The Hull-White Model Introduction

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1 Abstract

The project aims to present one factor Hull-White model from the derivation of its theory to practical usage in calibration to market data and derivatives pricing. The idea is to provide a comprehensive, step-by-step exploration of the model from scratch to it's basic usage, so that the reader can get to know the model with everything clearly stated in one paper. Implementation of the code can be found in the GitHub repository stated below.

GitHub: https://github.com/Atarios17/Stochastic_Interest_Rates

2 Introduction

2.1 Model formulation

Hull-White model is a stochastic *short rate* model that is used for modeling stochastic behavior of interest rates and derivative products based on them (for example bond options or swaptions).

The term *short rate* describes the instantaneous spot interest rate available on the market at a given time t. Formally, it is the rate of return on an infinitesimally short–maturity zero–coupon bond, and hence determines the instantaneous growth of money in the risk–free account. In the Hull–White model, the short rate is modeled as a stochastic process that evolves over time

Mathematical formulation of the short rate (marked by $r(t)$) starts with assumption that it is a stochastic process with following dynamics

$$dr(t) = \mu(r, t)\, dt + \sigma(r, t)\, d\tilde{W}(t),$$

where $\tilde{W}_t$ is Wiener process under objective measure P.

The beginning is therefore very similar to the Black-Scholes model and typically after stating above definition/assumption, we would create portfolio consisting of derivative instrument and Δ amount of underlying, then set Δ to make the portfolio risk-free and obtain derivatives pricing formula adequate to the Black-Scholes one.

However, the main and very important difference is that for short rate models the stochastic underlying is **not tradable** (as there is no such thing as *buying* or *selling* interest rate). Therefore it requires a bit different approach to obtain the derivatives pricing equation.

2.2 Term structure equation and it's derivation

To deal with underlying being non-tradable, the portfolio will be created from very natural instruments closely related to interest rates: bonds. To do this, another assumptions is made that bonds can be priced and their value depends on three factors: moment of pricing t, bond maturity T and short rate at time of pricing $r(t)$. Bond price is therefore assumed to be a function F

$$\text{Bond Price} = F(t, r(t), T)$$

with condition $F(T, r(T), T) = 1$ for all $r(T)$. We rather think of t and $r(t)$ as variables, while T is more of an parameter. Also, stating t is enough to express that we will use $r(t)$ at the same time, so function F will be often expressed just as $F^T(t)$ (visualizing T as parameter) or even shorter just as F^T (when t is understandable from the context). Bond prices are also commonly called by letter p with only t and T arguments.

To summarize, for a moment we introduced couple of names for bond prices, however we will only use F^T to make incoming derivation clearly readable and after that we will abandon calling bond prices F and we'll switch to commonly used $p(t,T)$. We have following equivalent names

$$F(t, r(t), T) = p(t, T) = F^T(t) = F^T$$

and as stated above, we'll stick to using F^T for below derivation.

Above assumption on bond prices is enough to derive pricing formula. First, let's write down bond price dynamics using Itô formula:

$$dF^T = F_t^T dt + F_r^T dr(t) + \frac{1}{2} F_{rr}^T d\langle r \rangle_t^2,$$

where:

$$dr(t) = \mu\left(r, t\right) dt + \sigma\left(r, t\right) d\tilde{W}(t),$$
$$d\langle r \rangle_t^2 = \sigma^2(r, t) dt,$$

which after substitution and grouping results in

$$dF^T = \left(F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2(r(t), t) F_{rr}^T\right) dt + \sigma(r(t), t) F_r^T d\tilde{W}(t).$$

It will appear convenient to substitute

$$\alpha_T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2(r(t), t) F_{rr}^T}{F^T}$$

$$\sigma_T = \frac{\sigma(r(t),t)F_r^T}{F^T}$$

leading as of now to

$$dF^T = F^T\alpha_T dt + F^T\sigma_T d\tilde{W}(t) = F^T \cdot \left(\alpha_T dt + \sigma_T d\tilde{W}(t)\right).$$

Having dynamics of bond prices, we can now move to portfolio creation. The idea is to make it consist of two bonds, one with maturity S and another with maturity T and try to make it risk-free choosing the right proportions. It's therefore natural to look at portfolio relatively, by spending u_T (ratio) of initial capital V_t for T-Bonds and u_S of initial capital for S-Bonds such that $u_T + u_S = 1$. Since F^T is price of T-Bond, for $V_t \cdot u_T$ we are able to buy (and own in portfolio) $V_t \cdot u_T/F^T$ of these bonds (and consequently we have $V_t \cdot u_S/F^S$ of S-Bonds). Increment of such portfolio is therefore given by

$$dV_t = V_t \cdot \left(\frac{u_T}{F^T} \cdot dF^T + \frac{u_S}{F^S} \cdot dF^S\right).$$

Now if we substitute dF^T and dF^S, we can see convenience of previously introduced α_T and σ_T which now partially simplify with divisions by F^T and F^S giving (after grouping)

$$dV_t = V_t\left(u_T\alpha_T + u_S\alpha_S\right)dt + V_t\left(u_T\sigma_T + u_S\sigma_S\right)d\tilde{W}(t).$$

Now, that's the moment when we are able to remove stochastic term by choosing the proportions in our portfolio for which

$$u_T\sigma_T + u_S\sigma_S = 0.$$

To ensure this, we combine above with condition that $u_T + u_S = 1$ to get simple system of two equations

$$\begin{cases} u_T + u_S = 1 \\ u_T\sigma_T + u_S\sigma_S = 0 \end{cases}$$

that can be solved to obtain searched portfolio proportions

$$u_T = \frac{-\sigma_S}{\sigma_T - \sigma_S},$$
$$u_S = \frac{\sigma_T}{\sigma_T - \sigma_S},$$

and results in

$$dV_t = V_t\left(\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right)dt.$$

Short term risk-free value increment is described by short-rate at time t so simply by modeled $r(t)$

$$V_t r(t)dt = V_t\left(\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right)dt.$$

Therefore short rate must fulfill

$$r(t) = \frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S}$$

or if we split T and S terms

$$\frac{\alpha_S - r(t)}{\sigma_S} = \frac{\alpha_T - r(t)}{\sigma_T}.$$

Now that's the final point of the derivation: since α_T describes expected rate of return from T-Bond (if we go up to dF^T we can see that expected value would be $F^T \alpha_T dt$) the above fraction is expected excess of bond return over risk-free rate divided by it's volatility. It's also known as market price of risk and we've just shown that in short rate model it's the same for bond of each maturity (regardless of choice of T and S). The market price of risk can be therefore marked with single function $\lambda(t)$

$$\lambda(t) = \frac{\alpha_T - r(t)}{\sigma_T},$$

as it holds for all parameters T and all variables t and $r(t)$ with probability 1. Now if we substitute what we assigned under α_T and σ_T we obtain the term structure equation for bonds

$$F_t^T + \frac{1}{2}\sigma(r(t),t)^2 F_{rr}^T + (\mu(r(t),t) - \lambda(t)\sigma(r(t),t))F_r^T - r(t)F^T = 0.$$

Actually, nowhere in our argument above we recalled that F^T is price of a bond (we've only assumed that price of the derivative F is based on variable $r(t)$, t and parameter T). The above argument is therefore valid for any derivative matching this assumption and therefore we can write generally **Term Structure Equation (TSE):**

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2(r(t),t)\frac{\partial^2 V_t}{\partial r^2} + (\mu(r(t),t) - \lambda(t)\sigma(r(t),t))\frac{\partial V}{\partial r} - r(t)V_t = 0$$

with condition that at maturity V_T is equal to its payoff depending on $r(T)$. Model that holds it is called Term Structure Model.

2.3 Affine Term structure Model

To finish specification of the Hull-White model we are yet to specify functions $\mu(r(t),t)$ and $\sigma(r(t),t)$. In full generalization the Term Structure Equation on its own doesn't ensure *nice* behavior of the short rate model, but specific choice of the two functions may lead to some desired properties and derivative instruments having closed-form formulas.

One of the desired properties is bond price formula in the following form

$$p(t,T) = e^{A(t,T)-r(t)B(t,t)}$$

Hull-White model (along with other popular short rate models) decides to have this property. Term Structure Model for which above property holds is called Affine Term Structure Model and many named short rate models choose to be Affine Term Structure.

