Simplex Algorithm Implementation

Àlex Batlle Casellas (39413434J) & Jose Pérez Cano(26053623M)

November 2020

1 Data sets

We were assigned the 7th and 47th data sets. These data sets can be found (albeit a little modified to be readable by our code) in the `data` directory within the source files. If you want to use a different data set, please abide by the format presented in the file `data_model`.

2 Description

We have implemented the Primal Simplex Algorithm in `C++` using both phase I and phase II within the same code, so there is no need to run a problem twice through the algorithm to get a solution. With regards to taxation and the treatment of degeneracy, we have implemented both the Bland rule and the most-negative-reduced-cost rule, but in the code we only use the Bland rule as it guarantees the convergence of the algorithm even with degenerate problems. One may change this by modifying `src/simplex.cc`.

We provide the executables for Linux, MacOS Catalina and Microsoft Windows 10. If you have another operating system simply go to the terminal, enter the directory of the project in the terminal and type `make clean` and `make`. It handles all the libraries automatically. If you want to see the code, the simplex algorithm is in `src/simplex.cc` while the other files are matrix and vector functions to simplify the code.

3 Solutions

Below we present the solutions to the eight problems we were assigned, together with some important information from each iteration. The first four problems are Àlex's (Student 7), and the four last problems are Jose's (Student 47).

3.1 PL1

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]         Phase I
[CppLEX]  iter 1: q = 0, rq = -234.000, B(p) = 21, θ* = 0.967, z=2754.800
[CppLEX]  iter 2: q = 1, rq = -295.700, B(p) = 20, θ* = 2.786, z=1931.068
[CppLEX]  iter 3: q = 2, rq = -424.703, B(p) = 29, θ* = 0.110, z=1884.442
[CppLEX]  iter 4: q = 4, rq = -189.573, B(p) = 27, θ* = 1.021, z=1690.876
[CppLEX]  iter 5: q = 3, rq = -94.185, B(p) = 2, θ* = 0.556, z=1638.515
[CppLEX]  iter 6: q = 5, rq = -307.183, B(p) = 4, θ* = 1.605, z=1145.480
[CppLEX]  iter 7: q = 6, rq = -42.480, B(p) = 25, θ* = 0.693, z=1116.024
[CppLEX]  iter 8: q = 8, rq = -1230.171, B(p) = 26, θ* = 0.185, z=888.625
[CppLEX]  iter 9: q = 2, rq = -45.311, B(p) = 28, θ* = 1.373, z=826.432
[CppLEX]  iter 10: q = 9, rq = -212.541, B(p) = 23, θ* = 0.181, z=788.039
[CppLEX]  iter 11: q = 4, rq = -206.645, B(p) = 0, θ* = 1.084, z=564.133
[CppLEX]  iter 12: q = 10, rq = -159.571, B(p) = 2, θ* = 2.162, z=219.114
[CppLEX]  iter 13: q = 0, rq = -232.968, B(p) = 9, θ* = 0.051, z=207.198
[CppLEX]  iter 14: q = 7, rq = -412.035, B(p) = 0, θ* = 0.085, z=172.039
[CppLEX]  iter 15: q = 11, rq = -502.595, B(p) = 22, θ* = 0.178, z=82.349
[CppLEX]  iter 16: q = 0, rq = -159.947, B(p) = 24, θ* = 0.515, z=0.000
```

```
[CppLEX]  iter 17: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z = 0.000
[CppLEX]  Found initial SBF at iteration 17
[CppLEX]        Phase II
[CppLEX]  iter 18: q = 9, rq = −21.356, B(p) = 0, θ* = 0.670, z=169.095
[CppLEX]  iter 19: q = 13, rq = −74.552, B(p) = 11, θ* = 0.988, z=95.456
[CppLEX]  iter 20: q = 0, rq = −53.987, B(p) = 8, θ* = 0.272, z=80.756
[CppLEX]  iter 21: q = 2, rq = −24.179, B(p) = 6, θ* = 1.361, z=47.837
[CppLEX]  iter 22: q = 14, rq = −0.437, B(p) = 9, θ* = 50.563, z=25.740
[CppLEX]  iter 23: q = 15, rq = −0.004, B(p) = 14, θ* = 88.742, z=25.364
[CppLEX]  iter 24: q = 16, rq = −0.390, B(p) = 5, θ* = 37.315, z=10.801
[CppLEX]  iter 25: q = 6, rq = −49.943, B(p) = 3, θ* = 1.254, z=−51.844
[CppLEX]  iter 26: q = 5, rq = −7.409, B(p) = 2, θ* = 1.227, z=−60.932
[CppLEX]  iter 27: q = 14, rq = −0.221, B(p) = 15, θ* = 103.796, z=−83.837
[CppLEX]  iter 28: q = 17, rq = −1.012, B(p) = 14, θ* = 53.340, z=−137.803
[CppLEX]  iter 29: q = 18, rq = −0.330, B(p) = 0, θ* = 116.796, z=−176.342
[CppLEX]  iter 30: q = 2, rq = −22.951, B(p) = 1, θ* = 0.862, z=−196.129
[CppLEX]  iter 31: q = 19, rq = −0.826, B(p) = 6, θ* = 152.832, z=−322.373
[CppLEX]  iter 32: q = 12, rq = −0.897, B(p) = 5, θ* = 0.726, z=−323.024
[CppLEX]  iter 33: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z=−323.024
[CppLEX]  Optimal solution found, iteration: 33, z=−323.024
[CppLEX] End of ASP


