s_1, & & s_2, & & s_3 & \geq 0
\end{array}$$

Augmented Matrix:

$$\begin{bmatrix}
1 & -1 & -2 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 4 \\
0 & 1 & -2 & 0 & 1 & 0 & 2 \\
0 & -2 & 1 & 0 & 0 & 1 & 2
\end{bmatrix}$$

Fundamental Steps of the Simplex Method

Is the current BFS Optimal?

Can we increase the value of z by increasing the value of a non-basic variable?

If we increase x or y, we will have to increase z to satisfy the constraint in Row 0, $z - x - 2y = 0$.

Which non-basic variable should we increase?

A one-unit increase in x will give us a one-unit increase in z.

A two-unit increase in y will give us a two-unit increase in z.

How much can we increase y?

- **Row-1 constraint:** $x + y + s_1 = 4$

  $x + y + s_1 = 4 \Rightarrow s_1 = 4 - x - y$

  $x = 0 \Rightarrow s_1 = 4 - y$

  $s_1 \geq 0 \Rightarrow y \leq 4$

- **Row-2 constraint:** $x - 2y + s_2 = 2$

  $x - 2y + s_2 = 2 \Rightarrow s_2 = 2 - x + 2y$

  $x = 0 \Rightarrow s_2 = 2 + 2y$

  $s_2 \geq 0 \Rightarrow y \geq -1$

- **Row-3 constraint:** $-2x + y + s_3 = 2$

  $-2x + y + s_3 = 2 \Rightarrow s_3 = 2 + 2x - y$

  $x = 0 \Rightarrow s_3 = 2 - y$

  $s_3 \geq 0 \Rightarrow y \leq 2$

We can increase y to 2 if we decrease s_3 to 0 at the same time.

To increase y and decrease s_3, we pivot on Row 3, Column 3 of the augmented matrix:

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & -2 & 0 & 1 & 0 & 2 \\ 0 & -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 & 0 & 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 0 & 0 & 1 & 2 & 6 \\ 0 & -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 0 & 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 0 & 0 & 1 & 2 & 6 \\ 0 & -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \equiv \begin{aligned} z - 5x + 2s_3 &= 4 \\ 3x + s_1 - s_3 &= 2 \\ -3x + s_2 + 2s_3 &= 6 \\ -2x + y + s_3 &= 2 \end{aligned}$$

Is the current BFS optimal?

$$\begin{bmatrix} 1 & -5 & 0 & 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 0 & 0 & 1 & 2 & 6 \\ 0 & -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \equiv \begin{array}{rcl} z - 5x + 2s_3 & = & 4 \\ 3x + s_1 - s_3 & = & 2 \\ -3x + s_2 + 2s_3 & = & 6 \\ -2x + y + s_3 & = & 2 \end{array}$$

If we increase x we will have to increase z to satisfy the constraint in row 0, $z - 5x + 2s_3 = 4$. Therefore, the basis is might not be optimal.

Which non-basic variable should we increase?

A one-unit increase in x will give us a five-unit increase in z.
A one-unit increase in s_3 will give us a two-unit *decrease* in z.
Increase x.

How much can we increase x?

- **Row-1 constraint:** $3x + s_1 - s_3 = 2$

  $3x + s_1 - s_3 = 2 \Rightarrow s_1 = 2 - 3x + s_3$

  $s_3 = 0 \Rightarrow s_1 = 2 - 3x$

  $s_1 \geq 0 \Rightarrow x \leq \frac{2}{3}$

- **Row-2 constraint:** $-3x + s_2 + 2s_3 = 6$

  $-3x + s_2 + 2s_3 = 6 \Rightarrow s_2 = 6 + 3x - 2s_3$

  $s_3 = 0 \Rightarrow s_2 = 6 + 3x$

  $s_2 \geq 0 \Rightarrow x \geq -2$ (we can increase s_2 to compensate for any increase in x)

- **Row-3 constraint:** $-2x + y + s_3 = 2$

  $-2x + y + s_3 = 2 \Rightarrow y = 2 + 2x - s_3$

  $s_3 = 0 \Rightarrow y = 2 + 2x$

  $y \geq 0 \Rightarrow x \geq -1$

We can increase x to $\frac{2}{3}$ if we decrease s_1 to 0 at the same time. x becomes basic and s_1 becomes non-basic.

z	x	y	s_1	s_2	s_3		
1	-5	0	0	0	2	4	(z)
0	3	0	1	0	-1	2	(s_1)
0	-3	0	0	1	2	6	(s_2)
0	-2	1	0	0	1	2	(y)

Pivot on column 2, the x column, row 1 (the only row where s_1 has a non-zero entry).

$$\begin{bmatrix} 1 & -5 & 0 & 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 1 & 0 & -1 & 2 \\ 0 & -3 & 0 & 0 & 1 & 2 & 6 \\ 0 & -2 & 1 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} & 0 & \frac{1}{3} & \frac{22}{3} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{10}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} & 0 & \frac{1}{3} & \frac{22}{3} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{10}{3} \end{bmatrix} \equiv \begin{array}{rcl} z + \frac{5}{3}s_1 + \frac{1}{3}s_3 & = & \frac{22}{3} \\ x + \frac{1}{3}s_1 - \frac{1}{3}s_3 & = & \frac{2}{3} \\ s_1 + s_2 + s_3 & = & 8 \\ y + \frac{2}{3}s_1 + \frac{1}{3}s_3 & = & \frac{10}{3} \end{array}$$

After the two pivots, the LP is now in the following form:

$$
\begin{aligned}
\max z & \\
\text{s.t.} \quad z + \tfrac{5}{3}s_1 + \tfrac{1}{3}s_3 &= \tfrac{22}{3} \\
x + \tfrac{1}{3}s_1 - \tfrac{1}{3}s_3 &= \tfrac{2}{3} \\
s_1 + s_2 + s_3 &= 8 \\
y + \tfrac{2}{3}s_1 + \tfrac{1}{3}s_3 &= \tfrac{10}{3} \\
x, y, s_1, s_2, s_3 &\geq 0
\end{aligned}
$$

The BFS $z = \frac{22}{3}$, $x = \frac{2}{3}$, $y = \frac{10}{3}$, $s_2 = 8$, $s_1 = s_3 = 0$ is optimal. Why?