ϵ-Approximate Coded Matrix Multiplication is Nearly Twice as Efficient as Exact Multiplication

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***Abstract*—We study coded distributed matrix multiplication from an approximate recovery viewpoint. We consider a system of P computation nodes where each node stores $1/m$ of each multiplicand via linear encoding. Our main result shows that the matrix product can be recovered with ϵ relative error from any m of the P nodes for any $\epsilon > 0$. We obtain this result through a careful specialization of MatDot codes—a class of matrix multiplication code previously developed in the context of exact recovery ($\epsilon = 0$). Since previous results showed that the MatDot code is tight for a class of linear coding schemes for exact recovery, our result shows that allowing for mild approximations leads to a system that is nearly twice as efficient as exact reconstruction. Moreover, we develop an optimization framework based on alternating minimization that enables the discovery of new codes for approximate matrix multiplication.**

Theorem proofs and other missing details are provided in the extended version of the paper [1].

I. Introduction

Coded computing has emerged as a promising paradigm to resolving straggler and security bottlenecks in large-scale distributed computing platforms [2]–[25]. The foundations of this paradigm lie in novel code constructions for elemental computations such as matrix operations and polynomial computations, and fundamental limits on their performance. In this paper, we show that the state-of-the-art fundamental limits for such elemental computations grossly underestimate the performance by focusing on *exact* recovery of the computation output. By allowing for mild *approximations* of the computation output, we show significant improvements in terms of the trade-off between fault-tolerance and the degree of redundancy.

Consider a distributed computing system with P nodes for performing the matrix multiplication $\mathbf{AB}$. If each node is required to store a fraction $1/m$ of both matrices, the best known recovery threshold is equal to $2m-1$ achieved by the MatDot code [4]. Observe the contrast between distributed coded *computation* with distributed data *storage*, where a maximum distance separable (MDS) code ensures that if each node stores a fraction $1/m$ of the data, then the data can be recovered from any m nodes[1] [26]. Indeed, the recovery threshold of m is crucial to the existence of practical codes that bring fault-tolerance to large-scale data storage systems with relatively minimal overheads (e.g., single parity and Reed-Solomon codes [27]).

The contrast between data storage and computation is even more pronounced when we consider the generalization of matrix-multiplication towards multi-variate polynomial evaluation $f(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell)$ where each node is allowed to store a fraction $1/m$ of each of $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_\ell$; in this case, the technique of Lagrange coded-computing [6] demonstrates that the recovery threshold is $d(m-1)+1$, where d is the degree of the polynomial. Note that a recovery threshold of m is only obtained for the special case of degree $d=1$ polynomials, i.e., elementary linear transformations that were originally studied in [28]. While the results of [4], [29] demonstrate that the amount of redundancy is much less than previously thought for degree $d>1$ computations, these codes still require an overwhelming amount of additional redundancy—even to tolerate a single failed node—when compared to codes for distributed storage.

[1]This essentially translates to the Singleton bound being tight for a sufficiently large alphabet

A. Summary of Results

Our paper is the result of the search for an analog of MDS codes—in terms of the amount of redundancy required—for coded-computation of polynomials with degree greater than 1. We focus on the case of coded matrix multiplication where the goal is to recover the matrix product $\mathbf{C} = \mathbf{AB}$. We consider a distributed computation system of P worker nodes similar to [3], [4]; we allow each worker to store an m-th fraction of matrices of $\mathbf{A}$, $\mathbf{B}$ via linear transformations (encoding). The workers output the product of the encoded matrices. A central master/fusion node collects the output of a set $\mathcal{S}$ of non-straggling workers and aims to decode $\mathbf{C}$ with a relative error of ϵ. The recovery threshold $K(m,\epsilon)$ is the cardinality of the largest minimal subset $\mathcal{S}$ that allows for such recovery. It has been shown in [4], [29] that, for natural classes of linear encoding schemes, $K(m,0) = 2m-1$.

Our main result shows that the MatDot code with a specific set of evaluation points is able to achieve $K(m,\epsilon) = m$, remarkably, for *any* $\epsilon > 0$. A simple converse shows that the our result is tight for $0 < \epsilon < 1$ for unit norm matrices. Our results mirrors several results in classical information theory (e.g., almost lossless data compression), where allowing ϵ-error for any $\epsilon > 0$ leads to surprisingly significant improvements in performance. We believe that these results open up a new avenue in coded computing research via revisiting existing code constructions and allowing for an ϵ-error.

A second contribution of our paper is the development of an optimization formulation that enables the discovery of new coding schemes for approximate computing. We show that the optimization can be solved through an alternating minimization algorithm that has simple, closed-form iterations as well as provable convergence to a local minimum. We demonstrate

through numerical examples that our optimization approach finds approximate computing codes with favourable trade-offs between approximation error and recovery threshold.

B. Related Work

The study of coded computing for elementary linear algebra operations, starting from [5], [28], is an active research (see surveys [23]–[25]). Notably, the recovery thresholds for matrix multiplication were established via achievability and converse results respectively in [3], [4], [29]. The Lagrange coded computing framework of [6] generalized the systematic MatDot code construction of [4] to the context of multi-variate polynomial evaluations and established a tight lower bound on the recovery threshold. These works focused on exact recovery of the computation output.

References [30], [31] studied the idea of gradient coding from an approximation viewpoint, and demonstrated that improvements in recovery threshold is possible. However, in contrast with our results, the error obtained either did not correct all possible error patterns with a given recovery threshold (i.e., they considered a probabilistic erasure model), or the relative error of their approximation was lower bounded, unlike our main results. The references that are most relevant to our work are [21], [22], [32]. Like us, these references aim to improve the recovery threshold of coded matrix multiplication by allowing for a relative error of ϵ. These references use random linear coding (i.e., sketching) techniques to obtain a recovery threshold $\overline{K}(\epsilon, \delta, m)$ where δ is the probability of failing to recover the matrix product with a relative error of ϵ; the problem statement of [32] is particularly similar to ours. Our results can be viewed as strict improvement, as we are able to obtain a recovery threshold of m even with $\delta = 0$, whereas the recovery threshold is at least $2m - 1$ for $\delta = 0$ in these references. A related line of work in [33], [34] study coded polynomial evaluation beyond exact recovery and note techniques to improve the quality of the approximation. Yet, we are the first to establish the strict gap in the recovery thresholds for ϵ-error computations versus exact computation for matrix multiplication, which is a canonical case of degree 2 polynomial evaluation.

II. System Model and Problem Statement

A. Notations

We define $[n] \triangleq \{1, 2, \cdots, n\}$. We use bold fonts for vectors and matrices. $A[i, j]$ denotes the (i, j)-th entry of an $M \times N$ matrix $\mathbf{A}$ ($i \in [M], j \in [N]$) and $v[i]$ is the i-th entry of a length-N vector $\mathbf{v}$ ($i \in [N]$).

B. System Model

We consider a distributed computing system with a master node and P worker nodes. At the beginning of the computation, a master node distributes appropriate tasks and inputs to worker nodes. Worker nodes perform the assigned task and send the result back to the master node. Worker nodes are prone to failures or delay (stragglers). Once the master node receives results from a sufficient number of worker nodes, it produces the final output.

We are interested in distributed matrix multiplication, where the goal is to compute

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}. \tag{1}$$

We assume $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ are matrices with a bounded norm, i.e.,

$$||\mathbf{A}||_F \leq \eta \quad \text{and} \quad ||\mathbf{B}||_F \leq \eta, \tag{2}$$

where $||\cdot||_F$ denotes Frobenius norm. We further assume that worker nodes have memory constraints such that each node can hold only an m-th fraction of $\mathbf{A}$ and an m-th fraction of $\mathbf{B}$ in memory. To meet the memory constraint, we break down $\mathbf{A}, \mathbf{B}$ into small equal-sized sub-blocks as follows[2]:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{1,1} & \cdots & \mathbf{A}_{1,q} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{p,1} & \cdots & \mathbf{A}_{p,q} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_{1,1} & \cdots & \mathbf{B}_{1,p} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_{q,1} & \cdots & \mathbf{B}_{q,p} \end{bmatrix}, \tag{3}$$

where $pq = m$.

To mitigate failures or stragglers, a master node encodes redundancies through linear encoding. The i-th worker node receives encoded inputs $\widetilde{\mathbf{A}}_i$ and $\widetilde{\mathbf{B}}_i$ such that:

$$\widetilde{\mathbf{A}}_i = f_i(\mathbf{A}_{1,1}, \cdots, \mathbf{A}_{p,q}), \widetilde{\mathbf{B}}_i \quad = g_i(\mathbf{B}_{1,1}, \cdots, \mathbf{B}_{p,q}),$$

where

$$f_i : \underbrace{\mathbb{R}^{\frac{n}{p} \times \frac{n}{q}} \times \cdots \times \mathbb{R}^{\frac{n}{p} \times \frac{n}{q}}}_{pq=m} \to \mathbb{R}^{\frac{n}{p} \times \frac{n}{q}}, \tag{4}$$

$$g_i : \underbrace{\mathbb{R}^{\frac{n}{q} \times \frac{n}{p}} \times \cdots \times \mathbb{R}^{\frac{n}{q} \times \frac{n}{p}}}_{m} \to \mathbb{R}^{\frac{n}{q} \times \frac{n}{p}}. \tag{5}$$

We assume that f_i, g_i are linear, i.e., their outputs are linear combinations of m inputs. For example, we may have

$$f_i(\mathbf{V}_1, \cdots, \mathbf{V}_m) = \gamma_{i,1}\mathbf{V}_1 + \cdots + \gamma_{i,m}\mathbf{V}_m \tag{6}$$

for some $\gamma_{i,j} \in \mathbb{R}$ ($j \in [m]$).