B* = {2, 4, 13, 7, 17, 12, 18, 16, 10, 19}
xB* = [1.34833, 1.24684, 3.10074, 4.53625, 110.546, 0.726318,
168.06, 167.496, 6.64308, 252.772]
```

3.2 PL2

The problem returned as an unfeasible problem after 6 Simplex phase I iterations.

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]        Phase I
[CppLEX]  iter 1: q = 0, rq = −285.000, B(p) = 29, θ* = 0.045, z=1041.239
[CppLEX]  iter 2: q = 1, rq = −262.985, B(p) = 28, θ* = 0.155, z=1000.523
[CppLEX]  iter 3: q = 14, rq = −1.000, B(p) = 24, θ* = 211.662, z=788.860
[CppLEX]  iter 4: q = 16, rq = −1.000, B(p) = 26, θ* = 128.612, z=660.249
[CppLEX]  iter 5: q = 17, rq = −1.000, B(p) = 27, θ* = 20.508, z=639.741
[CppLEX]  We determined the problem was unfeasible at iteration 6
[CppLEX] End of ASP
```

3.3 PL3

The problem found an optimum after 20 Simplex phase I iterations and 39 phase II iterations. The found solution is $xB* = [17.0718, 2.42863, 291.377, 4.20169, 14.7844, 1.00908, 2.69688, 351.529, 2.70776, 298.141]$ for the basis $B* = \{16, 7, 17, 0, 18, 9, 13, 19, 3, 15\}$

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]        Phase I
[CppLEX]  iter 1: q = 0, rq = −44.000, B(p) = 25, θ* = 0.386, z=2784.000
[CppLEX]  iter 2: q = 1, rq = −350.000, B(p) = 26, θ* = 0.888, z=2473.216
[CppLEX]  iter 3: q = 2, rq = −701.117, B(p) = 21, θ* = 0.298, z=2264.337
[CppLEX]  iter 4: q = 3, rq = −814.919, B(p) = 0, θ* = 0.036, z=2235.137
[CppLEX]  iter 5: q = 4, rq = −722.910, B(p) = 23, θ* = 1.178, z=1383.720
[CppLEX]  iter 6: q = 0, rq = −17.486, B(p) = 22, θ* = 0.183, z=1380.512
[CppLEX]  iter 7: q = 5, rq = −150.885, B(p) = 0, θ* = 0.599, z=1290.111
[CppLEX]  iter 8: q = 6, rq = −187.897, B(p) = 27, θ* = 0.189, z=1254.583
[CppLEX]  iter 9: q = 0, rq = −2076.220, B(p) = 3, θ* = 0.076, z=1097.215
[CppLEX]  iter 10: q = 7, rq = −268.481, B(p) = 2, θ* = 0.184, z=1047.728
```

