Assignment3 Exercise1

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January 30, 2024

Derivations of the Importance Weights

To observing a hidden Markov model (HMM), we typically assume:

1. Markov Property: The future state depends only on the current state. y_i is independent of $x_{1:i-1}$ when conditioned on x_i.
2. Observational Independence: Observations are independent given the current state. y_i is independent of $y_{1:i-1}$ when conditioned on x_i.
3. Causality, y_i and x_i are independent of $x_{i+1:n}$.
4. Initial State Distribution: $p(x_1)$.
5. State Transition Model: $q(x_n|x_{n-1})$.
6. Observation Model: $p(y_n|x_n)$.

$p(y_{1:n}|x_{1:n})$: The likelihood of observations given the states. Due to the independence assumption, this breaks down into a product of individual observation likelihoods:

$$p(y_{1:n}|x_{1:n}) = \prod_{i=1}^{n} p(y_i|x_i)$$

$p(x_{1:n})$: The prior probability of the state sequence. Using the Markov property:

$$p(x_{1:n}) = p(x_1) \prod_{i=2}^{n} p(x_i|x_{i-1})$$

$q(x_{1:n})$: The proposal distribution from which states are sampled. It can be factored similarly:

$$q(x_{1:n}) = q(x_1) \prod_{i=2}^{n} q(x_i|x_{i-1})$$

Given $w_n = \frac{p(y_{1:n}|x_{1:n})p(x_{1:n})}{q(x_{1:n})}$, substitute the above expressions:

$$w_n = \frac{\prod_{i=1}^{n} p(y_i|x_i) \cdot p(x_1) \prod_{i=2}^{n} p(x_i|x_{i-1})}{q(x_1) \prod_{i=2}^{n} q(x_i|x_{i-1})}$$

$$w_n = \frac{p(y_n|x_n)p(x_n|x_{n-1})}{q(x_n|x_{n-1})} \cdot \frac{\prod_{i=1}^{n-1} p(y_i|x_i) \cdot p(x_1) \prod_{i=2}^{n-1} p(x_i|x_{i-1})}{q(x_1) \prod_{i=2}^{n-1} q(x_i|x_{i-1})}$$

The first part of the product is $\alpha(x_n) = \frac{p(y_n|x_n)p(x_n|x_{n-1})}{q(x_n|x_{n-1})}$, and the second part is w_{n-1}.

Therefore, we have shown that under the simplifying assumptions of a hidden Markov model, the importance weight w_n can be expressed as $w_n = \alpha(x_n)w_{n-1}$, where $\alpha(x_n)$ is defined as $\frac{p(y_n|x_n)p(x_n|x_{n-1})}{q(x_n|x_{n-1})}$ and w_{n-1} is the importance weight.