Worker nodes are oblivious of the encoding/decoding process and simply perform matrix multiplication on the inputs they receive. In our case, each worker node computes

$$\widetilde{\mathbf{C}}_i = \widetilde{\mathbf{A}}_i \cdot \widetilde{\mathbf{B}}_i, \tag{7}$$

and returns the $\frac{n}{p} \times \frac{n}{p}$ output matrix $\widetilde{\mathbf{C}}_i$ to the master node.

Finally, when the master node receives outputs from a subset of worker nodes, say $\mathcal{S} \subseteq [P]$, it performs decoding:

$$\widehat{\mathbf{C}}_{\mathcal{S}} = d_{\mathcal{S}}((\widetilde{\mathbf{C}}_i)_{i \in \mathcal{S}}), \tag{8}$$

where $\{d_{\mathcal{S}}\}_{\mathcal{S} \subseteq [P]}$ is a set of predefined decoding functions that take $|\mathcal{S}|$ inputs from $\mathbb{R}^{\frac{n}{p} \times \frac{n}{p}}$ and outputs an n-by-n matrix. Note that we do not restrict the decoders $d_{\mathcal{S}}$ to be linear.

C. Approximate Recovery Threshold

Let $\mathbf{f}$ and $\mathbf{g}$ be vectors of encoding functions:

$$\mathbf{f} = [f_1 \quad \cdots \quad f_P], \mathbf{g} = [g_1 \quad \cdots \quad g_P], \tag{9}$$

and let $\mathbf{d}$ be a length-2^P vector of decoding functions $d_{\mathcal{S}}$ for all subsets $\mathcal{S} \subseteq [P]$. Then, we say that the ϵ-approximate recovery threshold of $\mathbf{f}, \mathbf{g}, \mathbf{d}$ is K if

$$|\widehat{C}_{\mathcal{S}}[i, j] - C[i, j]| \leq \epsilon \quad (i, j \in [n]), \tag{10}$$

[2]We limit ourselves to splitting the input matrices into a grid of submatrices. Splitting into an arbitrary shape is beyond the scope of this work.

for every $\mathcal{S} \subseteq [P]$ such that $|\mathcal{S}| \geq K$, and any $\mathbf{A}$ and $\mathbf{B}$ that satisfy the norm constraints (2). We denote this recovery threshold as $K(m, \epsilon, \mathbf{f}, \mathbf{g}, \mathbf{d})$. Moreover, let $K^*(m, \epsilon)$ be defined as the minimum of $K(m, \epsilon, \mathbf{f}, \mathbf{g}, \mathbf{d})$ over all possible linear functions $\mathbf{f}$, $\mathbf{g}$ and all possible decoding functions $\mathbf{d}$, i.e.,

$$K^*(m, \epsilon) \triangleq \min_{\mathbf{f},\mathbf{g},\mathbf{d}} K(m, \epsilon, \mathbf{f}, \mathbf{g}, \mathbf{d}). \tag{11}$$

Note that parameters p and q are embedded in $\mathbf{f}$ and $\mathbf{g}$. Through an achievability scheme in [4] and a converse in [29], for exact recovery, the optimal threshold has been characterized to be $2m - 1$:

Theorem 1 (Adaptation of Theorem 2 in [29] and Theorem III.1 in [4])**.** *Under the system model given in Section II-B*

$$K^*(m, \epsilon = 0) = 2m - 1. \tag{12}$$

Our main result is the following:

Theorem 2. *Under the system model given in Section II-B, the optimal ϵ-approximate recovery threshold is:*

$$K^*(m, \epsilon) = m. \tag{13}$$

(Achievability – Theorem 3)
For any $0 < \epsilon < \min(2, 3\eta^2\sqrt{2m-1})$, the ϵ-approximate MatDot codes in Construction 2 achieves:

$$K(m, \epsilon, \mathbf{f}_{\epsilon\text{-MatDot}}, \mathbf{g}_{\epsilon\text{-MatDot}}, \mathbf{d}_{\epsilon\text{-MatDot}}) = m.$$

(Converse – Theorem 4)
For all $0 < \epsilon < \eta^2$, $K^*(m, \epsilon) \geq m$.

III. Theoretical Characterization of $K^*(m, \epsilon)$

In this section, we first propose the construction of ϵ-approximate MatDot codes that can achieve the recovery threshold of m for ϵ approximation error. Then, we show the converse result which shows that the recovery threshold cannot be smaller than m for sufficiently small ϵ.

A. Approximate MatDot Codes

We briefly introduce the construction of MatDot codes and then we show that a simple adaptation of MatDot codes can be used for approximate coded computing.

Construction 1 (MatDot Codes [4])**.** *Define polynomials $p_{\mathbf{A}}(x)$ and $p_{\mathbf{B}}(x)$ as follows:*

$$p_{\mathbf{A}}(x) = \sum_{i=1}^{m} \mathbf{A}_i x^{i-1}, p_{\mathbf{B}}(x) = \sum_{j=1}^{m} \mathbf{B}_j x^{m-j}. \tag{14}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_P$ be P distinct elements in $\mathbb{R}$. The i-th worker receives encoded versions of matrices:

$$\widetilde{\mathbf{A}}_i = p_{\mathbf{A}}(\lambda_i) = \mathbf{A}_1 + \lambda_i \mathbf{A}_2 + \cdots + \lambda_i^{m-1}\mathbf{A}_m,$$
$$\widetilde{\mathbf{B}}_i = p_{\mathbf{B}}(\lambda_i) = \mathbf{B}_m + \lambda_i \mathbf{B}_{m-1} + \cdots + \lambda_i^{m-1}\mathbf{B}_1,$$

and then computes matrix multiplication on the encoded matrices:

$$\widetilde{\mathbf{C}}_i = \widetilde{\mathbf{A}}_i \widetilde{\mathbf{B}}_i = p_{\mathbf{A}}(\lambda_i) p_{\mathbf{B}}(\lambda_i) = p_{\mathbf{C}}(\lambda_i).$$

The polynomial $p_{\mathbf{C}}(x)$ has degree $2m-2$ and has the following form:

$$p_{\mathbf{C}}(x) = \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbf{A}_i \mathbf{B}_j x^{m-1+(i-j)}. \tag{15}$$

Once the master node receives outputs from $2m-1$ successful worker nodes, it can recover the coefficients of $p_{\mathbf{C}}(x)$ through polynomial interpolation, and then recover $\mathbf{C} = \sum_{i=1}^{m} \mathbf{A}_i \mathbf{B}_i$ as the coefficient of x^{m-1} in $p_{\mathbf{C}}(x)$. □

The recovery threshold of MatDot codes is $2m-1$ because the output polynomial $p_{\mathbf{C}}(x)$ is a degree-$(2m-2)$ polynomial and we need $2m-1$ points to recover all of the coefficients of $p_{\mathbf{C}}(x)$. However, in order to recover $\mathbf{C}$, we only need the coefficient of x^{m-1} in $p_{\mathbf{C}}(x)$. The key idea of Approximate MatDot Codes is to choose the evaluation points carefully to reduce this overhead. In fact, we have to choose evaluation points in a small interval that is proportional to ϵ.

Construction 2 (ϵ-Approximate MatDot codes)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be matrices in $\mathbb{R}^{n\times n}$ that satisfy $||\mathbf{A}||_F, ||\mathbf{B}||_F \leq \eta$. Let $\epsilon \in \mathbb{R}$ be a constant such that*

$$0 < \epsilon < \min(2, 3\eta^2\sqrt{2m-1}). \tag{16}$$

Then, ϵ-Approximate MatDot code is a MatDot code defined in Construction 1 with evaluation points $\lambda_1, \ldots, \lambda_P$ that satisfy:

$$|\lambda_i| < \frac{\epsilon}{6\eta^2\sqrt{2m-1}(m-1)m}, \quad i \in [P]. \tag{17}$$

We then show that this construction has the approximate recovery threshold of m.

Theorem 3. *For any $0 < \epsilon < \min(2, 3\eta^2\sqrt{2m-1})$, the ϵ-approximate MatDot codes in Construction 2 achieves:*

$$K(m, \epsilon, \mathbf{f}_{\epsilon\text{-MatDot}}, \mathbf{g}_{\epsilon\text{-MatDot}}, \mathbf{d}_{\epsilon\text{-MatDot}}) = m,$$

where $\mathbf{f}_{\epsilon\text{-MatDot}}, \mathbf{g}_{\epsilon\text{-MatDot}}, \mathbf{d}_{\epsilon\text{-MatDot}}$ are encoding and decoding functions specified by Construction 2.

While we defer the full proof to [1], we provide an intuitive explanation of the above theorem.

B. An insight behind Approximate MatDot Codes

Let $S(x)$ be a polynomial of degree $2m-1$ and let $P(x)$ be a polynomial of degree m. Then, $S(x)$ can be written as:

$$S(x) = P(x)Q(x) + R(x), \tag{18}$$

and the degree of Q and R are both at most $m-1$. Now, let $\lambda_1, \ldots, \lambda_m$ be the roots of $P(x)$. Then,

$$S(\lambda_i) = R(\lambda_i). \tag{19}$$

If we have m evaluations at these points, we can exactly recover the coefficients of the polynomial $R(x)$.