```
[CppLEX]  iter 11: q = 8, rq = −188.134, B(p) = 6, θ* = 1.093, z=842.155
[CppLEX]  iter 12: q = 11, rq = −1101.338, B(p) = 20, θ* = 0.070, z=765.332
[CppLEX]  iter 13: q = 2, rq = −96.019, B(p) = 7, θ* = 0.415, z=725.498
[CppLEX]  iter 14: q = 3, rq = −39.473, B(p) = 5, θ* = 1.707, z=658.133
[CppLEX]  iter 15: q = 6, rq = −360.932, B(p) = 24, θ* = 0.771, z=379.917
[CppLEX]  iter 16: q = 5, rq = −129.133, B(p) = 3, θ* = 1.873, z=138.091
[CppLEX]  iter 17: q = 7, rq = −60.710, B(p) = 2, θ* = 0.882, z=84.564
[CppLEX]  iter 18: q = 10, rq = −452.139, B(p) = 29, θ* = 0.165, z=9.799
[CppLEX]  iter 19: q = 3, rq = −44.322, B(p) = 28, θ* = 0.221, z=0.000
[CppLEX]  iter 20: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z = 0.000
[CppLEX]  Found initial SBF at iteration 20
[CppLEX]         Phase II
[CppLEX]  iter 21: q = 9, rq = −83.692, B(p) = 0, θ* = 1.635, z=175.330
[CppLEX]  iter 22: q = 12, rq = −86.223, B(p) = 8, θ* = 1.175, z=74.031
[CppLEX]  iter 23: q = 0, rq = −29.461, B(p) = 1, θ* = 0.700, z=53.399
[CppLEX]  iter 24: q = 13, rq = −144.085, B(p) = 0, θ* = 1.222, z=−122.649
[CppLEX]  iter 25: q = 8, rq = −204.665, B(p) = 4, θ* = 0.073, z=−137.573
[CppLEX]  iter 26: q = 15, rq = −0.175, B(p) = 8, θ* = 33.945, z=−143.519
[CppLEX]  iter 27: q = 0, rq = −45.701, B(p) = 11, θ* = 0.521, z=−167.338
[CppLEX]  iter 28: q = 16, rq = −0.110, B(p) = 0, θ* = 115.386, z=−180.020
[CppLEX]  iter 29: q = 8, rq = −13.638, B(p) = 5, θ* = 0.300, z=−184.110
[CppLEX]  iter 30: q = 17, rq = −0.222, B(p) = 8, θ* = 43.172, z=−193.691
[CppLEX]  iter 31: q = 0, rq = −18.367, B(p) = 15, θ* = 2.110, z=−232.452
[CppLEX]  iter 32: q = 5, rq = −45.134, B(p) = 12, θ* = 0.374, z=−249.327
[CppLEX]  iter 33: q = 19, rq = −0.531, B(p) = 5, θ* = 42.508, z=−271.913
[CppLEX]  iter 34: q = 15, rq = −0.405, B(p) = 10, θ* = 283.393, z=−386.567
[CppLEX]  iter 35: q = 5, rq = −23.171, B(p) = 16, θ* = 0.216, z=−391.581
[CppLEX]  iter 36: q = 18, rq = −0.642, B(p) = 6, θ* = 4.564, z=−394.509
[CppLEX]  iter 37: q = 10, rq = −2.350, B(p) = 5, θ* = 0.554, z=−395.812
[CppLEX]  iter 38: q = 16, rq = −0.090, B(p) = 10, θ* = 17.072, z=−397.351
[CppLEX]  iter 39: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z=−397.351
[CppLEX]  Optimal solution found, iteration: 39, z=−397.351
[CppLEX] End of ASP