To find condition for which term structure is affine, we can take above $p(t,T)$, calculate its derivatives and substitute it to the Term Structure Equation as it must be satisfied. Doing so leads to

$$\frac{d}{dt}A(t,T) - (1 + \frac{d}{dt}B(t,T))r(t) - \mu(r(t),t)B(t,T) + \frac{1}{2}\sigma^2(r(t),t)B^2(t,T) = 0$$

Finding condition for functions $\mu(r(t),t)$ and $\sigma(r(t),t)$ under which the above equation is fulfilled in full generality would be very complex (as it depends on form of $\mu(r(t),t)$ and $\sigma^2(r(t),t)$ functions). Therefore in bibliography simplification is made as only functions for which $\mu(r(t),t)$ and $\sigma^2(r(t),t)$ are affine (i.e. linear functions of $r(t)$) are considered

$$\begin{cases} \mu(r(t),t) = a(t)r(t) + b(t) \\ \sigma(r(t),t) = \sqrt{c(t)r(t) + d(t)} \end{cases} .$$

Substitution of such $\mu(r(t),t)$ and $\sigma(r(t),t)$ allows us to group terms that are multiplied by $r(t)$ and the ones that are not

$$\begin{aligned} &\frac{d}{dt}A(t,T) - b(t)B(t,T) + \frac{1}{2}d(t)B^2(t,T) \\ &\qquad - \left(1 + \frac{d}{dt}B(t,T) + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T)\right) \cdot r(t) = 0 \end{aligned}$$

Since the equation must hold for all t, T and $r(T)$ we can observe that the bracket multiplied by $r(t)$ must be equal to 0 (as the rest is dependent only on t and T, so shifting $r(T)$ cannot change the value). When the bracket is 0, only first part of equation remains giving us system of two equations which must hold

$$\begin{cases} \frac{d}{dt}B(t,T) + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T) = -1 \\ \frac{d}{dt}A(t,T) = b(t)B(t,T) - \frac{1}{2}d(t) \cdot B^2(t,T) \end{cases}$$

The additional condition is $A(T,T) = 0$ and $B(T,T) = 0$, since we want $p(T,T) = 1$. The specific choice of $a(t)$, $b(t)$, $c(t)$ and $d(t)$ functions is now clearly a model choice. Next section enlists couple of standard named models, including Hull-White on which we will focus on.

2.4 Named Models

Not to picture Hull-White as the only Affine Term Structure model (which uses above simplifications), below we state a list of multiple named short-rate models that use the same framework.

Vasicek Model

$$dr(t) = (\beta - \alpha \cdot r(t))\, dt + \sigma \cdot dW_t$$

for

$$\alpha/\beta/\sigma \rightarrow \text{constants},\ \ \sigma > 0$$

Cox, Ingersoll and Ross (CIR) Model

$$dr(t) = (\beta - \alpha \cdot r(t))\, dt + \sqrt{\gamma \cdot r}\ dW_t$$

for

$$\alpha/\beta/\gamma \rightarrow \text{constants}$$

Ho & Lee Model

$$dr(t) = \theta(t) \cdot dt + \sigma \cdot dW_t$$

for

$$\sigma \rightarrow \text{constant},\ \ \sigma > 0$$

Hull-White Model

In this paper we will focus only on the Hull-White model which is extension of the Vasicek model. Dynamics of the short-rate under it are given by

$$dr(t) = (\theta(t) - \alpha \cdot r(t))dt + \sigma \cdot dW(t),$$

for

$$\alpha/\sigma \rightarrow \text{constants},\ \ \sigma > 0,\ \ \alpha > 0$$

and $\theta(t)$ is a deterministic function of t.

The form of the model is now specified, but it still raises the obvious question

What exactly should we use as α, σ and $\theta(t)$ parameters?

The answer is: they are determined by current market data. Therefore, to obtain them, we will need to calibrate our model to current market prices. However, to be able to do it, we need to know pricing formulas for couple of financial instruments, which are derived in the following section.

3 Pricing Formulas

To calibrate and use Hull-White model we need to know pricing formula for Bond prices (which will be used for calibration of $\theta(t)$) and formula for at least one instrument based on short-rate volatility, which we want to use to calibrate α and σ.

The choice of instruments for calibration of α and σ should be in line with how we plan to use the Hull-White model. Namely, if we want to price typology X (which is based on volatility) then it's best to calibrate model to market data of typology X.

In case of this paper we choose to calibrate our model to Swaptions, which requires us to know pricing formula for Swaptions and Bond Options (as Bond Options price is present in formula for swaptions price). It means that by the way we will also cover possibility of calibrating model to bond options.

3.1 Bonds

Formula for **zero-coupon** Bond prices is consequence of Hull-White being Affine Term Structure model. As explained in previous section value of bond in such model is given by

$$p(t,T) = e^{A(t,T) - r_t \cdot B(t,T)}$$

where

$$\begin{cases} \frac{d}{dt}B(t,T) + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T) = -1 \\ \frac{d}{dt}A(t,T) = b(t)B(t,T) - \frac{1}{2}d(t)B^2(t,T). \end{cases}$$

In our case

$$a \equiv -\alpha,\ b(t) = \theta(t),\ c(t) \equiv 0,\ d(t) \equiv \sigma,$$

so we have

$$\begin{cases} \frac{d}{dt}B(t,T) - \alpha B(t,T) = -1 \\ \frac{d}{dt}A(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma B^2(t,T). \end{cases}$$

The first equation is simply differential equation for $B(t,T)$ which can be easily solved to get

$$B(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right).$$

This result can be substituted to second equation and then simply integrated. Then we obtain

$$A(t,T) = -\int_t^T \theta(s)B(s,T)ds + \frac{\sigma^2}{2\alpha^2}\left(T - t + \frac{2}{\alpha}e^{-\alpha(T-t)} - \frac{1}{2\alpha}e^{-2\alpha(T-t)} - \frac{3}{2\alpha}\right).$$

To summarize, in Hull-White model bond price is given by

$$p(t,T) = e^{A(t,T) - r_t \cdot B(t,T)}$$

where

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

and

$$A(t,T) = -\int_t^T \theta(s)B(s,T)ds + \frac{\sigma^2}{2\alpha^2}\left(T - t + \frac{2}{\alpha}e^{-\alpha(T-t)} - \frac{1}{2\alpha}e^{-2\alpha(T-t)} - \frac{3}{2\alpha}\right).$$

3.2 Bond Options

Before calculating value of European Call Bond Option we need to consider certain difficulties that may arise along the way. In general, to calculate bond option price we need to introduce vast theory concerning martingale measure and change of the numeraire. We will briefly introduce the basic ideas.

Consider a given financial market (not necessarily a bond market) with the usual locally risk free asset B, and a risk neutral martingale measure Q. We know that measure is a martingale measure only relative to some chosen numeraire asset, and we recall that the risk neutral martingale measure, with the money account B as numeraire, has the property of martingalizing all processes of the form $S(t)/B(t)$ where S is the arbitrage free price process of any (non-dividend paying) traded asset. Now, let us consider the pricing problem for a contingent claim X, in a model with a stochastic short rate r. Using the standard risk neutral valuation formula we know that the price at $t = 0$ of X is given by

$$\Pi(0; X) = \mathbf{E}^Q[e^{-\int_0^T r(s)ds} \cdot X]$$

The problem with this formula from a computational point of view is that in order to compute the expected value we have to get hold of the joint distribution (under Q) of the two stochastic variables: $\int_0^T r(s)ds$ and X, and finally we have to integrate with respect to that distribution. This turns out to be rather hard work.

Let us now make the (extremely unrealistic) assumption that r and X are independent under Q. Then the expectation above splits, and we have the formula

$$\Pi(0; X) = \mathbf{E}^Q[e^{-\int_0^T r(s)ds}] \cdot \mathbf{E}^Q[X],$$

which we may rewrite as

$$\Pi(0; X) = p(0,T) \cdot \mathbf{E}^Q[X].$$

The drawback with the argument above is that, in most concrete cases, r and X are not independent under Q. Fortunately, there exists a general pricing formula, a special case of which reads as

$$\Pi(0; X) = p(0, T) \cdot \mathbf{E}^T[X].$$

Here $\mathbf{E}^T$ denotes expectation w.r.t. the so called forward neutral measure Q^T, which we will discuss below.

Suppose that we are given a specified bond market model with a fixed (money account) martingale measure Q. For a fixed time of maturity T we now choose the **zero coupon bond** maturing at T as our new numeraire.

For a fixed T, the $T - forward$ measure Q^T is defined as the martingale measure for the numeraire process $p(t, T)$. For any $T - claim$ X we have

$$\Pi(t; X) = p(t, T)\mathbf{E}^T[X|F_t],$$

where $\mathbf{E}^T$ denotes integration w.r.t. Q^T.