Recall that we only need the coefficient of x^{m-1} in MatDot codes. Letting $P(x) = x^m$, $S(x)$ can be written as:

$$S(x) = x^m Q(x) + R(x). \tag{20}$$

Since the lower order terms are all in $R(x)$, the coefficient of x^{m-1} in $R(x)$ is equal to the coefficient of x^{m-1} in $S(x)$. Thus, recovering the coefficients of $R(x)$ is sufficient for MatDot decoding. However, x^m has only one root, 0. For approximate decoding, we can use points close to 0 as evaluation points to make $x^m \approx 0$. Then, we have:

$$S(\lambda_i) = \lambda_i^m Q(\lambda_i) + R(\lambda_i) \approx R(\lambda_i). \tag{21}$$

When $|S(\lambda_i) - R(\lambda_i)|$ is small, we can use m evaluations of $S(\lambda_i)$'s to approximately interpolate $R(x)$. Moreover, when λ_i is small, we can also bound $|S(\lambda_i) - R(\lambda_i)|$ when Q has

a bounded norm. In our case, S has a bounded norm due the norm constraints (2) on the input matrices and, thus, Q must have a bounded norm since the higher-order terms in S are solely determined by Q.

C. Converse

We have shown that for *any* matrices $\mathbf{A}$ and $\mathbf{B}$, and with a recovery threshold of m, MatDot codes can achieve arbitrarily small error. We now show a converse indicating that for a recovery threshold of $m-1$, there exists matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ where the error cannot be made arbitrarily small for any type of encoding.

Theorem 4. *Under the system model given in Section II, for any* $0 < \epsilon < \eta^2$,

$$K^*(m,\epsilon) \geq m. \tag{22}$$

Proof Sketch. The idea of the proof is to choose a particular matrix $\mathbf{A}$ such that its encoded version $\boldsymbol{f}_{\mathcal{S}}(\mathbf{A}) = \mathbf{0}$, which would make the computation output of each worker to be $\mathbf{0}$. Then, via an appropriate choice of $\mathbf{B}$, we use the triangle inequality to lower bound the norms of the sums of $(\boldsymbol{d}_{\mathcal{S}}(\mathbf{0}) - \mathbf{AB})$ or $(\boldsymbol{d}_{\mathcal{S}}(\mathbf{0}) + \mathbf{AB})$ which leads to the desired bound. □

IV. An Optimization Approach to Approximate Coded Computing

The construction of Approximate MatDot code shows the theoretical possibility that the approximate recovery threshold can be brought down to m from $2m-1$. In this section, we propose another approach to find an approximate coded computing strategy. As we are not aiming for zero error, we pose the question as an optimization problem where the difference between the original matrix and the reconstructed matrix is minimized. The goal of optimization is to find ϵ such that $K^*(m,\epsilon) \leq k$ for a given k, within the space of linear encoding and decoding functions.

In Section IV-A, we illustrate how designing a code is framed as an optimization problem through a simple example, and in Section IV-B, we state the optimization objective and method for a general case. Finally, we show the results of our optimization algorithm in Section IV-C.

A. A simple example

Consider an example of $m = 2, k = 2, P = 3$. The input matrices are split into:

$$\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2], \quad \mathbf{B} = \begin{bmatrix}\mathbf{B}_1 \\ \mathbf{B}_2\end{bmatrix}. \tag{23}$$

As f_i's and g_i's are linear encoding functions, let

$$\boldsymbol{\alpha}^{(i)} = \begin{bmatrix}\alpha_1^{(i)} \\ \alpha_2^{(i)}\end{bmatrix}, \boldsymbol{\beta}^{(i)} = \begin{bmatrix}\beta_1^{(i)} \\ \beta_2^{(i)}\end{bmatrix} \tag{24}$$

be the encoding coefficients for $\mathbf{A}$ and $\mathbf{B}$ for the i-th node. The i-th worker node receives encoded inputs:

$$\widetilde{\mathbf{A}}^{(i)} = \alpha_1^{(i)}\mathbf{A}_1 + \alpha_2^{(i)}\mathbf{A}_2, \;\; \widetilde{\mathbf{B}}^{(i)} = \beta_1^{(i)}\mathbf{B}_1 + \beta_2^{(i)}\mathbf{B}_2.$$

The matrix product output at the i-th worker node is:

$$\begin{aligned}\widetilde{\mathbf{C}}^{(i)} = \widetilde{\mathbf{A}}^{(i)}\widetilde{\mathbf{B}}^{(i)} &= \alpha_1^{(i)}\beta_1^{(i)} \cdot \mathbf{A}_1\mathbf{B}_1 + \alpha_1^{(i)}\beta_2^{(i)} \cdot \mathbf{A}_1\mathbf{B}_2 \\ &+ \alpha_2^{(i)}\beta_1^{(i)} \cdot \mathbf{A}_2\mathbf{B}_1 + \alpha_2^{(i)}\beta_2^{(i)} \cdot \mathbf{A}_2\mathbf{B}_2.\end{aligned}$$

The recovery threshold $k = 2$ implies that with any two $\widetilde{\mathbf{C}}^{(i)}$, $\widetilde{\mathbf{C}}^{(j)}$, $i \neq j,\ i,j \in [3]$, the master node can recover:

$$\begin{aligned}\mathbf{C} &= \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 \\ &= 1 \cdot \mathbf{A}_1\mathbf{B}_1 + 0 \cdot \mathbf{A}_1\mathbf{B}_2 + 0 \cdot \mathbf{A}_2\mathbf{B}_1 + 1 \cdot \mathbf{A}_2\mathbf{B}_2.\end{aligned}$$

For illustration, assume that nodes $i = 1$ and $j = 2$ responded first. For linear decoding, our goal is to determine decoding coefficients $d_1, d_2 \in \mathbb{R}$ that yield

$$\mathbf{C} = d_1\widetilde{\mathbf{C}}^{(1)} + d_2\widetilde{\mathbf{C}}^{(2)}.$$

For the previous equality to hold for any $\mathbf{A}$ and $\mathbf{B}$, the coefficients must satisfy:

$$\begin{aligned}[1 \quad 0 \quad 0 \quad 1] = d_1 &\begin{bmatrix}\alpha_1^{(1)}\beta_1^{(1)} & \alpha_1^{(1)}\beta_2^{(1)} & \alpha_2^{(1)}\beta_1^{(1)} & \alpha_2^{(1)}\beta_2^{(1)}\end{bmatrix} \\ + d_2 &\begin{bmatrix}\alpha_1^{(2)}\beta_1^{(2)} & \alpha_1^{(2)}\beta_2^{(2)} & \alpha_2^{(2)}\beta_1^{(2)} & \alpha_2^{(2)}\beta_2^{(2)}\end{bmatrix}\end{aligned} \tag{25}$$

By reshaping the length-4 vectors in (25) into 2×2 matrices and denoting the identity matrix by $\mathbf{I}_{2\times 2}$, (25) is equivalent to

$$\mathbf{I}_{2\times 2} = \sum_{i=1}^{2} d_i\boldsymbol{\alpha}^{(i)}\boldsymbol{\beta}^{(i)T}. \tag{26}$$

Encoding coefficients $\boldsymbol{\alpha}^{(i)}$'s, $\boldsymbol{\beta}^{(i)}$'s and the decoding coefficients d_i's that satisfy the equality in (26) would guarantee exact recovery for any input matrices $\mathbf{A}$ and $\mathbf{B}$. However, we are interested in *approximate* recovery, which means that we want the LHS and RHS in (26) to be approximately equal. Hence, the goal of optimization is to find encoding and decoding coefficients that minimize the difference between LHS and RHS in (26). One possible objective function for this is:

$$||\mathbf{I}_{2\times 2} - \sum_{i=1}^{2} d_i\boldsymbol{\alpha}^{(i)}\boldsymbol{\beta}^{(i)T}||_F^2. \tag{27}$$

Recall that this is for the scenario where the third node fails and the first two nodes are successful. There are $\binom{3}{2} = 3$ scenarios where two nodes out of three nodes are successful. For the final objective function, we have to add such loss function for each of these three scenarios. We formalize this next.