B* = {16, 7, 17, 0, 18, 9, 13, 19, 3, 15}
xB* = [17.0718, 2.42863, 291.377, 4.20169, 14.7844, 1.00908,
2.69688, 351.529, 2.70776, 298.141]
```

3.4 PL4

The problem returned as an unlimited problem after 18 Simplex phase I iterations and 56 phase I + phase II iterations.

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]         Phase I
[CppLEX]  iter 1: q = 0, rq = −429.000, B(p) = 28, θ* = 7.614, z=3891.750
[CppLEX]  iter 2: q = 1, rq = −163.125, B(p) = 30, θ* = 8.789, z=2458.125
[CppLEX]  iter 3: q = 2, rq = −370.688, B(p) = 31, θ* = 0.266, z=2359.418
[CppLEX]  iter 4: q = 3, rq = −214.276, B(p) = 33, θ* = 0.673, z=2215.187
[CppLEX]  iter 5: q = 4, rq = −86.202, B(p) = 2, θ* = 1.581, z=2078.929
[CppLEX]  iter 6: q = 5, rq = −132.108, B(p) = 0, θ* = 0.890, z=1961.290
[CppLEX]  iter 7: q = 2, rq = −1763.920, B(p) = 4, θ* = 0.055, z=1864.461
[CppLEX]  iter 8: q = 6, rq = −163.130, B(p) = 2, θ* = 0.087, z=1850.210
[CppLEX]  iter 9: q = 7, rq = −108.081, B(p) = 5, θ* = 0.884, z=1754.712
[CppLEX]  iter 10: q = 8, rq = −749.787, B(p) = 29, θ* = 0.730, z=1207.418
[CppLEX]  iter 11: q = 0, rq = −157.894, B(p) = 7, θ* = 0.906, z=1064.433
[CppLEX]  iter 12: q = 2, rq = −452.255, B(p) = 24, θ* = 0.553, z=814.558
[CppLEX]  iter 13: q = 7, rq = −147.485, B(p) = 26, θ* = 2.795, z=402.314
```

```
[CppLEX]  iter 14: q = 9, rq = −83.188, B(p) = 1, θ* = 2.908, z=160.420
[CppLEX]  iter 15: q = 10, rq = −68.734, B(p) = 25, θ* = 0.319, z=138.497
[CppLEX]  iter 16: q = 1, rq = −10.054, B(p) = 32, θ* = 1.211, z=126.317
[CppLEX]  iter 17: q = 11, rq = −161.802, B(p) = 27, θ* = 0.781, z=−0.000
[CppLEX]  iter 18: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z = 0.000
[CppLEX]  Found initial SBF at iteration 18
[CppLEX]        Phase II
[CppLEX]  iter 19: q = 5, rq = −5.058, B(p) = 10, θ* = 1.292, z=−757.310
[CppLEX]  iter 20: q = 12, rq = −22.503, B(p) = 1, θ* = 1.543, z=−792.024
[CppLEX]  iter 21: q = 4, rq = −5.615, B(p) = 3, θ* = 0.251, z=−793.433
[CppLEX]  iter 22: q = 10, rq = −10.465, B(p) = 6, θ* = 0.152, z=−795.028
[CppLEX]  iter 23: q = 3, rq = −2.138, B(p) = 11, θ* = 1.605, z=−798.459
[CppLEX]  iter 24: q = 14, rq = −0.974, B(p) = 3, θ* = 115.995, z=−911.435
[CppLEX]  iter 25: q = 1, rq = −9.697, B(p) = 2, θ* = 0.707, z=−918.292
[CppLEX]  iter 26: q = 6, rq = −75.387, B(p) = 9, θ* = 1.261, z=−1013.362
[CppLEX]  iter 27: q = 2, rq = −115.675, B(p) = 4, θ* = 0.805, z=−1106.436
[CppLEX]  iter 28: q = 16, rq = −0.466, B(p) = 10, θ* = 80.530, z=−1143.955
[CppLEX]  iter 29: q = 15, rq = −0.410, B(p) = 5, θ* = 2.748, z=−1145.081
[CppLEX]  iter 30: q = 17, rq = −0.730, B(p) = 15, θ* = 23.847, z=−1162.491
[CppLEX]  iter 31: q = 9, rq = −205.996, B(p) = 16, θ* = 0.133, z=−1189.802
[CppLEX]  iter 32: q = 10, rq = −20.778, B(p) = 9, θ* = 0.100, z=−1191.870
[CppLEX]  iter 33: q = 18, rq = −0.283, B(p) = 10, θ* = 11.420, z=−1195.098
[CppLEX]  iter 34: q = 19, rq = −0.437, B(p) = 18, θ* = 11.564, z=−1200.155
[CppLEX]  iter 35: q = 15, rq = −0.436, B(p) = 7, θ* = 23.215, z=−1210.287
[CppLEX]  iter 36: q = 20, rq = −0.915, B(p) = 2, θ* = 58.813, z=−1264.128
[CppLEX]  iter 37: q = 16, rq = −0.557, B(p) = 15, θ* = 307.975, z=−1435.580
[CppLEX]  iter 38: q = 2, rq = −469.316, B(p) = 0, θ* = 0.062, z=−1464.909
[CppLEX]  iter 39: q = 9, rq = −12.757, B(p) = 2, θ* = 0.021, z=−1465.177
[CppLEX]  iter 40: q = 18, rq = −0.730, B(p) = 9, θ* = 4.549, z=−1468.495
[CppLEX]  iter 41: q = 7, rq = −0.807, B(p) = 17, θ* = 0.798, z=−1469.139
[CppLEX]  iter 42: q = 15, rq = −1.830, B(p) = 12, θ* = 8.213, z=−1484.168
[CppLEX]  iter 43: q = 17, rq = −0.546, B(p) = 7, θ* = 44.197, z=−1508.298
[CppLEX]  iter 44: q = 21, rq = −0.440, B(p) = 15, θ* = 74.711, z=−1541.169
[CppLEX]  iter 45: q = 9, rq = −27.315, B(p) = 16, θ* = 1.617, z=−1585.328
[CppLEX]  iter 46: q = 22, rq = −2.175, B(p) = 18, θ* = 201.477, z=−2023.555
[CppLEX]  iter 47: q = 0, rq = −1372.954, B(p) = 9, θ* = 0.012, z=−2040.287
[CppLEX]  iter 48: q = 5, rq = −15.003, B(p) = 0, θ* = 0.028, z=−2040.711
[CppLEX]  iter 49: q = 7, rq = −241.197, B(p) = 5, θ* = 0.012, z=−2043.691
[CppLEX]  iter 50: q = 2, rq = −24.174, B(p) = 8, θ* = 2.559, z=−2105.542
[CppLEX]  iter 51: q = 15, rq = −3.273, B(p) = 2, θ* = 498.130, z=−3735.753
[CppLEX]  iter 52: q = 0, rq = −297.410, B(p) = 7, θ* = 1.541, z=−4194.146
[CppLEX]  iter 53: q = 16, rq = −7.215, B(p) = 1, θ* = 526.948, z=−7996.272
[CppLEX]  iter 54: q = 7, rq = −376.927, B(p) = 0, θ* = 2.347, z=−8880.787
[CppLEX]  iter 55: q = 23, rq = −11.292, B(p) = 7, θ* = 329.000, z=−12596.000
[CppLEX]  We determined the problem was unlimited at the iteration 56
[CppLEX] End of ASP


B* = {16, 17, 23, 14, 22, 15, 6, 19, 21, 20}
xB* = [1161, 9018, 329, 3338, 6855, 3639, 134, 7958, 8602, 986]
```