The option under consideration is a European call on S with date of maturity T and strike price K. We are thus considering the T-claim

$$X = max[S(T) - K, 0].$$

In the Hull–White model the price, at t, of a European option with strike price K, and time of maturity T_1, on a bond maturing at T_2 is given by the formula

$$\Pi(t, X) = p(t, T_2)N(d_1) - Kp(t, T_1)N(d_2), \quad \text{for Call Option}$$
$$\Pi(t, X) = Kp(t, T_1)N(-d_2) - p(t, T_2)N(-d_1), \quad \text{for Put Option}$$

where

$$d_2 = \frac{ln(\frac{p(t,T_2)}{Kp(t,T_1)}) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}},$$
$$d_1 = d_2 + \sqrt{\Sigma^2},$$
$$\Sigma^2 = \frac{\sigma^2}{2\alpha^3}(1 - e^{-2\alpha T_1})(1 - e^{-\alpha(T_2 - T_1)})^2.$$

3.3 Swaptions (Jamshidian's trick)

Swaption prices, unfortunately, have no closed formula under Hull-White model. However, with bond options price formula in hand it is possible to split swaption price into sum of bond option prices using the numerical Jamshidian's Trick.

To present it, we will focus on a receiver swaption. Lets mark that we are considering option maturing at time T to enter a swap starting at time T, maturing at time S and receiving fixed rate r_K with payment frequency δ (expressed in years) and notional N. We assume that the same curve is used for forecasting in floating leg and for discounting. The option will be exercised only if fixed leg will be worth more at time T than floating leg (which we are potentially going to pay). We can express payoff of the receiver swaption at time T as

$$Swaption\_Payoff(T) = max\Big(Fixed\_Leg(T) - Floating\_Leg(T), 0\Big),$$

where $Fixed\_Leg(T)$ and $Floating\_Leg(T)$ obviously mean values of fixed and floating legs at time T. The first step is to present what they're equal to.

Fixed leg consists of fixed flows which are equal to notional multiplied by payment frequency and fixed rate ($N \cdot \delta \cdot r_K$). Thus $Fixed\_Leg(T)$ will be sum of the flows discounted by zero coupon bond to time T. Let's mark that T_i mean time of i-th payment ($T = T_0 < T_1 < \cdots < T_n = S$), then we have

$$Fixed\_Leg(T) = \sum_{i=1}^{n} N \cdot \delta_i \cdot r_K \cdot p(T, T_i)$$

Regarding floating leg, we assume LIBOR approach (i.e. floating rate is fixed at start of the accrual). It means that floating leg flows can be presented as forward rates for given accruals (let's mark as $r_{T_i,T_{(i+1)}}$ forward rate from T_i to T_{i+1}) also multiplied by discount factor, payment frequency and notional

$$Floating\_Leg(T) = \sum_{i=1}^{n} N \cdot \delta_i \cdot r_{T_{i-1},T_i} \cdot p(T, T_i),$$

but we can express it in much nicer form when we substitute formula for forward rate

$$r_{T_{i-1},T_i} = \frac{1}{\delta_i}\left(\frac{p(T,T_{i-1})}{p(T,T_i)} - 1\right),$$

which simplifies with payments frequency and discount factor

$$\delta_i \cdot r_{T_{i-1},T_i} \cdot p(T,T_i) = \delta_i \cdot \frac{1}{\delta_i}\left(\frac{p(T,T_{i-1})}{p(T,T_i)} - 1\right) \cdot p(T,T_i) = p(T,T_{i-1}) - p(T,T_i).$$

We just obtained that

$$Floating\_Leg(T) = N \cdot \sum_{i=1}^{n} p(T, T_{i-1}) - p(T, T_i)$$

and we can notice that terms of the sum cancel out leaving only the first and last ones

$$\sum_{i=1}^{n} p(T, T_{i-1}) - p(T, T_i) = p(T, T_0) - \cancel{p(T, T_1)} + \cancel{p(T, T_1)} - \ \dots \ + \cancel{p(T, T_{n-1})} - p(T, T_n),$$

and because $T = T_0$, we have obtained that

$$Floating\_Leg(T) = N \cdot (p(T, T_0) - p(T, T_n)) = N \cdot (1 - p(T, T_n)) .$$

Altogether it enables us to express swaption payoff as follows

$$Swaption\_payoff(T) = max\left(\sum_{i=1}^{n} N \cdot \delta_i \cdot r_K \cdot p(T, T_i) - N \cdot (1 - p(T, T_n)) , 0\right)$$

and we can take notional out of maximum

$$Swaption\_payoff(T) = N \cdot max\left(\sum_{i=1}^{n} \delta_i \cdot r_K \cdot p(T, T_i) + p(T, T_n) - 1, 0\right).$$

The Key Idea of the Jamshidian's Trick

First, we need to realize that above is indeed a payoff i.e. function applied to underlying random variable (so above payoff is indeed a random value). How so? Our random variable short-rate defines what is value of the zero-coupon bonds at given time and above ones are in the future maturity time T. In other words lets notice that

$$p(T, T_i) = p(T, r_T, T_i),$$

so the value of above payoff depends on random realization of r_T.

Now the Jamshidian's Trick is to break down swaption payoff into sum of bond option payoffs. The key idea is as follows: let's take the 1 that is subtracted in payoff and break it down into pieces that can be matched with sum terms (and additional term $p(T, T_n)$).

How to do it? Jamshidian notices that since $p(T, r_T, T_i)$ is a monotonic function with respect to r_T, among all possible r_T values there is one (that we will call r^*) for which sum of positive terms in payoff is equal to 1. Mathematically, due to monotonicity of $p(T, r_T, T_i)$ (w.r.t. r_T) there exists r^* such that

$$\sum_{i=1}^{n} \delta_i \cdot r_K \cdot p(T, r^*, T_i) + p(T, r^*, T_n) = 1.$$

Due to complexity of $p(T, r^*T_i)$ terms it's not possible to solve the r^* analytically, but it can be done numerically. This is exactly the place where we need numerical interference and so we assume that the above r^* is now found numerically with a computer (for example in python using a scipy solver). Values $p(T, r^*, T_i)$ are then deterministic.

Once we got the r^* we can transcribe number 1 subtracted in payoff and substitute it to obtain

$$\begin{aligned} Swaption\_payoff(T) = N \cdot max\Bigg(&\sum_{i=1}^{n} \delta_i \cdot r_K \cdot p(T, r_T, T_i) + p(T, r_T, T_n) \\ &- \Bigg(\sum_{i=1}^{n} \delta_i \cdot r_K \cdot p(T, r^*, T_i) + p(T, r^*, T_n) \Bigg), 0 \Bigg), \end{aligned}$$

where we can group corresponding terms

$$\begin{aligned} Swaption\_payoff(T) = N \cdot max\Bigg(&\sum_{i=1}^{n} \delta_i \cdot r_K \cdot (p(T, r_T, T_i) - p(T, r^*, T_i)) \\ &+ p(T, r_T, T_n) - p(T, r^*, T_n), 0 \Bigg). \end{aligned}$$

Now, the crucial observation is that even with r_T being random variable, there are only two options: r_T realization will be either lower (equal) or higher than r^*, and (once again) because $p(T, r_T, T_i)$ is monotonic w.r.t. r_T it will cause all bond values to be (accordingly) higher or lower than their r^* matches

$$\begin{cases} r_T \leq r^* \quad \Rightarrow \quad \forall i \quad p(T, r_T, T_i) \geq p(T, r^*, T_i) \\ r_T > r^* \quad \Rightarrow \quad \forall i \quad p(T, r_T, T_i) < p(T, r^*, T_i). \end{cases}$$

Thanks to that observation, we can change maximum of sums into sum of maxima. If r_T is lower (equal) than r^* then payoff is positive and no maximum is applied but if r_T is higher than r^* then instead of applying maximum to whole sum, we can apply maximum separately to each subtraction, as due to above each one of them will be negative and will cancel out when treated with maximum. Therefore, we have

$$\begin{aligned} Swaption\_payoff(T) = N \cdot \sum_{i=1}^{n} \delta_i \cdot r_K \cdot max\Big(p(T, r_T, T_i) - p(T, r^*, T_i), 0 \Big) + \\ + N \cdot max\Big(p(T, r_T, T_n) - p(T, r^*, T_n), 0 \Big). \end{aligned}$$

Thanks to that breakdown we can now easily notice that each maximum is simply payoff of bond option with strike being $p(T, r^*, T_n)$ and the ones from sum are multiplied by $\delta_i \cdot r_K$. Finally we obtain

$$V_{Swaption} = N \cdot r_K \sum_{i=1}^{n} V_{Bond\_Option}(T, T_i, p(T, r^*, T_i)) \; + N \cdot V_{Bond\_Option}(T, S, p(T, r^*, S)),$$

where $V_{Bond\_Option}(T, S, K)$ means value of bond option with maturity at time T, for bond starting at T and maturing at S, with strike K and notional equal to 1.