B. Optimization Formulation

We formulate the optimization framework for arbitrary values of m, k and P. We denote the encoding coefficients for the i-th node as:

$$\boldsymbol{\alpha}^{(i)} = [\alpha_1^{(i)}, \cdots, \alpha_m^{(i)}]^T, \qquad \boldsymbol{\beta}^{(i)} = [\beta_1^{(i)}, \cdots, \beta_m^{(i)}]^T.$$

Let $\mathcal{P}_k([P]) = \{\mathcal{S} : \mathcal{S} \subseteq [P], |\mathcal{S}| = k\}$ and let $\mathcal{S}_p$ be the p-th set in $\mathcal{P}_k([P])$. In other words, $\mathcal{P}_k([P])$ is a set of all failure scenarios with k successful nodes out of P nodes. Then, we define $\mathbf{d}^{(p)}$ as the vector of decoding coefficients when $\mathcal{S}_p$ is the set of successful workers. We define our optimization problem as follows:

Optimization for Approximate Coded Computing:

$$\min_{\substack{\boldsymbol{\alpha}^{(i)},\,\boldsymbol{\beta}^{(i)},\,\mathbf{d}^{(p)} \\ i=1,\ldots,n, \\ p=1,\ldots,\binom{P}{k}}} \sum_{p=1}^{\binom{P}{k}} ||\mathbf{I}_{m\times m} - \sum_{i\in\mathcal{S}_p} d_i^{(p)}\boldsymbol{\alpha}^{(i)}\boldsymbol{\beta}^{(i)T}||_F^2. \tag{28}$$

Symbol	Dimension	Expression
$\boldsymbol{\mathcal{A}}$	$m \times P$	$\left[\boldsymbol{\alpha}^{(1)} \quad \cdots \quad \boldsymbol{\alpha}^{(P)}\right]$
$\boldsymbol{\mathcal{B}}$	$m \times P$	$\left[\boldsymbol{\beta}^{(1)} \quad \cdots \quad \boldsymbol{\beta}^{(P)}\right]$
$\mathbf{Z}^{(\text{full})}$	$P \times P$	$(\boldsymbol{\mathcal{A}}^T\boldsymbol{\mathcal{A}}) \odot (\boldsymbol{\mathcal{B}}^T\boldsymbol{\mathcal{B}})$
$\mathbf{z}^{(\text{full})}$	P	$[\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\beta}^{(i)}]_{i=1,\ldots,P}$
$\mathbf{Z}^{(p)}$	$k \times k$	$\mathbf{Z}^{(\text{full})}\vert_{i\in\mathcal{S}_p, j\in\mathcal{S}_p}$
$\mathbf{z}^{(p)}$	k	$\mathbf{z}^{(\text{full})}\vert_{i\in\mathcal{S}_p}$
$\mathbf{Y}$	$P \times P$	$\left[\sum_{p:i,j\in\mathcal{S}_p} d_i^{(p)} d_j^{(p)}\right]_{\substack{i=1,\ldots,P \\ j=1,\ldots,P}}$
$\mathbf{y}$	P	$\left[\sum_{p:i\in\mathcal{S}_p} d_i^{(p)}\right]_{i=1,\ldots,P}$
$\mathbf{Y}_{\boldsymbol{\mathcal{A}}}$, $\mathbf{Y}_{\boldsymbol{\mathcal{B}}}$	$P \times P$	$\mathbf{Y}_{\boldsymbol{\mathcal{A}}} = \mathbf{Y} \odot (\boldsymbol{\mathcal{A}}^T\boldsymbol{\mathcal{A}})$, $\mathbf{Y}_{\boldsymbol{\mathcal{B}}} = \mathbf{Y} \odot (\boldsymbol{\mathcal{B}}^T\boldsymbol{\mathcal{B}})$

TABLE I
SUMMARY OF NOTATIONS USED IN PROPOSITION 1 AND ALGORITHM 1

Notice that (28) is a non-convex problem, but it is convex with respect to each coordinate, i.e., with respect to $\{\boldsymbol{\alpha}^{(i)} : i \in [n]\}$, $\{\boldsymbol{\beta}^{(i)} : i \in [n]\}$, and $\{\mathbf{d}^{(p)} : p \in \left[\binom{P}{k}\right]\}$. Hence, we propose an alternating minimization algorithm that minimizes for $\mathbf{d}^{(p)}$, $\boldsymbol{\alpha}^{(i)}$, and $\boldsymbol{\beta}^{(i)}$ sequentially. Each minimization step is a quadratic optimization with a closed-form solution, which we describe in the following proposition. The notation used in the proposition and in Algorithm 1 is summarized in Table I

Proposition 1. *The stationary points of the objective function given in* (28) *satisfy*

$$\textit{(i)}\ \ \mathbf{Z}^{(p)} \cdot \mathbf{d}^{(p)} = \mathbf{z}^{(p)} \quad \textit{for}\ \ p = 1, \ldots, \binom{n}{k},$$

$$\textit{(ii)}\ \ \mathbf{Y}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{A}} = \text{diag}(\mathbf{y})\boldsymbol{\mathcal{B}},$$

$$\textit{(iii)}\ \ \mathbf{Y}_{\boldsymbol{\mathcal{A}}}\boldsymbol{\mathcal{B}} = \text{diag}(\mathbf{y})\boldsymbol{\mathcal{A}},$$

where $\text{diag}(\mathbf{y})$ *is an* n*-by-*n *matrix which has* y_i *on the* i*-th diagonal and* 0 *elsewhere.*

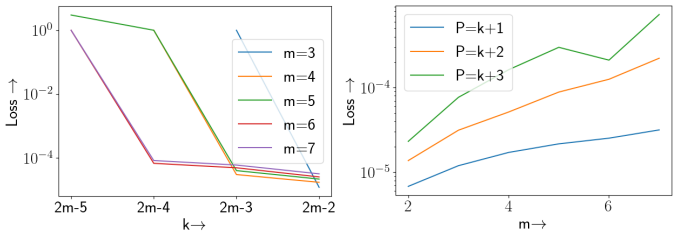
Algorithm 1 presents an alternating minimization procedure for computing a local minimum of (28). The algorithm sequentially solves conditions (i)–(iii) in Proposition 1. Since each step corresponds to minimizing (28) for one of the variables $\mathbf{d}^{(p)}$, $\boldsymbol{\mathcal{A}}$, and $\boldsymbol{\mathcal{B}}$, the resulting objective is non-increasing in the algorithm's iterations and converges to a local minimum.

Algorithm 1: Alternating Quadratic Minimization

Input: Positive Integers m, k and P $(P > k)$;
Output: $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\mathbf{d}^{(p)}$ $(p = 1, \ldots, P)$;
Initialize: Random $m \times P$ matrices $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$;
while *num_iter* < *max_iter* **do**
  Compute $\mathbf{Z}^{(\text{full})}$ and $\mathbf{z}^{(\text{full})}$ from $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$;
  for $p \leftarrow 1$ **to** P **do**
    Solve for $\mathbf{d}^{(p)} : \mathbf{Z}^{(p)}\mathbf{d}^{(p)} = \mathbf{z}^{(p)}$
  end
  Compute $\mathbf{Y}$ and $\mathbf{y}$, and $\mathbf{Y}_{\boldsymbol{\mathcal{B}}}$;
  $\boldsymbol{\mathcal{A}} \leftarrow \boldsymbol{\mathcal{A}}^*$, $\boldsymbol{\mathcal{A}}^*$: solution of $\mathbf{Y}_{\mathcal{B}} \cdot \boldsymbol{\mathcal{A}} = \text{diag}(\mathbf{y}) \cdot \boldsymbol{\mathcal{B}}$;
  Compute $\mathbf{Y}_{\boldsymbol{\mathcal{A}}}$;
  $\boldsymbol{\mathcal{B}} \leftarrow \boldsymbol{\mathcal{B}}^*$, $\boldsymbol{\mathcal{B}}^*$: solution of $\mathbf{Y}_{\boldsymbol{\mathcal{A}}} \cdot \boldsymbol{\mathcal{B}} = \text{diag}(\mathbf{y}) \cdot \boldsymbol{\mathcal{A}}$;
end

We next show how the optimization objective in (28) is related to the relative error of the computation output. Let $\ell^{(p)}$ be the loss function for the p-th scenario, i.e.,

$$\ell^{(p)} = ||\mathbf{I}_{m\times m} - \sum_{i\in\mathcal{S}_p} d_i^{(p)} \boldsymbol{\alpha}^{(i)} \boldsymbol{\beta}^{(i)T}||_F^2. \tag{29}$$



(a) Loss vs. k for $P = k + 1$ (b) Loss vs. m for $k = 2m - 2$

Fig. 1. Summary of results of running Algorithm 1 for 100,000 iterations. The y-axis is the loss function given in (28).

Theorem 5. *The error between the decoded result from the nodes in* $\mathcal{S}_p$*,* $\widehat{\mathbf{C}}_{\mathcal{S}_p}$*, and the true result* $\mathbf{C}$ *can be bounded as:*

$$||\mathbf{C} - \widehat{\mathbf{C}}_{\mathcal{S}_p}||_F \leq \sqrt{\ell^{(p)}} \cdot m \cdot \eta^2. \tag{30}$$

C. Optimization Results

We summarize the results of running Algorithm 1 for various combinations of parameters, m, k, P in Fig. 1. We demonstrate the best result out of 20 random initializations; for each trial, we ran the optimization for 100,000 iterations. From Fig.1(a), we can see that the loss function is very small ($\sim 10^{-4}$) for $k < 2m - 1$. However, the loss increases as k gets closer to m, reaching ~ 1. This could be due to the optimization algorithm more easily getting stuck at a local minimum as k approaches m. In Fig. 1(b), we observe that the loss for $k = 2m - 2$ remains small for different m values. The loss function increases as m and P get bigger because the number of terms we add in the loss function increases with m and P.

V. DISCUSSION AND FUTURE WORK

This paper opens new directions for coded computing by showing the power of approximations. Specifically, an open research direction is the investigation of related coded computing frameworks (e.g., polynomial evaluations) to examine the gap between ϵ-error and 0-error recovery thresholds. Since our constructions requires evaluation points close to 0 (Section III), we potentially get ill-conditioned encoding matrices. An open direction of future work is to explore numerically stable coding schemes building on recent works, e.g. [35]–[37], focusing on ϵ-error instead of exact computation.