3.5 PL5

The problem found an optimum after 22 Simplex phase I iterations and 35 phase II iterations. The found solution is $xB* = [0.753349, 5.02245, 3.2615, 1.80148, 2.10122, 729.43, 266.26, 121.359, 0.402646, 0.0646541]$ for the basis $B* = \{0, 13, 1, 4, 6, 19, 15, 18, 5, 11\}$

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]        Phase I
```

```
[CppLEX]  iter 1: q = 0, rq = −205.000, B(p) = 21, θ* = 0.757, z=2269.786
[CppLEX]  iter 2: q = 1, rq = −54.286, B(p) = 0, θ* = 0.883, z=2221.833
[CppLEX]  iter 3: q = 2, rq = −104.667, B(p) = 28, θ* = 0.757, z=2142.577
[CppLEX]  iter 4: q = 3, rq = −408.539, B(p) = 2, θ* = 0.465, z=1952.627
[CppLEX]  iter 5: q = 4, rq = −348.229, B(p) = 23, θ* = 0.521, z=1771.289
[CppLEX]  iter 6: q = 5, rq = −205.179, B(p) = 25, θ* = 0.504, z=1667.959
[CppLEX]  iter 7: q = 0, rq = −1329.369, B(p) = 22, θ* = 0.252, z=1332.659
[CppLEX]  iter 8: q = 6, rq = −515.866, B(p) = 27, θ* = 0.209, z=1224.741
[CppLEX]  iter 9: q = 2, rq = −694.973, B(p) = 0, θ* = 0.072, z=1174.566
[CppLEX]  iter 10: q = 7, rq = −219.848, B(p) = 26, θ* = 0.644, z=1033.014
[CppLEX]  iter 11: q = 0, rq = −143.728, B(p) = 2, θ* = 0.626, z=943.085
[CppLEX]  iter 12: q = 8, rq = −1211.579, B(p) = 3, θ* = 0.019, z=919.554
[CppLEX]  iter 13: q = 2, rq = −229.574, B(p) = 1, θ* = 0.363, z=836.235
[CppLEX]  iter 14: q = 9, rq = −32.553, B(p) = 6, θ* = 2.593, z=751.816
[CppLEX]  iter 15: q = 10, rq = −331.577, B(p) = 24, θ* = 1.182, z=359.803
[CppLEX]  iter 16: q = 3, rq = −110.382, B(p) = 7, θ* = 0.900, z=260.510
[CppLEX]  iter 17: q = 6, rq = −90.016, B(p) = 0, θ* = 0.705, z=197.089
[CppLEX]  iter 18: q = 11, rq = −108.926, B(p) = 29, θ* = 1.525, z=30.982
[CppLEX]  iter 19: q = 7, rq = −139.013, B(p) = 9, θ* = 0.060, z=22.696
[CppLEX]  iter 20: q = 12, rq = −73.755, B(p) = 7, θ* = 0.277, z=2.259
[CppLEX]  iter 21: q = 0, rq = −167.125, B(p) = 20, θ* = 0.014, z=−0.000
[CppLEX]  iter 22: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z = 0.000
[CppLEX]  Found initial SBF at iteration 22
[CppLEX]        Phase II
[CppLEX]  iter 23: q = 1, rq = −26.435, B(p) = 6, θ* = 0.998, z=101.105
[CppLEX]  iter 24: q = 7, rq = −252.543, B(p) = 5, θ* = 0.325, z=18.991
[CppLEX]  iter 25: q = 6, rq = −106.812, B(p) = 10, θ* = 0.474, z=−31.599
[CppLEX]  iter 26: q = 13, rq = −217.066, B(p) = 2, θ* = 0.371, z=−112.099
[CppLEX]  iter 27: q = 5, rq = −141.027, B(p) = 8, θ* = 1.285, z=−293.330
[CppLEX]  iter 28: q = 15, rq = −0.068, B(p) = 12, θ* = 60.970, z=−297.455
[CppLEX]  iter 29: q = 18, rq = −0.300, B(p) = 15, θ* = 23.656, z=−304.555
[CppLEX]  iter 30: q = 9, rq = −9.877, B(p) = 3, θ* = 0.581, z=−310.294
[CppLEX]  iter 31: q = 17, rq = −0.039, B(p) = 0, θ* = 157.608, z=−316.380
[CppLEX]  iter 32: q = 19, rq = −0.315, B(p) = 7, θ* = 338.310, z=−423.099
[CppLEX]  iter 33: q = 0, rq = −18.459, B(p) = 17, θ* = 0.722, z=−436.426
[CppLEX]  iter 34: q = 15, rq = −0.225, B(p) = 9, θ* = 266.260, z=−496.397
[CppLEX]  iter 35: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z=−496.397
[CppLEX]  Optimal solution found, iteration: 35, z=−496.397
[CppLEX] End of ASP


