Assignment1 Exercise3

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1 Exercise 3.1

The Gaussian pdf is given by:

$$p(x) = N(x|\mu, \sigma^2)$$

With the purpose of deriving the Log-Normal PDF, we need to find the probability density function of y, denoted as $q(y)$.

Using the Change of Variables Formula, which states:

$$q(y) = p(x)\left|\frac{dx}{dy}\right|$$

Perform the following transformations:

$$y = e^x$$

The inverse transformation of y:

$$x = \ln(y)$$

This transformation is bijective since each value of x map to a unique value of y and vice versa.

Now, differentiate x with respect to y:

$$\frac{dx}{dy} = \frac{1}{y}$$

Substitute the above transformation into Change of Variables Formula:

$$q(y) = N(\ln(y)|\mu, \sigma^2)\left|\frac{1}{y}\right|$$

From probability density function of Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Substitute the expression for $p(x)$ into the equation:

$$q(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}}$$

The resulting expression is the probability density function of a Log-Normal distribution, μ and σ^2. are parameters.

This is the Log-Normal pdf derived from the Gaussian pdf using the given transformation.

2 Exercise 3.2

From the problem description in 3.2, the following data are available

$$
\begin{aligned}
p(e=1) &: \varepsilon = 0.0001 \\
p(b=1) &: \beta = 0.0001 \\
\alpha_b &: 0.99 \\
\alpha_e &: 0.01 \\
f &: 0.001
\end{aligned}
$$

Exhaust all possible permutations:

$$
\begin{aligned}
p(a=1|b=1,e=1) &: \text{Alarm rings when suffer a burglary and an earthquake} \\
p(a=1|b=1,e=0) &: \text{Alarm rings when suffer a burglary but no earthquake} \\
p(a=1|b=0,e=1) &: \text{Alarm rings when suffer an earthquake but no burglary} \\
p(a=1|b=0,e=0) &: \text{Alarm rings but no burglary and no earthquake}
\end{aligned}
$$

$$
\begin{aligned}
p(a=1|b=1,e=1) &= 1-(1-f)(1-\alpha_b)(1-\alpha_e) = 0.9901099 \\
p(a=1|b=1,e=0) &= 1-(1-f)(1-\alpha_b) = 0.99001 \\
p(a=1|b=0,e=1) &= 1-(1-f)(1-\alpha_e) = 0.01099 \\
p(a=1|b=0,e=0) &= f = 0.001
\end{aligned}
$$

According to the Law of Total Probability

$$p(a,b,e,c,r) = p(b)\cdot p(e)\cdot p(a|b,e)\cdot p(r|e)\cdot p(c|a)$$

For ease of presentation, let A denote an event in which the alarm ring and you receive a phone call from a neighbor, and B denote an earthquake occurs, the radio broadcasts the news of the earthquake. And it is knowable $p(A) = p(c=1|a=1) = 1, p(B) = p(r=1|e=1) = 1$. That means if your neighbor call you, the alarm must have gone off, and if the news reports an earthquake, there must be one. Therefore, we have the following expression.

According to Bayes' Theorem, since $p(a=1) = 0.02$

$$p(b,e|a=1) = \frac{p(a=1|b,e)\cdot p(b)\cdot p(e)}{p(a=1)}$$

$$p(b=0,e=0|a=1) = 0.4993$$
$$p(b=1,e=0|a=1) = 0.4947$$
$$p(b=0,e=1|a=1) = 0.0055$$
$$p(b=1,e=1|a=1) = 0.0005$$

there is full probability formula, we can calculate $p(b|e=1,a=1)$

$$P(b=1|a=1) = P(b=1,e=0|a=1) + P(b=1,e=1|a=1) = 0.495$$

$$P(b=0|a=1) = P(b=0,e=0|a=1) + P(b=0,e=1|a=1) = 0.505$$

$$p(a=1) = p(a=1|A) \cdot p(A) + p(a=1|B) \cdot p(B) + p(a=1|\neg A, \neg B) \cdot p(\neg A, \neg B)$$

We could eventually calculate the probability that there is a burglar in your home when there is an earthquake, that is, a,b,e,c,r are all 1:

According to Bayes' Theorem, since $p(e=1|a=1) = 0.0060$

$$p(b=1|e=1,a=1) = \frac{p(e=1,a=1|b=1) \cdot p(b=1)}{p(e=1,a=1)}$$

Substituting the above values

$$p(b=0|e=1,a=1) = 0.92$$

$$p(b=1|e=1,a=1) = 0.08$$

There is an 8% chance that a burglar is in your house.

3 Exercise 3.3

The probability mass function of the Dirichlet distribution is given by:

$$P(x_1, x_2, \ldots, x_{27}|\theta) = \frac{\Gamma(\sum_{k=1}^{27}\alpha_k)}{\prod_{k=1}^{27}\Gamma(\alpha_k)} \prod_{k=1}^{27} \frac{\Gamma(x_k+\alpha_k)}{\Gamma(\alpha_k)}$$

According to the Bayes' Theorem,

$$p(x_{2001} = e|D) = \int p(x_{2001} = e|\theta)p(\theta|D)d\theta$$

$\theta \sim Dir(\alpha_1, ..., \alpha_{27})$follows a Dirichlet distribution with $a_k = 10$ for all k and we observed e 260 times in 2000 samples,

$$p(x_{2001} = e|\theta) = \frac{\Gamma(\sum_{k=1}^{27}\alpha_k)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_{27})} \prod_{k=1}^{27} \frac{\Gamma(x_k+\alpha_k)}{\Gamma(\alpha_k)} \approx 0.1189$$

We observed e 260 times, a 100 times and p 87 times in 2000 samples, to compute $p(x_{2001} = e, x_{2002} = a|D)$,

$$p(x_{2001} = e, x_{2002} = a|D) = \int p(x_{2001} = e, x_{2002} = a|\theta)p(\theta|D)d\theta \approx 0.002070$$