Additionally, very important note, above we've derived whole trick using receiver swaption, and we can see that during the moment of substituting r^* terms for 1 the call-put relation reverses between swaption and later created bond options. Namely, receiver swaption breaks down into call options, while payer swaption breaks down into put options

$$V_{Receiver\_Swaption} = N \cdot r_K \sum_{i=1}^{n} V_{Call\_Bond\_Option}(T_i) \; + N \cdot V_{Call\_Bond\_Option}(S)$$

and

$$V_{Payer\_Swaption} = N \cdot r_K \sum_{i=1}^{n} V_{Put\_Bond\_Option}(T_i) \; + N \cdot V_{Put\_Bond\_Option}(S),$$

where we've used shorter notation $V_{Call\_Bond\_Option}(T_i) = V_{Call\_Bond\_Option}(T, T_i, p(T, r^*, T_i))$.

4 Calibration of the model

4.1 Goal

The Hull-White model has three parameters that we need to calibrate before using it: constant parameters α and σ, and function $\theta(t)$. Goal is to calibrate them to market data for risk-free Bonds and Swaptions (or Bond Options) using pricing formulas derived above.

4.2 Market Data used

We choose 2025-06-18 as our calibration date. As market data for risk-free bonds, we are going to use USD Treasury Bond rates from publicly available data at government site. For simplification we are treating rates as **zero-coupon** bond rates and the rates will be revaluated into market bond prices with details stated below under Implementation subsection.

Swaptions and Bond Options, however, are products not traded on the market, but rather OTC (Over-the-counter in private transactions) and access to such data can be bought from market data providers like ICE, so for them we are going to use artificial data, which is nonetheless trying to capture magnitude and shape of real life data (we generated it with ATM strikes corresponding to forward rates concluded from yield curves and volatilities being drawn from parabolic equations with random parameters and some normal distribution disorders.

Market Data for Swaptions and bond options is usually only set of volatilities for given strikes and maturities of options and their underlying. In calibration however, we will need to have prices of the assets. For this case, the usual apporach is to calculate values of the Swaptions and Bond Option using Black Model. The values calculated with Black Model are then treated (together with volatilities) as market data and used for calibration of Hull-White Model. As reference for calculating Swaptions prices with Black model we have used quantpie site and simply Wikipedia site of Black Model.

4.3 Treasury bonds and $\theta(t)$ function

First part of calibration makes sure that risk-free bonds priced by the model will match prices of risk-free bonds available on the market. To ensure that, the bond price formula is equated to market bond prices. Let's mark

$$Z^*(t,T) \text{ - Market risk-free bond price at time } t \text{ with maturity } T$$

Having price formula and market prices we get the following equation for all maturities T available in the market data

$$Z^*(t, T) = e^{A(t,T) - rB(t,T)}$$

which we solve to obtain the first condition for our parameters. Applying logarithm and solving for function $\theta(t)$ gives

$$\theta^*(t) = -\frac{\partial^2}{\partial t^2} \log\left(Z_M\left(t^*; t\right)\right) - \alpha \cdot \frac{\partial}{\partial t} \log\left(Z_M\left(t^*; t\right)\right) + \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t - t^*)}\right)$$

where by $\theta^*(t)$ we mean function $\theta(t)$ calibrated to market risk-free bonds and t^* marks the calibration time.

The above condition for $\theta(t)$ is not yet complete. First of all, it gives us values of the function only for a couple of points, based on maturities available on the market. Secondly, to obtain the above closed-form formula, we have differentiated equation and, in effect, we have obtained the second derivative of logarithm from bond prices.

The latter is the case, because market bond prices are actually assumed to be a function so we are then able to differenciate them and have $\theta(t)$ funciton defined for all arguments. Therefore, we are required to turn market bond prices $Z^*(t, T)$ into a function, which sets our first step in calibration to be interpolation of $Z^*(t, T)$ from couple of available points for all T arguments.

4.4 Interpolation of market bond prices

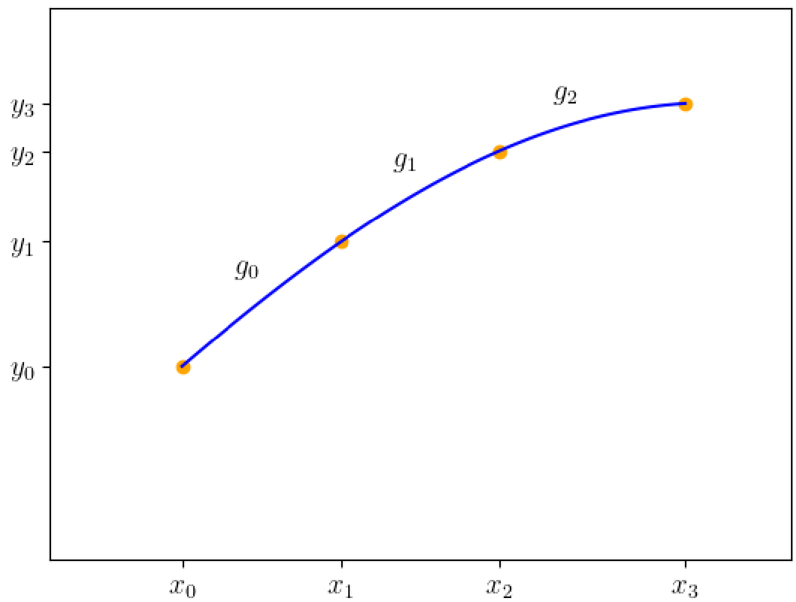
4.4.1 Approach

Interpolation can be done in multiple ways, but since it's not the main focus of our work, we will limit ourselves to two methods: Cubic Spline and Hermite Cubic Spline. We'll shortly describe them and then jump into implementation and usage.

4.4.2 Cubic Spline

Let us say that we know $n+1$ points $(x_0, y_0), \ldots, (x_n, y_n)$. We would like to be able to calculate the value of $y \in (y_i, y_j)$ for $x \in (x_i, x_j)$, where $i \neq j$. Let us denote the interpolating function by g, that is $g(x) = y$.

Figure 1: Cubic Spline Interpolation



The g function is a spline where each piece is a third-degree polynomial, we get

$$g_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

from that equation we get $4n$ unknown parameters.

Now we define constraints which guarantee smoothness and continuity at the knot points

1. $$g_i(x_i) = y_i \text{ and } g_i(x_{i+1}) = y_{i+1} \quad (2n \text{ constraints}),$$

2. $$g_i^{'}(x_{i+1}) = g_{i+1}^{'}(x_{i+1}) \quad (n - 1 \text{ constraints}),$$

3. $$g_i^{''}(x_{i+1}) = g_{i+1}^{''}(x_{i+1}) \quad (n - 1 \text{ constraints}).$$

We are still missing 2 constraints. There are different ways of achieving them, but before that, to simplify our writing, lets introduce $h_i = x_{i+1} - x_i$ and $m_i = y_{i+1} - y_i$. Now let us find out what we get by applying constraints listed above. By inserting h_i to our equations we get i.

$$g_i(x_i) = d_i,$$

ii.

$$a_i h_i^3 + b_i h_i^2 + c_i h_i = m_i,$$

iii.

$$3a_i h_i^2 + 2b_i h_i + c_i = c_{i+1},$$

iv.