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APPENDIX A
PROOF OF PROPOSITION 1

For simple demonstration, let us focus on the case where $\mathcal{S}_p = [1, \dots, k]$ and expand the term inside the sum. In the following equations, we will omit the superscript (p) for simplification.

$$||\mathbf{I}_{m\times m} - \sum_{i\in\mathcal{S}_p} d_i \boldsymbol{\alpha}^{(i)}\boldsymbol{\beta}^{(i)T}||_F^2 \tag{31}$$

$$= \mathrm{Tr}\left(\left(\mathbf{I} - \sum_{i=1}^{k} d_i \mathbf{X}_i\right)^T \left(\mathbf{I} - \sum_{i=1}^{k} d_i \mathbf{X}_i\right)\right) \tag{32}$$

$$(\mathbf{X}_i = \boldsymbol{\alpha}^{(i)}\boldsymbol{\beta}^{(i)T})$$

$$= \mathrm{Tr}\Bigg(\mathbf{I} - \sum_{i=1}^{k} d_i \mathbf{X}_i^T - \sum_{i=1}^{k} d_i \mathbf{X}_i + \left(\sum_{i=1}^{k} d_i \mathbf{X}_i\right)^T \left(\sum_{j=1}^{k} d_j \mathbf{X}_j\right)\Bigg) \tag{33}$$

$$= m - 2\sum_{i=1}^{k} d_i \,\mathrm{Tr}(\mathbf{X}_i) + \sum_{i=1}^{k}\sum_{j=1}^{k} d_i d_j \,\mathrm{Tr}(\mathbf{X}_i^T \mathbf{X}_j) \tag{34}$$

$$= m - 2\sum_{i=1}^{k} d_i \boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\beta}^{(i)} + \sum_{i=1}^{k}\sum_{j=1}^{k} d_i d_j (\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\alpha}^{(j)})(\boldsymbol{\beta}^{(i)} \cdot \boldsymbol{\beta}^{(j)}) \tag{35}$$

$$= m - 2\mathbf{d}\cdot\mathbf{z} + \mathbf{d}^T\mathbf{Z}\mathbf{d}, \tag{36}$$

where $\mathbf{d}$ and $\mathbf{z}$ are length-k column vectors: $\mathbf{d} = [d_1, \dots, d_k]$ and $\mathbf{z} = [\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\beta}^{(i)}]_{i=1,\dots,k}$. $\mathbf{Z}$ is a $k \times k$ matrix:

$$\mathbf{Z} = (\boldsymbol{\mathcal{A}}_k^T \boldsymbol{\mathcal{A}}_k) \odot (\boldsymbol{\mathcal{B}}_k^T \boldsymbol{\mathcal{B}}_k), \tag{37}$$

where $\boldsymbol{\mathcal{A}}_k = [\boldsymbol{\alpha}_1 \;\; \boldsymbol{\alpha}_2 \;\; \dots \;\; \boldsymbol{\alpha}_k]$ and $\boldsymbol{\mathcal{B}}_k = [\boldsymbol{\beta}_1 \;\; \boldsymbol{\beta}_2 \;\; \dots \;\; \boldsymbol{\beta}_k]$.

The partial derivative with respect to $\mathbf{d}$ can be represented as:

$$\frac{\partial}{\partial \mathbf{d}} L = -2\mathbf{z} + 2\mathbf{Z}\mathbf{d}. \tag{38}$$

Thus, the optimal $\mathbf{d}^*$ can be obtained by solving:

$$\mathbf{Z}\mathbf{d} = \mathbf{z}. \tag{39}$$

Note that this can be easily generalized to any $\mathbf{d}^{(p)} = [d_i]_{i\in\mathcal{S}_p}$. It only requires using different $\mathbf{Z}$ and $\mathbf{z}$ as follows:

$$\mathbf{Z} = (\boldsymbol{\mathcal{A}}^{(p)T} \boldsymbol{\mathcal{A}}^{(p)}) \odot (\boldsymbol{\mathcal{B}}^{(p)T} \boldsymbol{\mathcal{B}}^{(p)}), \;\; \mathbf{z}^{(p)} = [\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\beta}^{(i)}]_{i\in S_p},$$

where $\boldsymbol{\mathcal{A}}^{(p)} = [\boldsymbol{\alpha}_i]_{i\in S_p}$, $\boldsymbol{\mathcal{B}}^{(p)} = [\boldsymbol{\beta}_i]_{i\in S_p}$.

To obtain the gradient with respect to $\boldsymbol{\alpha}^{(i)}$'s and $\boldsymbol{\beta}^{(i)}$'s, let us expand the loss function given in (28) again. We now want to include the outer sum:

$$L = \sum_{p=1,\dots,\binom{n}{k}} ||\mathbf{I}_{m\times m} - \sum_{i\in\mathcal{S}_p} d_i^{(p)} \boldsymbol{\alpha}^{(i)}\boldsymbol{\beta}^{(i)T}||_F^2 \tag{40}$$

$$= \sum_{p=1,\dots,\binom{n}{k}} \Big(m - 2\sum_{i\in\mathcal{S}_p} d_i^{(p)} \boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\beta}^{(i)} + \sum_{i\in\mathcal{S}_p}\sum_{j\in\mathcal{S}_p} d_i^{(p)} d_j^{(p)} (\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\alpha}^{(j)})(\boldsymbol{\beta}^{(i)} \cdot \boldsymbol{\beta}^{(j)})\Big) \tag{41}$$

$$= \binom{n}{k} \cdot m - 2\sum_{i\in[n]} \Big(\sum_{p:i\in\mathcal{S}_p} d_i^{(p)}\Big) \boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\beta}^{(i)} + \sum_{i\in[n]}\sum_{j\in[n]} \Big(\sum_{p:i,j\in\mathcal{S}_p} d_i^{(p)} d_j^{(p)}\Big) (\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\alpha}^{(j)})(\boldsymbol{\beta}^{(i)} \cdot \boldsymbol{\beta}^{(j)}) \tag{42}$$

$$= \binom{n}{k} \cdot m - 2\sum_{i\in[n]} y_i \cdot \boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\beta}^{(i)} + \sum_{i\in[n]}\sum_{j\in[n]} Y_{i,j} (\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\alpha}^{(j)})(\boldsymbol{\beta}^{(i)} \cdot \boldsymbol{\beta}^{(j)}) \tag{43}$$

In the last line, we let $y_i = \sum_{p:i\in\mathcal{S}_p} d_i^{(p)}$ and $Y_{i,j} = \sum_{p:i,j\in\mathcal{S}_p} d_i^{(p)} d_j^{(p)}$. Now, the gradient of L with respect to $\boldsymbol{\alpha}^{(i)}$ can be written as:

$$\frac{\partial}{\partial \boldsymbol{\alpha}^{(i)}} L = -2y_i \boldsymbol{\beta}^{(i)} + 2\sum_{j\in[n]} Y_{i,j} (\boldsymbol{\beta}^{(i)} \cdot \boldsymbol{\beta}^{(j)}) \boldsymbol{\alpha}^{(j)}. \tag{44}$$

Following the notation that $\boldsymbol{\mathcal{A}} = \begin{bmatrix}\boldsymbol{\alpha}^{(1)} & \dots & \boldsymbol{\alpha}^{(n)}\end{bmatrix}$ and $\boldsymbol{\mathcal{B}} = \begin{bmatrix}\boldsymbol{\beta}^{(1)} & \dots & \boldsymbol{\beta}^{(n)}\end{bmatrix}$, this can be written in a matrix form:

$$\frac{\partial}{\partial \boldsymbol{\mathcal{A}}} L = -2\mathrm{diag}(\mathbf{y})\boldsymbol{\mathcal{B}}^T + 2\mathbf{Y}_{\boldsymbol{\mathcal{B}}}\boldsymbol{\mathcal{A}}^T, \tag{45}$$

where $\mathbf{y} = [y_i]_{i=1,\dots,n}$ is a column vector of length n and $\mathbf{Y}_{\mathcal{B}} = [Y_{i,j}(\boldsymbol{\beta}^{(i)} \cdot \boldsymbol{\beta}^{(j)})]_{i,j=1,\dots,n} = \mathbf{Y} \odot (\boldsymbol{\mathcal{B}}^T\boldsymbol{\mathcal{B}})$ is an $n \times n$ matrix. Thus, the optimal $\boldsymbol{\mathcal{A}}^*$ can be obtained by solving:

$$\mathbf{Y}_{\mathcal{B}} \cdot \boldsymbol{\mathcal{A}} = \mathrm{diag}(\mathbf{y}) \cdot \boldsymbol{\mathcal{B}}. \tag{46}$$

Similarly, the optimal $\boldsymbol{\mathcal{B}}^*$ can be obtained by solving:

$$\mathbf{Y}_{\mathcal{A}} \cdot \boldsymbol{\mathcal{B}} = \mathrm{diag}(\mathbf{y}) \cdot \boldsymbol{\mathcal{A}}, \tag{47}$$

whee $\mathbf{Y}_{\mathcal{A}} = \mathbf{Y} \odot (\boldsymbol{\mathcal{A}}^T\boldsymbol{\mathcal{A}})$.