B* = {0, 13, 1, 4, 6, 19, 15, 18, 5, 11}
xB* = [0.753349, 5.02245, 3.2615, 1.80148, 2.10122, 729.43,
266.26, 121.359, 0.402646, 0.0646541]
```

3.6 PL6

The problem found an optimum after 24 Simplex phase I iterations and 32 phase II iterations. The found solution is $xB* = [1.30562, 0.50931, 4.73536, 3.48015, 13.1357, 0.900593, 1.90388, 525.086, 679.748, 1.33937]$ for the basis $B* = \{7, 13, 3, 12, 15, 8, 6, 18, 16, 11\}$

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]        Phase I
[CppLEX]  iter 1: q = 0, rq = −318.000, B(p) = 21, θ* = 0.241, z=1553.241
[CppLEX]  iter 2: q = 1, rq = −281.414, B(p) = 27, θ* = 0.090, z=1528.025
[CppLEX]  iter 3: q = 2, rq = −269.687, B(p) = 1, θ* = 0.269, z=1455.361
[CppLEX]  iter 4: q = 3, rq = −379.188, B(p) = 22, θ* = 0.082, z=1424.190
[CppLEX]  iter 5: q = 5, rq = −500.663, B(p) = 2, θ* = 0.328, z=1260.172
[CppLEX]  iter 6: q = 6, rq = −919.063, B(p) = 25, θ* = 0.015, z=1246.582
```

```
[CppLEX]  iter 7: q = 2, rq = −148.482, B(p) = 20, θ* = 0.011, z=1244.983
[CppLEX]  iter 8: q = 7, rq = −92.418, B(p) = 2, θ* = 0.020, z=1243.101
[CppLEX]  iter 9: q = 8, rq = −522.477, B(p) = 7, θ* = 0.024, z=1230.580
[CppLEX]  iter 10: q = 9, rq = −2358.496, B(p) = 6, θ* = 0.002, z=1226.766
[CppLEX]  iter 11: q = 2, rq = −2675.560, B(p) = 3, θ* = 0.027, z=1154.168
[CppLEX]  iter 12: q = 4, rq = −437.698, B(p) = 28, θ* = 0.009, z=1150.175
[CppLEX]  iter 13: q = 1, rq = −499.027, B(p) = 26, θ* = 0.400, z=950.390
[CppLEX]  iter 14: q = 3, rq = −78.261, B(p) = 2, θ* = 0.072, z=944.762
[CppLEX]  iter 15: q = 6, rq = −151.111, B(p) = 1, θ* = 0.231, z=909.906
[CppLEX]  iter 16: q = 7, rq = −193.782, B(p) = 8, θ* = 0.360, z=840.141
[CppLEX]  iter 17: q = 1, rq = −252.494, B(p) = 24, θ* = 1.500, z=461.419
[CppLEX]  iter 18: q = 2, rq = −131.093, B(p) = 4, θ* = 0.164, z=439.860
[CppLEX]  iter 19: q = 8, rq = −126.414, B(p) = 2, θ* = 0.248, z=408.466
[CppLEX]  iter 20: q = 10, rq = −116.743, B(p) = 8, θ* = 0.457, z=355.173
[CppLEX]  iter 21: q = 11, rq = −290.726, B(p) = 29, θ* = 0.554, z=194.140
[CppLEX]  iter 22: q = 8, rq = −92.484, B(p) = 9, θ* = 1.806, z=27.096
[CppLEX]  iter 23: q = 12, rq = −182.368, B(p) = 23, θ* = 0.149, z=0.000
[CppLEX]  iter 24: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z = 0.000
[CppLEX]  Found initial SBF at iteration 24
[CppLEX]        Phase II
[CppLEX]  iter 25: q = 2, rq = −73.006, B(p) = 5, θ* = 0.690, z=−102.704
[CppLEX]  iter 26: q = 16, rq = −0.401, B(p) = 10, θ* = 77.680, z=−133.876
[CppLEX]  iter 27: q = 5, rq = −22.573, B(p) = 2, θ* = 0.445, z=−143.924
[CppLEX]  iter 28: q = 18, rq = −0.371, B(p) = 5, θ* = 221.645, z=−226.197
[CppLEX]  iter 29: q = 9, rq = −38.866, B(p) = 0, θ* = 0.991, z=−264.725
[CppLEX]  iter 30: q = 13, rq = −71.309, B(p) = 9, θ* = 0.530, z=−302.537
[CppLEX]  iter 31: q = 15, rq = −0.162, B(p) = 1, θ* = 13.136, z=−304.659
[CppLEX]  iter 32: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z=−304.659
[CppLEX]  Optimal solution found, iteration: 32, z=−304.659
[CppLEX] End of ASP


B* = {7, 13, 3, 12, 15, 8, 6, 18, 16, 11}
xB* = [1.30562, 0.50931, 4.73536, 3.48015, 13.1357, 0.900593,
1.90388, 525.086, 679.748, 1.33937]
```