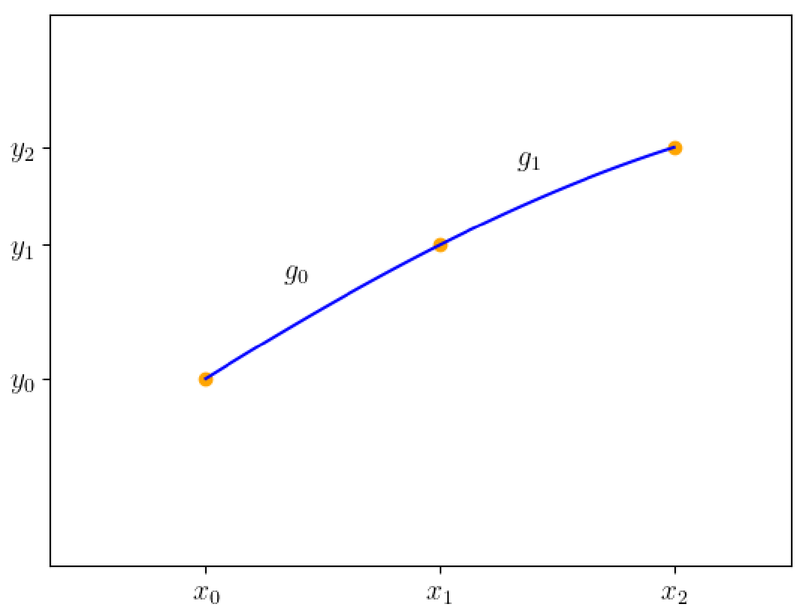
$$3a_i h_i + b_i = b_{i+1}.$$

From the above system of equations we solve ii for c_i and iv for a_i in terms of b_i. Next we plug in the results into iii to obtain

$$\frac{1}{3}b_i h_i + \frac{2}{3}b_{i+1}(h_{i+1} + h_i) + \frac{1}{3}b_{i+2}h_{i+1} = \frac{m_{i+1}}{h_{i+1}} - \frac{m_i}{h_i}.$$

Now we get back to the two missing conditions. We will get them from "not-a-knot" condition.

Figure 2: Cubic Spline Interpolation



It says that the two polynomials interpolating points (x_0, y_0) with (x_1, y_1) and (x_1, y_1) with (x_2, y_2) are equal. Thus we get to following equation

$$g_0(x) = g_1(x).$$

In order to implement the condition above we require

$$g_0^{'''}(x) = g_1^{'''}(x),$$

Which is equivalent to $g_0(x_1) = g_1(x_1)$ since we already have conditions for $g_0^{'}(x_1) = g_1^{'}(x_1)$ and $g_0^{''}(x_1) = g_1^{''}(x_1)$.

4.4.3 Cubic Hermite Spline

We have two points that we want to connect P_0 and P_1. Furthermore, we know the derivatives v_0 and v_1 at these points. Let P denote the third-degree polynomial with which we want to connect these points. We get the following system of equations

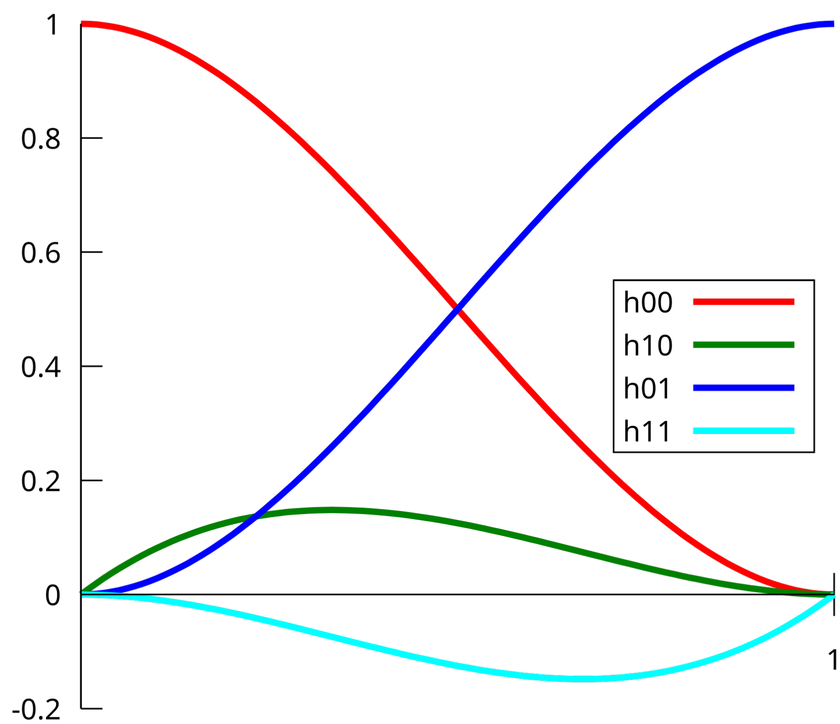
$$\begin{aligned} P(0) &= P_0, \\ P(1) &= P_1, \\ P'(0) &= v_0, \\ P'(1) &= v_1, \\ P(t) &= at^3 + bt^2 + ct + d. \end{aligned}$$

From the above system of equations we can easily determine $c = v_0$ and $d = P_0$. After determining the remaining coefficients we can write our system of equations in matrix form

$$P(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ v_0 \\ P_1 \\ v_1 \end{bmatrix}$$

By multiplying the matrix containing the powers of t and the characteristic matrix, we get the basis functions for each of the control points. The graph below shows what such functions look like.

Figure 3: The four Hermite basis functions



4.5 Implementation

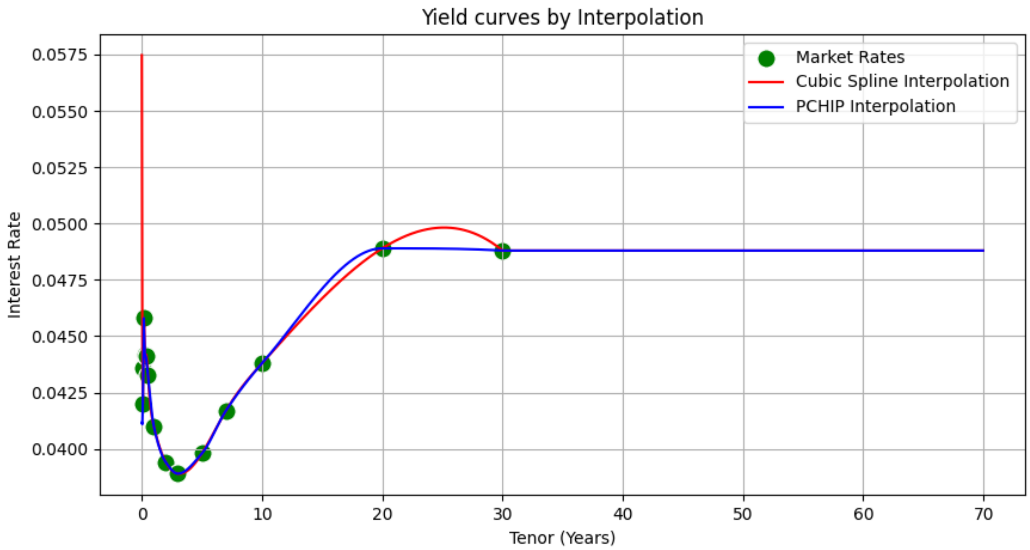
Both interpolation methods, i.e. *Cublic Spline Interpolation* and *PCHIP* (Piecewise Cubic Hermite Interpolation), have been implemented with *scipy.interpolate*. To achieve a flat point above the end points, we have overwritten the behavior of calling the interpolated function in the base *scipy* class. We decided for a constant rate because the extrapolation with polynomial cause the rate to go to $\pm\infty$.

Our starting point is following set of market rates taken from the government site:

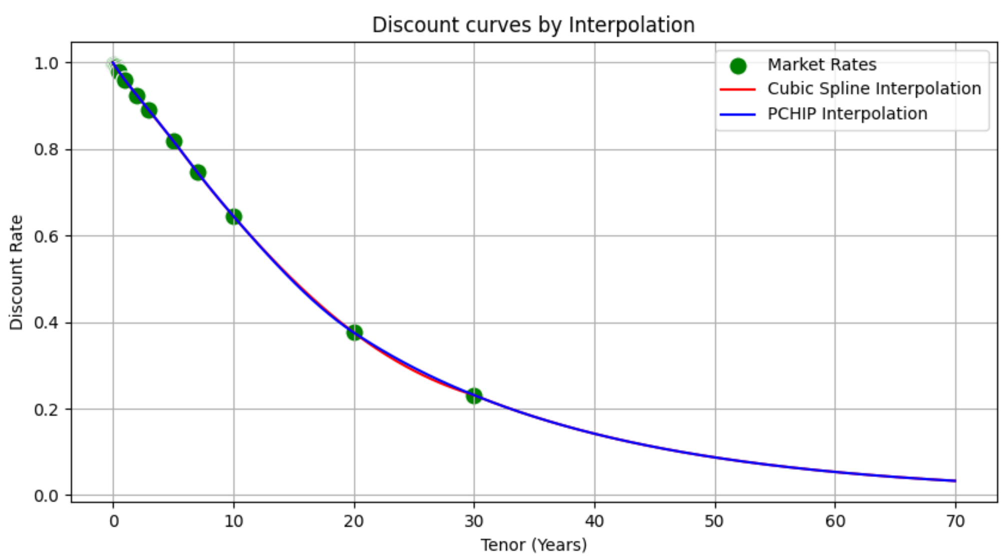
T	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	5	7	10	20	30
Rate	4.2	4.36	4.58	4.42	4.41	4.33	4.1	3.94	3.89	3.98	4.17	4.38	4.89	4.88

Table 1: Risk-free Market Rates

The plots below show and compare results of the interpolation with Cubic and Hermite Splines.