APPENDIX B
PROOF OF THEOREM 3

Let f be a $(k-1)$-degree polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1}, \tag{48}$$

and we use $\mathrm{vec}(f)$ to denote the row vector representation of the coefficients of f, i.e.,

$$\mathrm{vec}(f) \triangleq [a_0 \quad a_1 \quad \cdots \quad a_{k-1}] \triangleq \mathbf{a}. \tag{49}$$

Also, let $\boldsymbol{\lambda} \triangleq [\lambda_j]_{j\in[m]} \in \mathbb{R}^m$ be m distinct evaluation points. Then, we define a Vandermonde matrix for $\boldsymbol{\lambda}$ of degree k as:

$$\mathsf{Vander}(\boldsymbol{\lambda}, k) = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_m \\ \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \cdots & \lambda_m^{k-1} \end{bmatrix}. \tag{50}$$

The evaluations of f at the points $\boldsymbol{\lambda}$ can be written as

$$[f(\lambda_1) \quad f(\lambda_2) \quad \cdots \quad f(\lambda_m)] = \mathbf{a}\cdot\mathbf{V}, \tag{51}$$

where $\mathbf{V} = \mathsf{Vander}(\boldsymbol{\lambda}, k)$. When $m < k$, the null space of $\mathbf{V}$ can be conveniently expressed in terms of elementary symmetric polynomials.

Definition 1 (Elementary Symmetric Polynomial)**.** Let $\mathbf{x} = (x_1, \cdots, x_n)$. For $l \in \{0, 1, \ldots, n\}$, the elementary symmetric polynomials in n variables $e_l : \mathbb{R}^n \to \mathbb{R}$ are given by

$$e_l(\mathbf{x}) \triangleq \begin{cases} \sum\limits_{\substack{S\subseteq[n] \\ |S|=l}} \prod\limits_{i\in S} x_i, & \text{if } l = 1, \ldots, n, \\ 1, & \text{if } l = 0. \end{cases} \tag{52}$$

In particular, $e_1(\mathbf{x}) = \sum_{i\in[n]} x_i$ and $e_n(\mathbf{x}) = \prod_{i\in[n]} x_i$.

Lemma 1. *For $m < k$, the left null space of $\mathbf{V}$ is spanned by $\{\mathbf{u}_i\}_{i\in[k-m]} \subset \mathbb{R}^k$, where $\mathbf{u}_i = \mathrm{vec}(p_i)$ for the polynomials p_i's defined as:*

$$p_i(x) \triangleq x^{i-1}\prod_{j=1}^{m}(x - \lambda_j). \tag{53}$$

Proof. First, note that $\mathbf{u}_i \in \mathsf{null}(\mathbf{V}^T)$:

$$\mathbf{u}_i \cdot \mathbf{V} = [p_i(\lambda_1) \quad \cdots \quad p_i(\lambda_m)] = \mathbf{0}. \tag{54}$$

Next, we show that $\mathsf{dim}(\mathsf{span}(\mathbf{u}_1, \cdots, \mathbf{u}_{k-m})) = k - m$. The coefficients of the Lagrange polynomial $p_1(x) = \prod_{j=1}^{m}(x-\lambda_j)$ can be written as:

$$\mathbf{u}_1 = [e_m(\boldsymbol{\lambda}) \quad \cdots \quad e_0(\boldsymbol{\lambda}) \quad 0 \quad 0 \quad \cdots 0], \tag{55}$$

with $k - m - 1$ trailing zeros, and

$$\begin{aligned} \mathbf{u}_2 &= [0 \quad e_m(\boldsymbol{\lambda}) \quad \cdots \quad e_0(\boldsymbol{\lambda}) \quad 0 \quad \cdots 0], \\ &\vdots \\ \mathbf{u}_{k-m} &= [0 \quad 0 \quad \cdots \quad 0 \quad e_m(\boldsymbol{\lambda}) \quad \cdots \quad e_0(\boldsymbol{\lambda})]. \end{aligned} \tag{56}$$

Let $\mathbf{U} \in \mathbb{R}^{(k-m)\times k}$ be the matrix obtained by concatenating $\mathbf{u}_i$ row-wise, i.e., the matrix with i-th row equal to $\mathbf{u}_i$. Note that $\mathbf{U} = [\mathbf{U}_1 \; \mathbf{U}_2]$, where $\mathbf{U}_2 \in \mathbb{R}^{(k-m)\times(k-m)}$ is a lower-triangular matrix with diagonal entries equal to $e_0(\boldsymbol{\lambda}) = 1$. Consequently, $\mathbf{U}_2$ is full-rank (in particular, $\det(\mathbf{U}_2) = 1$). Therefore $\mathbf{U}$ is also full-rank and

$$\mathsf{rank}(\mathbf{U}) = \mathsf{dim}(\mathsf{span}(\mathbf{u}_1, \cdots, \mathbf{u}_{k-m})) = k - m.$$

□

We prove next a bound on the evaluation of the elementary symmetric polynomials $e_l, l \neq 0$ in terms of the ℓ_∞-norm of its entries.

Lemma 2. *Let $0 < \epsilon \leq 2$ and $\mathbf{x} \in \mathbb{R}^n$. If $\|\mathbf{x}\|_\infty \leq \epsilon/n$, then $|e_l(\mathbf{x})| \leq \epsilon$ for $l \in \{1, 2, \ldots, n\}$.*

Proof.

$$\begin{aligned} |e_l(\mathbf{x})| &= \Big|\sum_{\substack{S\in[n] \\ |S|=l}} \prod_{j\in S} x_j\Big| \\ &\leq \sum_{\substack{S\in[n] \\ |S|=l}} \Big|\prod_{j\in S} x_j\Big| \\ &\leq \sum_{\substack{S\in[n] \\ |S|=l}} \Big(\frac{\epsilon}{n}\Big)^l = \binom{n}{l}\cdot\Big(\frac{\epsilon}{n}\Big)^l \\ &\leq \frac{n^l}{l!}\cdot\frac{\epsilon^l}{n^l} = \frac{\epsilon^l}{l!} = \epsilon\cdot\prod_{k=2}^{l}\frac{\epsilon}{k} \\ &\leq \epsilon \end{aligned}$$

□

If $m = k$, the coefficients of f can be recovered *exactly* from $\{f(\lambda_j)\}_{j\in[m]}$ by inverting the linear system (51) as long as the evaluation points are distinct. When $m < k$ then, in general, $\mathbf{a}$ cannot be recovered exactly. In this case, the system (51) is undetermined: denoting the true (but unknown) coefficient vector by $\mathbf{a}^*$, any vector in the set

$$\{\mathbf{a}\} + \mathsf{null}(\mathbf{V}^T) = \{\mathbf{a} + \mathbf{n} \mid \mathbf{n} \in \mathsf{null}(\mathbf{V}^T)\} \tag{57}$$

will be consistent with the m evaluation points. Nevertheless, we show next that if the coefficients of f have bounded norm, i.e.,

$$\mathbf{a} \in \mathcal{B}_R \triangleq \{\mathbf{x} \in \mathbb{R}^k \text{ s.t. } ||\mathbf{x}||_2 \leq R\}, \tag{58}$$

then the first m coefficients $a_0, \ldots, a_{m-1}$ can be approximated with *arbitrary precision* by computing f at m distinct and sufficiently small evaluation points. This result is formally stated in Corollary 1, which is the main tool for proving the approximate coded computing recovery threshold.

Theorem 6. *Let $\boldsymbol{\lambda}$ be $m < k$ distinct evaluation points with corresponding $k \times m$ Vandermonde matrix $\mathbf{V}$ in (**??**) and $0 < \epsilon \leq \min(2, 3R)$. If $\|\boldsymbol{\lambda}\|_\infty < \frac{\epsilon}{3R(k-m)m}$, then for any $\mathbf{x} \in \mathcal{B}_R \cap \mathsf{null}(\mathbf{V}^T)$,*

$$|x[i]| \leq \epsilon \quad \textit{for} \quad i \in [m]. \tag{59}$$

Proof. Since $\mathbf{x} \in \mathsf{null}(\mathbf{V}^T)$, we can express $\mathbf{x}$ as:

$$\mathbf{x} = \alpha_1\mathbf{u}_1 + \cdots + \alpha_{k-m}\mathbf{u}_{k-m}, \tag{60}$$

for some $\alpha_1, \cdots, \alpha_{k-m} \in \mathbb{R}$. For a shorthand notation, we will use e_l for $e_l(\boldsymbol{\lambda})$, and let $e_l = 0$ if $l < 0$ or $l > m$. By substituting (55), (56) into (60), we get:

$$x[i] = \sum_{j=1}^{k-m} \alpha_j e_{m-i+j} \quad \text{for } \ i = 1, \ldots, k. \tag{61}$$

Since $e_0 = 1$,

$$x[k] = \alpha_{k-m} e_0 = \alpha_{k-m}. \tag{62}$$

Furthermore, because $\mathbf{x} \in \mathcal{B}_R$,

$$|x[k]| = |\alpha_{k-m}| \leq R. \tag{63}$$

Similarly,

$$x[k-1] = \alpha_{k-m-1} e_0 + \alpha_{k-m} e_1 = \alpha_{k-m-1} + \alpha_{k-m} e_1.$$

Now note that from Lemma 2,

$$|e_l(\boldsymbol{\lambda})| \leq \frac{\epsilon}{3R(k-m)} \triangleq \delta. \tag{64}$$

for $l \in [m]$. Thus,

$$\begin{aligned} |\alpha_{k-m-1}| &= |x[k-1] + \alpha_{k-m} e_1| \\ &\leq |x[k-1]| + |\alpha_{k-m} e_1| \\ &\leq R + R \cdot \delta \\ &= R(1+\delta). \end{aligned}$$