3.7 PL7

The problem returned as an unlimited problem after 13 Simplex phase I iterations and 36 phase I + phase II iterations.

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]        Phase I
[CppLEX]  iter 1: q = 0, rq = −649.000, B(p) = 33, θ* = 7.465, z=2492.444
[CppLEX]  iter 2: q = 1, rq = −348.667, B(p) = 24, θ* = 3.274, z=1350.959
[CppLEX]  iter 3: q = 2, rq = −243.729, B(p) = 31, θ* = 2.596, z=718.162
[CppLEX]  iter 4: q = 4, rq = −116.776, B(p) = 28, θ* = 0.193, z=695.634
[CppLEX]  iter 5: q = 5, rq = −116.161, B(p) = 27, θ* = 1.599, z=509.908
[CppLEX]  iter 6: q = 3, rq = −131.171, B(p) = 25, θ* = 0.405, z=456.734
[CppLEX]  iter 7: q = 7, rq = −131.880, B(p) = 26, θ* = 2.088, z=181.392
[CppLEX]  iter 8: q = 6, rq = −10.369, B(p) = 5, θ* = 1.204, z=168.912
[CppLEX]  iter 9: q = 8, rq = −59.757, B(p) = 32, θ* = 0.573, z=134.674
[CppLEX]  iter 10: q = 9, rq = −77.372, B(p) = 30, θ* = 0.744, z=77.093
[CppLEX]  iter 11: q = 5, rq = −10.973, B(p) = 1, θ* = 3.419, z=39.572
[CppLEX]  iter 12: q = 10, rq = −14.884, B(p) = 29, θ* = 2.659, z=−0.000
[CppLEX]  iter 13: q = 0, rq = 0.000, B(p) = 0, θ* = 0.000, z = 0.000
[CppLEX]  Found initial SBF at iteration 13
[CppLEX]        Phase II
[CppLEX]  iter 14: q = 11, rq = −71.272, B(p) = 0, θ* = 0.702, z=−614.284
```