We see that *Cubic Interpolation* spikes at the beginning, but after that it is fairly equal to *PCHIP Interpolation* up until tenor equal to 10. After that we observe different behavior for these methods. As mentioned, the curve gets flat for tenors greater than 30. The spike for *Cubic Interpolation* method is caused by a large increase in value (of the rate) relative to the argument (time).



The discount curve is defined as $e^{-r(t)\cdot t}$, where $r(t)$ is the interpolated market rates (yield curve above). In case of *discount curves* we observe minor differences in values for interpolated curves for tenor in interval $[10, 30]$.

To calibrate α and σ we apply Hull-White swaption pricing formula (or bond option pricing formula when calibrating to bond option data) to each record from the market data (i.e. set of arguments: option maturity, strike, swap / bond maturity) and we compare obtained prices with Black formula. Initially we provide start arguments $\alpha = 0.05$, $\sigma = 0.05$ and then the script numerically iterates over them to find the best fit. To measure the fit in our implementation we are minimizing the average price difference

$$\frac{1}{M}\sum_{i=1}^{M}(Hull\_White\_price_i - Black\_price_i)^2.$$

In our implementation we are using *scipy.optimize.minimize()* function to perform the fitting. Results are presented in the table below.

	α	σ
PCHIP	0.0408	0.0241

Table 2: α and σ values calibrated with swaption data

Important Notice: Although the bibliography suggests calibrating α and σ to the whole volatility surface at once, it is not recommended. In practice calibration is performed individually for a given trade to provide price which fits market data most accurately. However, for the presentation purpose we will stick to the α and σ calibration described above.

Having α and σ, the $\theta(t)$ function can be now obtained with the formula stated in section 4.3

$$\theta^*(t) = -\frac{\partial^2}{\partial t^2}\log\left(Z_M\left(t^*;t\right)\right) - \alpha\cdot\frac{\partial}{\partial t}\log\left(Z_M\left(t^*;t\right)\right) + \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-t^*)}\right),$$

but since we set $Z_M(t) = e^{-r(t)\cdot t}$ (assuming $t^* = 0$), the formula can be simplified and expressed using interpolated yield curve since we have

$$log\left(Z_M(t)\right) = -r(t)\cdot t$$

that leads to

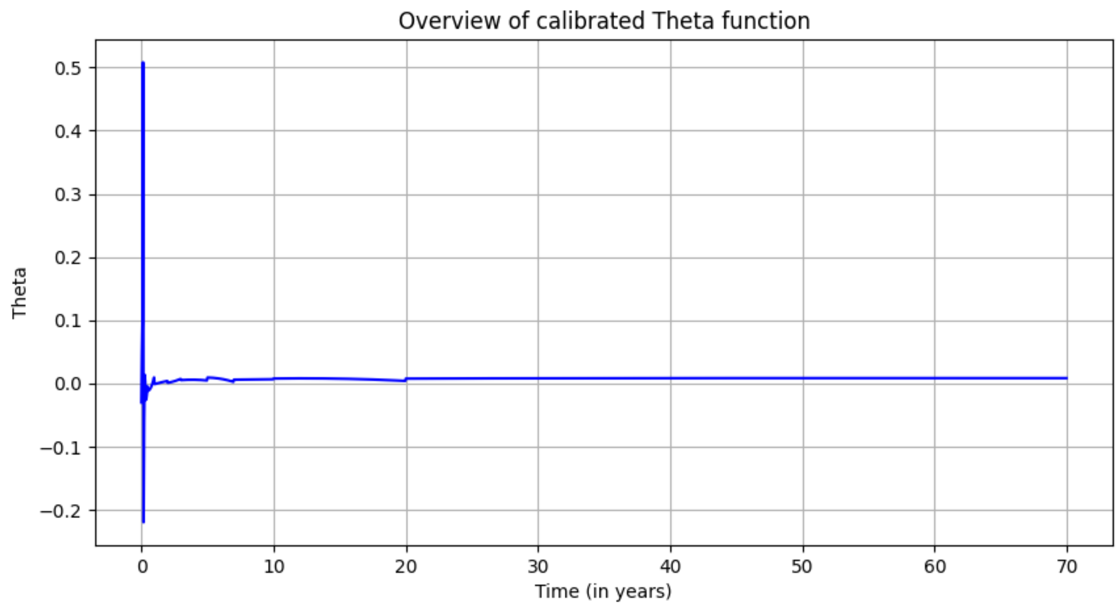
$$\frac{\partial}{\partial t}\log\left(Z_M\left(t\right)\right) = -t\cdot r'(t) - r(t)$$

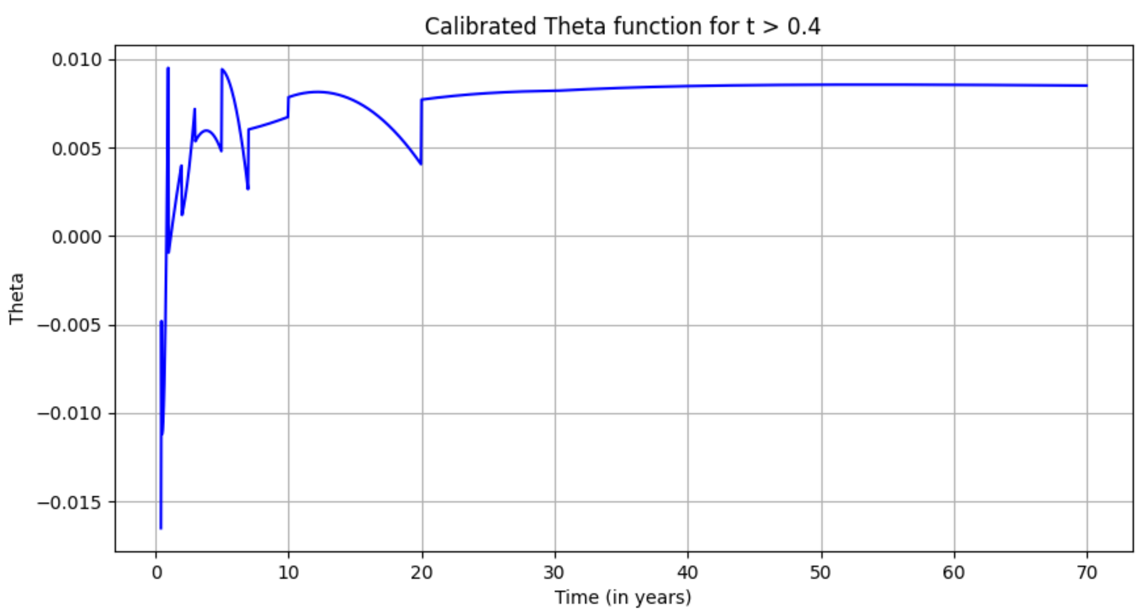
and

$$\frac{\partial^2}{\partial t^2}\log\left(Z_M\left(t^*;t\right)\right) = -t\cdot r''(t) - 2r'(t),$$

which is what we used for creation of $\theta(t)$.

Since $\theta(t)$ function is obtained with derivatives of yield curve (that is interpolated with splines), it will behave very spiky and "chaotic" by design. Namely, in between each interpolation node, $\theta(t)$ will visibly change its behavior, especially at the shorter tenors which are more dense. Nevertheless, such function ensures that model prices bonds in line with market data.