By repeating the same argument up to $x[m+1]$, for $l = 1, \ldots, k-m$, we obtain:

$$|\alpha_l| \leq R(1+\delta)^{k-m-l} \tag{65}$$

$$\leq R(1 + \frac{1}{k-m})^{k-m-l} \tag{66}$$

$$\leq R(1 + \frac{1}{k-m})^{k-m} \tag{67}$$

$$\leq 3R. \tag{68}$$

The inequality (66) follows from the assumption that $\epsilon \leq 3C$ and the last inequality holds because for a positive integer n:

$$(1 + \frac{1}{n})^n \leq 3 - \frac{1}{n} < 3.$$

Now, for $i \in [m]$, $x[i]$ can be written as:

$$|x[i]| = |\sum_{j=1}^{k-m} \alpha_j e_{m-i+j}| \tag{69}$$

$$\leq \sum_{j=1}^{k-m} |\alpha_j| |e_{m-i+j}| \tag{70}$$

$$\leq \sum_{j=1}^{k-m} |\alpha_j| \delta \tag{71}$$

$$\leq (k-m) \cdot 3R \cdot \delta = \epsilon. \tag{72}$$

The inequality in (71) holds because $m - l + j \neq 0$ for $l \in [m]$, and thus $|e_{m-l+j}| \leq \delta$. □

Corollary 1. *Consider a set $\{f(\lambda_j)\}_{j \in [m]}$ of $m < k$ evaluations of f at distinct points $\boldsymbol{\lambda}$. If $0 < \epsilon < \min(2, 3R)$ and $\|\boldsymbol{\lambda}\|_\infty < \frac{\epsilon}{6R(k-m)m}$, then for any two coefficient vectors $\mathbf{a}, \mathbf{b} \in \mathcal{B}_R$ that satisfy (51) (i.e., that are consistent with the evaluations), we have*

$$|a[i] - b[i]| \leq \epsilon \text{ for } i \in [m]. \tag{73}$$

Proof. Under the assumptions of the corollary,

$$\mathbf{n} \triangleq \mathbf{a} - \mathbf{b} \in \mathsf{null}(\mathbf{V}^T).$$

Moreover, the triangle inequality yields

$$\|\mathbf{n}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \leq 2R.$$

I.e., $\mathbf{n} \in \mathcal{B}_{2R} \cap \mathsf{null}(\mathbf{V}^T)$. The result follows by a direct application of Theorem 6. □

For an n-by-n matrix $\mathbf{C}$, the polynomial $p_{\mathbf{C}}(x)$ is essentially a set of n^2 polynomials, having one polynomial for each $C[i,j]$ $(i,j \in [n])$. For decoding, we have to interpolate each of those n^2 polynomials. Let us denote $p_{C[i,j]}(x)$ as the (i,j)-th polynomial for $C[i,j]$ and let the row vector representation of the coefficients of $p_{C[i,j]}(x)$ as $\mathbf{p}_{[i,j]}$.

Lemma 3. *If $||\mathbf{A}||_F \leq \eta$ and $||\mathbf{B}||_F \leq \eta$, then,*

$$||\mathbf{p}_{[i,j]}||_2 \leq \sqrt{2m-1}\eta^2. \tag{74}$$

Proof. Throughout the proof, $||\cdot||$ denotes a Frobenius norm for a matrix and a 2-norm for a vector. Let $\mathbf{P}_l$ be the coefficient of x^{l-1} in $p_{\mathbf{C}}(x)$ for $l \in [2m-1]$, which can be written as:

$$\mathbf{P}_l = \sum_{\substack{1 \leq i,j \leq m \\ j-i=m-l}} \mathbf{A}_i \mathbf{B}_j \tag{75}$$

$$= \begin{cases} \sum_{1 \leq i \leq l} \mathbf{A}_i \mathbf{B}_{i+m-l}, & \text{if } l \leq m \\ \sum_{l+1-m \leq i \leq m} \mathbf{A}_i \mathbf{B}_{i+m-l}, & \text{otherwise.} \end{cases} \tag{76}$$

Let us focus on the case when $l \leq m$ as the argument extends naturally for $l > m$. For $l \leq m$, $\mathbf{P}_l$ can be rewritten as:

$$\mathbf{P}_l = \sum_{1 \leq i \leq l} \mathbf{A}_i \mathbf{B}_{i+m-l} = [\mathbf{A}_1 \quad \cdots \quad \mathbf{A}_l] \cdot \begin{bmatrix} \mathbf{B}_{m-l+1} \\ \vdots \\ \mathbf{B}_m \end{bmatrix}. \tag{77}$$

As these matrices are submatrices of $\mathbf{A}$ and $\mathbf{B}$,

$$||\,[\mathbf{A}_1 \quad \cdots \quad \mathbf{A}_l]\,|| \leq \eta, \ \ ||\begin{bmatrix} \mathbf{B}_{m-l+1} \\ \vdots \\ \mathbf{B}_m \end{bmatrix}|| \leq \eta. \tag{78}$$

Since $||\mathbf{XY}|| \leq ||\mathbf{X}|| \cdot ||\mathbf{Y}||$, we have:

$$||\sum_{1 \leq i \leq l} \mathbf{A}_i \mathbf{B}_{i+m-l}|| \leq \eta^2. \tag{79}$$

We can apply the same argument for $l > m$ and show:

$$||\mathbf{P}_l|| = ||\sum_{\substack{1 \leq i,j \leq m \\ j-i=m-l}} \mathbf{A}_i \mathbf{B}_j|| \leq \eta^2 \quad \text{for } l \in [2m-1]. \tag{80}$$

Finally,

$$||\mathbf{p}_{[i,j]}|| = ||\,[P_1[i,j] \quad P_2[i,j] \quad \cdots \quad P_{2m-1}[i,j]]\,|| \tag{81}$$

$$= \sqrt{\sum_{l=1}^{2m-1} P_l[i,j]^2} \leq \sqrt{\sum_{l=1}^{2m-1} ||\mathbf{P}_l||^2} \tag{82}$$

$$\leq \sqrt{2m-1}\eta^2. \tag{83}$$

□

Algorithm 1 (Decoding of Approximate MatDot codes)**.** *Let $\boldsymbol{\lambda}^{(succ)}$ be a length-K vector with evaluation points at K successful worker nodes:*

$$\boldsymbol{\lambda}^{(succ)} = [\lambda_{i_1}, \cdots, \lambda_{i_K}], \tag{84}$$

and let $\mathbf{V}^{(succ)} = \mathsf{Vander}(\boldsymbol{\lambda}^{(succ)}, 2m-2)$. Finally, we denote $\mathbf{y}^{(succ)}_{[i,j]}$ as the evaluations of $p_{C[i,j]}(x)$ at $\boldsymbol{\lambda}^{(succ)}$, i.e.,

$$\mathbf{y}^{(succ)}_{[i,j]} = \begin{bmatrix} \widetilde{C}_{i_1}[i,j] & \cdots \widetilde{C}_{i_K}[i,j] \end{bmatrix}. \tag{85}$$

For decoding $C[i,j]$, we solve the following optimization:

$$\widehat{\mathbf{a}} = \underset{\mathbf{a}\mathbf{V}^{(succ)} = \mathbf{y}^{(succ)}_{[i,j]}}{\operatorname{argmin}} ||\mathbf{a}||_2. \tag{86}$$

If $||\mathbf{a}||_2 > \sqrt{2m-1}\eta^2$, declare failure. Otherwise, $\widehat{C}[i,j] = \widehat{a}[m]$.

Proof of Theoerem 3: First, note that the solution of the equation (86) and the true polynomial coefficients $\mathbf{p}_{[i,j]}$ both lie in $\mathcal{B}_{\sqrt{2m-1}\eta^2}$. As

$$||\lambda^{(\text{succ})}||_\infty < \frac{\epsilon}{6\eta^2\sqrt{2m-1}(m-1)m}, \tag{87}$$

by construction, Corollary 1 gives:

$$|\widehat{a}[l] - p_{[i,j]}[l]| \leq \epsilon \quad \text{for } l \in [m]. \tag{88}$$

Hence,

$$|\widehat{C}[i,j] - C[i,j]| = |\widehat{a}[m] - p_{[i,j]}[m]| \leq \epsilon. \tag{89}$$

APPENDIX C

Proof of Theorem 4. We show a contradiction, i.e., assume $K(m,\epsilon) = m-1,\ \forall \epsilon < \eta^2$. We need to show that there exist matrices $\mathbf{A}, \mathbf{B}$ such that $\epsilon \geq \eta^2$ for a recovery threshold of $m-1$.

Consider f_i and g_i defined in the system model (4) and (5). Let $\boldsymbol{f} = \{f_i\}_{i=1}^P$ and $\boldsymbol{g} = \{g_i\}_{i=1}^P$ be encoding functions for $\mathbf{A}$ and $\mathbf{B}$ respectively. Consider any set $\mathcal{S}$ of $m-1$ nodes. Let $\boldsymbol{f}_\mathcal{S}$, $\boldsymbol{g}_\mathcal{S}$ denote the restriction of $\boldsymbol{f}$, $\boldsymbol{g}$ to the nodes corresponding to $\mathcal{S}$ respectively.