```
[CppLEX]  iter 15: q = 12, rq = −530.699, B(p) = 7, θ* = 0.128, z=−682.364
[CppLEX]  iter 16: q = 0, rq = −122.696, B(p) = 10, θ* = 0.556, z=−750.529
[CppLEX]  iter 17: q = 14, rq = −0.598, B(p) = 12, θ* = 30.862, z=−768.982
[CppLEX]  iter 18: q = 1, rq = −14.617, B(p) = 4, θ* = 0.857, z=−781.508
[CppLEX]  iter 19: q = 13, rq = −5.074, B(p) = 3, θ* = 0.540, z=−784.247
[CppLEX]  iter 20: q = 15, rq = −0.842, B(p) = 8, θ* = 76.173, z=−848.384
[CppLEX]  iter 21: q = 3, rq = −39.595, B(p) = 13, θ* = 0.599, z=−872.095
[CppLEX]  iter 22: q = 16, rq = −0.586, B(p) = 2, θ* = 7.622, z=−876.562
[CppLEX]  iter 23: q = 13, rq = −12.624, B(p) = 0, θ* = 1.893, z=−900.455
[CppLEX]  iter 24: q = 17, rq = −0.162, B(p) = 3, θ* = 137.888, z=−922.769
[CppLEX]  iter 25: q = 18, rq = −0.157, B(p) = 5, θ* = 377.931, z=−982.164
[CppLEX]  iter 26: q = 0, rq = −6.959, B(p) = 11, θ* = 1.674, z=−993.813
[CppLEX]  iter 27: q = 5, rq = −11.301, B(p) = 16, θ* = 0.343, z=−997.687
[CppLEX]  iter 28: q = 7, rq = −35.379, B(p) = 9, θ* = 0.158, z=−1003.288
[CppLEX]  iter 29: q = 16, rq = −0.025, B(p) = 5, θ* = 28.667, z=−1004.002
[CppLEX]  iter 30: q = 19, rq = −0.101, B(p) = 6, θ* = 241.891, z=−1028.543
[CppLEX]  iter 31: q = 4, rq = −2.234, B(p) = 0, θ* = 2.758, z=−1034.704
[CppLEX]  iter 32: q = 11, rq = −0.036, B(p) = 4, θ* = 3.690, z=−1034.837
[CppLEX]  iter 33: q = 20, rq = −0.528, B(p) = 13, θ* = 450.739, z=−1273.039
[CppLEX]  iter 34: q = 4, rq = −19.829, B(p) = 11, θ* = 6.207, z=−1396.124
[CppLEX]  iter 35: q = 21, rq = −2.530, B(p) = 4, θ* = 1651.167, z=−5573.700
[CppLEX]  We determined the problem was unlimited at the iteration 36
[CppLEX] End of ASP


B* = {18, 17, 14, 19, 1, 20, 7, 16, 15, 21}
xB* = [9749.07, 11529.5, 7898.62, 1558.76, 123.167, 3722.11,
1.56, 2173.95, 2811.24, 1651.17]
```

3.8 PL8

The problem returned as an unfeasible problem after 16 Simplex phase I iterations.

```
[CppLEX] Start of the ASP with Bland Rule
[CppLEX]         Phase I
[CppLEX]  iter 1: q = 0, rq = −499.000, B(p) = 21, θ* = 0.126, z=995.908
[CppLEX]  iter 2: q = 2, rq = −714.851, B(p) = 28, θ* = 0.050, z=960.009
[CppLEX]  iter 3: q = 8, rq = −86.441, B(p) = 0, θ* = 0.294, z=934.613
[CppLEX]  iter 4: q = 10, rq = −171.520, B(p) = 20, θ* = 0.011, z=932.710
[CppLEX]  iter 5: q = 0, rq = −266.484, B(p) = 2, θ* = 0.030, z=924.601
[CppLEX]  iter 6: q = 11, rq = −216.501, B(p) = 8, θ* = 0.112, z=900.321
[CppLEX]  iter 7: q = 2, rq = −154.390, B(p) = 0, θ* = 0.013, z=898.294
[CppLEX]  iter 8: q = 14, rq = −1.000, B(p) = 24, θ* = 420.260, z=478.034
[CppLEX]  iter 9: q = 0, rq = −46.775, B(p) = 2, θ* = 0.018, z=477.194
[CppLEX]  iter 10: q = 15, rq = −1.000, B(p) = 25, θ* = 86.506, z=390.688
[CppLEX]  iter 11: q = 2, rq = −37.731, B(p) = 0, θ* = 0.013, z=390.192
[CppLEX]  iter 12: q = 17, rq = −1.000, B(p) = 27, θ* = 43.659, z=346.533
[CppLEX]  iter 13: q = 0, rq = −34.983, B(p) = 2, θ* = 0.018, z=345.905
[CppLEX]  iter 14: q = 19, rq = −1.000, B(p) = 29, θ* = 31.637, z=314.268
[CppLEX]  iter 15: q = 21, rq = −1.492, B(p) = 11, θ* = 6.015, z=305.294
[CppLEX]  We determined the problem was unfeasible at iteration 16
[CppLEX] End of ASP
```