5 Monte Carlo Simulations

Beside the analytical approach we also implemented the Monte Carlo simulations. In case of Hull-White model we have to simulate the whole path of $r(t)$ since the discount factor depends on integration of the $r(t)$ path. In our simulations we will generate $N = 25000$ paths of r. Let us recall the formula for the dynamics of the short rate in Hull-White model

$$dr(t) = (\theta(t) - \alpha \cdot r(t))dt - \sigma dW(t).$$

To simulate $r(t)$, we will use Euler discretization, that is

$$r(t + \Delta t) = r(t) + \theta(t)\Delta t + \alpha r(t) + \sigma\sqrt{\Delta t}Z,$$

where $Z \sim N(0, 1)$. To keep results fairly accurate, we assume $\Delta t = 0.0025$. We will designate $r_i(t)$ as the i-th simulated path at time t. In the next subsections we provide details of the price simulation of each instrument.

5.1 Bond Price

Let T_{bond} be the maturity of the bond. To find it's value we need to calculate

$$\mathbf{E}\left[e^{-\int_0^{T_{bond}} r(t)dt}\right].$$

Translating to numerical language, we need to calculate bond price for each path and take the average

$$Bond\ Price = \frac{1}{N}\sum_{i=1}^{N} e^{-\int_0^{T_{bond}} r_i(t)dt}.$$

For path integration we use *scipy.integrate.trapezoid()* function.

5.2 Bond Option Price

Let T denote the maturity of the Bond Option. To find it's value we need to calculate

$$\mathbf{E}\left[e^{-\int_0^{T} r(t)dt} \cdot payoff(r(T))\right].$$

Firstly we calculate the price of the Bond for time T

$$S(r(T)) = e^{A(T,T_{bond})-r(T)B(T,T_{bond})},$$

where T_{bond} stands for Bond maturity. Then the payoff for strike K

$$payoff(r(T)) = max(S(r(T)) - K, 0) \quad \text{for Call Option,}$$
$$payoff(r(T)) = max(K - S(r(T)), 0) \quad \text{for Put Option.}$$

Knowing that we just have to discount it for $t = 0$ (integral under expectation). Therefore we substitute the point $r_i(T)$ for each trajectory to calculate the payoff and discount. The last step is to take the average of the result for each path

$$Bond\ Option\ Price = \frac{1}{N}\sum_{i=1}^{N} e^{-\int_0^T r_i(t)dt} \cdot payoff(r_i(T)).$$

5.3 Swaption Price

Simulating Swaption prices is adequate to above Bond Option pricing. Let's remind that we are considering option maturing at time T to enter a swap starting at time T, maturing at time T_n to be receiving/paying fixed rate r_K with payments frequency δ (expressed in years) and notional N. Also let T_i describe moment of i-th payment. As discussed in swaption pricing section, the payoff of receiver swaption can be expressed as follows (this time emphasising dependency on r_T)

$$Receiver\_Swaption\_payoff(r_T) = N \cdot max\left(\sum_{i=1}^{n} \delta_i \cdot r_K \cdot p(T, r_T, T_i) + p(T, r_T, T_n) - 1, 0\right),$$

while payoff or payer swaption is

$$Payer\_Swaption\_payoff(r_T) = N \cdot max\left(1 - \sum_{i=1}^{n} \delta_i \cdot r_K \cdot p(T, r_T, T_i) - p(T, r_T, T_n), 0\right).$$

Once again we simply need to use our simulations to input final trajectory value to calculate the payoff and discount it. The price is the mean of the results

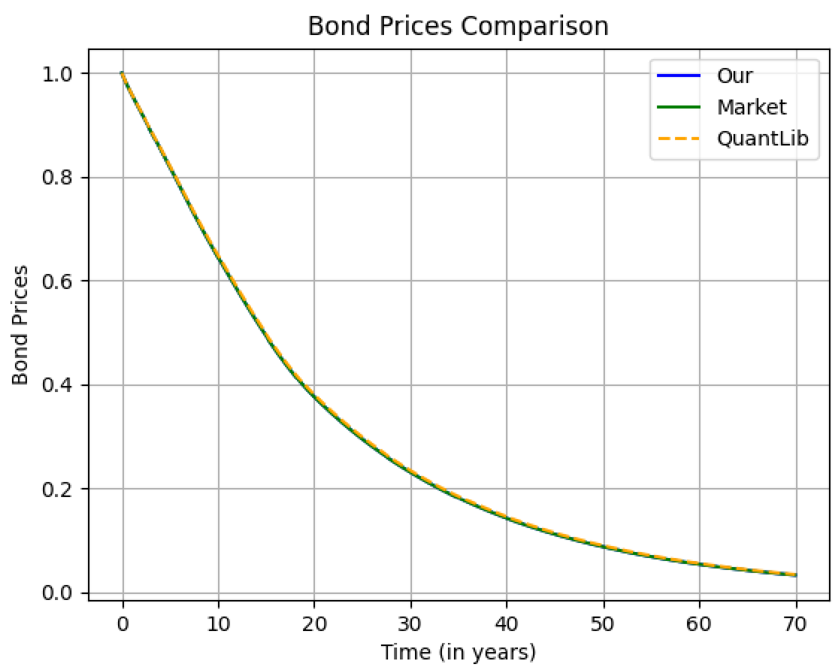
$$Swaption\ Price = \frac{1}{N}\sum_{i=1}^{N} e^{-\int_0^{T_{option}} r_i(t)dt} \cdot payoff(r_i(T)).$$

6 Model Benchmark (with QuantLib)

In the final section we will compare our implementation of the Hull-White model with the QuantLib implementation to benchmark the results. For the plots below, we will be using α and σ calibrated with swaptions market data and we choose *PCHIP* as the interpolation method for the yield curve.

6.1 Bond Prices

We start with comparison of the bond prices.

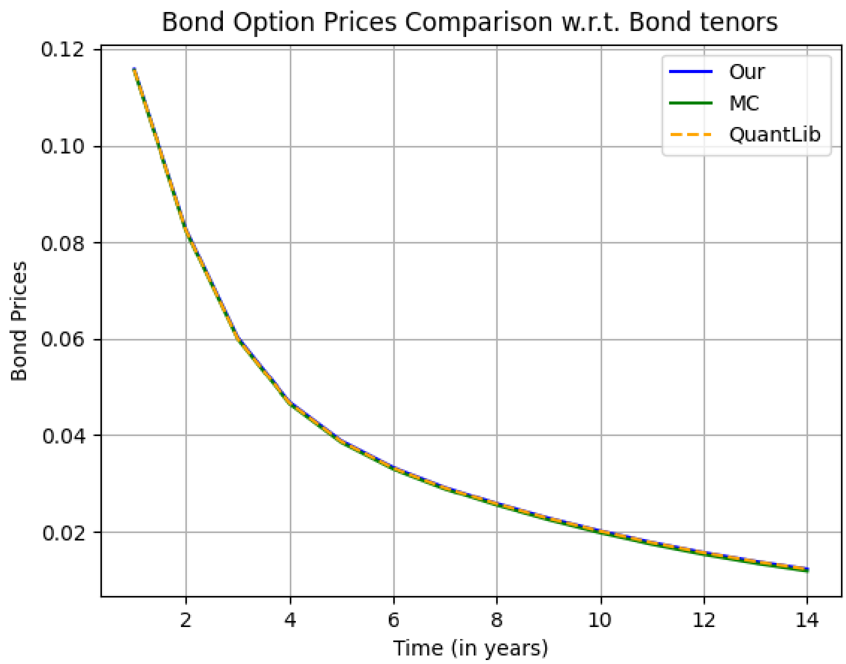


We can see that the curves corresponding to our version and the one from QuantLib overlap each other, proving accuracy of our implementation for bond prices.

6.2 Bond Option Prices

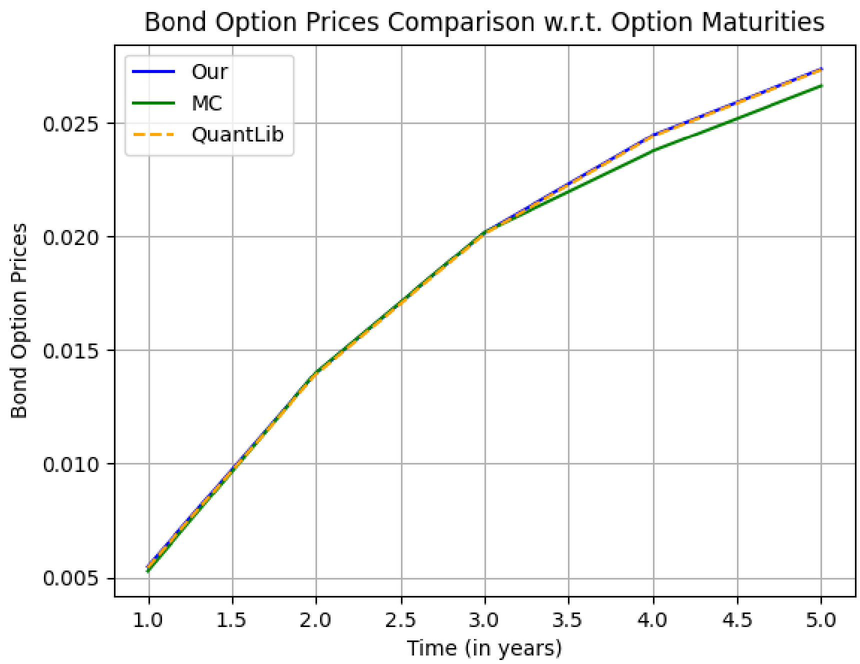
Secondly, lets take a look at Bond Options prices comparison. We will check them from perspective of option maturity and underlying maturity, to check whether prices behave in line with QuantLib.