Let $\boldsymbol{C_i}$ = output of i^{th} node, $i \in \mathcal{S}$.

$$\boldsymbol{C_i} = \left(f_i(\{\mathbf{A}_{j,k}\}_{j=1,k=1}^{p,q})\right)\left(g_i(\{\mathbf{B}_{j,k}\}_{j=1,k=1}^{q,p})\right)$$

Let $\boldsymbol{C} = \{\boldsymbol{C_i}\}_{i\in\mathcal{S}}$.

Let $d_\mathcal{S}(\cdot\,; \boldsymbol{f}_\mathcal{S}, \boldsymbol{g}_\mathcal{S})$ denote any decoding function corresponding to the $m-1$ nodes in $\mathcal{S}$ that takes $\boldsymbol{C}$ and gives an estimate of $\mathbf{AB}$. To show a contradiction, we show that there exist matrices $\mathbf{A}, \mathbf{B}$, such that

$$||d_\mathcal{S}(\boldsymbol{C}) - \mathbf{AB}||_F \geq \eta^2 > 0$$

Let **vectorize**$(\cdot)$ be a function that outputs a column-wise vectorization of the input matrix. Let $\boldsymbol{Q} \in \mathbb{R}^{p\times q}$ such that $\sigma_{\max}(\boldsymbol{Q}) = 1$ and, **vectorize**$(\boldsymbol{Q})$ is a null vector of $\boldsymbol{f}_\mathcal{S}$ i.e., $\boldsymbol{f}_\mathcal{S}(\textbf{vectorize}(\boldsymbol{Q}) \otimes \boldsymbol{D}) = \mathbf{0},\ \forall \boldsymbol{D} \in \mathbb{R}^{\frac{n}{p}\times\frac{n}{q}}$. Note that we can scale any such $\boldsymbol{Q}$, such that its maximum singular value is 1.

Let $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ be some constant matrices with bounded frobenius norms.

We set

$$\mathbf{A} = \boldsymbol{Q} \otimes \bar{\mathbf{A}},\ \bar{\mathbf{A}} \in \mathbb{R}^{\frac{n}{p}\times\frac{n}{q}}$$

and

$$\mathbf{B}^{(b)} = b\boldsymbol{Q}^T \otimes \bar{\mathbf{B}},\ \bar{\mathbf{B}} \in \mathbb{R}^{\frac{n}{q}\times\frac{n}{p}},\ b \in \mathbb{R}\backslash\{0\}$$

Note: $\mathbf{AB}^{(b)} = b\bar{\mathbf{A}}\bar{\mathbf{B}}$. Also observe that, $||\bar{\mathbf{A}}||_F \leq \eta$ and $||\bar{\mathbf{B}}||_F \leq \frac{\eta}{|b|}$.

Let

$$\boldsymbol{C}_i^{(b)} = \left(f_i(\{\mathbf{A}_{j,k}\}_{j=1,k=1}^{p,q})\right)\left(g_i(\{\mathbf{B}_{j,k}^{(b)}\}_{j=1,k=1}^{q,p})\right)$$

Let $\boldsymbol{C}^{(b)} = \{\boldsymbol{C}_i^{(b)}\}_{i\in\mathcal{S}}$. By construction $\boldsymbol{C}^{(b)} = \mathbf{0} \implies d_\mathcal{S}(\boldsymbol{C}^{(1)}) = d_\mathcal{S}(\boldsymbol{C}^{(-1)}) = d_\mathcal{S}(\mathbf{0})$. Then by triangle inequality,

$$\begin{aligned} ||d_\mathcal{S}(\boldsymbol{C}^{(1)}) - \mathbf{AB}^{(1)}||_F + ||d_\mathcal{S}(\boldsymbol{C}^{(-1)}) - \mathbf{AB}^{(-1)}||_F \\ \geq ||\mathbf{AB}^{(1)} - \mathbf{AB}^{(-1)}||_F \\ \geq 2||\mathbf{AB}^{(1)}||_F \end{aligned}$$

$$\max(||d_\mathcal{S}(\mathbf{0}) - \mathbf{AB}||_F, ||d_\mathcal{S}(\mathbf{0}) + \mathbf{AB}||_F) \geq ||\mathbf{AB}^{(1)}||_F$$

$$\begin{aligned} ||\mathbf{AB}^{(1)}||_F &= ||(\boldsymbol{Q}\otimes\bar{\mathbf{A}})(\boldsymbol{Q}^T\otimes\bar{\mathbf{B}})||_F \\ &= ||(\boldsymbol{Q}\boldsymbol{Q}^T)\otimes(\bar{\mathbf{A}}\bar{\mathbf{B}})||_F \\ &= \sqrt{\text{Tr}((\boldsymbol{Q}\boldsymbol{Q}^T\otimes\bar{\mathbf{B}}^T\bar{\mathbf{A}}^T)(\boldsymbol{Q}\boldsymbol{Q}^T\otimes\bar{\mathbf{A}}\bar{\mathbf{B}}))} \\ &= \sqrt{\text{Tr}((\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{Q}^T)\otimes(\bar{\mathbf{B}}^T\bar{\mathbf{A}}^T\bar{\mathbf{A}}\bar{\mathbf{B}}))} \\ &= \sqrt{\text{Tr}(\boldsymbol{Q}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{Q}^T)}\sqrt{\text{Tr}(\bar{\mathbf{B}}^T\bar{\mathbf{A}}^T\bar{\mathbf{A}}\bar{\mathbf{B}})} \\ &= ||\boldsymbol{Q}\boldsymbol{Q}^T||_F||\bar{\mathbf{A}}\bar{\mathbf{B}}||_F \end{aligned}$$

We can find matrices $\bar{\mathbf{A}}, \bar{\mathbf{B}}$ such that $||\bar{\mathbf{A}}\bar{\mathbf{B}}||_F = \eta^2$ due to Lemma 4.

$$\begin{aligned} ||\mathbf{AB}^{(1)}||_F &= ||\boldsymbol{Q}\boldsymbol{Q}^T||_F\,\eta^2 \\ &\geq ||\boldsymbol{Q}\boldsymbol{Q}^T||_2\,\eta^2 \\ &= ||\boldsymbol{Q}||_2^2\,\eta^2 \\ &= \eta^2 \end{aligned}$$

Therefore

$$\max(||d_\mathcal{S}(\mathbf{0}) - \mathbf{AB}||_F, ||d_\mathcal{S}(\mathbf{0}) + \mathbf{AB}||_F) \geq \eta^2$$

□

Therefore, we can say that there exists matrices $\mathbf{A}$ and ($\mathbf{B}^{(1)}$ or $\mathbf{B}^{(-1)}$) such that given $\eta = 1$, the decoding error is ≥ 1, when recovery threshold is set to $m-1$, showing that the decoding error cannot be taken arbitrarily small.

Lemma 4. *Choose $\bar{\mathbf{A}} = \boldsymbol{x}\boldsymbol{y}^T$ and $\bar{\mathbf{B}} = \boldsymbol{y}\boldsymbol{z}^T$, then $||\bar{\mathbf{A}}\bar{\mathbf{B}}||_F = ||\bar{\mathbf{A}}||_F||\bar{\mathbf{B}}||_F$.*

Proof.

$$\begin{aligned} ||\bar{\mathbf{A}}\bar{\mathbf{B}}||_F &= ||\boldsymbol{y}||^2||\boldsymbol{x}\boldsymbol{z}^T||_F \\ &= ||\boldsymbol{y}||^2\sqrt{\text{Tr}\left(\boldsymbol{z}\boldsymbol{x}^T\boldsymbol{x}\boldsymbol{z}^T\right)} \\ &= ||\boldsymbol{y}||^2||\boldsymbol{x}||||\boldsymbol{z}|| \\ ||\bar{\mathbf{A}}||_F||\bar{\mathbf{B}}||_F &= \sqrt{\text{Tr}\left(\boldsymbol{y}\boldsymbol{x}^T\boldsymbol{x}\boldsymbol{y}^T\right)}\sqrt{\text{Tr}\left(\boldsymbol{z}\boldsymbol{y}^T\boldsymbol{y}\boldsymbol{z}^T\right)} \\ &= ||\boldsymbol{y}||^2||\boldsymbol{x}||||\boldsymbol{z}|| \end{aligned}$$

□

APPENDIX D
PROOF OF THEOREM 5

Proof. Let $\mathbf{E} = \mathbf{I}_{m\times m} - \sum_{i\in\mathcal{S}_p} d_i^{(p)} \boldsymbol{\alpha}^{(i)}\boldsymbol{\beta}^{(i)T}$. $||\cdot||$ here represent Frobenius norm.

$$
\begin{aligned}
||\mathbf{C} - \widehat{\mathbf{C}}_{\mathcal{S}_p}|| &= ||\sum_{i,j} E[i,j]\mathbf{A}_i\mathbf{B}_j|| && (90)\\
&\leq \sum_{i,j} |E[i,j]|||\mathbf{A}_i\mathbf{B}_j|| && (91)\\
&\leq \sum_{i,j} |E[i,j]|||\mathbf{AB}|| && (92)\\
&\leq m||\mathbf{E}||\cdot||\mathbf{AB}|| && (93)\\
&\leq m||\mathbf{E}||\cdot||\mathbf{A}||\cdot||\mathbf{B}|| && (94)\\
&= m\sqrt{\ell^{(p)}}\eta^2. && (95)
\end{aligned}
$$

□