Let's begin with variable bond tenor ranging from 1 to 14 years. We assume that option maturity is 3 years.



Here we can also see that both curves overlap over the entire bond tenor section and are also in line with Monte Carlo simulations.

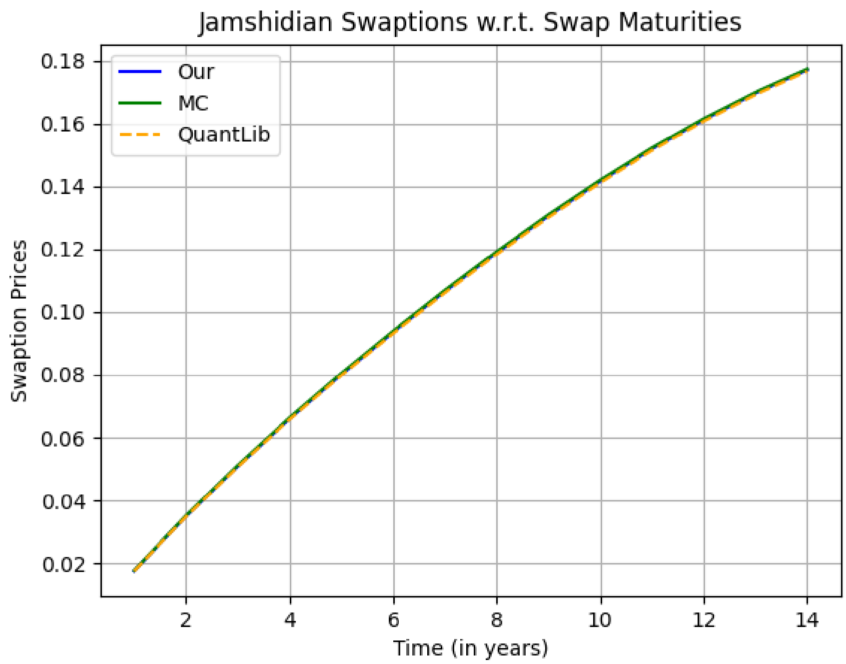
The next step is an analogous comparison with fixed bond tenor and option maturities ranging from 1 to 5 years.



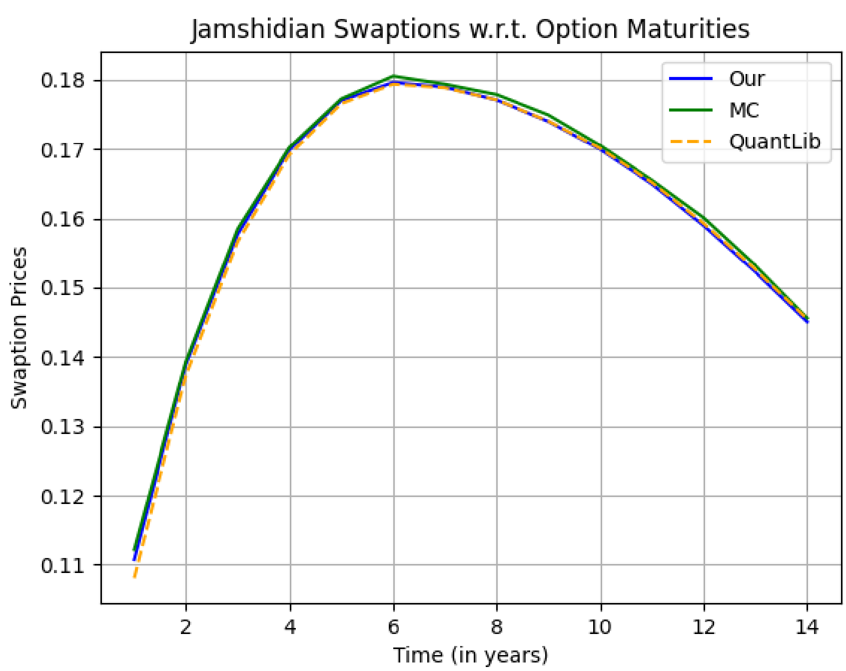
Just like before values from our implementation and QuantLib match each other. For Monte Carlo we get some discrepancy for option maturity greater than 3 years which is due to randomness of Monte Carlo simulations.

6.3 Swaption Prices

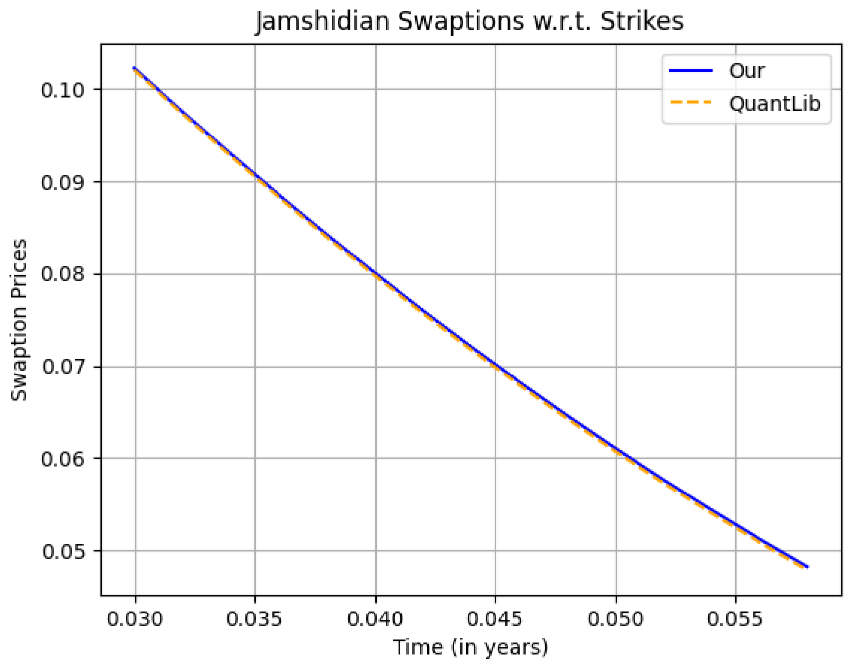
Finally we will present similar graphs for the swaptions pricing. First graph will present variable maturity swaps, where the option maturity is set at 5 years, and the swap frequency is set at 0.25 years (quarterly payments). Swap maturities range from 1 to 14 years.



The second one will set swap tenor at 5 years and option maturity range from 1 to 14 years.



The last plot checks whether pricing behaves well with respect to strike shifting, which is crucial in Jamshidian's Trick.



We can see that swaptions are no exception and all prices match QuantLib very well.

References

[1] Wilmott, Paul (2000). *Paul Wilmott on Quantitative Finance* (2nd ed.). John Wiley & Sons, Hoboken. Chapters 30–32.

[2] Björk, Tomas (2009). *Arbitrage Theory in Continuous Time* (3rd ed.). Oxford University Press. Chapters 23–26.

[3] Masutha, Ndinae Nico (2018). *Pricing Swaptions on Amortising Swaps*. MPhil thesis, Mathematical Finance, University of Cape Town.

[4] Holmer, Freya (2023). *The Continuity of Splines*. Youtube

[5] Chasnov, Jeffery (2021). *Cubic Spline Interpolation (Part A) | Lecture 44 | Numerical Methods for Engineers*. Youtube

[6] Chasnov, Jeffery (2021). *Cubic Spline Interpolation (Part B) | Lecture 45 | Numerical Methods for Engineers*. Youtube

[7] *Simulating the short rate in the Hull-White model*. quant.stackexchange